■ Mathematical Method for Physicists in Mathematica

Complex Variable theory

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Contents

0. Complex Numbers

0.1 Sum and Product

Complex number z is defined by $z = x + i y$ for some real number x and y. By tapping ESCiiESC, you can use imaginary unit i.

```
z1 = x1 + ⅈ y1
Element[{x1, y1}, Reals]
z2 = x2 + ⅈ y2
Element[{x2, y2}, Reals]
```

x1 + ⅈ y1

(x1 | y1) ∈ ℝ

x2 + ⅈ y2

(x2 | y2) ∈ ℝ

Then the sum of two complex number z_1 and z_2 is,

```
Simplify[z1 + z2]
```

x1 + x2 + ⅈ (y1 + y2)

How about product?

```
ComplexExpand[z1 z2]
```

x1 x2 - y1 y2 + i (x2 y1 + x1 y2)

<Remark0.1>

- **ComplexExpand[...]** expands (...) assuming that all variables are real. So it is useful when you calculate the product of complex numbers.

0.2 Properties

The properties of addition and multiplication of complex numbers are same as for real numbers.

• The commutative laws

$$z_1 + z_2 = z_2 + z_1, \qquad z_1 z_2 = z_2 z_1$$

• The associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

• The distributive law

$$z(z_1 + z_2) = zz_1 + zz_2$$

• There exists additive identity 0 and the multiplicative identity 1.

$$z + 0 = z \quad \text{and} \quad z \cdot 1 = z$$

• There exists additive inverse and the multiplicative inverse.

Let $-z = (u + i v)$ be the additive inverse of complex number $z = (x + i y)$. Then $(x + i y) + (u + i v) = 0$

$(x + u) + i(y + v) = 0$ thus, $u = -x$ and $v = -y$

Hence, $(-x - i y)$ is the additive inverse of complex number $(x + i y)$.

For any nonzero complex number $z = (x + i y)$, there is a number z^{-1} such that $z z^{-1} = 1$.

Let $z^{-1} = (u + i v)$ then, $(x + i y)(u + i v) = 1$

$(x u - y v) + i (x v + y u) = 1$ thus, $(x u - y v) = 1$ and $(y u + x v) = 0$

$$\texttt{LinearSolve}\left[\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$\left\{\left\{\frac{x}{x^2 + y^2}\right\}, \left\{-\frac{y}{x^2 + y^2}\right\}\right\}$$

Therefore, $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$

So the multiplicative inverse of $z = (x + i y)$ is $z^{-1} = \left(\frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2}\right)$

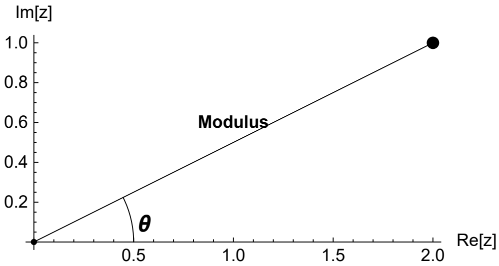
0.3 Vectors and Moduli

You can display a complex number graphically in Argand Diagram(complex plane).
First, define your own ArgandPlot[] function.

```
In[12]:= ArgandPlot[z_List] :=
    Show[Graphics[
        {PointSize[0.03], (Point[{Re[#], Im[#]}]) & /@ N[z],
       PointSize[0.015], Point[{0, 0}]},
        Axes → Automatic,
        AspectRatio → Automatic,
        AxesLabel → {"Re[z]", "Im[z]"}
    ]]
```

Then take a look at case of $2 + i$

```
Show[{ArgandPlot[{2 + i}],
  Graphics[{Line[{{0, 0}, {2, 1}}], Text[Style["Modulus", Bold], {1, 0.6}], Text[
    Style["θ", Bold, Medium], {0.55, 0.1}], Circle[{0, 0}, 0.5, {0, ArcTan[0.5]}]}]}]
```



The modulus or absolute value of complex number ($x+i\,y$) is defined as the nonnegative number $\sqrt{x^2+y^2}$.

In this case, the modulus is

$$\sqrt{2^2+1^2}$$

$$\sqrt{5}$$

or

```
r = Abs[2 + i]
```

$$\sqrt{5}$$

And the angle θ is called principal value of argument (Arg). θ can be determined from the equation $\frac{y}{x} = \tan\theta$. So the general value or argument (arg) is

$$(\text{arg}) = (\text{Arg}) + 2\,n\pi$$

The θ of above figure is

```
Print[{N[ArcTan[1/2]] Text["rad"], N[ArcTan[1/2]] * 180/π Text["degree"]}]
```

{0.463648 rad, 26.5651 degree}

or

```
θ = N[Arg[2 + i]]
```

0.463648

With modulus and argument, you can display a complex number not only in the Cartesian coordinate, but also in the polar coordinate.

$x + i\,y = r\cos\theta + i\,r\sin\theta = r\,(\cos\theta + i\sin\theta) = r\,e^{i\,\theta}$

For the case of 2+i,

```
r Exp[i θ]
```

```
2. + 1. i
```

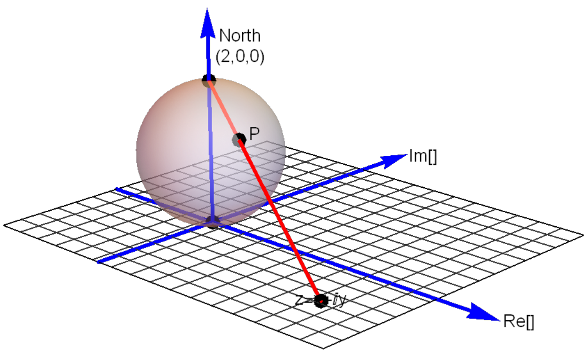
▶ convert to exponential 2. + 1. i

$2.23607\ e^{i\ 0.463648}$

```
Abs[2.23607 e^(i 0.463648)]
```

```
2.23607
```

Not only Argand diagram(complex plane) but also **stereographic projection** can illustrate the complex number by mapping whole complex plane(0 to z_∞) to unit sphere located at the origin of the complex plane(and vice versa). Below figure is sufficient to understand the concept of stereographic projection.



In the figure, the point on sphere $P = (X, Y, Z)$ projected to the point on complex plane $(x, y, 0)$

$$(X, Y, Z-2) = s\,(x, y, -2), \quad \begin{cases} X = s\,x \\ Y = s\,y \\ Z = -2\,s + 2 \end{cases} \quad \text{where } X^2 + Y^2 + (Z-1)^2 = 1$$

Then s will be the solution of $s^2(x^2 + y^2) + (-2\,s + 1)^2 = 1 \longrightarrow s^2\ \mathrm{abs}(z)^2 + (-2\,s + 1)^2 = 1$

```
Solve[Abs[z]^2 s^2 + (-2 s + 1)^2 == 1, s]
```

$$\left\{\{s \to 0\}, \left\{s \to \frac{4}{4 + \mathrm{Abs}[z]^2}\right\}\right\}$$

For the solution $s \to 0$ means north pole, however what we need is the solution $s \to \frac{4}{4+\mathrm{Abs}[z]^2}$

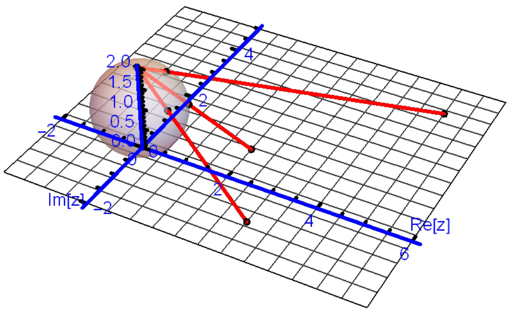
$$\text{Then,} \begin{cases} X = \frac{4\,x}{4+\mathrm{Abs}[z]^2} \\ Y = \frac{4\,y}{4+\mathrm{Abs}[z]^2} \\ Z = \frac{2\,\mathrm{Abs}[z]^2}{4+\mathrm{Abs}[z]^2} \end{cases}$$

It means that all point of infinity(z_∞) mapped to the north pole of the unit sphere.

Making your own function will be helpful.

```
In[13]:= StereoGraphic[z_List] :=
    Show[{Graphics3D[
        {{Opacity[0.5], Ball[{0, 0, 1}]},
     Red, Thick, Line[{{0, 0, 2}, {Re[#], Im[#], 0}}] & /@ N[z],
     Black, PointSize[0.015], (Point[{Re[#], Im[#], 0}]) & /@ N[z],
     (Point[{4 Re[#]/(4 + (Abs[#])^2), 4 Im[#]/(4 + (Abs[#])^2), 2 (Abs[#])^2/(4 + (Abs[#])^2)}]) & /@ N[z],
      PointSize[0.015], Point[{0, 0, 0}], Point[{0, 0, 2}]},
        Boxed → False,
        Axes → Automatic,
    AxesStyle → Directive[{Thick, Blue}],
        AxesOrigin → {0, 0, 0},
        AspectRatio → Automatic,
        PlotRange → Automatic,
        AxesLabel → {"Re[z]", "Im[z]"}]}]

Show[{StereoGraphic[{2 + i, 3 - i, 5 + 4 i}],
  Plot3D[0, {x, -2, 6}, {y, -2, 5}, PlotStyle → None]}]
```



On the sphere, we lose Euclidean geometry.

The stereographic projection was known to Hipparchus, Ptolemy and probably earlier to the Egyptians. One of its most important uses was the representation of celestial charts, since the astronomical phenomena were important in those days.
It is believed that already the map created in 1507 by Gualterius Lud was in stereographic projection, as were later the maps of Jean Roze (1542), Rumold Mercator (1595), and many others.

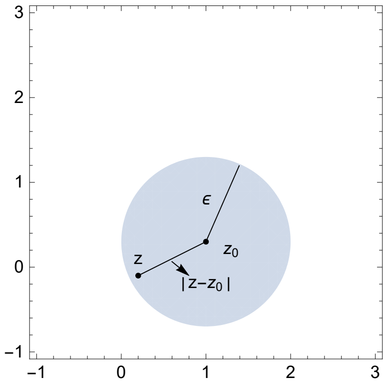
<Remark 0.2>

- **Arg[...]** shows that argument(θ) of the complex number , (arg) = (Arg) + 2 $n\,\pi$
- **Abs[...]** gives you a absolute value(modulus) of given complex number

0.4 Regions In The Complex Plane

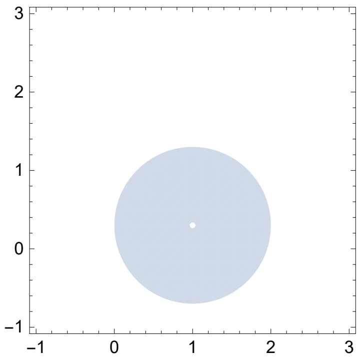
In this section, we will talk about set of complex numbers or points in the complex plane and their closeness to one another with visual examples.

- ***ϵ neighborhood*** **$|z - z_0| < \epsilon$ of a given point z_0**



It does not contain the points on contour

- ***Deleted neighborhood or punctured disk*** $0 < |z - z_0| < \epsilon$



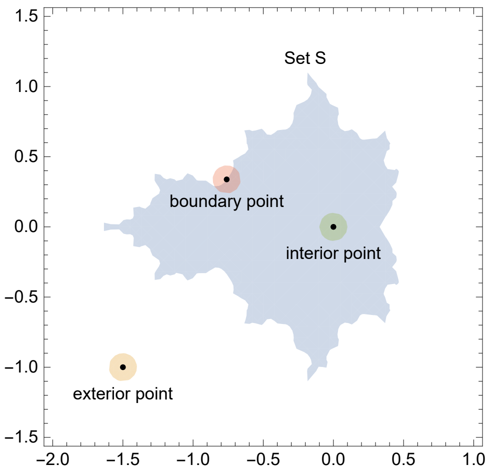
It contains of all points in z in an ϵ neighborhood of z_0 except for z_0 itself.

- ***Interior point* or *exterior point* of set S**

When neighborhood of z_0 contains only points of S $\longrightarrow$ "z_0 is an interior point of a set S"
When neighborhood of z_0 contains no points of S $\longrightarrow$ "z_0 is an exterior point of a set S"

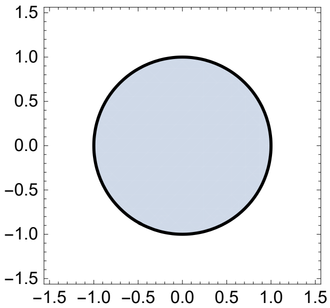
- ***Boundary point of set S***

If point z_0 is neither exterior point nor interior point, then it is a boundary point.



- ***Boundary of S***

Totality of all boundary points



The circle $|z| = 1$ is the boundary of $|z| < 1$ and $|z| \leq 1$

- ***An open set***

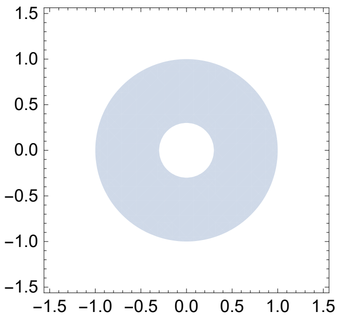
When the set contains none of its boundary points, it is called an open set. $|z| < 1$ is an open set since it does not contain its boundary, $|z| = 1$.

- ***A closed set***

A closed set contains all of its boundary set. $|z| \leq 1$ is a neat example.

- ***Closure* of set S**

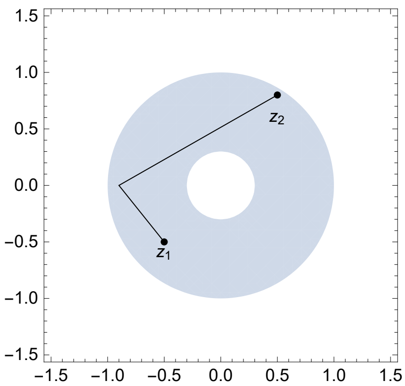
Closed set consisting of all points in S together with the boundary of S. $|z| < 1$ is an open set and $|z| \leq 1$ is its closure.



Some sets are neither open nor closed $0.3 < |z| \leq 1$ for instance, it contains $|z| = 1$ but it does not contain $|z| = 0.3$

- ***Connected* set**

An open set is connected if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end that lies entirely in S



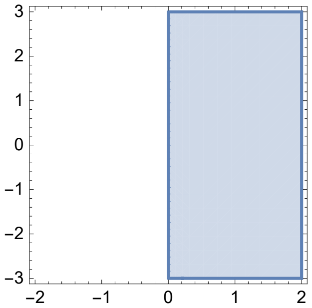
The open set $0.3 < |z| < 1$ is connected. A nonempty open set that is connected is called a *domain*. A domain together with some, none, or all of its boundary points is called *region.*

$\boxed{\boxed{\text{Domain}}} \subset \boxed{\boxed{\text{Region}}}$

- ***Bounded set***

A set S is bounded if every point of S lies inside some circle $|z| = R$. $0.3 < |z| \le 1$ are bounded region

```
RegionPlot[Re[x + ⅈ y] > 0, {x, -2, 2}, {y, -3, 3}]
```

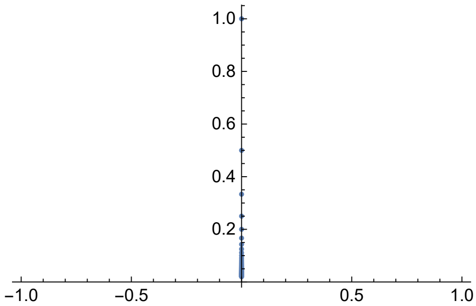


and $\mathrm{Re}[z] \ge 0$ is unbounded.

- ***Accumulation point* of a set S**

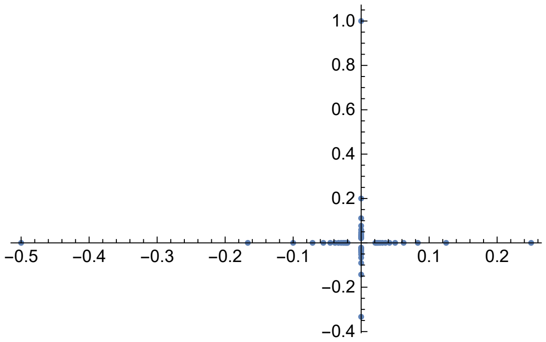
A point z_0 is said to be an accumulation point of a set S if each punctured disk(deleted neighborhood) of z_0 contains at least one point of S.
Let me show you some examples.

```
zn[n_] := ⅈ / n;
ListPlot[Table[{Re[zn[n]], Im[zn[n]]}, {n, 50}], PlotRange → All]
```



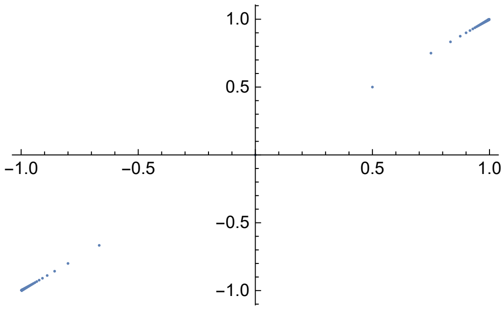
The origin is the only accumulation point of $z_n = i/n$

```
zn[n_] := ⅈ^n / n;
ListPlot[Table[{Re[zn[n]], Im[zn[n]]}, {n, 50}], PlotRange → All]
```



Also, $z_n = i^n/n$ has the origin as its only accumulation point

```
zn[n_] := (-1)^n (1 + i) (n - 1) / n;
ListPlot[Table[{Re[zn[n]], Im[zn[n]]}, {n, 500}], PlotRange → All]
```



$z_n = (-1)^n\,(1+i)\,\frac{(n-1)}{n}$ has two accumulation points; $(1+i)$, $(-1-i)$

<Remark 0.3>

- **RegionPlot[...]** shows you a region that satisfies the given condition (...)

1. Analytic Functions

1.1 Complex Functions

Let’s consider functions of complex variable . Suppose that
$w = u + i\,v$ is the value of a function f at $z = x + i\,y$

$$u + i\,v = f(x + i\,y)$$

Each of the real numbers u and v depends on the real variables x and y. So $f(z)$ can be expressed in terms of a pair of real valued functions of the real variables x and y.

$$f(z) = u(x, y) + i\,v(x, y)$$

In the polar coordinates, $\left(z = r\,e^{i\,\theta}\right)$

$$f(z) = f\left(r\,e^{i\,\theta}\right) = u(r, \theta) + i\,v(r, \theta)$$

If the imaginary part of $f(z)$, v is zero, then the f is a real-valued function of a complex variable.

Exercise.

Write the function $f(z) = \frac{1}{z^2+1}$ in the form $f(z) = u(x, y) + i\,v(x, y)$

Let $z = x + i\,y$ then, $f(z) = \frac{1}{(x+i\,y)^2}$

```
ComplexExpand[1/(x + i y)^2]
```

$$\frac{x^2}{\left(x^2+y^2\right)^2} - \frac{2\,i\,x\,y}{\left(x^2+y^2\right)^2} - \frac{y^2}{\left(x^2+y^2\right)^2}$$

Exercise.

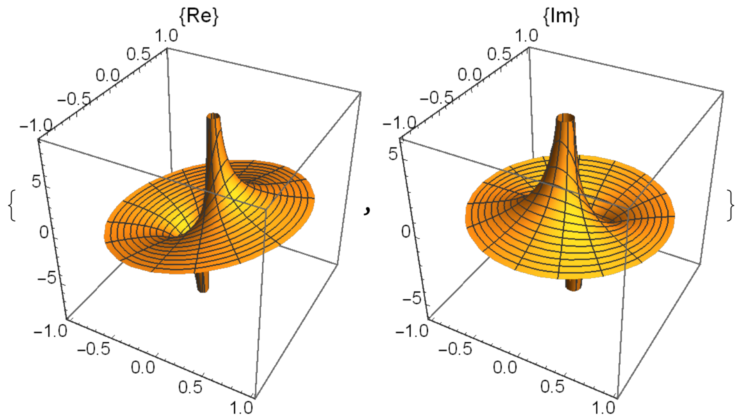
Write the function $f(z) = z + \frac{1}{z}$ in the form $f(z) = u(r, \theta) + i\, v(r, \theta)$

Let $z = r\, e^{i\,\theta}$ then, $f(z) = r\, e^{i\,\theta} + \frac{1}{r} e^{-i\,\theta}$

```
ComplexExpand[r e^(ⅈ θ) + 1/r e^(-ⅈ θ)]
```

$$\frac{\text{Cos}[\theta]}{r} + r\,\text{Cos}[\theta] + i\left(-\frac{\text{Sin}[\theta]}{r} + r\,\text{Sin}[\theta]\right)$$

```
{ParametricPlot3D[{r Cos[θ], r Sin[θ], Cos[θ]/r + r Cos[θ]},
  {r, 0, 1}, {θ, 0, 2 π}, PlotLabel → {Re}, BoxRatios → {1, 1, 1}],
 ParametricPlot3D[{r Cos[θ], r Sin[θ], -Sin[θ]/r + r Sin[θ]},
  {r, 0, 1}, {θ, 0, 2 π}, PlotLabel → {Im}, BoxRatios → {1, 1, 1}]}
```



1.2 Cauchy Riemann Condition

The derivative of function of complex variables is similar to that of its little brother, real function.

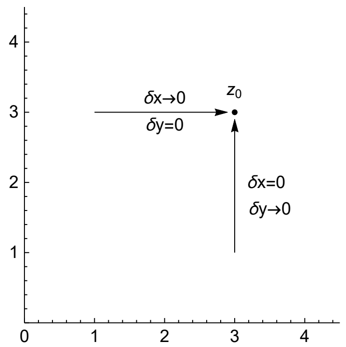
$$\lim_{\delta\to 0} \frac{f(z+\delta z)-f(z)}{(z+\delta z)-(z)} = \lim_{\delta\to 0} \frac{\delta f(z)}{\delta z} = \frac{d f}{d z} = f'(z)$$

Now find the value of limit.

From $z = x + i\,y$, $\delta z = \delta x + i\,\delta y$. Also, from $f = u + i\,v$, $\delta f = \delta u + i\,\delta v$. So that

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y}$$

Now take the limit by two different approaches as shown in following figure.



- $[\delta x \to 0,\ \delta y = 0]$

$$\lim_{\delta z\to 0}\frac{\delta f(z)}{\delta z}=\lim_{\delta x\to 0}\frac{\delta u+i\,\delta v}{\delta x+i\,\delta y}=\lim_{\delta x\to 0}\frac{\delta u+i\,\delta v}{\delta x}=\lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x}+i\,\frac{\delta v}{\delta x}\right)=\frac{\partial u}{\partial x}+i\,\frac{\partial v}{\partial x}$$

- $[\delta x=0,\ \delta y\to 0]$

$$\lim_{\delta z\to 0}\frac{\delta f(z)}{\delta z}=\lim_{\delta y\to 0}\frac{\delta u+i\,\delta v}{\delta x+i\,\delta y}=\lim_{\delta y\to 0}\frac{\delta u+i\,\delta v}{i\,\delta y}=\lim_{\delta y\to 0}\left(-i\,\frac{\delta u}{\delta y}+\frac{\delta v}{\delta y}\right)=-i\,\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

If there exists the derivative of $f(z)$, then the two limits must be same.

$$\frac{\partial u}{\partial x}+i\,\frac{\partial v}{\partial x}=-i\,\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y},\qquad \begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}\\[2mm] \dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y}\end{cases}$$

These are called the Cauchy-Riemann condition. Following diagram will help you to memorize Cauchy-Riemann condition;

$\dfrac{\partial u}{\partial x}$	$-\dfrac{\partial u}{\partial y}$
$\dfrac{\partial v}{\partial x}$	$\dfrac{\partial v}{\partial y}$

Since $\frac{\partial\partial u}{\partial x\partial x}+\frac{\partial\partial u}{\partial y\partial y}=\frac{\partial\partial v}{\partial x\partial y}-\frac{\partial\partial v}{\partial y\partial x}=0$ and $\frac{\partial\partial v}{\partial x\partial x}+\frac{\partial\partial v}{\partial y\partial y}=-\frac{\partial\partial u}{\partial x\partial y}+\frac{\partial\partial u}{\partial y\partial x}=0$,

Thus,

$\nabla^2 u=0$ and $\nabla^2 v=0$ So $u(x, y)$ and $v(x, y)$ are the solution of Laplace equation. In that case, $u(x, y)$ and $v(x, y)$ are referred to as harmonic functions.

<Remark 1.1>

- Cauchy-Riemann condition for $f(z)$ holds and the partial derivative of $u(x, y)$ and $v(x, y)$ is continuous $\Longleftrightarrow$ The derivative df/dz exists.

Exercise.
Find the Cauchy-Riemann condition for polar coordinates

Using $f\left(r\,e^{i\,\theta}\right)=R(r,\,\theta)\,e^{i\,\Theta(r,\theta)}$

$$f\left(r\,e^{i\,\theta}\right)=R(r,\,\theta)\,e^{i\,\Theta(r,\theta)}=R\cos\Theta+i\,R\sin\Theta$$

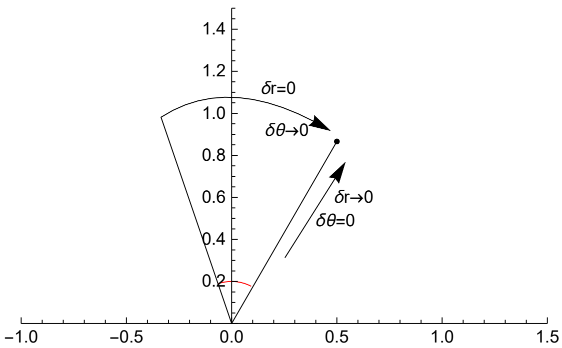
Then the increment of z and f are

$$\delta z=e^{i\,\theta}\,\delta r+i\,r\,e^{i\,\theta}\,\delta\theta=e^{i\,\theta}(\delta r+i\,r\,\delta\theta)\quad\text{and}$$

$$\delta f=\frac{\partial f}{\partial r}\,\delta r+\frac{\partial f}{\partial\theta}\,\delta\theta=\frac{\partial}{\partial r}\left(R\,e^{i\,\Theta}\right)\delta r+\frac{\partial}{\partial\theta}\left(R\,e^{i\,\Theta}\right)\delta\theta$$

$$\delta f=e^{i\,\Theta}\left(\frac{\partial R}{\partial r}+i\,R\,\frac{\partial\Theta}{\partial r}\right)\delta r+e^{i\,\Theta}\left(\frac{\partial R}{\partial\theta}+i\,R\,\frac{\partial\Theta}{\partial\theta}\right)\delta\theta$$

Now consider following routes.



- $[\delta\theta \to 0,\ \delta r = 0]$

$$\lim_{\delta z\to 0}\frac{\delta f(z)}{\delta z} =$$
$$\lim_{\delta z\to 0}\frac{e^{i\theta}(\delta R + i R\,\delta\Theta)}{e^{i\theta}(\delta r + i r\,\delta\theta)} = \lim_{\delta\theta\to 0}\frac{\delta R + i R\,\delta\Theta}{i r\,\delta\theta} = \lim_{\delta\theta\to 0}\left(-i\frac{\delta R}{r\,\delta\theta} + \frac{R\,\delta\Theta}{r\,\delta\theta}\right) = \frac{R}{r}\frac{\partial\Theta}{\partial\theta} - i\frac{1}{r}\frac{\partial R}{\partial\theta}$$

- $[\delta\theta = 0,\ \delta r \to 0]$

$$\lim_{\delta z\to 0}\frac{\delta f(z)}{\delta z} =$$
$$\lim_{\delta z\to 0}\frac{e^{i\theta}(\delta R + i R\,\delta\Theta)}{e^{i\theta}(\delta r + i r\,\delta\theta)} = \lim_{\delta r\to 0}\frac{\delta R + i R\,\delta\Theta}{\delta r} = \lim_{\delta r\to 0}\left(\frac{\delta R}{\delta r} + i\frac{R\,\delta\Theta}{\delta r}\right) = \frac{\partial R}{\partial r} + i R\frac{\partial\Theta}{\partial r}$$

If there exists derivative of $f(z)$, then $\frac{R}{r}\frac{\partial\Theta}{\partial\theta} - i\frac{1}{r}\frac{\partial R}{\partial\theta} = \frac{\partial R}{\partial r} + i R\frac{\partial\Theta}{\partial r}$

Hence, the Cauchy-Riemann condition for polar coordinate is, $\begin{cases} \frac{R}{r}\frac{\partial\Theta}{\partial\theta} = \frac{\partial R}{\partial r} \\ -\frac{1}{r}\frac{\partial R}{\partial\theta} = R\frac{\partial\Theta}{\partial r} \end{cases}$

1.3 Analytic Functions

Definitions.

- If $f(z)$ is differentiable and single-valued in a region of the complex plane, then $f(z)$ is an ***analytic function*** in that region. It is also called ***'holomorphic'*** and ***'regular'***
- $f(z)$ is called an ***entire function*** if it is analytic everywhere in the complex plane.
- If $f'(z)$ does not exist at $z = z_0$, then z_0 is labeled a ***singular point***

Exercise.
Show that z^2 is analytic.

Let $z = (x + i\,y)$ and remember Cauchy-Riemann condition.

```
u[x_, y_] := ComplexExpand[Re[(x + ⅈ y)^2]];
v[x_, y_] := ComplexExpand[Im[(x + ⅈ y)^2]];
u[x, y]
v[x, y]
```

$x^2 - y^2$

$2\,x\,y$

```
D[u[x, y], x] == D[v[x, y], y] && D[u[x, y], y] == D[-v[x, y], x]
```

True

We see that $f(z) = z^2$ satisfies the Cauchy – Riemann conditions . From the Remark 1.1 if the complex function $f(z)$ holds Cauchy-Riemann condition and the partial derivative of $u(x, y)$ and $v(x, y)$ are continuous, then $f(z)$ is differentiable.
Are they continuous? Yes! Plus, $f(z) = z^2$ is *a* single valued function. So we can conclude that $f(z) = z^2$ is analytic function.

1.4 Derivative of Analytic Function

From

$$\lim_{\delta z\to 0}\frac{\delta f(z)}{\delta z} = \lim_{\delta x\to 0}\frac{\delta u + i\,\delta v}{\delta x + i\,\delta y} = \lim_{\delta x\to 0}\frac{\delta u + i\,\delta v}{\delta x} = \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x} + i\,\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x},$$

$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial u + i\,\partial v}{\partial x} = \frac{\partial f}{\partial x}$ which leads to

$$[f(z)\,g(z)]' = \frac{d}{dz}[f(z)\,g(z)] = \left(\frac{\partial}{\partial x}\right)[f(z)\,g(z)] = \left(\frac{\partial f}{\partial x}\right)g(z) + f(z)\left(\frac{\partial g}{\partial x}\right) = f'(z)\,g(z) + f(z)\,g'(z)$$

Since $f(z) = z$ is analytic throughout complex plane, its derivative is $f'(z) = \frac{\partial (x + i\,y)}{\partial x} = 1$

Then $(d/dz\,(z\,z)) = z'\,z + z\,z' = z + z = 2\,z$. Therefore, $\frac{d\,z^n}{dz} = n\;z^{n-1}$

Exercise.
Find derivative of $f(z) = \frac{\sin z}{z}$

Let $z = (x + i\,y)$

```
u[x_, y_] := ComplexExpand[Re[Sin[(x + ⅈ y)] / (x + ⅈ y)]];
v[x_, y_] := ComplexExpand[Im[Sin[(x + ⅈ y)] / (x + ⅈ y)]];
Reduce[D[u[x, y], x] == D[v[x, y], y] && D[u[x, y], y] == D[-v[x, y], x]]
```

True

$f(z)$ satisfies the Cauchy-Riemann condition.

```
Simplify[D[u[x, y], x]]
Simplify[D[v[x, y], x]]
```

$$\frac{1}{(x^2 + y^2)^2}\left(\mathrm{Cosh}[y]\,\left(x\,(x^2 + y^2)\,\mathrm{Cos}[x] + (-x^2 + y^2)\,\mathrm{Sin}[x]\right) - y\,\left(2\,x\,\mathrm{Cos}[x] + (x^2 + y^2)\,\mathrm{Sin}[x]\right)\,\mathrm{Sinh}[y]\right)$$

$$-\frac{1}{(x^2 + y^2)^2}\left(y\,\mathrm{Cosh}[y]\,\left((x^2 + y^2)\,\mathrm{Cos}[x] - 2\,x\,\mathrm{Sin}[x]\right) + \left((x^2 - y^2)\,\mathrm{Cos}[x] + x\,(x^2 + y^2)\,\mathrm{Sin}[x]\right)\,\mathrm{Sinh}[y]\right)$$

The two derivatives are continuous and $f(z)$ is single valued. Thus $f(z)$ is analytic. Since $f(z)$ is analytic, its derivative is identical to that of real domain.

```
D[Sin[z] / z, z]
```

$$\frac{\mathrm{Cos}[z]}{z} - \frac{\mathrm{Sin}[z]}{z^2}$$

Thus, $f'(z) = \frac{\text{Cos}[z]}{z} - \frac{\text{Sin}[z]}{z^2}$. ■

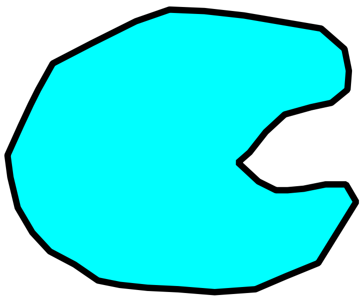
2. Cauchy’s Integral Theorem

2.1 Contour Integrals

...continued from 0.4 *Regions in the complex plane

- **Simply connected region**

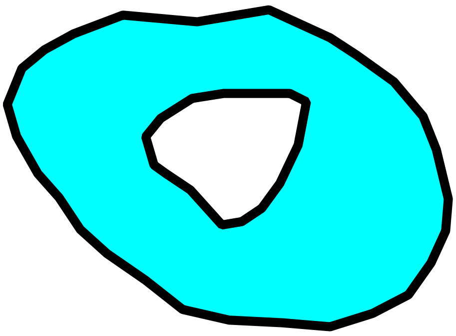
A region is simply connected if every closed curve within it can be shrunk continuously to a point that is within the region



Simply connected region has no holes inside the region.

- **Multiply connected region**

A region which is connected but not simply connected.

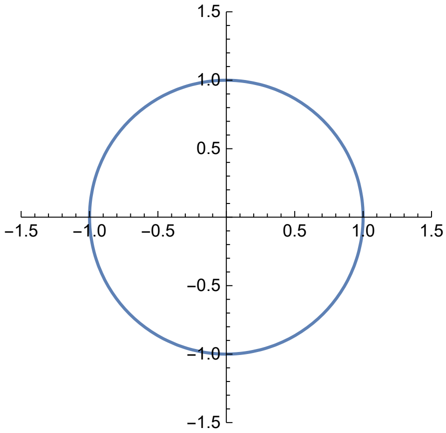


< Theorem 1 > Cauchy’s integral theorem

- If $f(z)$ is a analytic function at all point of a simply connected region in the complex plane and if C is a closed contour within that region, then $\oint_C f(z)\, dz = 0$

Check this out for $f(z) = z^n$, $\left(\text{let } z = r\, e^{i\theta}\right)$ with following path (unit circle).

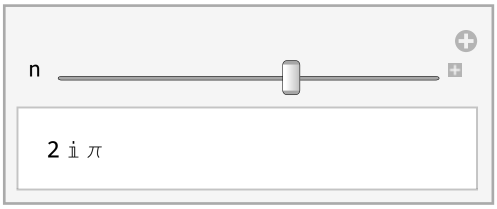
```
In[14]:= ff[z_, n_] := z^n;
z[r_, θ_] := r ⅇ^(ⅈ θ)
```



$$\oint_C f(z)\,dz = \int_{\theta=0}^{2\pi} r^n e^{i n\theta}\, i\, r e^{i\theta}\, d\theta = i\, r^{n+1}\int_{\theta=0}^{2\pi} e^{i(n+1)\theta}\, d\theta = i\, r^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} = 0 \text{ for } n \neq -1$$

```
Manipulate[∫_0^{2π} ff[z[r, θ], n] ⅈ r ⅇ^{ⅈ θ} ⅆθ, {n, -6, 2, 1}]
```

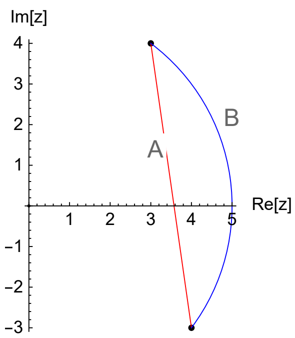
```
Table[∫_0^{2π} ff[z[r, θ], n] ⅈ r ⅇ^{ⅈ θ} ⅆθ, {n, -6, 2, 1}]
```



{0, 0, 0, 0, 0, 2 ⅈ π, 0, 0, 0}

Only when $n = -1$, the value of integral is $2\pi i$. For $n = -1$, $f(z) = 1/z$ that has singularity point at $z = 0$. Then the given region contains singularity point, that is to say, it is no longer a simply connected set. Therefore $f(z) = 1/z$ does not satisfying Cauchy's integral theorem.

Exercise.
Calculate $\int_{3+4i}^{4-3i} (4z^2 - 3iz)\, dz$ on the following two paths.



For line contour,

$$\int_{3+4i}^{4-3i} \left(4z^2 - 3iz\right) dz$$

$$\frac{76}{3} - \frac{707\, i}{3}$$

For curve contour

Remember $z[r,\ \theta] = r\, e^{i\,\theta}$. So the dz is $i\, r\, e^{i\,\theta}\, d\theta = i z\, d\theta$.

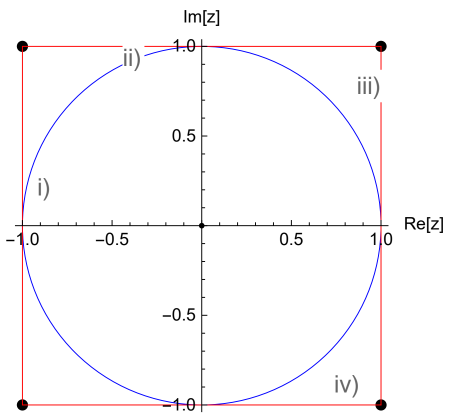
```
z[r_, θ_] := r ⅇ^(ⅈ θ);
∫_ArcTan[4/3]^ArcTan[-3/4] (4 z[5, θ]^2 - 3 ⅈ z[5, θ]) ⅈ z[5, θ] ⅆθ
```

$$\frac{76}{3} - \frac{707\, i}{3}$$

Since the integral about the closed contour AB is zero, integrals about two contours, A and B are same.

Exercise.

Evaluate $\oint_C (x^2 - i\, y^2)\, dz$ on following two contours



Let’s consider the rectangular contour first. Cartesian coordinate will be more convenient.

$dz \to dx + i\, dy$

From 1st to 4th contour,

```
∫_-1^1 (1 - ⅈ y^2) ⅈ ⅆy + ∫_-1^1 (x^2 - ⅈ) ⅆx + ∫_1^-1 (1 - ⅈ y^2) ⅈ ⅆy + ∫_1^-1 (x^2 - ⅈ) ⅆx
```

0

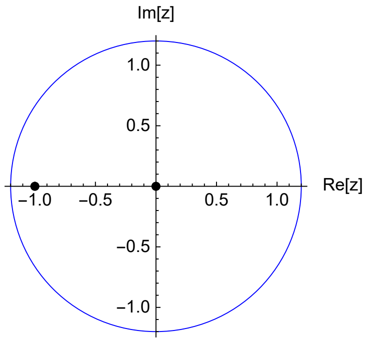
Now the clockwise circular contour. $dz \to i z\, d\theta$ (x is the real part of z and y is the imaginary part of z)

```
z[r_, θ_] := ComplexExpand[r ⅇ^(ⅈ θ)];
∫_2π^0 (Re[z[1, θ]]^2 - ⅈ Im[z[1, θ]]^2) ⅈ z[1, θ] ⅆθ
```

0

Result of two contours are identical. However, the complex function $(x^2 - i\, y^2)$ is $f(z) = \left(\frac{z+z^*}{2}\right)^2 + i\left(\frac{z-z^*}{2}\right)^2$. We already know that z^* is not an analytic function that the Cauchy’s integral theorem does not apply. Then, why these two results are same? Because of symmetry.

Exercise.
Show that $\oint_C \frac{dz}{z^2+z}$ = 0 in which the contour C is a circle defined by |z|>1



Two singular points are inside the region. We can enjoy the genius *Mathematica*.(Don't forget the $dz = i\, z\, d z$)

```
z[r_, θ_] := r ⅇ^(ⅈ θ);
∫_0^(2π) 1/(z[2, θ]^2 + z[2, θ]) ⅈ z[2, θ] ⅆθ
0
```

Without the help of *Mathematica,* we can cope with this problem on our own. Consider that $\frac{1}{z(z+1)}$ can be expanded into $\frac{1}{z} - \frac{1}{z+1}$. Now we have two integrals ; $\oint_C \frac{1}{z} dz$ and $-\oint_C \frac{1}{z+1} dz$.
We already know the result of $\oint_C \frac{1}{z} dz$; $2\pi i$.
Then how about $-\oint_C \frac{1}{z+1} dz$? It depends on the contour. When contour contains the singular point, say, $z = -1$, then the integral has the same value of $-\oint_C \frac{1}{z} dz$; $-2\pi i$. Because the Cauchy's theorem said that "the integral of an analytic function over a closed path has a value that remains unchanged over all possible continuous deformations of the contour". For containing-no-singular point case, obviously the value of integral is zero.

$$\oint_C \frac{1}{z+1} dz = \begin{cases} 2\pi i \text{ (when the contour contains } z = -1) \\ 0 \text{ (when the contour does not contain } z = -1) \end{cases}$$

Now we can solve the problem!

$$\oint_C \frac{1}{z(z+1)} dz = \oint_C \frac{1}{z} dz - \oint_C \frac{1}{z+1} dz = \begin{cases} 2\pi i - 2\pi i = 0 \text{ (when the contour contains } z = -1) \\ 2\pi i - 0 = 2\pi i \text{ (when the contour does not contain } z = -1) \end{cases}$$

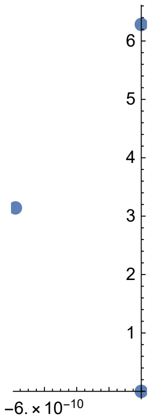
```
∫_0^(2π) 1/(z[0.2, θ]^2 + z[0.2, θ]) ⅈ z[0.2, θ] ⅆθ
0. + 6.28319 ⅈ
```

Mathematica thinks so too!
Few integrals around the $||z|| = 1$ are;

```
Table[∫_0^(2π) 1/(z[r, θ]^2 + z[r, θ]) ⅈ z[r, θ] ⅆθ, {r, 0.9, 1.1, .1}]
{1.23358 × 10^-16 + 6.28319 ⅈ, -7.79634 × 10^-10 + 3.14159 ⅈ, 0. + 6.28319 × 10^-6 ⅈ}
```

```
Chop[{1.23358 × 10^-16 + 6.28319 ⅈ, -7.79634 × 10^-10 + 3.14159 ⅈ, 0. + 6.28319 × 10^-6 ⅈ}]
{0. + 6.28319 ⅈ, -7.79634 × 10^-10 + 3.14159 ⅈ, 0. + 6.28319 × 10^-6 ⅈ}
```

```
ListPlot[(Tooltip[{Re[#1], Im[#1]}] &) /@
   {0. + 6.28319 ⅈ, -7.79634 × 10^-10 + 3.14159 ⅈ, 0. + 6.28319 × 10^-6 ⅈ},
 PlotStyle → PointSize[Large], AspectRatio → 3]
```



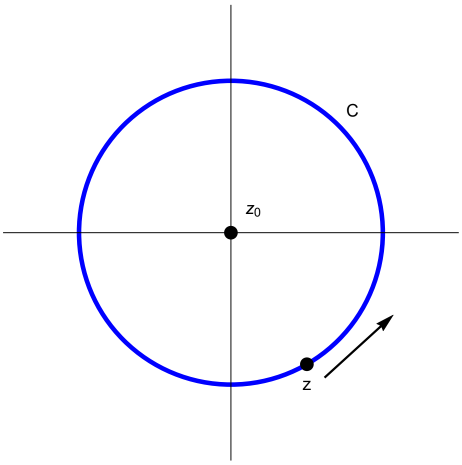

```
z[r_, θ_] := r ⅇ^(ⅈ θ);
```

$$\int_0^{2\pi} \frac{(\mathrm{Sin}[z[a+1,\theta]])^2}{(z[a+1,\theta]-a)^4}\, i\, z[a+1,\theta]\, d\theta$$

$$\mathrm{ConditionalExpression}\left[0,\ \mathrm{Re}[a] \le -\frac{1}{2}\right]$$

2.2 Cauchy’s Integral Formula

Consider the analytic function $f(z)$ and the counterclockwise contour $(\theta : 0 \longrightarrow 2\pi)$



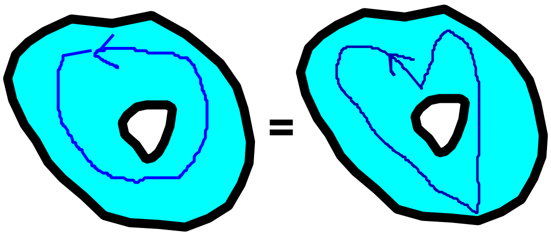
where z_0 can be any point in the interior region. Then there are useful result :

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

Now prove it! Let z be the $z_0 + r e^{i\theta}$ and $dz = i r e^{i\theta} d\theta$ then,

$$\oint_C \frac{f(z)}{z-z_0}\, dz = \int_0^{2\pi} \frac{f(z_0 + r e^{i\theta})}{z_0 + r e^{i\theta} - z_0}\, i r e^{i\theta}\, d\theta = \int_0^{2\pi} f(z_0 + r e^{i\theta})\, i\, d\theta$$

and we can take the limit $r \to 0$ since the integral doesn’t matter the shape of contour at the multiply connected region.



In multiply connected region, no matter how contour shapes,
it gives the same result.

Then $\oint_C \frac{f(z)}{z-z_0} dz = \int_0^{2\pi} f(z_0 + r e^{i\theta}) i\, d\theta = \int_0^{2\pi} f(z_0) i\, d\theta = 2\pi i f(z_0)$. If z_0 is outside of the region, then the integrand is analytic so that the contour integral results to zero.

By differentiating z_0 , we will get following result;

<Remark 2.1> Derivatives of Cauchy's integral formula

$$\begin{pmatrix} f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz \\ f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz \\ f''(z_0) = \frac{1\cdot 2}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3} dz \\ \vdots \\ f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \end{pmatrix}$$

Further Applications - Cauchy's Inequality

If $f(z) = \sum a_n z^n$ is analytic and bounded, $|f(z)| \le M$ on a circle of radius r about the origin, then

$$|a_n| r^n \le M$$

gives upper bound for the coefficients of its Taylor expansion.

Let $M(r)$ be the maximum value of $|f(z)|$ in the given contour, $|z| = r$, .

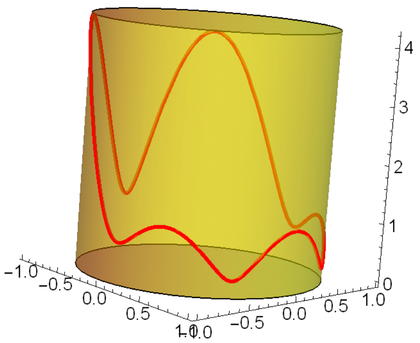
and use the Cauchy integral for $a_n = \frac{f^{(n)}(z)}{n!}$

First, consider specific case with visual assistant.

```
z[u_] := e^(i u);
f[u_] := 1/z[u]^2 - 1/z[u] + z[u] - (z[u])^2 + z[u]^3;
Max[ComplexExpand[Abs[f[u]]], u]
```

$\text{Max}\left[u, \sqrt{\text{Cos}[3u]^2 + (2\,\text{Sin}[u] - 2\,\text{Sin}[2u] + \text{Sin}[3u])^2}\right]$

```
Show[{ParametricPlot3D[{Cos[u], Sin[u], ComplexExpand[Abs[f[u]/z[u]^2]]}, {u, 0, 2 π},
   BoxRatios → {1, 1, 1}, PlotRange → All, PlotStyle → Red], Graphics3D[
   {Yellow, Opacity[0.5], Cylinder[{{0, 0, 0}, {0, 0, 4.2}}, 1]}]}, Boxed -> False]
```



In that case, $\frac{f(z)}{z^2}$ (Red curve) $\leq \frac{M(r)}{r^2}$ (Yellow cylinder) . Which means that $\oint \frac{f(z)}{z^2} \, dz \leq \frac{M(r)}{r^2} 2 \pi r$.

In general, $\frac{1}{2\pi} \left| \oint \frac{f(z)}{z^{n+1}} \, dz \right| \leq \frac{M(r)}{2 \pi r^{n+1}} 2 \pi r$. Where $a_n = \frac{f^{(n)}(z)}{n!} = \frac{1}{n!} \frac{n!}{2 \pi i} \oint \frac{f(z)}{z^{n+1}} \, dz = \frac{-i}{2\pi} \oint \frac{f(z)}{z^{n+1}} \, dz$.

Then, $| a_n | = \frac{1}{2\pi} \left| \oint \frac{f(z)}{z^{n+1}} \, dz \right| \leq M(r) \frac{2 \pi r}{2 \pi r^{n+1}}$ which leads to $| a_n | r^n \leq M$ (Cauchy's inequality)

Exercise.

Show that $\frac{1}{2 \pi i} \oint z^{m-n-1} \, dz$, ($m$ and n integrers) is a representation of the $\delta_{m\,n}$

By the Cauchy integral, $\frac{1}{2 \pi i} \oint \frac{z^m}{z^{n+1}} \, dz = \frac{1}{2 \pi i} \frac{2 \pi i}{n!} \left| \frac{d^n}{d z^n} z^m \right|_{z=0} = \frac{1}{n!} \left| \frac{d^n}{d z^n} z^m \right|_{z=0}$ where,

```
D[z^m, {z, n}]
```

```
z^(m-n) FactorialPower[m, n]
```

$\frac{1}{n!} \left| \frac{d^n}{d z^n} z^m \right|_{z=0} = \begin{cases} 0 & (m > n) \\ 0 & (m < n) \\ 1 & (m = n) \end{cases}$ Therefore, $\frac{1}{2 \pi i} \oint z^{m-n-1} \, dz$ represents $\delta_{m\,n}$

Exercise.

Assuming that f(z) is analytic on and within a closed contour C and that the point z_0 is within C, show that $\oint_C \frac{f'(z)}{z - z_0} \, dz = \oint_C \frac{f(z)}{(z-z_0)^2} \, dz$.

→ Cauchy integral.

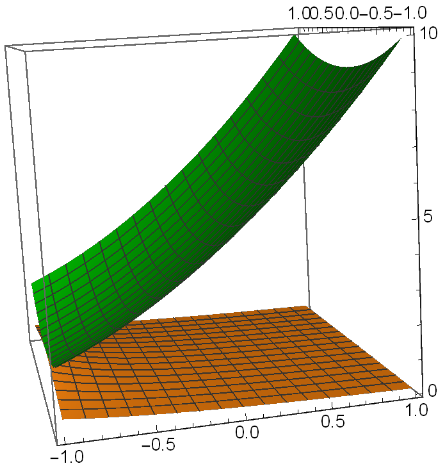
$\oint_C \frac{f(z)}{(z-z_0)^2} = 2 \pi i f'(z_0) = \oint_C \frac{f'(z)}{z-z_0} \, dz$

Exercise.

A function $f(z)$ is analytic within a closed contour C (and continuous on C). If $f(z) \neq 0$ within C and $| f(z) | \leq M$ on C, show that $| f(z) | \leq M$ for all points within C.

For specific case with visual representation, let's see $f(z) = (1 + x + i y)^2$.

```
z[x_, y_] := 2 + x + ⅈ y;
Show[{ParametricPlot3D[{x, y, ComplexExpand[Abs[z[x, y]^2]]}, {x, -1, 1},
    {y, -1, 1}, BoxRatios → {1, 1, 1}, PlotRange → All, PlotStyle → Green],
  ParametricPlot3D[{x, y, ComplexExpand[Abs[1 / z[x, y]^2]]}, {x, -1, 1},
    {y, -1, 1}, BoxRatios → {1, 1, 1}, PlotRange → All]}]
```



(1) Since $f(z)$ is analytic and continuous in given contour, its real and imaginary part is the solution of Laplace's equation, that is to say, its real and imaginary part do not have local maximum or minimum interior of given region.

(2) $\Longrightarrow$ See <Theorem 2>. maximum modulus principle. ■

Exercise.
Evaluate $\oint_C \frac{dz}{z(2z+1)}$ for the unit circle contour.

$\oint_C \frac{dz}{z(2z+1)} = \oint_C dz\left(\frac{1}{z} - \frac{1}{z+1/2}\right) = 2\pi i - 2\pi i = 0$. If you want to check the answer,

```
z[θ_] := ⅇ^(ⅈ θ);
∫_0^(2 π) 1/(z[θ] (2 z[θ] + 1)) ⅆθ
0
```

Correct!

Exercise.
Evaluate $\oint_C \frac{f(z)\,dz}{z(2z+1)^2}$ for the unit circle contour.

Let's make a partial fraction.

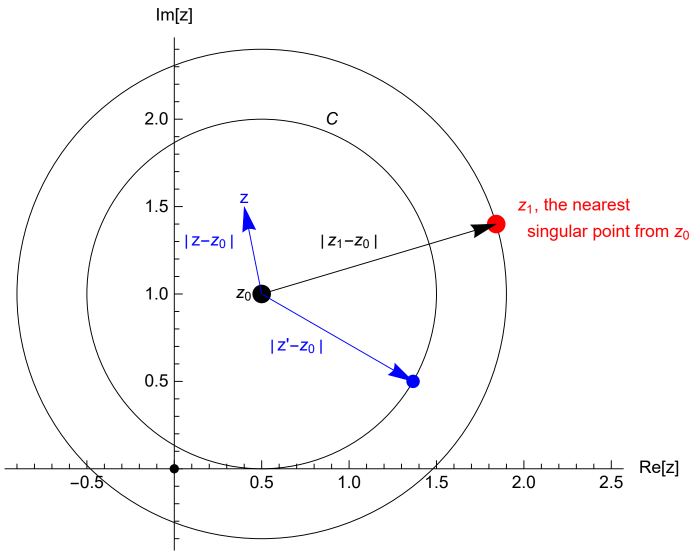
$\oint_C \frac{f(z)\,dz}{z(2z+1)^2} = \oint_C f(z)\,dz\left(\frac{1}{z} - \frac{1}{z+1/2}\right)\frac{1}{2z+1} = \oint_C f(z)\,dz\left(\frac{1}{z(2z+1)} - \frac{1}{2(z+1/2)^2}\right) = \oint_C f(z)\,dz\left(\frac{1}{z} - \frac{1}{z+1/2} - \frac{1}{2(z+1/2)^2}\right)$

By Cauchy integral, $\oint_C f(z)\,dz\left(\frac{1}{z} - \frac{1}{z+1/2} - \frac{1}{2(z+1/2)^2}\right) = \left\{f(0) - f\left(-\frac{1}{2}\right) - \frac{1}{2}f'\left(-\frac{1}{2}\right)\right\}2\pi i$

3. Laurent Expansion

3.1 Taylor Expansion

Take a deep look at following figure.



Contour C can be any path between z_0 and its closest singular point z_1. Suppose we are trying to expand $f(z)$ about $z = z_0$.

From the Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)[1-(z-z_0)/(z'-z_0)]} \end{aligned}$$

Where z is any point interior to C and z' is *a* point on the C.

Now consider the binomial theorem.

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=0}^{\infty} t^n$$

We already know that above series is convergent for $|t| < 1$. Since $\frac{|z-z_0|}{|z'-z_0|} < 1$, replace t into $\frac{z-z_0}{z'-z_0}$.

$$\frac{1}{1-(z-z_0)/(z'-z_0)} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

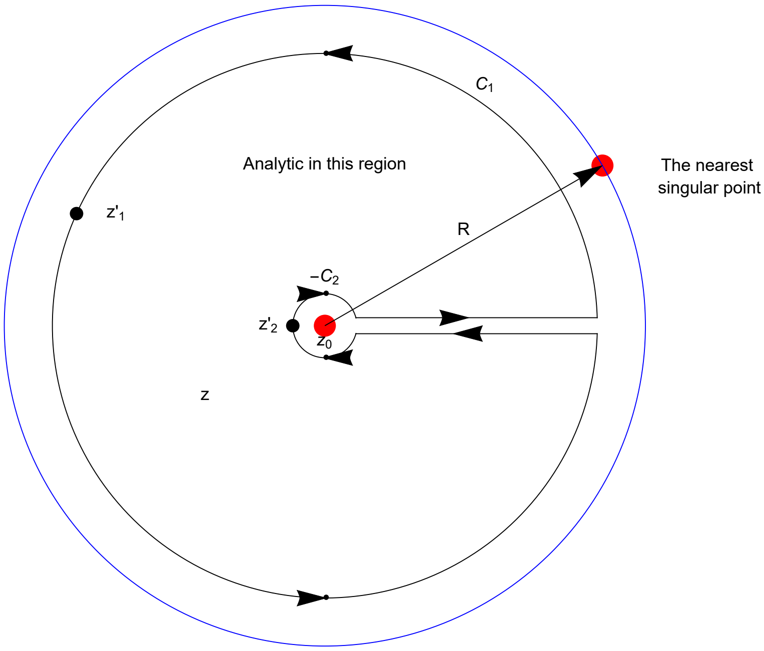
Then, $f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)[1-(z-z_0)/(z'-z_0)]}$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}} =$$

$$\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\, 2\pi i \frac{f^{(n)}(z_0)}{n!} = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{(Taylor expansion)}$$

3.2 Laurent Series

Consider the following figure.



Then the annular region for Laurent series is (where z is any point between C_2 and C_1)

$| z'_1 - z_0 | > | z - z_0 |$ and $| z'_2 - z_0 | < | z - z_0 |$

From Cauchy's integral formula, $f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z} dz'$ where $\oint_C = \oint_{C_1} - \oint_{C_2}$ so,

$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z} dz' = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{z'-z} dz' - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{z'-z} dz'$

Let's go slowly down!

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{z'-z} dz' - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{z'-z} dz' \\
&= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{(z'-z_0)-(z-z_0)} dz' - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{(z'-z_0)-(z-z_0)} dz' \\
&= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{(z'-z_0)\{1-(z-z_0)/(z'-z_0)\}} dz' - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{-(z-z_0)\{1-(z'-z_0)/(z-z_0)\}} dz' \\
&= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{(z'-z_0)\{1-(z-z_0)/(z'-z_0)\}} dz' + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{(z-z_0)\{1-(z'-z_0)/(z-z_0)\}} dz' \\
&= \frac{1}{2\pi i}\oint_{C_1} \sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n \frac{f(z')}{(z'-z_0)} dz' + \frac{1}{2\pi i}\oint_{C_2} \sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n \frac{f(z')}{(z-z_0)} dz' \\
&= \frac{1}{2\pi i}\oint_{C_1} \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}} f(z') dz' + \frac{1}{2\pi i}\oint_{C_2} \sum_{n=0}^{\infty}\frac{(z'-z_0)^n}{(z-z_0)^{n+1}} f(z') dz' \\
&= \frac{1}{2\pi i}\oint_{C_1} \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}} f(z') dz' + \frac{1}{2\pi i}\oint_{C_2} \sum_{n=1}^{\infty}\frac{(z'-z_0)^{n-1}}{(z-z_0)^{n}} f(z') dz' \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{C_1} \frac{f(z') dz}{(z'-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n} \oint_{C_2} (z'-z_0)^{n-1} f(z') dz' \\
&= S_1 + S_2
\end{aligned}
$$

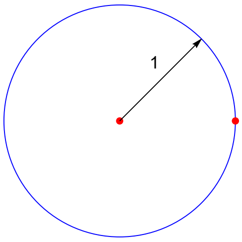
$$
\text{Where}\begin{cases} S_1 = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{C_1} \frac{f(z') dz'}{(z'-z_0)^{n+1}} \\ S_2 = \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n} \oint_{C_2} (z'-z_0)^{n-1} f(z') dz' \end{cases}
$$

S_1 is convergent for $| z - z_0 | <$ $| z' - z_0 |$ (interior of C_1) and S_2 is convergent for $| z - z_0 | > | z' - z_0 |$ (exterior of C_2)

We can combine this two series into one series and this is called Laurent series.

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \text{ where } a_n=\frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

For instance, take a look at $f(z)=\frac{1}{z(z-1)}$.



The distance from $z=0$ to closest singularity point $z=1$ is 1. That is, when we make the Laurent expansion about $z_0=0$, the convergent area is $0<R<1$

$\frac{1}{z(z-1)}=\frac{1}{z-1}-\frac{1}{z}=-\frac{1}{1-z}-\frac{1}{z}=-\frac{1}{z}-(1+z+z^2+z^3+\ldots)=-\frac{1}{z}-1-z-z^2-z^3-\ldots=-\sum_{n=-1}^{\infty} z^n$

From $a_n=\frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$, replace $f(z')$ into $\frac{1}{z'(z'-1)}$ where $z_0=0$. Then we have

$$a_n=\frac{1}{2\pi i}\oint \frac{dz'}{(z')^{n+2}(z'-1)}=\begin{cases}-1 & \text{for } n\geq -1\\ 0 & \text{for } n<-1\end{cases}$$

Since $\frac{1}{z'-1}=-\frac{1}{1-z'}=-\sum_{m=0}^{\infty}(z')^m$,

$a_n=\frac{1}{2\pi i}\oint \frac{dz'}{(z')^{n+2}(z'-1)}=-\frac{1}{2\pi i}\oint \sum_{m=0}^{\infty}(z')^m\frac{dz'}{(z')^{n+2}}=-\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint (z')^{m-n-2}\,dz'$

We have already shown that $\frac{1}{2\pi i}\oint z^{m-n-1}\,dz$ (m and n integrers) is a representation of the $\delta_{m\,n}$.

Thus, $a_n=-\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint (z')^{m-n-2}\,dz'=-\sum_{m=0}^{\infty}\delta_{m,n+1}=\begin{cases}-1 & \text{for } n\geq -1\\ 0 & \text{for } n<-1\end{cases}$

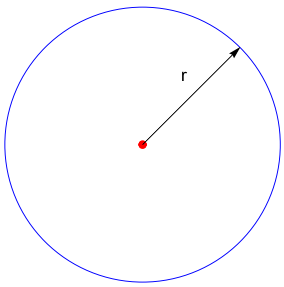
which is double validation for a_n

Exercise.
Develop the Taylor expansion of $\ln(z+1)$

By binomial theorem, $\frac{1}{1+z}=1-z+z^2-\ldots=\sum_{n=0}^{\infty}(-1)^n z^n$.

Then integrate both side. $\ln(z+1)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$

Exercise.
A function $f(z)$ is analytic on and within the unit circle. Also, $|f(z)|<1$ for $|z|\leq 1$ and $f(0)=0$. Show that $|f(z)|<|z|$ for $|z|\leq 1$.



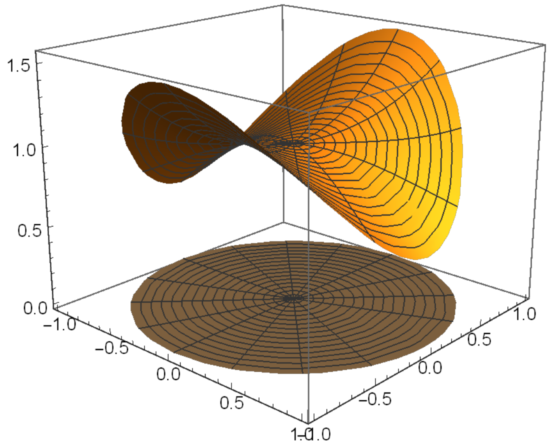
First, $g(z) = \frac{f(z)}{z}$ is analytic on $0 < |z| < 1$ and let $g(0) = \lim_{z\to 0} \frac{f(z)}{z} = f'(0)$ then,

$g(z)$ is analytic on $|z| < 1$. Now we need maximum modulus principle.

<Theorem 2> Maximum Modulus Principle

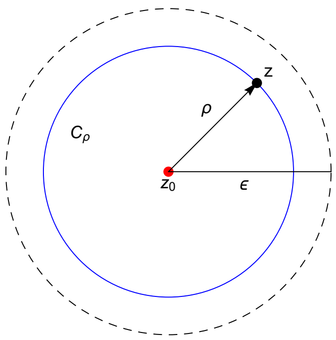
- If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \le |f(z_0)|$ for all points z in it.

```
Show[{ParametricPlot3D[
   {r Cos[θ], r Sin[θ], ComplexExpand[Abs[ComplexExpand[Cos[z[r, θ]]]]]},
   {θ, 0, 2 π}, {r, 0, 1}, PlotRange → All],
  ParametricPlot3D[{r Cos[u], r Sin[u], 0}, {u, 0, 2 π}, {r, 0, 1}, PlotStyle → Gray]}]
```



For example , we can think of $|\cos(z)|$, above potato chip. As you can see, there isn't local maximum *inside* the domain (in this case, interior of unit disk)

First, we will show that $f(z)$ has the constant value $f(z_0)$ when $|f(z)| \le |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \epsilon$ where f is analytic. Consider the following neighborhood of z_0



Let $f(z)$ satisfies the stated conditions and then let

z_1 be any point other than z_0 in the given neighborhood. ρ would be the distance between z_1 and z_0.

If C_ρ denotes the positively oriented circle $|z - z_0| = \rho$, the Cauchy integral formula tells us that

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(z)}{z - z_0}\, dz \quad \text{where } z = z_0 + \rho e^{i\theta}\ (0 \le \theta \le 2\pi)$$

$$\text{Then, } f(z_0) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}}\, \rho e^{i\theta}\, d\theta = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$$

$$\text{Therefore, } f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$$

From $f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$, we obtain inequality, $|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta$. (Do not forget that!)

On the other hand the condition says that $\left|f(z_0 + \rho e^{i\theta})\right| \le |f(z_0)|$ which means that

$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, d\theta = |f(z_0)|$

Then, $\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \le |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta$

Hence, $|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta$

$$2\pi\, |f(z_0)| = \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta$$

$$\int_0^{2\pi} |f(z_0)|\, d\theta = \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta$$

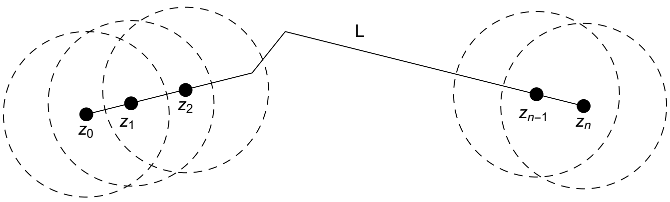
$$0 = \int_0^{2\pi} (|f(z_0)| - |f(z_0 + \rho e^{i\theta})|)\, d\theta$$

That is, $|f(z_0)| - |f(z_0 + \rho e^{i\theta})| = 0, \quad |f(z_0)| = |f(z_0 + \rho e^{i\theta})|$

Now we have shown that $|f(z_0)| = |f(z)|$ for all point z on the circle $|z - z_0| = \rho$

Since the ρ can be any value between 0 and the ϵ $(0 < \rho < \epsilon)$, consequently $|f(z_0)| = |f(z)|$ everywhere in the neighborhood $|z - z_0| < \epsilon$.

Now it's time to use this lemma to prove the maximum modulus principle



Draw a polygonal line L lying in domain D and extending form z_0 to any other point P in D.

Also, d represents the shortest distance from points on L to the boundary of D.

Next, there is a finite sequence of points $\{z_0, z_1, z_2, \ldots, z_{n-1}, z_n\}$ along L such that z_n coincides with the point P and $|z_k - z_{k-1}| < d \quad (k = 1, 2, \ldots, n)$

And then forming a finite sequence of neighborhoods; $N_0, N_1, N_2, \ldots, N_{n-1}, N_n$ (above figure) where each N_k has center z_k and radius d.

Now f is analytic in each of these neighborhoods and the center of each neighborhood N_k ($k = 1, 2, \ldots, n$) lies in the neighborhood N_{k-1}.

Let's look at the statement of Theorem again.

"That is, there is no point z_0 in the domain such that $|f(z)| \le |f(z_0)|$ for all points z in it."

Since $|f(z)|$ was assumed to have a maximum value in D at z_0, it also has a maximum value in N_0 at that point.

Hence, $f(z)$ has the constant value $f(z_0)$ throughout N_0.

Since z_1 is in the neighborhood N_0, $f(z) = f(z_1) = f(z_0)$ for z in N_1.

z_2 is in the neighborhood N_1 so $f(z) = f(z_2) = f(z_1) = f(z_0)$ for z in N_2. … Continuing in this manner, we eventually reach the neighborhood N_n and arrive at the fact that $f(z_n) = f(z_0)$ where z_n is any point other than z_0 in D. Then we may conclude that $f(z) = f(z_0)$ for every point z in D. That is to say, no maximum value in the domain D. ■

Finally, our journey is over. Let's get back to the Exercise.

We have $g(z) = \begin{cases} \frac{f(z)}{z} & z \ne 0 \\ f'(0) & z = 0 \end{cases}$ which is analytic in domain $|z| \le 1$. Also $|f(z)| < 1$ for $|z| \le 1$

Then, at the contour of $|z| = r$, $(r < 1)$ the maximum modulus theorem implies that
$\left|g(z)\right| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} < \frac{1}{r}$.
The theorem also tells us that when there is a given closed disk $|z| \le r$, for any point z in disk, there exists z_r on the boundary
such that $\left|g(z)\right| \le \left|g(z_r)\right| = \left|\frac{f(z_r)}{z_r}\right| = \frac{|f(z_r)|}{|z_r|} < \frac{1}{r}$. As $r \longrightarrow 1$, we will get $|g(z)| < 1$.
In short, $\frac{|f(z)|}{|z|} < 1 \longrightarrow |f(z)| < |z|$ for $|z| \le 1$. This exercise is sometimes called Schwarz's theorem. ■

Exercise.
Obtain the Laurent expansion of $z\, e^z/(z-1)$ about $z = 1$.

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-1)^n$, $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')\, dz'}{(z'-z_0)^{n+1}} = \frac{1}{2\pi i} \oint_C \frac{z'\, e^{z'}\, dz'}{(z'-1)^{n+2}}$

$a_n = \frac{1}{2\pi i} \oint_C \frac{z'\, e^{z'}\, dz'}{(z'-1)^{n+2}} = \frac{D^{(n+1)}(z\, e^z|_{z=1}}{(n+1)!} = \frac{[(n+1)\, e^z + z\, e^z]_{z=1}}{(n+1)!} = \frac{(n+2)\, e}{(n+1)!}$

$\{a_{-2} = 0,\ a_{-1} = e,\ a_0 = 2\, e,\ a_1 = 3/2\ e,\ a_2 = 2/3\ e,\ a_3 = 5/24\, e,\ a_4 = 1/20\ e,\ \ldots\}$
Therefore,
$z\, e^z/(z-1) = \sum_{n=-1}^{\infty} a_n (z-1)^n = e\left[\frac{1}{z-1} + 2 + \frac{3}{2}(z-1) + \frac{3}{2}(z-1)^2 + \frac{5}{24}(z-1)^3 + \frac{1}{20}(z-1)^4 + \ldots\right]$
Or, let $z = (z-1) + 1$.
Then,
$\{(z-1)+1\}\, e^{(z-1)+1}/(z-1) = e\left(1 + \frac{1}{z-1}\right) e^{(z-1)} = e\left(1 + \frac{1}{z-1}\right) \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} = e\left(\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} + \sum_{n=0}^{\infty} \frac{(z-1)^{n-1}}{n!}\right)$

$e\left(\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} + \sum_{n=0}^{\infty} \frac{(z-1)^{n-1}}{n!}\right) =$

$e\left(\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} + \sum_{n=0}^{\infty} \frac{(z-1)^n}{(n+1)!} + \frac{1}{z-1}\right) = e\left(\sum_{n=0}^{\infty} \frac{(n+1)\,(z-1)^n}{(n+1)!} + \sum_{n=0}^{\infty} \frac{(z-1)^n}{(n+1)!} + \frac{1}{z-1}\right) = e\left(\sum_{n=0}^{\infty} \frac{(n+2)\,(z-1)^n}{(n+1)!} + \frac{1}{z-1}\right)$

Therefore, $z\, e^z/(z-1) = \frac{e}{z-1} + e\left[\sum_{n=0}^{\infty} \frac{(n+2)\,(z-1)^n}{(n+1)!}\right]$
Computer is faster than hand .

```
Series[z Exp[z] / (z - 1), {z, 1, 4}]
```

$\frac{e}{z-1} + 2\, e + \frac{3}{2}\, e\, (z-1) + \frac{2}{3}\, e\, (z-1)^2 + \frac{5}{24}\, e\, (z-1)^3 + \frac{1}{20}\, e\, (z-1)^4 + O[z-1]^5$

Exercise.
Obtain the Laurent expansion of $(z-1)\, e^{1/z}$ about $z = 0$.

$(z-1)\, e^{1/z} = (z-1) \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} =$

$\sum_{n=0}^{\infty} \frac{z^{-n+1}}{n!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = z + \left[\sum_{n=0}^{\infty} \frac{z^{-n}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}\right] = z + \left[\sum_{n=0}^{\infty} \frac{z^{-n}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{(n+1)\, z^{-n}}{(n+1)!}\right] = z - \sum_{n=0}^{\infty} \frac{n\, z^{-n}}{(n+1)!}$

```
Series[(z - 1) Exp[1 / z], {z, 0, 4}]
```

$e^{\frac{1}{z} + O[z]^5}\, (-1 + z + O[z]^5)$

3.3 Singularities

As we can see at $f(z) = \frac{1}{z-z_0}$, if $f(z)$ is not analytic at $z = z_0$ but analytic at all neighborhood of z_0, then we call that a point z_0 is an isolated singular point of function $f(z)$. Let's say that we have expanded $f(z)$ in the Laurent series about z_0 ;

$f(z) = \cdots\cdots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)^1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\cdots$

Then there are the following definitions.

Definitions.

- **If all the a_{-n}, (n : $1 \sim \infty$) 's are zero, $f(z)$ is analytic at $z = z_0$ and we call z_0 a <u>regular point</u>.**
- **If $f(z)$ has finite negative power, $(z - z_0)^{-n}$, where n is integer, then $f(z)$ is said to have a <u>pole of order n</u> at $z = z_0$. A pole of order 1 is also called <u>simple pole</u>.**
- **If $f(z)$ has infinite negative power, $f(z)$ has an <u>essential singularity</u> at $z = z_0$**
- **The coefficient a_{-1} of $1/(z - z_0)$ is called the residue of $f(z)$ at $z = z_0$**

One way to identify a pole of $f(z)$ is to examine $\lim_{z \to z_0} (z - z_0)^n f(z_0)$ for various integers n. The smallest integer for which this limit exists gives the order of the pole at $z = z_0$. We already know $z e^z / (z-1)$ has a pole of order 1 from previous Exercise. Let's check!

```
Table[Limit[(z - 1)^n z ⅇ^z / (z - 1), z → 1], {n, 0, 3}]
```

```
{Indeterminate, ⅇ, 0, 0}
```

So $z = 1$ is a pole of order 1 (simple pole) of $z e^z / (z-1)$

Essential singularities are often identified directly from Laurent expansion.
$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$ for instance, it has $z = 0$ for essential singularity.

We can map $f(z)$ into the other complex plane w. In short, if we want to know about the behavior of $f(z)$ as $z \longrightarrow \infty$ then investigate the behavior of $f(1/t)$ as $t \longrightarrow 0$ by the mapping $(z \longrightarrow 1/t)$. Consider $\sin(z)$.

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2n+1)}}{(2n+1)!}$$

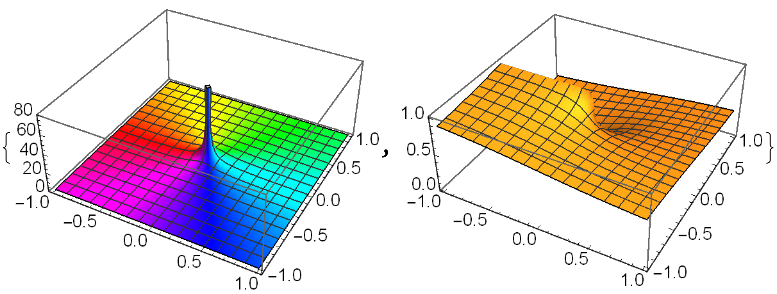
As z goes to the infinity, replace z by $1/t$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2n+1)}}{(2n+1)!} \longrightarrow \sin(1/t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\, t^{(2n+1)}}$$

where $\sin(1/t)$ has a essential singularity at $t = 0$. Hence, we can conclude that $\sin(z)$ has an essential singularity at $z = \infty$

I want to show some poles by *Mathematica*'s wonderful visualization. First, simple pole ;$z = 0$ at $1/z$.

```
{Plot3D[Abs[#], {x, -1, 1}, {y, -1, 1}, PlotRange → {0, 80}, PlotPoints → 40,
    ColorFunction → Function[{x, y, z}, Hue[Rescale[Arg[#], {-π, π}]]],
    ColorFunctionScaling → False],
   Plot3D[Rescale[Arg[#], {-π, π}], {x, -1, 1}, {y, -1, 1}]} &[(x + ⅈ y)^-1]
```

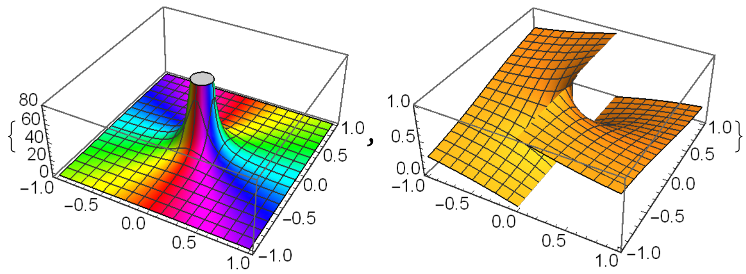


Note that as the value of the argument is increasing , the color changes from red to purple.

With ColorFunction, left hand side graph not only shows modulus of complex function but also shows phase variation.

Now let’s see the pole of 2nd order!

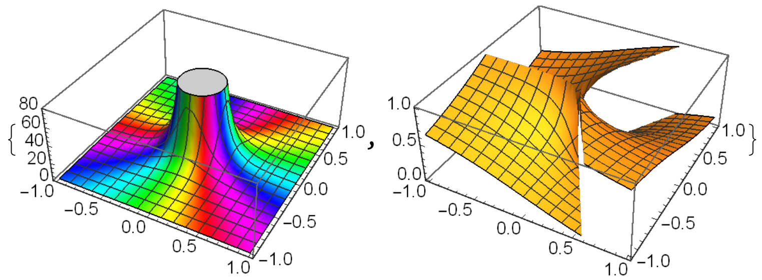
```
{Plot3D[Abs[#], {x, -1, 1}, {y, -1, 1}, PlotRange → {0, 80}, PlotPoints → 40,
    ColorFunction → Function[{x, y, z}, Hue[Rescale[Arg[#], {-π, π}]]],
    ColorFunctionScaling → False],
  Plot3D[Rescale[Arg[#], {-π, π}], {x, -1, 1}, {y, -1, 1}]} &[(x + ⅈ y)^-2]
```



It blows up more strongly at the singularity and there is a variation of phase in the function of 4π as one loops the singularity.

How about pole of 3rd order?

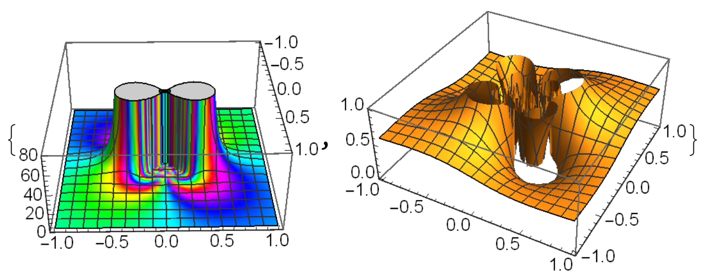
```
{Plot3D[Abs[#], {x, -1, 1}, {y, -1, 1}, PlotRange → {0, 80}, PlotPoints → 40,
    ColorFunction → Function[{x, y, z}, Hue[Rescale[Arg[#], {-π, π}]]],
    ColorFunctionScaling → False],
  Plot3D[Rescale[Arg[#], {-π, π}], {x, -1, 1}, {y, -1, 1}]} &[(x + ⅈ y)^-3]
```



I hope you like the picture, if you don’t like, then email me and tell me what’s wrong with that picture.

Now let’s see essential singularity ; $z = 0$ at e^{-1/z^2}

```
{Plot3D[Abs[#], {x, -1, 1}, {y, -1, 1}, PlotRange → {0, 80}, PlotPoints → 40,
    ColorFunction → Function[{x, y, z}, Hue[N[Rescale[Arg[#], {-π, π}]]]],
    ColorFunctionScaling → False],
  Plot3D[N[Rescale[Arg[#], {-π, π}]], {x, -1, 1}, {y, -1, 1}]} &[Exp[-(x + ⅈ y)^-2]]
```



As you can see, it has rapid variations in phase around the singular point $z = 0$

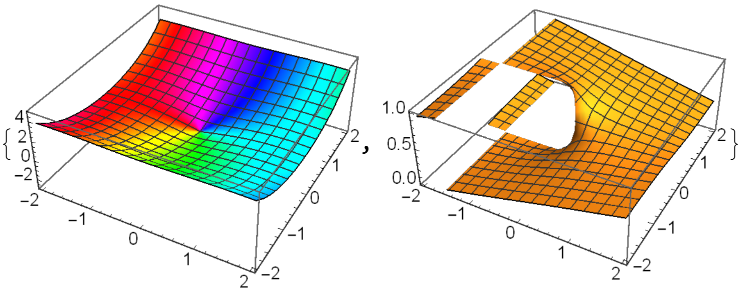
Simply change it and write it again later.

In[16]:=
```
ComplexPlot3D =
 {Plot3D[Abs[#1], #2, #3, #4, PlotPoints → 40,
    ColorFunction → Function[{x, y, z}, Hue[N[Rescale[Arg[#], {-π, π}]]]],
    ColorFunctionScaling → False], Plot3D[N[Rescale[Arg[#], {-π, π}]], #2, #3]} &
```

Out[16]=
```
{Plot3D[Abs[#1], #2, #3, #4, PlotPoints → 40,
   ColorFunction → Function[{x, y, z}, Hue[N[Rescale[Arg[#1], {-π, π}]]]],
   ColorFunctionScaling → False], Plot3D[N[Rescale[Arg[#1], {-π, π}]], #2, #3]} &
```

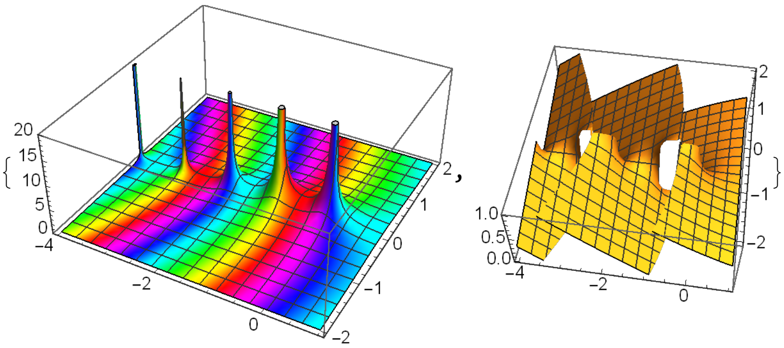
ComplexPlot of some special functions...

```
ComplexPlot3D[Sin[x + i y], {x, -2, 2}, {y, -2, 2}, PlotRange → {-3, 5}]
```



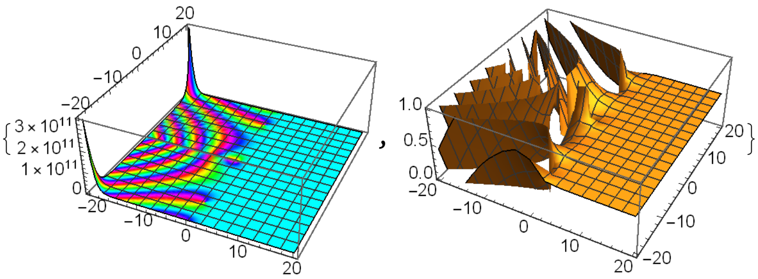
Have you ever seen this guy? ; $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ ..maybe spring semester?

```
ComplexPlot3D[Gamma[x + i y], {x, -4, 1}, {y, -2, 2}, PlotRange → {0, 20}]
```

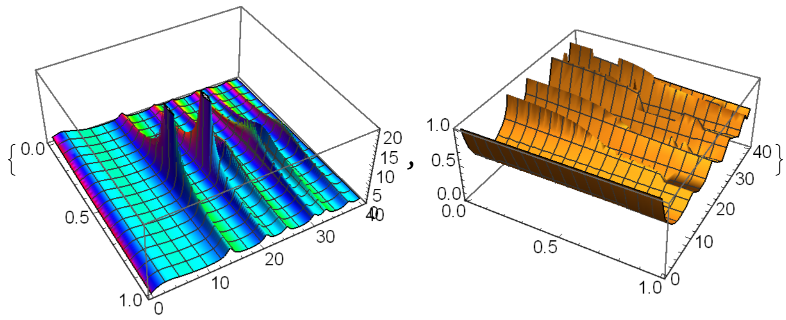


Riemann zeta function is interesting.

```
ComplexPlot3D[Zeta[x + i y], {x, -20, 20}, {y, -20, 20}, PlotRange → All]
```



```
ComplexPlot3D[1/Zeta[x + i y], {x, 0, 1}, {y, 0, 40}, PlotRange → {0, 20}]
```



Note that $1/\zeta(z)$ is diverge when Re[z] = 1/2 ⟹ Riemann hypothesis

Exercise.

A function $f(z)$ can be represented by $f(z) = \frac{f_1(z)}{f_2(z)}$, in which $f_1(z)$ and $f_2(z)$ are analytic, where $\begin{cases} f_1(z_0) \neq 0 \\ f_2(z_0) = 0, \quad f_2{}'(z_0) \neq 0 \end{cases}$.

Show that a_{-1} of $f(z)$ at $z = z_0$ is given by $a_{-1} = \frac{f_1(z)}{f_2{}'(z)}$.

From $f_2(z_0) = 0$, $f_2{}'(z_0) \neq 0$, $f_2(z)$ can be expressed as $f_2(z) = (z - z_0)\, f_{2'}(z)$. Remember that coefficient of Laurent series is

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')\, dz'}{(z'-z_0)^{n+1}}$$

Therefore, $a_{-1} = \frac{1}{2\pi i} \oint_C \frac{f(z')\, dz'}{(z'-z_0)^{-1+1}} = \frac{1}{2\pi i} \oint_C \frac{f_1(z')}{f_2(z')}\, dz' = \frac{1}{2\pi i} \oint_C \frac{f_1(z')}{(z-z_0)\, f_{2'}(z)}\, dz' = \frac{1}{2\pi i} \oint_C \frac{f_1(z')/f_{2'}(z)}{(z-z_0)}\, dz' = \frac{f_1(z_0)}{f_{2'}(z_0)}$

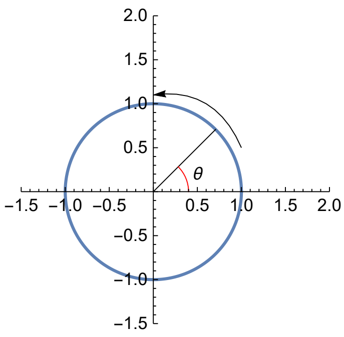
Since $f_2(z) = (z - z_0)\, f_{2'}(z)$, $f_2{}'(z) = f_{2'}(z) + (z - z_0)\, f_{2'}{}'(z)$, $f_{2'}(z_0) = f_2{}'(z_0)$. Hence, $a_{-1} = \frac{f_1(z_0)}{f_2{}'(z_0)}$ and this is the residue of $f(z) = \frac{f_1(z)}{f_2(z)}$ at $z = z_0$

<Remark 3. 1>

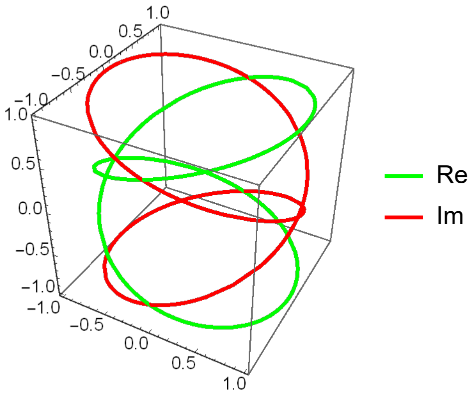
- Functions that have no singularities in the finite complex plane are called entire functions (e.g. e^z, $\sin(z)$, $\cos(z)$)
- A function that is analytic throughout the finite complex plane except for isolated poles is called meromorphic (e.g. ratio of two polynomials, $\tan(z)$, $\cot(z)$)

Not only poles and essential singularities, but there are also singularities uniquely associated with multivalued functions. By choosing specific value of multivalued function $f(z)$ then we can assign to $f(z)$ values at nearby points in a way that causes continuity in $f(z)$

Consider $f(z) = z^{1/2} = e^{i\theta/2}$ starting and ending at $z_0 = +1$

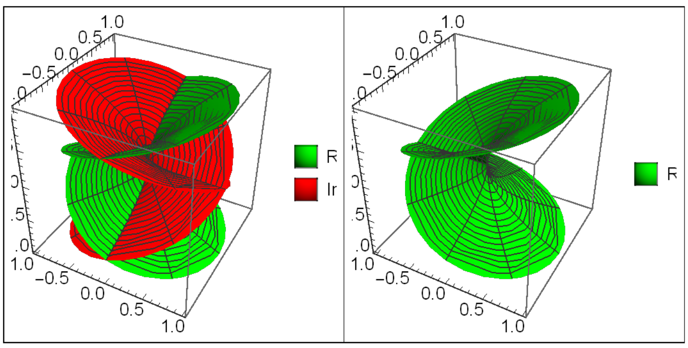



```
f[r_, θ_] := √r ⅇ^(ⅈ θ/2);
ParametricPlot3D[{{ Cos[θ], Sin[θ], Re[f[1, θ]]}, { Cos[θ], Sin[θ], Im[f[1, θ]]}},
 {θ, 0, 4 π}, PlotRange → All, PlotStyle → {Green, Red}, PlotLegends → {"Re", "Im"}]
```

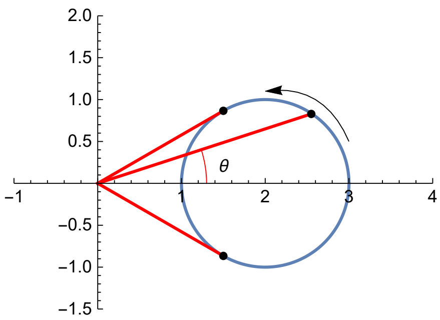


Note that $f(z)$ has the multiple value at $z_0 = +1$; $+1$ and -1 As you can see above figure, *Mathematica* choose +1.
As the path progresses, it is forming a closed loop. Notice that $e^{i 2\pi/2} = -1$ and $e^{i 4\pi/2} = 1$. Following figures are collection of paths consisting of passage around the radius r circle ($0 < r < 1$)
This figure is also called the Riemann surface for the function $f(z) = z^{1/2}$

```
f[r_, θ_] := √r ⅇ^(ⅈ θ/2);
GraphicsRow[
   {ParametricPlot3D[{{#1, #2, Re[f[r, θ]]}, {#1, #2, Im[f[r, θ]]}}, {θ, 0, 4 π},
     {r, 0, 1}, PlotRange → All, PlotStyle → {Green, Red}, PlotLegends → {"Re", "Im"}],
    ParametricPlot3D[{{#1, #2, Re[f[r, θ]]}}, {θ, 0, 4 π}, {r, 0, 1}, PlotRange → All,
     PlotStyle → {Green}, PlotLegends → {"Re"}]}, Frame → All] &[r Cos[θ], r Sin[θ]]
```

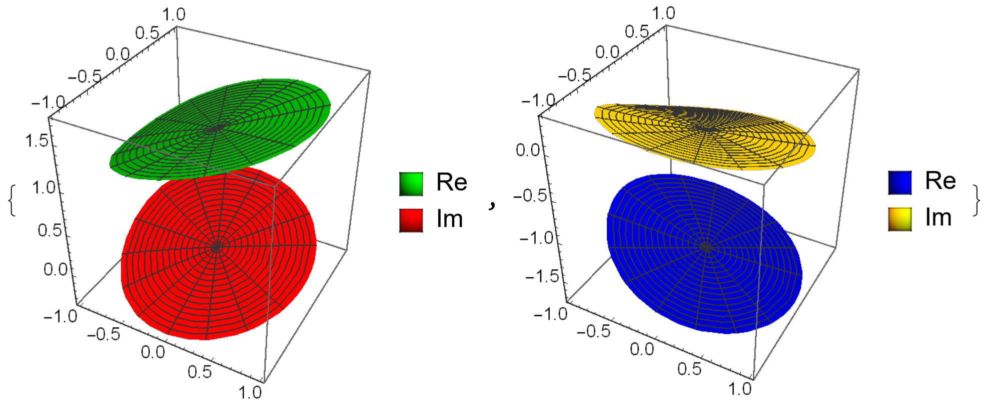


After that, now see what happens to the function $f(z)$ as we pass counterclockwise around a circle of unit radius centered at $z = +2$



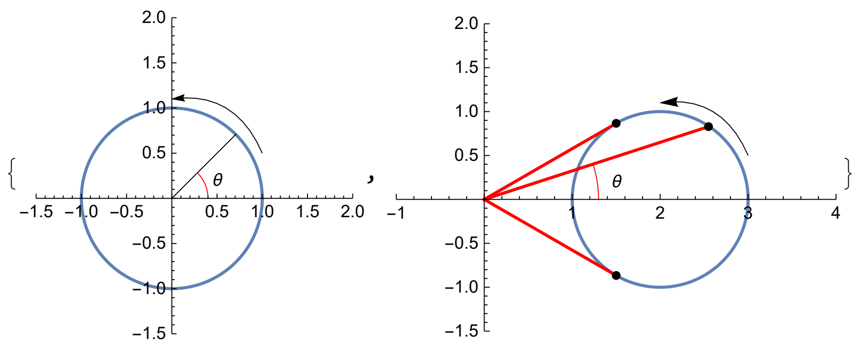
At $z = 3$, $f(z)$ is $+\sqrt{3}$ and $-\sqrt{3}$. Choose $\sqrt{3}$. Since the θ is oscillating from $\pi/6$ to $-\pi/6$, it dose not bring us to a different value of the multivalued function $z^{1/2}$. Following figures are representing the value of $f(z)$ as starting from different initial value; $\sqrt{3}$ or $-\sqrt{3}$

```
f1[r_, θ_] := (2 + r ⅇ^(ⅈ θ))^(1/2);
f2[r_, θ_] := - (2 + r ⅇ^(ⅈ θ))^(1/2);
{Show[{ParametricPlot3D[{{#1, #2, Re[f1[r, θ]]}, {#1, #2, Im[f1[r, θ]]}}, #3,
      #4, PlotRange → All, PlotStyle → {Green, Red}, PlotLegends → {Re, Im}]}],
   Show[{ParametricPlot3D[{{#1, #2, Re[f2[r, θ]]}, {#1, #2, Im[f2[r, θ]]}}, #3,
      #4, PlotRange → All, PlotStyle → {Blue, Yellow}, PlotLegends → {Re, Im}]}]} &[
 r Cos[θ], r Sin[θ], {θ, 0, 2 π}, {r, 0, 1}]
```



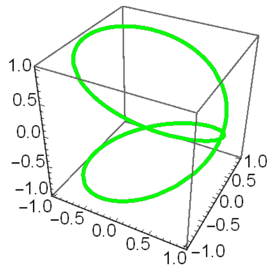
(where $f1$ and $f2$ are two branches of original function $f(z) = z^{1/2}$. Moreover, green & red one is principal branch. More details are given below.)

What’s the difference between two examples?

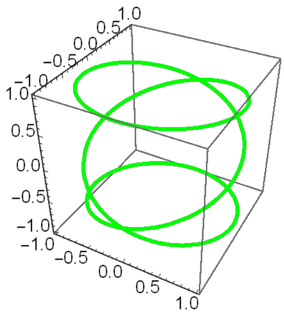


The first contains $z = 0$ and the second does not contain $z = 0$ where $z = 0$ is a singular point; $f(z) = z^{1/2}$ does not have a derivative at $z = 0$. We cannot decide the exact value of the derivative at z=0, which means that ambiguity in the function value will appear when the contour encircles such a singular point. This singular point is called **branch point.** The order of branch point is defined as the number of paths around it until the original value is reached.

```
{ParametricPlot3D[{{Cos[θ], Sin[θ], Im[ⅇ^(ⅈ θ/2)]}}, {θ, 0, 4 π}, PlotRange → All,
   PlotStyle → {Green}, PlotLegends → "z^(1/2); branch point of order 2 at z=0    "]
  ParametricPlot3D[{{Cos[θ], Sin[θ], Im[ⅇ^(ⅈ θ/3)]}}, {θ, 0, 6 π}, PlotRange → All,
   PlotStyle → {Green}, PlotLegends → "z^(1/3); branch point of order 3 at z=0"]}
```

{ 

$z^{1/2}$; branch point of order 2 at z=0



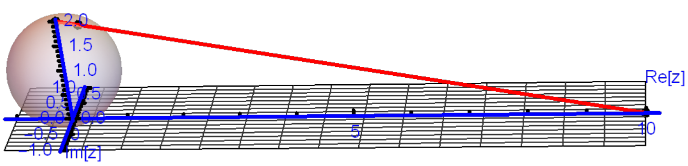
$z^{1/3}$; branch point of order 3 at z=0}

Now then, how can we restrict the multivalued function to the single-valued function? By drawing appropriate **branch cut!** Branch cut is a line that the path cannot cross. It must start from branch point and continue to infinity to another branch point. After drawing branch cut, the original multivalued function is successfully restricted to single-valued function and this single-valued function is called the **branch** of original function. Sometimes it is more convenient to agree on the branch to be used, and such a branch is called the **principal branch**(as I already mentioned), with the value of $f(z)$ on that branch called its **principal value**.

When you decide the branch cut, stereographic projection will be helpful. For the case of $z^{1/2}$, there are two branch point; $z = 0$, $z = \infty$. We have investigated that $z = 0$ is a branch point of $z^{1/2}$, however, how can we know about its behavior near z_∞? Answer: **By transformation $z \to 1/w$!** Thanks to $z \to 1/w$ transformation, we can understand the behavior near z_∞ of $z^{1/2}$ by searching near origin point.

$$\text{By } z \to 1/w \ , \ z^{1/2} \to w^{-1/2}$$

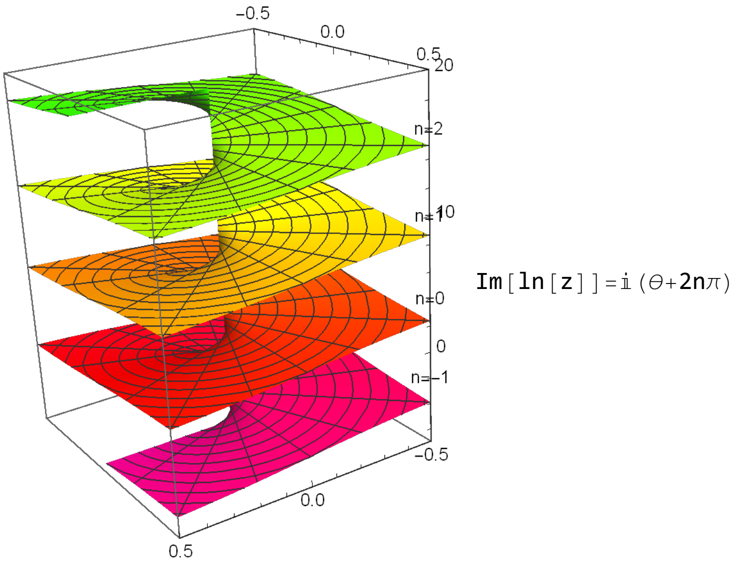
Still, $w^{-1/2}$ is a multivalued function, that is to say, $w = 0$ is a branch point. Therefore, z_∞ is also a branch point.



As you can see above figure, branch cut on the sphere is meridian of the sphere. The meridian is projected to whole positive Re[z] axis. Hence, on the complex plane, the branch cut will be infinite line from $z = 0$ to z_∞

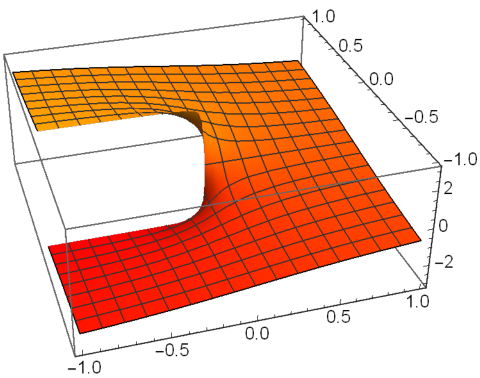
Exercise.
ln(z) has an infinite number of branches

Consider the polar representation of logarithm; $\ln(z) = \ln\left(r\, e^{i(\theta+2n\pi)}\right) = \ln(r) + i(\theta + 2n\pi)$ where n can be any integer. Following figure is imaginary part of ln(z) where n is from -1 to 2 (sometimes called the Riemann surface for the function ln z).



Since ln(z) has no derivative at $z = 0$ point, ln(z) is singular at $z = 0$ where is a branch point of ln(z). Furthermore, $z = 0$ is a branch point of infinite order (since n can be any value). Now we can draw appropriate branch cut; $z = 0$ to $z = \infty$. Once branch cut has drawn, it is restricted to single-valued function.

```
Plot3D[Im[Log[x + i y]], {x, -1, 1}, {y, -1, 1},
 ColorFunction → Function[{x, y, z}, Hue[0.1 z]]]
```

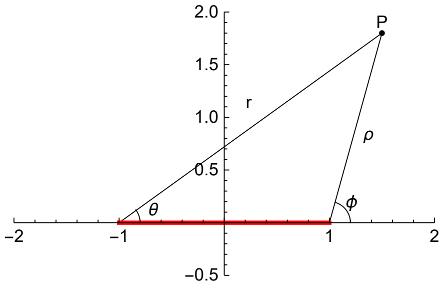


Principal branch of the $\ln(z) = \ln(r) + i(\theta + 2n\pi)$ is $n = 0$ case. The above figure is the principal value of ln(z). If you do not take any other action, *Mathematica* shows principal branch only.

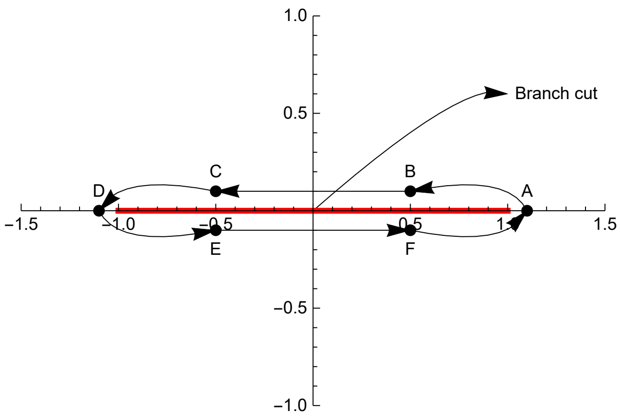
Exercise.

Multiple branch points of $f(z) = \left(z^2 - 1\right)^{1/2}$

Consider following branch cut and see what happens to $f(z)$ as we move around the vicinity of the branch



Assume that point P is our safari bus. During the journey, P has two coordinates;-
$-1 + r e^{i\theta}$, $1 + \rho e^{i\phi}$ for convenient. Then we have
$f(z) = (z^2 - 1)^{1/2} = (z-1)^{1/2}(z+1)^{1/2} = \left(\rho e^{i\phi}\right)^{1/2}\left(r e^{i\theta}\right)^{1/2} = r^{1/2}\rho^{1/2} e^{i(\theta+\phi)/2}$. Now this is the our exact route;



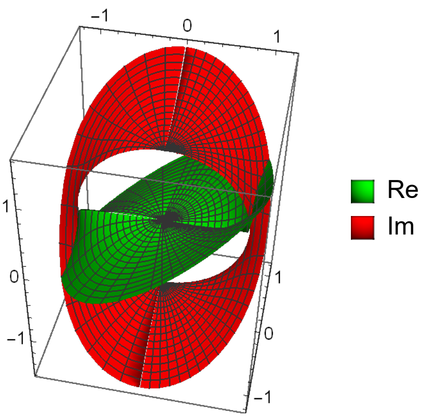
It starts at A and returns to A through B, C, D, E, F. Record the value of θ and ϕ at each point to check the change of the phase of $f(z)$, $e^{i(\theta+\phi)/2}$;

point	θ	ϕ	$\frac{\theta+\phi}{2}$	$e^{\frac{\theta+\phi}{2}}$
A	0	0	0	1
B	0	π	$\frac{\pi}{2}$	i
C	0	π	$\frac{\pi}{2}$	i
D	π	π	π	-1
E	2π	π	$\frac{3\pi}{2}$	$-i$
F	2π	π	$\frac{3\pi}{2}$	$-i$
A	2π	2π	2π	1

From this record, we can see that the phase at (B, C) and (E, F) are different from each other and the last A exceeds the first A by 2π, meaning that $f(z)$ is single-valued for the given route. Following figures help you to understand the behavior of f (z). Go ahead!

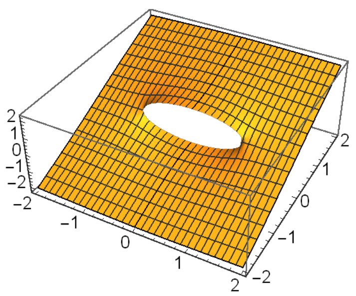
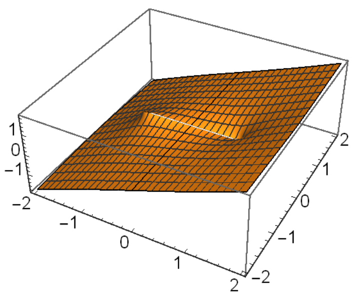
- By polar coordinate;

```
Show[{ParametricPlot3D[{{n Cos[θ], n Sin[θ], Re[fz[n, θ]]},
    {n Cos[θ], n Sin[θ], Im[fz[n, θ]]}}, {θ, 0, 0.5 π}, {n, 0, 1.2}, PlotRange → All,
   PlotStyle → {Green, Red}, PlotLegends → {Re, Im}], ParametricPlot3D[
   {{n Cos[θ], n Sin[θ], -Re[fz[n, θ]]}, {n Cos[θ], n Sin[θ], -Im[fz[n, θ]]}},
   {θ, 0.5 π, 1.5 π}, {n, 0, 1.2}, PlotRange → All, PlotStyle → {Green, Red}],
  ParametricPlot3D[{{n Cos[θ], n Sin[θ], Re[fz[n, θ]]},
    {n Cos[θ], n Sin[θ], Im[fz[n, θ]]}}, {θ, 1.5 π, 2 π},
   {n, 0, 1.2}, PlotRange → All, PlotStyle → {Green, Red}]}]
```



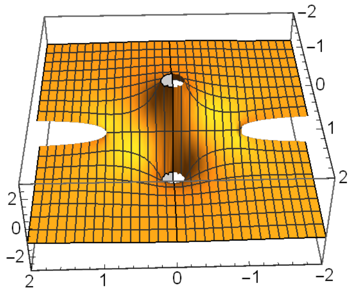
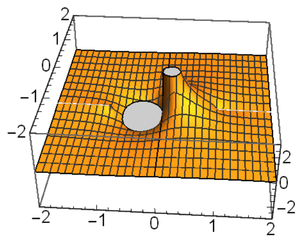
- By Cartesian coordinate;

```
{Show[Plot3D[Im[√((x + ⅈ y)^2 - 1)], {x, 0, 2}, {y, -2, 2}, PlotRange → All],
  Plot3D[-Im[√((x + ⅈ y)^2 - 1)], {x, 0, -2}, {y, -2, 2}, PlotLegends → ";Im"]],
 Show[Plot3D[Re[√((x + ⅈ y)^2 - 1)], {x, 0, 2}, {y, -2, 2}, PlotRange → All],
  Plot3D[-Re[√((x + ⅈ y)^2 - 1)], {x, 0, -2}, {y, -2, 2}, PlotLegends → ";Re"]]}
```

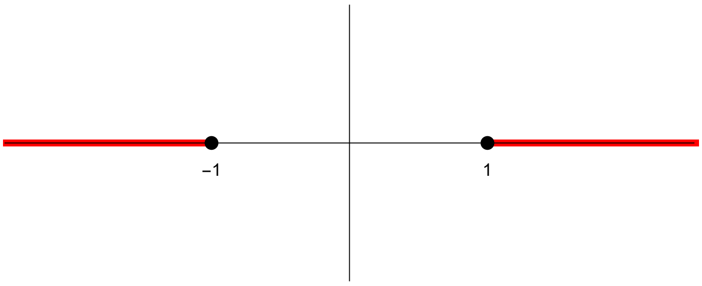
{  ;Im ,  ;Re }

Another way to create branch cuts is transformation $z \to 1/w$;

```
{Show[Plot3D[Im[√((x + ⅈ y)^-2 - 1)], {x, 0, 2},
   {y, -2, 2}, PlotRange → {{-2, 2}, {-2, 2}, {-3, 3}}],
  Plot3D[-Im[√((x + ⅈ y)^-2 - 1)], {x, 0, -2}, {y, -2, 2}, PlotLegends → ";Im"]],
 Show[Plot3D[Re[√((x + ⅈ y)^-2 - 1)], {x, 0, 2}, {y, -2, 2},
   PlotRange → {{-2, 2}, {-2, 2}, {-3, 3}}],
  Plot3D[-Re[√((x + ⅈ y)^-2 - 1)], {x, 0, -2}, {y, -2, 2}, PlotLegends → ";Re"]]}
```

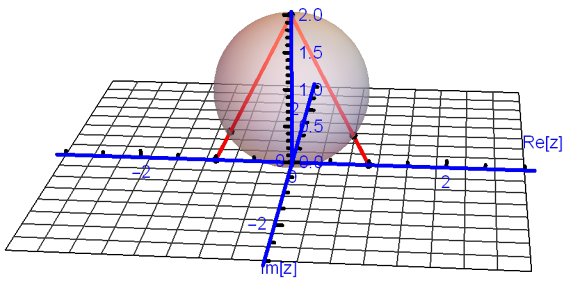
{  ";Im" ,  ";Re" }

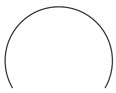
In this case, the branch cut is $\{x > 1 \text{ and } x < -1\}$;



*By stereographic projection,

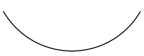
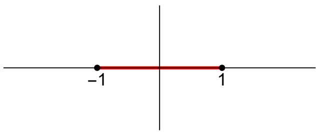
```
Show[{StereoGraphic[{-1, 1}], Plot3D[0, {x, -3, 3}, {y, -3, 3}, PlotStyle → None]}]
```

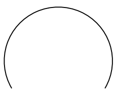
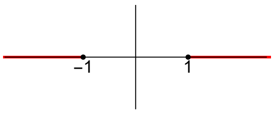


The branch cut can be  or  since branch cut is a curve between two branch points

(In this case z_∞ is not a branch point. Consider the transformation $z \to 1/w$

$(z^2-1)^{1/2} \to \frac{(1-w^2)^{1/2}}{w}$ near the $w=0$, it behaves like $1/w$ which is single valued)

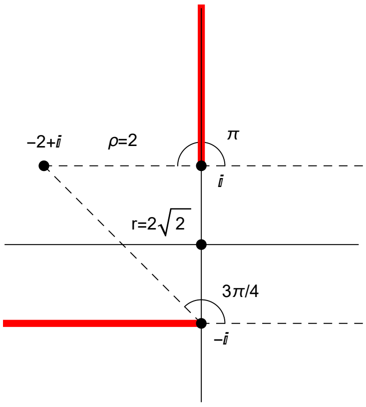
Projection of  will be  and

projection of  will be 

Exercise.

The function $F(z) = \ln(z^2+1)$ is made single-valued by straight-line branch cuts.

If $F(0) = -2\pi i$, find the value of $F(i - 2)$.



Red line is the branch cut of $F(z) = \ln(z^2 + 1)$

$F(z) = \ln(z^2 + 1) = \ln(z + i) + \ln(z - i)$ and $z = 0$ can be expressed as $\left\{i + e^{i(-\pi/2 + 2a\pi)} \text{ or } -i + e^{i(\pi/2 + 2b\pi)}\right\}$.

$F(0) = \ln\left(-i + e^{i(\pi/2+2b\pi)} + i\right) + \ln\left(i + e^{i(-\pi/2+2a\pi)} - i\right) =$

$$i(\pi/2 + 2b\pi) + i(-\pi/2 + 2a\pi) = 2(a + b)\pi i = -2\pi i$$

Therefore, $(a + b) = -1$.

$z = -2 + i$ can be represented as $\left\{i + \rho e^{i(\pi + 2a\pi)} \text{ or } -i + r e^{i(3\pi/4 + 2b\pi)}\right\}$

.

$F(i - 2) = \ln\left(-i + r e^{i(3\pi/4+2b\pi)} + i\right) + \ln\left(i + \rho e^{i(\pi+2a\pi)} - i\right) = i(3\pi/4 + 2b\pi) + i(\pi + 2a\pi) + \ln(r\rho)$

$= i(7\pi/4 - 2\pi) + \ln\left(4\sqrt{2}\right) = \ln\left(4\sqrt{2}\right) - \frac{\pi}{4} i$

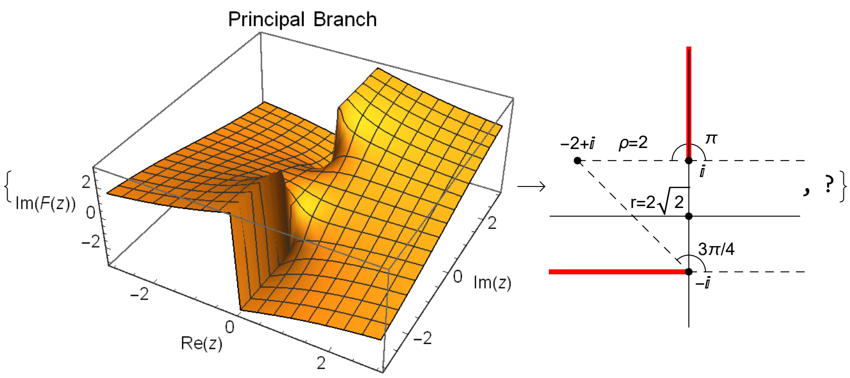
The computer is as smart as us.

```
F[z_Complex] := ComplexExpand[Log[z^2 + 1]];
F[z_Real] := Log[z^2 + 1]
```

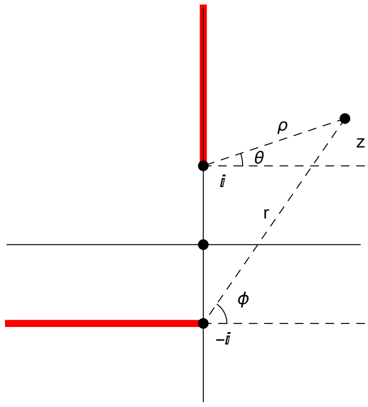
```
F[i - 2]
```

$-\frac{i\pi}{4} + \frac{\text{Log}[32]}{2}$

As I previously mentioned, *Mathematica* gives principal branch only, but the situation of given exercise is not a principal branch. How can we fit $F(z)$ into given branch cut?



Imagine that we are on a safari bus again.



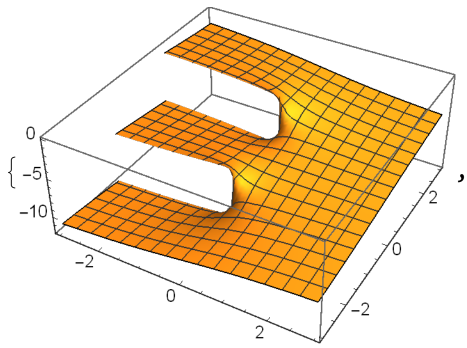
z can be represented by $-i + r\, e^{i(\theta+2a\pi)}$ or $i + \rho\, e^{i(\phi+2b\pi)}$ where $a + b = -1$ by given condition.

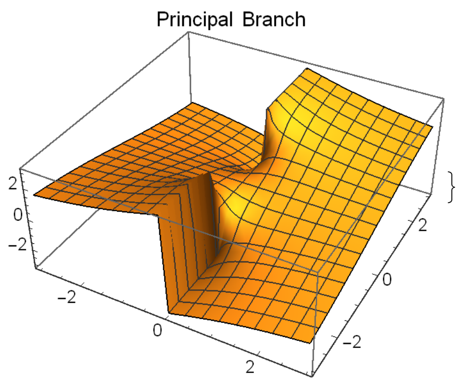
Then $F(z) = \ln(r) + i\,(\theta + 2\,b\,\pi) + \ln(\rho) + i(\phi + 2\,a\,\pi) = \ln(r\,\rho) + i\,(\theta + \phi - 2\,\pi)$

Therefore, the imaginary part of $F(z)$ is $(\theta + \phi - 2\,\pi)$ where
$\theta = \mathrm{Arg}\,[x + (y - 1)\,i]$, $\phi = \mathrm{Arg}\,[x + (y + 1)\,i]$

```
θ[x_, y_] := Arg[x + (y - 1) ⅈ];
ϕ[x_, y_] := Arg[x + (y + 1) ⅈ];
ImF[x_, y_] := θ[x, y] + ϕ[x, y] - 2 π
```

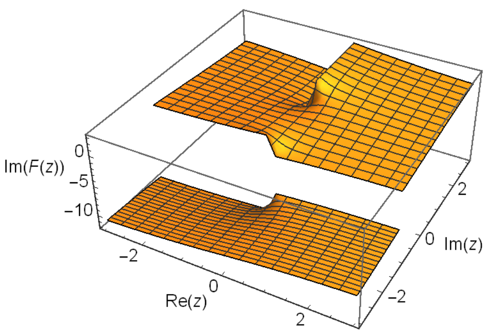
```
{Plot3D[ImF[x, y], {x, -3, 3}, {y, -3, 3}],
 Plot3D[Im[F[x + ⅈ y]], {x, -3, 3}, {y, -3, 3}, PlotLabel → "Principal Branch"]}
```





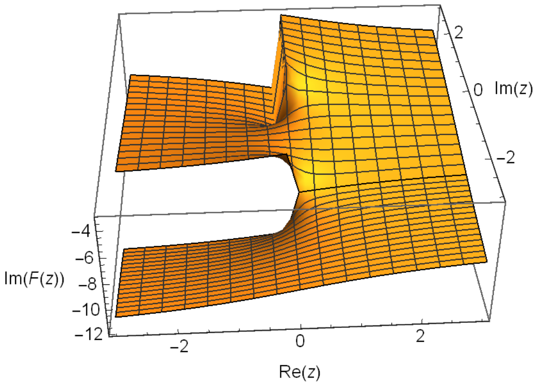
Branch point of two branches are identical; $-i$ and i but the branch cuts are in different. Given branch cut is appropriate ensemble of two. Let’s mix them!

```
Show[{Plot3D[Im[F[x + ⅈ y]], {x, -3, 3}, {y, -1, 3}, PlotRange → All,
   AxesLabel → {Re[z], Im[z], Im[F[z]]}], Plot3D[ImF[x, y], {x, -3, 3}, {y, -3, -1}]}]
```



Since raw $\mathrm{Im}[F[x + i\,y]]$ is an principal branch, we have to add the phase , $-2\,\pi$.

```
Show[{Plot3D[Im[F[x + ⅈ y]] - 2 π, {x, -3, 3}, {y, -1, 3}, PlotRange → All,
   AxesLabel → {Re[z], Im[z], Im[F[z]]}], Plot3D[ImF[x, y], {x, -3, 3}, {y, -3, -1}]}]
```



Finally we have the branch given in exercise. If you want to create the other branch, cut and paste with other branches as we did above. Various of branches can be made by this process.

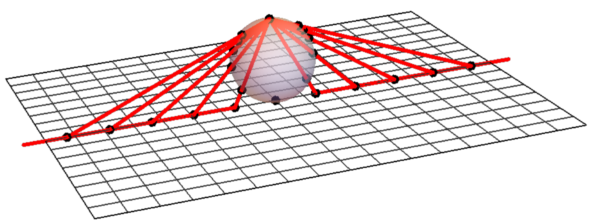
*Stereographic Projection

$F(z) = \ln(z^2+1) - (z \to 1/w) \to \ln(1/w^2+1) = \ln(1+w^2) - 2\ln w \Longrightarrow$ also a multivalued function near the point $w = 0$. $\Longrightarrow z_\infty$ is a branch point.

Then, branch points of $F(z)$ are $z = -i,\ i,\ z_\infty$

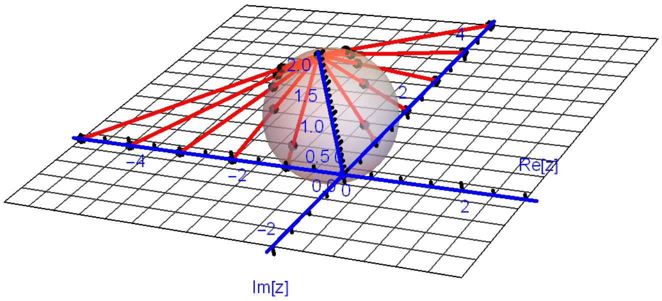
The principal branch will be $(-i, -\infty i),\ (i, \infty i)$

```
Show[{Graphics3D[{Thick,
    Red, Line[{{0, -1, 0}, {0, -6, 0}}], Line[{{0, 1, 0}, {0, 6, 0}}]}, Boxed -> False],
  StereoGraphic[{-5 ⅈ, -4 ⅈ, -3 ⅈ, -2 ⅈ, -ⅈ, ⅈ, 2 ⅈ, 3 ⅈ, 4 ⅈ, 5 ⅈ}],
  Plot3D[0, {x, -5, 5}, {y, -5.5, 5.5}, PlotStyle → None]}]
```



Rolling the ball slightly, we can construct the branch cut we need.

```
Show[{StereoGraphic[{-5, -4, -3, -2, -1, 0, i, 2 i, 3 i, 4 i, 5 i}],
  Plot3D[0, {x, -5, 3}, {y, -2, 5}, PlotStyle → None]}]
```



4. Calculus of Residues

4.1 Residue Theorem

Do you remember what residue means in complex analysis?; it means that coefficient of $1/(z-z_0)$ in the Laurent expansion.

$$f(z)=\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \;=\cdots+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots\;,\;\text{Residue}[f(z),\{z,z_0\}]=a_{-1}$$

If $f(z)$ has a simple pole at $z=z_0$,

$$f(z)\;=\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots$$

It is easy to find a_{-1} ; just multiply $(z-z_0)$ and replace take the limit $z\to z_0$

$$a_{-1}=\lim_{z\to z_0}(z-z_0)\left(\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots\right)=\lim_{z\to z_0}(z-z_0)f(z)$$

If there is a pole of order n , you have to add more processes. First multiply $(z-z_0)^n$ to change negative powers into positive powers;

$$(z-z_0)^n\left(\frac{a_{-n}}{(z-z_0)^n}+\cdots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots\right)=$$
$$\left(a_{-n}+\cdots+a_{-1}(z-z_0)^{n-1}+a_0(z-z_0)^n+a_1(z-z_0)^{n+1}+\cdots\right)$$

Next, get the $(n-1)$ th derivative of $(z-z_0)^n f(z)$ to eliminate $(a_{-n},\cdots,a_{-2})$

$$\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)=(n-1)!\,a_{-1}+\frac{n!}{1!}a_0(z-z_0)+\frac{(n+1)!}{2!}a_1(z-z_0)^2+\cdots$$

Then take the limit $z\to z_0$

$$\lim_{z\to z_0}\left[\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)\right]=(n-1)!\,a_{-1}\;,\quad a_{-1}=\frac{1}{(n-1)!}\lim_{z\to z_0}\left[\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)\right]$$

Sometimes it is easier to extend series expansion and check the coefficient of $1/(z-z_0)$. Let's see.

- Residue of $\frac{1}{4z+1}$ at $z=-1/4$; $\lim_{z=-1/4}\left(\frac{z+1/4}{4z+1}\right)$

$z=-1/4$ is evidently a simple pole

```
Limit[(z + 1/4) (1/(4 z + 1)), z → -1/4]
```

$$\frac{1}{4}$$

■ Residue of $\frac{\ln z}{z^2+4}$ at $z=2e^{i\pi/2}$; $\lim_{z=2e^{i\pi/2}}\left(\frac{(z-2e^{i\pi})\ln z}{z^2+4}\right)$

Limit$\left[(z-2\,\mathrm{Exp}[i\,\pi/2])\ (\mathrm{Log}[z]/(z^2+4)),\ z\to 2\,e^{i\,\pi/2}\right]$

$$\frac{1}{8}\left(\pi - 2\,i\,\mathrm{Log}[2]\right)$$

Let's see the series expansion of $\frac{\ln z}{z^2+4}$

Series$\left[(\mathrm{Log}[z]/(z^2+4)),\ \{z,\ 2\,i,\ 4\}\right]$

$$\frac{\pi-2\,i\,\mathrm{Log}[2]}{8\,(z-2\,i)}+\left(-\frac{1}{8}+\frac{i\,\pi}{32}+\frac{\mathrm{Log}[2]}{16}\right)+\frac{1}{128}\left(-8\,i-\pi+2\,i\,\mathrm{Log}[2]\right)(z-2\,i)+\left(\frac{5}{192}-\frac{i\,\pi}{512}-\frac{\mathrm{Log}[2]}{256}\right)(z-2\,i)^2+\frac{(64\,i+3\,\pi-6\,i\,\mathrm{Log}[2])\,(z-2\,i)^3}{6144}+\left(-\frac{1}{240}+\frac{i\,\pi}{8192}+\frac{\mathrm{Log}[2]}{4096}\right)(z-2\,i)^4+O[z-2\,i]^5$$

It's obvious that $\frac{\ln z}{z^2+4}$ has a simple pole and its residue is $(\pi-2\,i\ln(2))/8$.

■ Residue of $\frac{z}{\sin^2 z}$ at $z=\pi$

$\frac{1}{1!}$ **Limit**$\left[D\left[(z-\pi)^2\ (z/\mathrm{Sin}[z]^2),\ z\right],\ z\to\pi\right]$

1

Maybe you can expand $\frac{z}{\sin^2 z}$ by substitute $w=z-\pi$.

$$\frac{w+\pi}{\sin^2 w}=\frac{w+\pi}{\left(w-\frac{w^3}{3!}+\frac{w^5}{5!}+\cdots\right)^2}=\frac{w+\pi}{\left(w^2-\frac{w^4}{3}+\frac{2w^6}{45}+\cdots\right)}=\frac{w+\pi}{w^2\left(1-\frac{w^2}{3}+\frac{2w^4}{45}+\cdots\right)}=\left(\frac{1}{w}+\frac{\pi}{w^2}\right)\left\{1-\left(\frac{w^2}{3}-\frac{2w^4}{45}+\cdots\right)\right\}^{-1}$$
$$=\left(\frac{1}{w}+\frac{\pi}{w^2}\right)\left\{1+\left(\frac{w^2}{3}-\frac{2w^4}{45}+\cdots\right)+\left(\frac{w^2}{3}-\frac{2w^4}{45}+\cdots\right)^2+\cdots\right\}=\frac{\pi}{w^2}+\frac{1}{w}+\frac{\pi}{3}+\frac{w}{3}+\cdots$$
$$=\frac{\pi}{(z-\pi)^2}+\frac{1}{(z-\pi)}+\frac{\pi}{3}+\frac{(z-\pi)}{3}+\cdots$$

Series$\left[z/\mathrm{Sin}[z]^2,\ \{z,\ \pi,\ 4\}\right]$

$$\frac{\pi}{(z-\pi)^2}+\frac{1}{z-\pi}+\frac{\pi}{3}+\frac{z-\pi}{3}+\frac{1}{15}\pi\,(z-\pi)^2+\frac{1}{15}(z-\pi)^3+\frac{2}{189}\pi\,(z-\pi)^4+O[z-\pi]^5$$

$\frac{z}{\sin^2 z}$ has second order pole at $z=\pi$ and its residue is 1.

■ Residue of $f(z)=\frac{\cot\pi z}{z(z+2)}$ at $z=0$

If you are alone in a desert island or in an elevator, it is convenient to expand the $f(z)$ and search residue.

$$\frac{\cot\pi z}{z(z+2)}=\frac{\left(\frac{1}{\pi z}-\frac{\pi z}{3}-\frac{\pi^3 z^3}{45}+\cdots\right)}{2z(1+z/2)}=\frac{1}{2z}\left(\frac{1}{\pi z}-\frac{\pi z}{3}-\frac{\pi^3 z^3}{45}+\cdots\right)\left(1-\frac{z}{2}+\frac{z^2}{4}-\cdots\right)$$
$$=\frac{1}{2z}\left(\frac{1}{\pi z}-\frac{1}{2\pi}-\frac{(3-4\pi^2)z}{12\pi}+\cdots\right)=\frac{1}{2\pi z^2}-\frac{1}{4\pi z}-\frac{(3-4\pi^2)}{24\pi}+\cdots$$

It has 2nd order pole at $z=0$ and its residue is $-\frac{1}{4\pi}$

But, when you are in a cozy office with your computer, use computer as,

```
Series[Cot[π z] / (z (z + 2)), {z, 0, 4}]
```

$$\frac{1}{2\pi z^2}-\frac{1}{4\pi z}+\left(\frac{1}{8\pi}-\frac{\pi}{6}\right)+\left(-\frac{1}{16\pi}+\frac{\pi}{12}\right)z+\frac{\left(45-60\pi^2-16\pi^4\right)z^2}{1440\pi}+\frac{\left(-45+60\pi^2+16\pi^4\right)z^3}{2880\pi}+\frac{\left(945-1260\pi^2-336\pi^4-128\pi^6\right)z^4}{120960\pi}+O[z]^5$$

There is an even more powerful command in *Mathematica,* Residue[…] let's see

```
Residue[Cot[π z] / (z (z + 2)), {z, 0}]
```

$$-\frac{1}{4\pi}$$

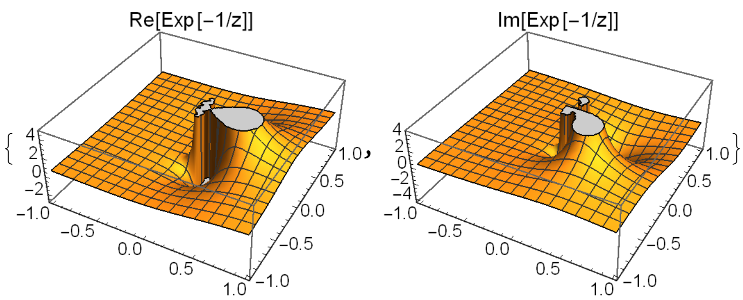
Is Residue[…] works for essential singularity? Let me show you.

- Residue of $f(z) = e^{-1/z}$ at $z = 0$

```
Residue[e^(-1/z), {z, 0}]
Residue[e^(-1/z), {z, 0}]
```

No, it does not work well.

```
{Plot3D[Exp[x / (x^2 + y^2)] Cos[y / (x^2 + y^2)], {x, -1, 1}, {y, -1, 1},
  PlotLabel → "Re[Exp[-1/z]]"], Plot3D[-Exp[x / (x^2 + y^2)] Sin[y / (x^2 + y^2)],
  {x, -1, 1}, {y, -1, 1}, PlotLabel → "Im[Exp[-1/z]]"]}
```



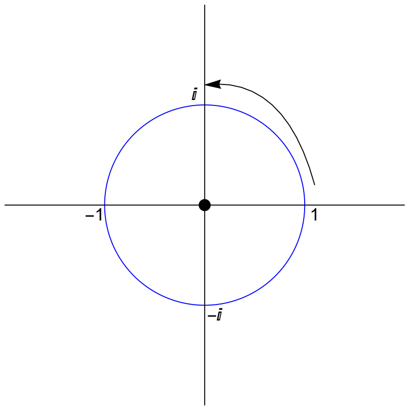
But do not be disappointed.

By Cauchy integral, we obtain

$$\begin{cases} a_n \oint (z-z_0)^n \, dz = 0\,, & n \neq -1 \\ a_{-1} \oint (z-z_0)^{-1} \, dz = 2\pi i\, a_{-1}\,, & n = -1 \end{cases}$$

$$\text{Therefore,} \quad \oint f(z)\, dz = 2\pi i\, a_{-1}$$

For appropriate contour, you can get a residue by Cauchy integral. We want to get residue of $f(z) = e^{-1/z}$ at $z = 0$.



Determine appropriate contour ; $1 \to i \to -1 \to -i \to 1$

Then take the numerical integration(faster than symbolic integration) $\frac{1}{2\pi i}\oint f(z)\, dz = a_{-1}$

```
NIntegrate[e^(-1/z), {z, 1, i, -1, -i, 1}] / (2 Pi i)
```

```
-1. - 8.49738 × 10^-14 i
```

Remove the noise

```
Chop[NIntegrate[e^(-1/z), {z, 1, i, -1, -i, 1}] / (2 Pi i)]
```

```
-1.
```

Is that correct? Let's see the series expansion of $e^{-1/z}$

$$e^{-1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}(-1/z)^n = 1 - \frac{1}{z} + \frac{1}{2!\, z^2} + \cdots$$

```
Series[e^-w, {w, 0, 4}] /. w → 1/z
```

$$1 - \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 - \frac{1}{6}\left(\frac{1}{z}\right)^3 + \frac{1}{24}\left(\frac{1}{z}\right)^4 + O\left[\frac{1}{z}\right]^5$$

Coefficient of $1/z$ is -1; correct!

Then why do we care about the residue? You might have noticed that residue is related to contour integral in the above example. Consider following integral

$$\oint_C \frac{dz}{(z-z_0)^n}$$

with C is small contour encircles z_0 therefore, in the C, $z = z_0 + \rho e^{i\theta}$

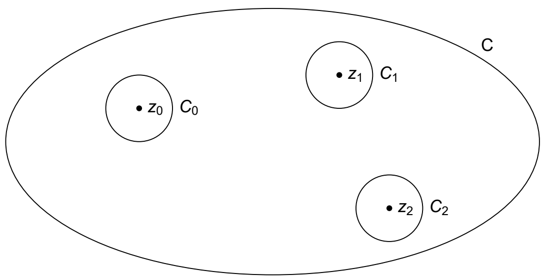
$$\oint_C \frac{dz}{(z-z_0)^n} = \int_{\theta=0}^{2\pi} \frac{i\rho e^{i\theta}\, d\theta}{(z_0+\rho e^{i\theta}-z_0)^n} = \int_{\theta=0}^{2\pi} \frac{i\rho e^{i\theta}\, d\theta}{(\rho e^{i\theta})^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases}$$

For function $f(z)$ with its Laurent series $\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)^1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$,

the contour integral around the $z = z_0$ is

$$\begin{aligned}\oint_C f(z)\, dz &= \oint_C \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)^1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots dz \\ &= \cdots + 0 + 2\pi i\, a_{-1} + 0 + \cdots = 2\pi i\, a_{-1}\end{aligned}$$

Sometimes the contour contains more than two poles



In this case,

$$\oint_C f(z)\, dz = 2\pi i \sum(\text{encircled residues})$$

This is the residue theorem. With residue theorem, we can replace the problem of evaluating contour integral into the problem of computing residues at the enclosed singular points.

<Remark 4.1>

- **Residue[$f(z)$, {z, z_0}]** gives you a residue of $f(z)$ at $z = z_0$ or
- **Chop[NIntegrate[$f(z)$, {z, proper contour}]/(2 Pi i)]** , **NResidue[$f(z)$, {z, z_0}]** gives residue numerically

4.2 Cauchy Principal Value

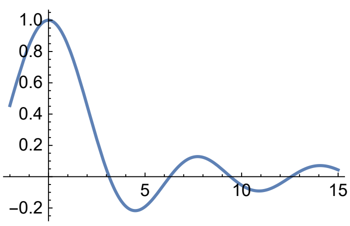
Definitions.

- **Cauchy principal value is the real integral of a $f(x)$ with an isolated singularity on the integration path at the point x_0**
- $$P \int_a^b f(x)\, dx = \lim_{\delta \to 0+} \left[\int_a^{x_0 - \delta} f(x)\, dx + \int_{x_0+\delta}^b f(x)\, dx \right]$$

For example, consider $\int_0^\infty \frac{\sin(x)}{x}\, dx$.

```
{Plot[Sin[x] / x, {x, -2, 15}], ∫_0^∞ Sin[x] / x ⅆx}
```

{



, $\frac{\pi}{2}$}

By exponential form,

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \int_0^\infty \frac{e^{ix} - e^{-ix}}{2ix}\, dx = \int_0^\infty \frac{e^{ix}}{2ix}\, dx - \int_0^\infty \frac{e^{-ix}}{2ix}\, dx$$

where $\frac{e^{\pm ix}}{2ix}$ has simple pole at $z = 0$. Therefore we cannot define above integral. However,

$$\lim_{\delta \to 0} \int_\delta^\infty \frac{e^{ix}}{2ix}\, dx - \int_\delta^\infty \frac{e^{-ix}}{2ix}\, dx$$

exists so long as δ is not precisely zero. We can change second integral as

$$\int_{\delta}^{\infty} \frac{e^{-i x}}{2 i x} dx \longrightarrow \begin{pmatrix} x \to -y \\ dx \to -dy \end{pmatrix} \longrightarrow \int_{-\delta}^{-\infty} \frac{e^{i y}}{-2 i y} - dy = -\int_{-\infty}^{-\delta} \frac{e^{i x}}{2 i x} dx$$

Then,

$$\lim_{\delta \to 0} \int_{\delta}^{\infty} \frac{e^{i x}}{2 i x} dx - \int_{\delta}^{\infty} \frac{e^{-i x}}{2 i x} dx = \lim_{\delta \to 0} \int_{\delta}^{\infty} \frac{e^{i x}}{2 i x} dx + \int_{-\infty}^{-\delta} \frac{e^{i x}}{2 i x} dx$$

By the definition of Cauchy principal value, we can write above integral as

$$\lim_{\delta \to 0+} \left[\int_{-\infty}^{-\delta} \frac{e^{i x}}{2 i x} dx + \int_{\delta}^{\infty} \frac{e^{i x}}{2 i x} dx \right] = P \int_{-\infty}^{\infty} \frac{e^{i x}}{2 i x} dx$$

Finally,

$$\int_{0}^{\infty} \frac{\sin(x)}{x} dx = P \int_{-\infty}^{\infty} \frac{e^{i x}}{2 i x} dx$$

Setting **PrincipalValue** option to **True**, you can get Cauchy principal value by *Mathematica*'s Integral command

```
{Integrate[e^(i x) / (2 i x), {x, -∞, ∞}], "→ does not return the correct answer"}
```

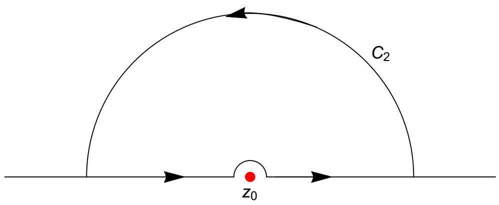
$$\left\{ \int_{-\infty}^{\infty} -\frac{i\, e^{i x}}{2 x} dx, \to \text{does not return the correct answer} \right\}$$

```
Integrate[e^(i x) / (2 i x), {x, -∞, ∞}, PrincipalValue → True] == ∫_0^∞ Sin[x] / x dx == π / 2
```

```
True
```

We already learned that getting Cauchy principal value of some functions by *Mathematica.* Then how does *Mathematica* knows it ? Consider some function $f(z)$ that has a simple pole at z_0 ;

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + \cdots$$



Let the integration over a tiny semicircle or radius δ $\begin{cases} I_{\text{over}} & \text{integrate from } \pi \text{ to } 0 \\ I_{\text{under}} & \text{integrate from } \pi \text{ to } 2\pi \end{cases}$

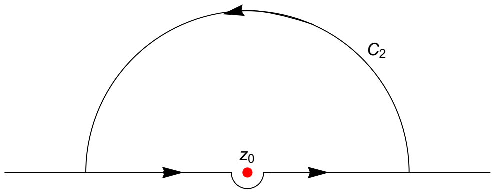
By the substitution $z \to z_0 + \delta e^{i\theta}$, $dz \to i \delta e^{i\theta} d\theta$,

$$I_{\text{over}} = \int_{\pi}^{0} i \delta e^{i\theta} d\theta \left[\frac{a_{-1}}{\delta e^{i\theta}} + a_0 + \cdots \right] = \int_{\pi}^{0} \left[i\, a_{-1} + i \delta e^{i\theta} a_0 + \cdots \right] d\theta = -i \pi a_{-1}$$

By the residue theorem, the value of integral of the given contour is $2\pi i$ (sum of total residues). In short,

$$\oint f(z)\, dz = P \int f(z)\, dz + I_{\text{over}} + \int_{C_2} f(z)\, dz = 2\pi i \sum \text{residues (other than at } z = z_0)$$

$$P \int f(z)\, dz = -I_{\text{over}} - \int_{C_2} f(z)\, dz + 2\pi i \sum \text{residues (other than at } z = z_0)$$



Similarly,

$$I_{\text{under}} = \int_{\pi}^{2\pi} i\,\delta e^{i\theta} d\theta\left[\frac{a_{-1}}{\delta e^{i\theta}} + a_0 + \cdots\right] = \int_{\pi}^{2\pi}\left[i\,a_{-1} + i\,\delta e^{i\theta} a_0 + \cdots\right] d\theta = i\,\pi\,a_{-1}$$

$$\oint f(z)\,dz = P\int f(z)\,dz + I_{\text{under}} + \int_{C_2} f(z)\,dz = 2\pi i \sum \text{residues}$$

$$P\int f(z)\,dz = -I_{\text{under}} - \int_{C_2} f(z)\,dz + 2\pi i \sum \text{residues}$$

Exercise.

Show that $P\int_0^{\infty} \frac{x^{-p}}{1-x}\,dx = -\pi\cot(p\,\pi)$ $(0 < p < 1)$

Hint: Pole expansion of $\cot(z) \longrightarrow \frac{1}{z} + \sum_{n=1}^{\infty} 2z\left(\frac{1}{z^2-(n\pi)^2}\right)$

```
Integrate[x^-p / (1 - x), {x, 0, ∞}, PrincipalValue → True]
```

```
Integrate[x^-p/(1 - x), {x, 0, ∞}, PrincipalValue → True]
```

It seems that *Mathematica* cannot solve this problem. So, prepare your pen and paper

By definition,

$$P\int_0^{\infty} \frac{x^{-p}}{1-x}\,dx = \lim_{\delta\to 0}\int_0^{1-\delta} \frac{x^{-p}}{1-x}\,dx + \int_{1+\delta}^{\infty} \frac{x^{-p}}{1-x}\,dx$$

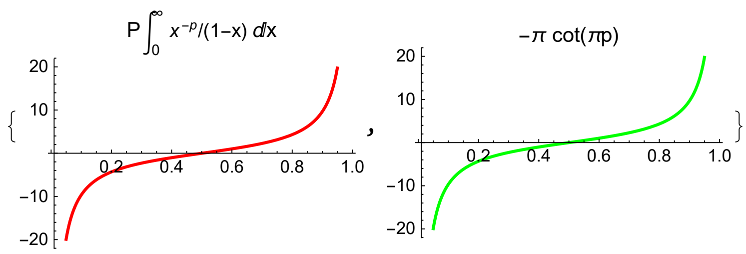
Maybe you need

$$\frac{1}{1-x} = \begin{cases} \sum_{n=0}^{\infty} x^n & \square \\ -\frac{1}{x}(1-1/x)^{-1} = -\sum_{n=0}^{\infty} x^{-n-1} & \square \end{cases}$$

After substitution,

$$\begin{aligned} P\int_0^{\infty} \frac{x^{-p}}{1-x}\,dx &= \lim_{\delta\to 0}\int_0^{1-\delta} \sum_{n=0}^{\infty} x^{n-p}\,dx + \int_{1+\delta}^{\infty} -\sum_{n=0}^{\infty} x^{-n-p-1}\,dx \\ &= \lim_{\delta\to 0}\sum_{n=0}^{\infty}\left[\int_0^{1-\delta} x^{n-p}\,dx - \int_{1+\delta}^{\infty} x^{-n-p-1}\,dx\right] \\ &= \lim_{\delta\to 0}\sum_{n=0}^{\infty}\left[\frac{(1-\delta)^{n-p+1}}{n-p+1} + \frac{(1+\delta)^{-n-p}}{-n-p}\right] \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{n-p+1} + \frac{1}{-n-p}\right] = \cdots + \frac{1}{-p-2} + \frac{1}{-p-1} + \frac{1}{-p} + \frac{1}{-p+1} + \frac{1}{-p+2} + \cdots \\ &= -\frac{1}{p} + \sum_{n=1}^{\infty} \frac{1}{-p-n} + \frac{1}{-p+n} = -\frac{1}{p} + \sum_{n=1}^{\infty} \frac{-2p}{p^2-n^2} \\ &= -\left(\frac{1}{p} + \sum_{n=1}^{\infty} \frac{2p}{p^2-n^2}\right) = -\pi\cot(\pi p) \quad (\text{with } 0 < p < 1) \quad \blacksquare \end{aligned}$$

```
{Plot[(NIntegrate[x^-p / (1 - x), {x, 0, 0.99}] + NIntegrate[x^-p / (1 - x), {x, 1.01, ∞}]),
  {p, 0.01, 0.99}, PlotStyle → Red, PlotLabel → "P∫_0^∞ x^-p/(1-x) ⅆx"],
 Plot[-π Cot[π p], {p, 0.01, 0.99}, PlotStyle → Green, PlotLabel → "-π cot(πp)"]}
```



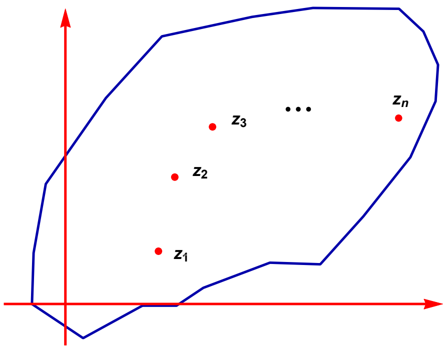
Almost same!

4.3 Pole Expansion of Meromorphic Functions

Meromorphic functions are referred to as analytic functions whose singularities are only isolated poles; no branch points ratio of two polynomials is good example of meromorphic function. These meromorphic functions can extended as pole expansion by Mittag-Leffler's theorem.

<Theorem 3> Mittag-Leffler's Theorem

■ $$f(z) = f(0) + \sum_{n=1}^{\infty} b_n\left(\frac{1}{z-z_n} + \frac{1}{z_n}\right)$$



Consider above contour C_N and some function $f(z)$ that has discrete simple pole at z_n , $n = 1, 2, 3, \cdots, N$ with respective residues b_n , $n = 1, 2, 3, \cdots, N$. The pole is ordered by its modulus; $0 < |z_1| \le |z_2| \le \cdots$.

We assume that $\lim_{z\to\infty} |f(z)/z| \longrightarrow 0$ Note that $f(z)$ must be analytic at $z = 0$.

Then there exists a contour integral

$$I_N = \oint_{C_N} \frac{f(w)}{w(w-z)} dw$$

Now $\frac{f(w)}{w(w-z)}$ has simple pole at $w = 0,\ z,\ z_n,\ \ n = 1, 2, 3, \cdots, N$.

Respective residues are decided as following;

$$\begin{cases} \lim_{w\to 0}\,(w-0)\,\frac{f(w)}{w(w-z)} = \frac{f(0)}{-z} & \text{at } \ w = 0 \\ \lim_{w\to z}\,(w-z)\,\frac{f(w)}{w(w-z)} = \frac{f(z)}{z} & \text{at } \ w = z \\ \lim_{w\to z_n}\,(w-z_n)\,\frac{f(w)}{w(w-z)} = \lim_{w\to z_n} \frac{(w-z_n)\left[\frac{b_n}{(w-z_n)} + \cdots\right]}{w(w-z)} = \lim_{w\to z_n} \frac{b_n + \cdots}{w(w-z)} = \frac{b_n}{z_n(z_n-z)} & \text{at } \ w = z_n \end{cases}$$

Since I_N means $2\pi i$ (sum of residues),

$$I_N = \oint_{C_N} \frac{f(w)}{w(w-z)} dw = 2\pi i \left(\frac{f(0)}{-z} + \frac{f(z)}{z} + \sum_{n=1}^{N} \frac{b_n}{z_n(z_n-z)} \right)$$

Taking the large N and then I_N goes to 0. Finally we have,

$$0 = 2\pi i \left(\frac{f(0)}{-z} + \frac{f(z)}{z} + \sum_{n=1}^{N} \frac{b_n}{z_n(z_n-z)} \right), \quad \frac{f(z)}{z} = \frac{f(0)}{z} - \sum_{n=1}^{N} \frac{b_n}{z_n(z_n-z)}$$

$$f(z) = f(0) - \sum_{n=1}^{N} \frac{b_n z}{z_n(z_n-z)} \quad , \quad f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z-z_n} + \frac{1}{z_n} \right)$$

$\tan(z)$ for instance,

$$\tan(z) = \frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}$$

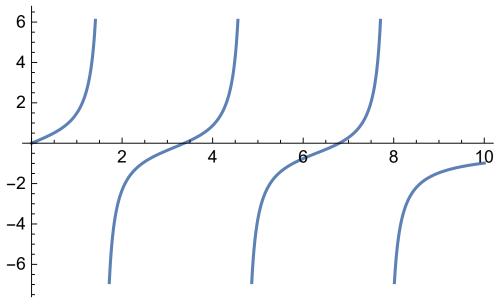
which has singular point at $z = \pm \frac{(2n+1)}{2}\pi$. Get the residue at each pole

$$b_n = \lim_{z\to(2n+1)\pi/2} \frac{[z-(2n+1)\pi/2]\sin(z)}{\cos(z)}$$
$$= \left[\frac{\sin(z) + [z-(2n+1)\pi/2]\cos(z)}{-\sin(z)} \right]_{z=(2n+1)\pi/2} = -1$$

By the Mittag-Leffler's theorem,

$$\tan(z) = \tan(0) + \sum_{n=0}^{\infty} (-1) \left(\frac{1}{z-(2n+1)\pi/2} + \frac{1}{(2n+1)\pi/2} \right) + \sum_{n=0}^{\infty} (-1) \left(\frac{1}{z+(2n+1)\pi/2} + \frac{1}{-(2n+1)\pi/2} \right)$$
$$= \sum_{n=0}^{\infty} (-1) \left(\frac{1}{z-(2n+1)\pi/2} + \frac{1}{z+(2n+1)\pi/2} \right) = \sum_{n=0}^{\infty} \left(\frac{2z}{[(2n+1)\pi/2]^2 - z^2} \right)$$
$$= 2z \left(\frac{1}{[\pi/2]^2 - z^2} + \frac{1}{[3\pi/2]^2 - z^2} + \frac{1}{[5\pi/2]^2 - z^2} + \cdots \right)$$

```
Plot[2 z (1/((π/2)^2 - z^2) + 1/((3 π/2)^2 - z^2) + 1/((5 π/2)^2 - z^2)), {z, 0, 10}]
```



Similar to its mama!

Exercise.
Find the pole expansion of $\cot(z)$

$\cot(z)$ has simple pole at $\pm n\pi$. So it has singularity at $z = 0$. Then how can we apply the Mittag-Leffler's theorem?

Maybe we consider bit modified version; analytic at $z = 0$

```
Series[Cot[z], {z, 0, 3}]
```

$$\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + O[z]^4$$

```
Residue[Cot[z], {z, 0}]
```

1

Since $f(z) = 1/z + \cdots$, subtract $1/z$ from $\cot(z)$, then it will be analytic at $z = 0$. Now let's see the slightly changed version, $\cot(z) - 1/z$ which is analytic at $z = 0$ and therefore we can apply the Mittag-Leffler's theorem.

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n\left(\frac{1}{z-z_n} + \frac{1}{z_n}\right)$$

The recipe tells that you need to obtain residue. Then go ahead!

$$b_n = \lim_{z\to n\pi}(z - n\pi)[\cot(z) - 1/z] = \lim_{z\to n\pi}\frac{(z-n\pi)[z\cos(z)-\sin(z)]}{z\sin(z)}$$

$$= \left[\frac{z\cos(z) - \sin(z) + (z-n\pi)(-z\sin(z))}{\sin(z) + z\cos(z)}\right]_{z=n\pi} = \left[\frac{z\cos(z)}{z\cos(z)}\right]_{z=n\pi} = 1$$

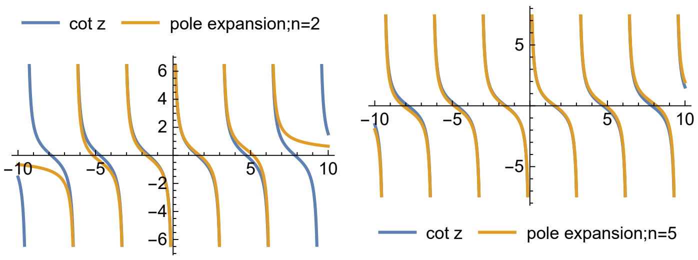
Then the theorem gives you

$$\cot(z) - \frac{1}{z} = 0 + \sum_{n=1}^{\infty} 1\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right) + \sum_{n=1}^{\infty} 1\left(\frac{1}{z+n\pi} - \frac{1}{n\pi}\right) = +\sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right) = \sum_{n=1}^{\infty} 2z\left(\frac{1}{z^2-(n\pi)^2}\right)$$

$$\text{Therefore,}\quad \cot(z) = \frac{1}{z} + \sum_{n=1}^{\infty} 2z\left(\frac{1}{z^2-(n\pi)^2}\right)$$

```
GraphicsGrid[{{Plot[{Cot[z], 1/z + Sum[2 z/(z² - n² π²), {n, 1, 2}]},
    {z, -10, 10}, PlotLegends → Placed[{"cot z", "pole expansion;n=2"}, Above]],
   Plot[{Cot[z], 1/z + Sum[2 z/(z² - n² π²), {n, 1, 5}]}, {z, -10, 10},
    PlotLegends → Placed[{"cot z", "pole expansion;n=5"}, Below]]}}]
```



Exercise.
Find the pole expansion of sec(z) and csc(z).

```
Series[Sec[z], {z, 0, 4}]
```

$$1 + \frac{z^2}{2} + \frac{5z^4}{24} + O[z]^5$$

Directly apply to Mittag-Leffler theorem sec(z) has singular point
$z_n = (n + 1/2)\pi$ n: all integer $-\infty \sim \infty$

$$b_n = \lim_{z\to(n+1/2)\pi}\frac{[z-(n+1/2)\pi]}{\cos(z)}$$

$$= \left[\frac{1}{-\sin(z)}\right]_{z=(n+1/2)\pi} = (-1)^{n+1}$$

By theorem,

$$\sec(z) = 1 + \sum_{n=-\infty}^{\infty} (-1)^{n+1} \left(\frac{1}{z-(n+1/2)\,\pi} + \frac{1}{(n+1/2)\,\pi}\right)$$

The second term is,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(n+1/2)\,\pi} = \cdots - \frac{2}{5\,\pi} + \frac{2}{3\,\pi} - \frac{2}{\pi} - \frac{2}{\pi} + \frac{2}{3\,\pi} - \frac{2}{5\,\pi} + \cdots = -\frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

$\arctan(x)$ function will help you. ; $\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\,k+1)} x^{2\,k+1}$

```
Series[ArcTan[x], {x, 0, 7}]
```

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + O[x]^8$$

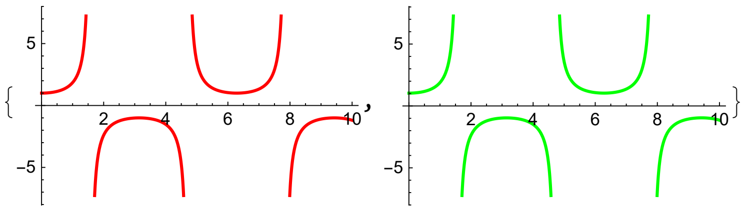
or you can obtain by on your own ; starting from $D(\arctan(x)) = 1/(1+x^2)$,
$\arctan(x) = \int (1+x^2)^{-1}\, dx$
therefore, $\arctan(x) = \int (1 - x^2 + x^4 - \cdots)\, dx = x - x^3/3 + x^5/5 - x^7/7 \cdots$ anyway,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(n+1/2)\,\pi} = -\frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right) = -\frac{4}{\pi}(\arctan(1)) = -\frac{4}{\pi}\left(\frac{\pi}{4}\right) = -1$$

Hence,

$$\sec(z) = 1 - 1 + \sum_{n=-\infty}^{\infty} (-1)^{n+1} \left(\frac{1}{z-(n+1/2)\,\pi}\right) = -\frac{\pi}{z^2-(1/2)^2\,\pi^2} + \frac{3\,\pi}{z^2-(3/2)^2\,\pi^2} + \cdots$$
$$= \pi \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2\,n+1)}{z^2-(n+1/2)^2\,\pi^2}$$

```
{Plot[Sec[z], {z, 0, 10}, PlotStyle → Red],
 Plot[π Sum[((-1)^(n+1) (2 n + 1))/(z^2 - (n + 1/2)^2 π^2), {n, 0, 8}], {z, 0, 10}, PlotStyle → Green]}
```



SInce csc(0) diverge, we have to apply modified version.

```
Series[Csc[x], {x, 0, 4}]
```

$$\frac{1}{x} + \frac{x}{6} + \frac{7\,x^3}{360} + O[x]^5$$

I'd like to use $\csc(x) - 1/x$, maybe you agree with me. Go get the residue now!

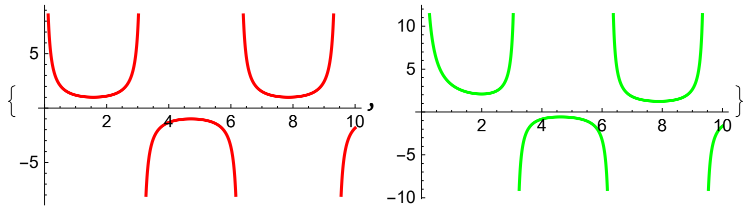
$$b_n = \lim_{z\to n\,\pi} [z - n\,\pi]\,(\cot(z) - 1/z) = \lim_{z\to n\,\pi} \frac{[z-n\,\pi]\,(z-\sin(z))}{z\sin(z)}$$
$$= \left[\frac{(z-\sin(z)) + (z-n\,\pi)\,(1-\cos(z))}{\sin(z) + z\cos(z)}\right]_{z=n\,\pi} = (-1)^n$$

By theorem,

$$\csc(z) - \frac{1}{z} = 0 + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z-n\,\pi} + \frac{1}{n\,\pi}\right) + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z+n\,\pi} - \frac{1}{n\,\pi}\right)$$
$$= \sum_{n=1}^{\infty} (-1)^n \frac{2\,z}{z^2-n^2\,\pi^2}$$

$$\therefore \csc(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2\,z}{z^2-n^2\,\pi^2}$$

```
{Plot[Csc[z], {z, 0, 10}, PlotStyle → Red],
 Plot[1/z + Sum[((-1)^n 2 z)/(z^2 - n^2 π^2), {n, 0, 8}], {z, 0, 10}, PlotStyle → Green]}
```



4.4 Counting Poles and Zeros

<Theorem 4> Counting Poles and Zeros

Now let's find out how to obtain information about the number of poles and zeros.
function $f(z)$ relative to point z_0 can be expressed as follows.

$$f(z) = (z - z_0)^{\mu}\, g(z) \quad \begin{cases} \mu > 0 & (z_0 \text{ is zero}) \\ \mu < 0 & (z_0 \text{ is pole}) \end{cases}$$

with $g(z)$ finite and nonzero at $z = z_0$. Then,

$$\frac{f'(z)}{f(z)} = \frac{\mu(z-z_0)^{\mu-1}\, g(z) + (z-z_0)^{\mu}\, g'(z)}{(z-z_0)^{\mu}\, g(z)} = \frac{\mu}{(z-z_0)} + \frac{g'(z)}{g(z)}$$

Since $g(z)$ is nonzero and finite at $z = z_0$, $g'(z)/g(z)$ cannot be singular. Therefore, $f'(z)/f(z)$ has simple pole at $z = z_0$ and its residue is μ. Applying it to powerful residue theorem, we have

$$\oint_C \frac{f'(z)}{f(z)}\, dz = 2\pi i \times (\text{sum of each } \mu)$$

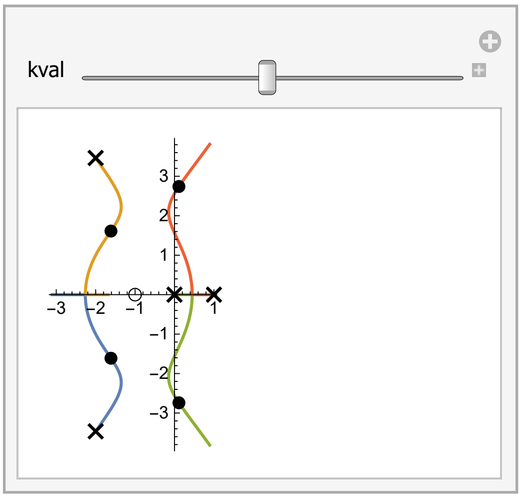
From $f(z) = (z - z_0)^{\mu}\, g(z)$ $\begin{cases} \mu > 0 & (z_0 \text{ is zero}) \; f \text{ has } \mu - \text{fold zero at } z = z_0 \longrightarrow \mu \text{ zeros} \\ \mu < 0 & (z_0 \text{ is pole}) \; f \text{ has } (-\mu) - \text{fold pole at } z = z_0 \longrightarrow -\mu \text{ poles} \end{cases}$,

$$\oint_C \frac{f'(z)}{f(z)}\, dz = 2\pi i\, (N_f - P_f)\,,$$

where $\begin{cases} N_f : \text{number of zeros of } f(z) \\ P_f : \text{number of poles of } f(z) \end{cases}$ within the region enclosed by C

For instance, $f(s) = \frac{(1+s)}{(-1+s)\, s\, (16+4\, s+s^2)}$, there are 3 poles and 1 zero in the entire plane. **RootLocusPlot[⋯]** shows you all poles and zeros $f(s)$ has.

```
Manipulate[RootLocusPlot[(k (1 + s) / ((-1 + s) s (16 + 4 s + s^2)))ᵀ, {k, 0, 80},
  PoleZeroMarkers → {Automatic, "ParameterValues" → kval}], {kval, 0, 80}]
```



where × 's are simple poles and o's are zeros. This diagram even shows you that variation of zeros as k changes. As the time goes, ●'s move to o . This method is used in control theory to see how the system changes. Maybe we can see it later or not. But it is evident that this graphic is fascinating. From the figure we have,

pole (x) P_f	zero (o) N_f
4	1

$\} \longrightarrow N_f - P_f = -3$

```
N[-3 * 2 π ⅈ]
0. - 18.8496 ⅈ
```

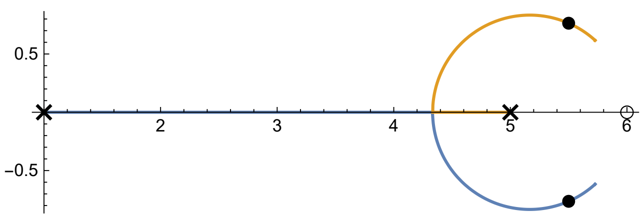
By contour integral, check! (Choose contour that encircles all of poles and zeros. I take the radius 10 circle)

```
Chop[NIntegrate[(D[(1 + s) / ((-1 + s) s (16 + 4 s + s^2)), s]) / ((1 + s) / ((-1 + s) s (16 + 4 s + s^2))),
  {s, 10, 10 ⅈ, -10, -10 ⅈ, 10}]]
0. - 18.8496 ⅈ
```

Correct!

Next, consider 'degenerated' case.

```
RootLocusPlot[(k (s - 6)^2 / ((s - 1) (s - 5))), {k, 0, 10}, PlotRange → All]
```



It seems that $P_f(x)$ is 2 and $N_f(o)$ is 1 . Contour integral may have the value

```
N[(1 - 2) * 2 π ⅈ]
0. - 6.28319 ⅈ
```

Check!

```
Chop[NIntegrate[D[(s - 6)^2/((s - 1) (s - 5)), s] / ((s - 6)^2/((s - 1) (s - 5))), {s, 7, 7 ⅈ, -7, -7 ⅈ, 7}]]
0
```

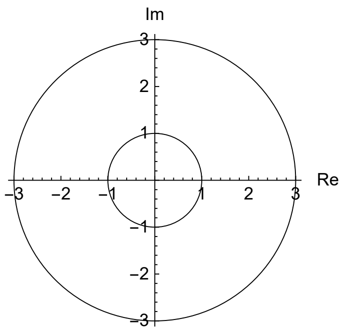
The reason why the two results are not the same is because of you haven't count the folded zero

Since $\frac{(s-6)^2}{(s-1)(s-5)}$ has two-fold zero at $s = 6$, the number of zeros N_f is 2. So

```
N[(2 - 2) * 2 π ⅈ]
0.
```

Now it makes sense.

Exercise.
Count the zeros of $F(z) = z^3 - 2z + 11$ with moduli between 1 and 3

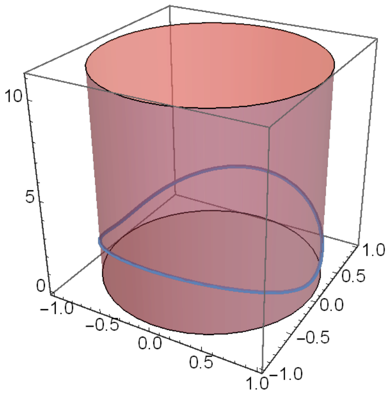


To solve this problem, you need **Rouché's theorem**

- If $f(z)$ and $g(z)$ are analytic in the region bounded by a curve C and $|f(z)| > |g(z)|$ on C , then $f(z)$ and $f(z) + g(z)$ have the same number of zeros in the region bounded by C

First, compute the number of zeros within $|z| = 1$. Let $f(z) = 11$ and $g(z) = z^3 - 2z$.

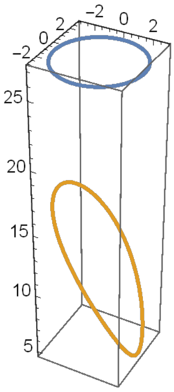
```
Show[ParametricPlot3D[{Cos[u], Sin[u], Abs[e^(3 i u) - 2 e^(i u)]},
  {u, 0, 2 π}, PlotLegends → "Abs[g(z)]\n and f(z)=11"],
 Graphics3D[{Opacity[0.4], Pink, Cylinder[{{0, 0, 0}, {0, 0, 11}}, 1]}],
 PlotRange → All, BoxRatios -> {1, 1, 1}]
```



Since $|g(z)| < 11$, $f(z) = 11$ and $f(z) + g(z) = z^3 - 2z + 11$ have the same number of zeros within the $|z| = 1$ where $f(z) = 1$ has no zero therefore, $F(z)$ has no zero in the $|z| = 1$.

Next, compute the number of zeros within $|z| = 3$. Let $f(z) = z^3$ and $g(z) = 11 - 2z$

```
ParametricPlot3D[
 {{3 Cos[u], 3 Sin[u], Abs[27 e^(3 i u)]}, {3 Cos[u], 3 Sin[u], Abs[11 - 6 e^(i u)]}},
 {u, 0, 2 π}, PlotLegends → {"Abs[f(z)]", " Abs[g(z)]"}]
```



Since $|g(z)| < |f(z)| = 27$, $f(z) = z^3$ and $f(z) + g(z) = z^3 - 2z + 11$ have the same number of zeros within the $|z| = 3$ where $f(z) = z^3$ has 3 zeros therefore, $F(z)$ has 3 zeros in the $|z| = 3$.

Hence, the number of zeros of $F(z) = z^3 - 2z + 11$ with moduli between 1 and 3 is 3. ■

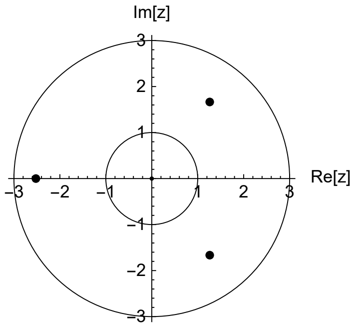
A computer makes this problem more simple.

```
N[Solve[z^3 - 2 z + 11 == 0, z]]
```

```
{{z → -2.52217}, {z → 1.26108 + 1.66463 i}, {z → 1.26108 - 1.66463 i}}
```

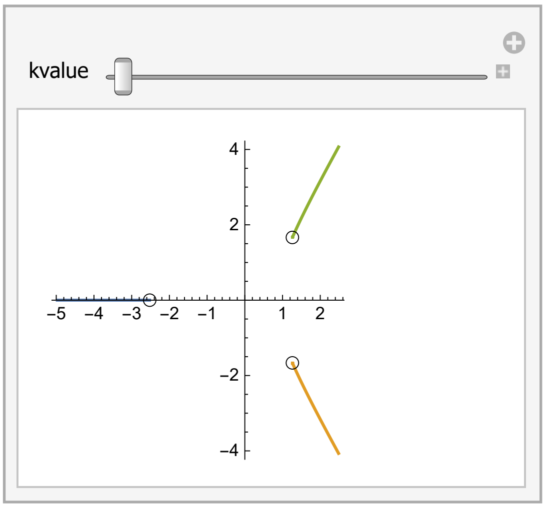
Display the zeros in Argand plane.

```
Show[
 ArgandPlot[z /. {{z → -2.52217}, {z → 1.26108 + 1.66463 ⅈ}, {z → 1.26108 - 1.66463 ⅈ}}],
 Graphics[{Circle[], Circle[{0, 0}, 3]}]]
```



Or use RootLocusPlot

```
Manipulate[RootLocusPlot[TransferFunctionModel[k (s³ - 2 s + 11), s], {k, 0, 10},
  PoleZeroMarkers → {Automatic, "ParameterValues" → kvalue}], {kvalue, 0, 10}]
```



pole (x) P_f	zero (o) N_f
0	3

Or use contour integral on proper path.

```
Chop[NIntegrate[(D[(s³ - 2 s + 11), s]) / (s³ - 2 s + 11), {s, 3, 3 ⅈ, -3, -3 ⅈ, 3}]] / (2 π ⅈ) -
 Chop[NIntegrate[(D[(s³ - 2 s + 11), s]) / (s³ - 2 s + 11), {s, 1, ⅈ, -1, -ⅈ, 1}]] / (2 π ⅈ)
```

```
3. + 0. ⅈ
```

<Remark 4.2>

- $\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i (N_f - P_f)$,

  where $\begin{cases} N_f : \text{number of zeros of } f(z) \\ P_f : \text{number of poles of } f(z) \end{cases}$ within the region enclosed by C

- **RootLocusPlot[...]** shows you that poles and zeros of the system. With **Manipulate[...]** , you can see how it changes as the time goes.

4.5 Product Expansion and Entire Functions

Since entire functions such as $\sin(z)$ or $\cos(z)$ do not have any pole, we cannot derive the pole expansion of those functions. Instead, we can derive the product expansion for them.

From,

$$\frac{f'(z)}{f(z)} = \frac{\mu(z-z_0)^{\mu-1}\,g(z) + (z-z_0)^{\mu}\,g'(z)}{(z-z_0)^{\mu}\,g(z)} = \frac{\mu}{(z-z_0)} + \frac{g'(z)}{g(z)}$$

If $f(z)$ is an entire function, then $f'(z)/f(z)$ will be meromorphic (singularities for only poles!) with all its pole simple.

Fortunately, we can use Mittag-Leffler theorem for function which has discrete simple poles

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n\left(\frac{1}{z-z_n} + \frac{1}{z_n}\right) \longrightarrow \frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left(\frac{1}{z-z_n} + \frac{1}{z_n}\right)$$

And integrate it

$$\begin{aligned}\int_0^z dz\,\frac{f'(z)}{f(z)} = \ln[f(z)] - \ln[f(0)] \quad &= \int_0^z dz\,\frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\int_0^z dz\left(\frac{1}{z-z_n} + \frac{1}{z_n}\right)\\ &= z\,\frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\ln(z-z_n) - \ln(-z_n) + \frac{z}{z_n}\right]\end{aligned}$$

Finally exponentiate them.

$$\begin{aligned}f(z)/f(0) &= \exp\left[z\,\frac{f'(0)}{f(0)}\right]\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)e^{z/z_n},\\ f(z) &= f(0)\,\exp\left[z\,\frac{f'(0)}{f(0)}\right]\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)e^{z/z_n}\end{aligned}$$

Exercise.
Derive the product expansion of $\sin(z)$.

Since $\sin(0) = 0$, we cannot directly apply above theorem. Thus we need to modify the original form.

Defining $f(z) = \sin(z)/z$, we can apply the theorem.

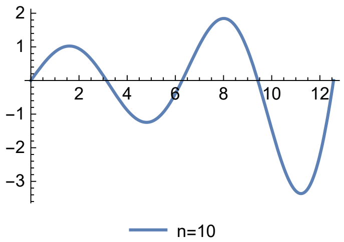
$$f(z) = \sin(z)/z = 1 - \frac{z^2}{3} + \cdots, \quad \begin{cases} f(0) & = 1 \\ f'(0) & = 0 \end{cases}$$

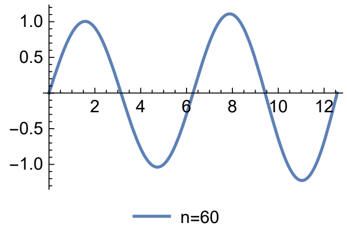
Since $f(z)$ has zeros at $z = \pm\, n\,\pi$, $z_n = \pm\, n\,\pi$ then,

$$\begin{aligned}f(z) = {}& \exp[0]\prod_{n=1}^{\infty}\left(1 - \frac{z}{n\,\pi}\right)e^{z/(n\,\pi)}\prod_{n=1}^{\infty}\left(1 + \frac{z}{n\,\pi}\right)e^{-z/(n\,\pi)}\\ = {}& \prod_{n=1}^{\infty}\left(1 - \frac{z}{n\,\pi}\right)\prod_{n=1}^{\infty}\left(1 + \frac{z}{n\,\pi}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\,\pi^2}\right)\end{aligned}$$

$$\therefore\ \sin(z) = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\,\pi^2}\right) \quad \blacksquare$$

```
GraphicsGrid[{{Plot[z (Product[(1 - z² / (n² π²)), {n, 1, 10}]),
    {z, 0, 4 π}, PlotLegends → Placed[{"n=10"}, Below]],
   Plot[z (Product[(1 - z² / (n² π²)), {n, 1, 60}]), {z, 0, 4 π},
    PlotLegends → Placed[{"n=60"}, Below]]}}]
```





Exercise.
Derive the product expansion of cos(z).

Luckily we can directly apply the theorem to cos(z) .

$$\cos(z) = 1 - \frac{z^2}{2} + \cdots, \quad \begin{cases} f(0) & = 1 \\ f'(0) & = 0 \end{cases}$$

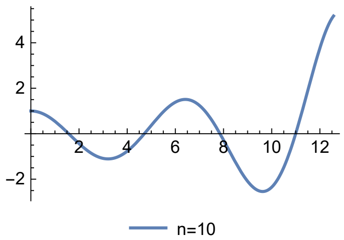
cos(z) has zeros at $\pm\left(\frac{2n-1}{2}\right)\pi$; $z_n = \pm\left(\frac{2n-1}{2}\right)\pi$

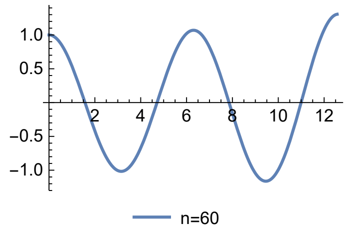
$$\cos(z) = \exp[0] \prod_{n=1}^{\infty}\left(1 - \frac{z}{(n-1/2)\pi}\right) e^{z/[(n-1/2)\pi]} \prod_{n=1}^{\infty}\left(1 + \frac{z}{(n-1/2)\pi}\right) e^{-z/[(n-1/2)\pi]}$$

$$= \prod_{n=1}^{\infty}\left(1 - \frac{z}{(n-1/2)\pi}\right) \prod_{n=1}^{\infty}\left(1 + \frac{z}{(n-1/2)\pi}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{(n-1/2)^2\pi^2}\right)$$

$$\therefore \cos(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{(n-1/2)^2\pi^2}\right) \quad \blacksquare$$

```
GraphicsGrid[
 {{Plot[(Product[(1 - z² / ((n - 1 / 2)² π²)), {n, 1, 10}]), {z, 0, 4 π}, PlotLegends →
     Placed[{"n=10"}, Below]], Plot[(Product[(1 - z² / ((n - 1 / 2)² π²)), {n, 1, 60}]),
    {z, 0, 4 π}, PlotLegends → Placed[{"n=60"}, Below]]}}]
```





4.6 Evaluation of Definite Integrals

Before we start,

<Remark 4.3>

- In *Mathematica,* you can calculate integral by typing **Integrate[integrand, {range}]** or just typing [ESC]dintt[ESC]

For most case, *Mathematica* gives you a right evaluation.

- **Trigonometric Integrals, Range (0, 2 π)**

Exercise.

Show that $\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a \pm b\sin\theta} = \frac{2\pi}{(a^2-b^2)^{1/2}}$, for $a > |b|$

$$\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \oint \frac{1/i\, z\, dz}{a \pm b/2\,(z+z^{-1})} = -i \oint \frac{dz}{a\, z \pm b/2\,(z^2+1)} = \mp i\, \frac{2}{b} \oint \frac{dz}{z^2 \pm (2\, a/b)\, z + 1} = \mp i\, \frac{2}{b} \oint \frac{dz}{(z-z_1)\,(z-z_2)}$$

with $|z_1| < 1 < |z_2|$

$$\text{For} \begin{cases} +(2\,a/b), & z_1, z_2 = -\frac{a}{b} \pm \sqrt{\left(\frac{a}{b}\right)^2 - 1} \longrightarrow \text{Residue}: \frac{2\pi i}{z_1-z_2} = \frac{2\pi i}{2\sqrt{(a/b)^2-1}} \\ -(2\,a/b), & z_2, z_1 = \frac{a}{b} \pm \sqrt{\left(\frac{a}{b}\right)^2 - 1} \longrightarrow \text{Residue}: \frac{2\pi i}{z_1-z_2} = \frac{2\pi i}{-2\sqrt{(a/b)^2-1}} \end{cases}$$

Therefore,

$$\mp i\, \frac{2}{b} \oint \frac{dz}{(z-z_1)\,(z-z_2)} = \mp i\, \frac{2}{b} \left(\pm \frac{2\pi i}{2\sqrt{(a/b)^2-1}} \right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Similarly,

$$\int_0^{2\pi} \frac{d\theta}{a \pm b\sin\theta} = \oint \frac{1/i\, z\, dz}{a \pm b/(2\,i)\,(z-z^{-1})} = -i \oint \frac{dz}{a\, z \pm b/(2\,i)\,(z^2-1)} = \mp i\, \frac{2\,i}{b} \oint \frac{dz}{z^2 \pm (2\, i\, a/b)\, z - 1} = \pm \frac{2}{b} \oint \frac{dz}{(z-z_1)\,(z-z_2)}$$

with $|z_1| < 1 < |z_2|$

$$\text{For} \begin{cases} +(2\,i\,a/b), & z_1, z_2 = -\frac{i\,a}{b} \pm \sqrt{\left(\frac{i\,a}{b}\right)^2 + 1} \longrightarrow \text{Residue}: \frac{2\pi i}{z_1-z_2} = \frac{2\pi i}{2\sqrt{-(a/b)^2+1}} \\ -(2\,i\,a/b), & z_2, z_1 = \frac{i\,a}{b} \pm \sqrt{\left(\frac{i\,a}{b}\right)^2 + 1} \longrightarrow \text{Residue}: \frac{2\pi i}{z_1-z_2} = \frac{2\pi i}{-2\sqrt{-(a/b)^2+1}} \end{cases}$$

Therefore,

$$\pm \frac{2}{b} \oint \frac{dz}{(z-z_1)\,(z-z_2)} = \pm \frac{2}{b} \left(\pm \frac{2\pi i}{2\sqrt{-(a/b)^2+1}} \right) = \pm \frac{2}{b} \left(\pm \frac{2\pi i}{2\, i\sqrt{(a/b)^2-1}} \right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Hence, $\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a \pm b\sin\theta} = \frac{2\pi}{(a^2-b^2)^{1/2}}$ ■

Exercise.

Show that $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi\, a}{(a^2-1)^{3/2}}$, $a > 1$

Since $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}$,

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\oint \frac{1/i\, z\, dz}{\left(a+\left(z+z^{-1}\right)/2\right)^2} = -\frac{i}{2}\oint \frac{1/z\, dz}{a^2+\left(z+z^{-1}\right)^2/4+a\left(z+z^{-1}\right)} = -\frac{i}{2}\oint \frac{1/z\, dz}{a^2+\left(z^2+z^{-2}+2\right)/4+a\left(z+z^{-\right.}}$$

$$= -\frac{i}{2}\oint \frac{dz}{a^2\, z+\left(z^3+z^{-1}+2\, z\right)/4+a\left(z^2+1\right)} = -\frac{i}{2}\oint \frac{dz}{z^3/4+a\, z^2+\left(a^2+1/2\right) z+a+z^{-1}/4}$$

```
Factor[z^3/4 + a z^2 + (a^2 + 1/2) z + a + 1/(4 z)]
```

$$\frac{\left(1+2\, a\, z+z^2\right)^2}{4\, z}$$

$$-\frac{i}{2}\oint \frac{dz}{z^3/4+a\, z^2+\left(a^2+1/2\right) z+a+z^{-1}/4} = -\frac{i}{2}\oint \frac{4\, z\, dz}{\left(1+2\, a\, z+z^2\right)^2} = -\frac{i}{2}\oint \frac{4\, z\, dz}{\left[(z-z_1)\,(z-z_2)\right]^2}$$

with $|z_1| < 1 < |z_2|$

Since $a > 1$, $\left|-a+\sqrt{a^2-1}\,\right| < 1 < \left|-a-\sqrt{a^2-1}\,\right|$.

So residue is

```
D[4 z/(z - (-a - Sqrt[a^2 - 1]))^2, z] /. z -> -a + Sqrt[a^2 - 1]
```

$$\frac{1}{-1+a^2} - \frac{-a+\sqrt{-1+a^2}}{\left(-1+a^2\right)^{3/2}}$$

```
Simplify[1/(-1 + a^2) - (-a + Sqrt[-1 + a^2])/(-1 + a^2)^(3/2)]
```

$$\frac{a}{\left(-1+a^2\right)^{3/2}}$$

Maybe using **Residue[ ]** will save your time.

```
Residue[1/(z^3/4 + a z^2 + (a^2 + 1/2) z + a + 1/(4 z)), {z, -a + Sqrt[a^2 - 1]}]
```

$$\frac{a}{\left(-1+a^2\right)^{3/2}}$$

Therefore,

$$-\frac{i}{2}\oint \frac{4\, z\, dz}{\left[(z-z_1)\,(z-z_2)\right]^2} = -\frac{i}{2}\left[2\,\pi\, i\, \frac{a}{\left(-1+a^2\right)^{3/2}}\right] = \frac{\pi\, a}{\left(-1+a^2\right)^{3/2}} \quad \blacksquare$$

Exercise.

Show that $\int_0^{2\pi} \frac{d\theta}{1-2\, t\cos\theta+t^2} = \frac{2\,\pi}{1-t^2}$, for $|t| < 1$.

What happens if $|t| > 1$ or $|t| = 1$?

$$\int_0^{2\pi} \frac{d\theta}{1-2\, t\cos\theta+t^2} = \int_0^{2\pi} \frac{d\theta}{\left(1+t^2\right)-2\, t\cos\theta}$$

$$\begin{cases} \left(1+t^2\right) > 2\, t & , \ |t| < 1 \\ \left(1+t^2\right) > 2\, t & , \ |t| > 1 \end{cases}$$

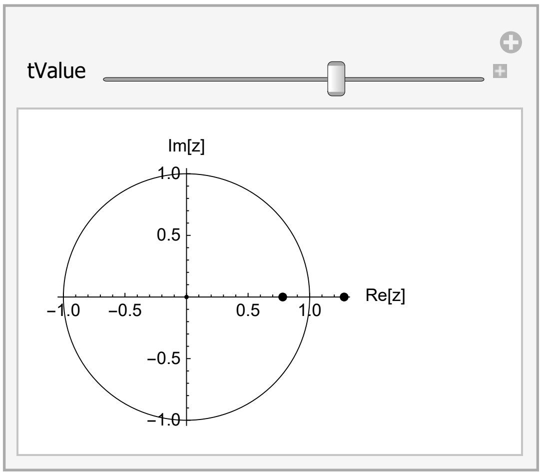
Remember $\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \frac{2\,\pi}{\left(a^2-b^2\right)^{1/2}}$, for $a > |b|$

$$\int_0^{2\pi}\frac{d\theta}{(1+t^2)-2t\cos\theta}=\begin{cases}\frac{2\pi}{((1+t^2)^2-(2t)^2)^{1/2}}=\frac{2\pi}{1-t^2} & (1+t^2)>2t \quad , \ |t|<1\\ \frac{2\pi}{((1+t^2)^2-(2t)^2)^{1/2}}=\frac{2\pi}{t^2-1} & (1+t^2)>2t \quad , \ |t|>1\\ \text{Having singluarity on the path} & , \ |t|=1\end{cases}$$

Replace the integral automatically.

$$\int_0^{2\pi}\frac{d\theta}{1-2t\cos\theta+t^2}=\oint\frac{1/i z\,dz}{1-t\,(z+z^{-1})+t^2}=-i\oint\frac{dz}{z-t\,(z^2+1)+t^2 z}$$

```
Manipulate[Show[{ArgandPlot[z /. Solve[-t z^2 + (t^2 + 1) z - t == 0, z] /. t → tValue],
   Graphics[{Circle[]}]}], {tValue, -5, 5}]
```



This shows you a change of singularity with t value.

Exercise.

$\int_0^{2\pi}\frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$

$$\int_0^{2\pi}\frac{\cos 3\theta\,d\theta}{5-4\cos\theta}=\oint\frac{(z^3+z^{-3})/(2iz)\,dz}{5-4/2\,(z+z^{-1})}=-\frac{i}{2}\oint\frac{(z^2+z^{-4})\,dz}{5-2\,(z+z^{-1})}=-\frac{i}{2}\oint\frac{(z^6+1)\,dz}{5z^4-2\,(z^5+z^3)}=-\frac{i}{2}\oint\frac{(z^6+1)\,dz}{-2z^3(z-2)\,(z-1/2)}$$

$$\text{Residue at } z=0 \longrightarrow \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{(z^6+1)}{(z-2)\,(1-2z)}\right]_{z=0}$$

```
(1/2!) D[(z^6 + 1) / (-2 (z - 2) (z - 1/2)), {z, 2}] /. z → 0
```

$-\frac{21}{8}$

$$\text{Residue at } z=1/2 \longrightarrow \left[\frac{(z^6+1)}{-2z^3(z-2)}\right]_{z=1/2}$$

```
(z^6 + 1) / (-2 z^3 (z - 2)) /. z → 1/2
```

$\frac{65}{24}$

Or,

```
Table[Residue[(z^6 + 1) / (-2 z^3 (z - 2) (z - 1/2)), {z, s}], {s, {0, 1/2}}]
```

$\left\{-\frac{21}{8}, \frac{65}{24}\right\}$

Sum of two residues ; $-\frac{21}{8} + \frac{65}{24} = \frac{1}{12}$ Therefore, $-\frac{i}{2} \oint \frac{(z^6+1)\,dz}{-2\,z^3(z-2)\,(z-1/2)} = -\frac{i}{2}\,2\,\pi\,i\left(\frac{1}{12}\right) = \frac{\pi}{12}$

$$\int_0^{2\pi} \frac{\mathrm{Cos}[3\,\theta]}{5 - 4\,\mathrm{Cos}[\theta]}\,d\theta$$

$\frac{\pi}{12}$

- **Integral, Range $-\infty$ to ∞**

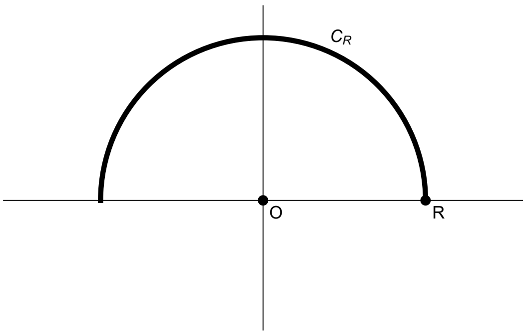
<Theorem 5>

Assumptions :

$$\left(\begin{array}{ll} a. & \text{a function } f(z) \text{ is analytic at all points in the upper half plane } y \geq 0 \\ & \text{except for a finite number of poles} \\ b. & C_R \text{ denotes a semicircle } z = R\,e^{i\,\theta}\ (0 \leq \theta \leq \pi) \\ c. & \text{In the limit } R \to \infty \text{ in the upper half plane, } f(z) \text{ vanishes more strongly} \\ & \text{than } 1/z\text{. In short, } \lim_{R\to\infty} z\,f(z) = 0 \end{array}\right)$$

Then, for every positive constant a ,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$



To prove this, let's write the z in the polar form.

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = \lim_{R\to\infty} \int_0^{\pi} f\left(R\,e^{i\,\theta}\right) i\,R\,e^{i\,\theta}\,d\theta \leq \lim_{R\to\infty} \int_0^{\pi} \left| f\left(R\,e^{i\,\theta}\right) i\,R\,e^{i\,\theta}\right|\,d\theta$$

$$\lim_{R\to\infty} \int_0^{\pi} \left| f\left(R\,e^{i\,\theta}\right) i\,R\,e^{i\,\theta}\right|\,d\theta \leq (\pi) \lim_{R\to\infty} \left| f\left(R\,e^{i\,\theta}\right) i\,R\,e^{i\,\theta}\right|$$

By assumption, $\lim_{R\to\infty} \left| f\left(R\,e^{i\,\theta}\right) i\,R\,e^{i\,\theta}\right|$ goes to zero.

$$\therefore \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

Hence, integral around the given contour is,

$$\oint f(z)\,dz = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 2\,\pi\,i\,(\sum \text{residues of } f(z))$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\,\pi\,i\,(\sum \text{residues of } f(z))$$

Exercise.

Evaluate $\int_0^\infty \frac{dx}{1+x^2}$

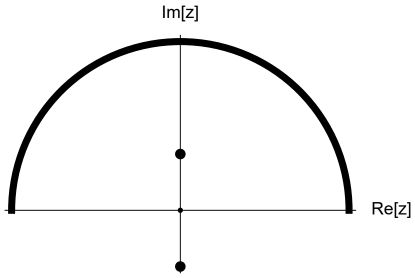
Since integrand is even, we can extend the range by

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^2}$$

Then we can consider the following integral with given contour

$$\oint \frac{dz}{1+z^2} = \int_{C_R} \frac{dz}{1+z^2} + \int_{-\infty}^\infty \frac{dx}{1+x^2}$$

```
Show[{ArgandPlot[z /. Solve[1 + z^2 == 0, z]],
  Graphics[{Thickness[0.02], Circle[{0, 0}, 3, {0, π}]}]}, Ticks → False]
```



Since $\int_{C_R}$ goes to zero,

$$\oint \frac{dz}{1+z^2} = \int_{-\infty}^\infty \frac{dx}{1+x^2} = 2\pi i\left(\sum \text{Residues of } \frac{1}{1+z^2}\right)$$

In the region there is only one pole; $z = i$ at simple pole

Residue at $z = i$: $\left[\frac{z-i}{1+z^2}\right]_{z=i} = \left[\frac{1}{z+i}\right]_{z=i} = \frac{1}{2i}$

Finally,

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = 2\pi i\left(\frac{1}{2i}\right) = \pi$$

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

```
Residue[1/(1 + z^2), {z, i}]
```

$-\frac{i}{2}$

```
∫_0^∞ 1/(1 + x^2) dx
```

$\frac{\pi}{2}$

■ Integrals with Complex Exponentials

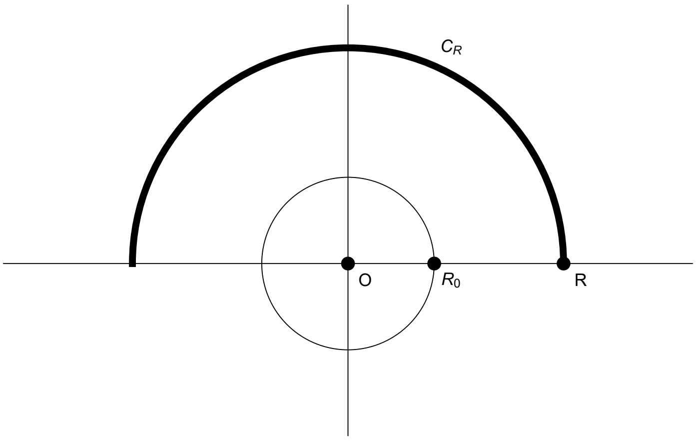
<Theorem 6 > Jordan's Lemma

Suppose that

$$\left(\begin{array}{l} a.\ \ a \text{ function } f(z) \text{ is analytic at all points in the upper half plane } y \geq 0 \\ \quad\ \text{that are exterior to a circle } |z| = R_0 \\ b.\ \ \ C_R \text{ denotes a semicircle } z = R\,e^{i\theta}\ (0 \leq \theta \leq \pi), \text{ where } R > R_0 \\ c.\ \ \ \text{for all points } z \text{ on } C_R, \text{ there is a positive constant } M_R \text{ such that} \\ \quad\ \ |f(z)| \leq M_R \text{ and } \lim_{R\to\infty} M_R = 0 \end{array}\right)$$

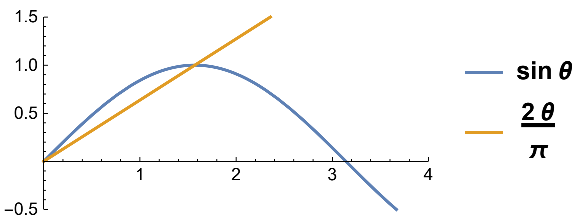
Then, for every positive constant a $(a > 0)$,

$$\lim_{R\to\infty} \int_{C_R} f(z)\, e^{i a z}\, dz = 0$$



Before we prove it , note that $\sin(\theta) \geq \frac{2\,\theta}{\pi}$ when $0 \leq \theta \leq \frac{\pi}{2}$

```
Plot[{Sin[θ], 2/π θ}, {θ, 0, 2 π}, PlotLegends → Automatic,
 PlotRange → {{0, 4}, {-0.5, 1.5}}, AspectRatio → 0.5]
```



As a result,

$$\sin(\theta) \geq 2\,\theta/\pi \ \longrightarrow\ e^{-R\sin\theta} \leq e^{-2R\theta/\pi} \quad \text{when } 0 \leq \theta \leq \frac{\pi}{2}$$

$$\int_0^{\pi/2} d\theta\, e^{-R\sin\theta} \leq \int_0^{\pi/2} d\theta\, e^{-2R\theta/\pi} = \frac{\pi}{2R}\left(1 - e^{-R}\right) \leq \frac{\pi}{2R} \quad (R > 0)$$

Since $e^{-R\sin\theta}$ is symmetric about the vertical line $\theta = \pi/2$,

$$\int_{\pi/2}^{\pi} d\theta\, e^{-R\sin\theta} \leq \frac{\pi}{2R} \quad (R > 0)$$

By adding them together,

$$\int_0^{\pi/2} d\theta\, e^{-R\sin\theta} + \int_{\pi/2}^{\pi} d\theta\, e^{-R\sin\theta} = \int_0^{\pi} d\theta\, e^{-R\sin\theta} \leq \frac{\pi}{2R} + \frac{\pi}{2R} = \frac{\pi}{R} \quad (R > 0)\,,$$

which is called Jordan’s inequality.

Now we are ready to prove the Jordan’s lemma !

By the assumption,

$$\int_{C_R} f(z)\, e^{i a z}\, dz = \begin{pmatrix} z \to R e^{i\theta} \\ dz \to i R e^{i\theta} d\theta \end{pmatrix} \Rightarrow \int_{\theta=0}^{\pi} f\left(R e^{i\theta}\right) e^{i a R e^{i\theta}}\, i R e^{i\theta}\, d\theta$$

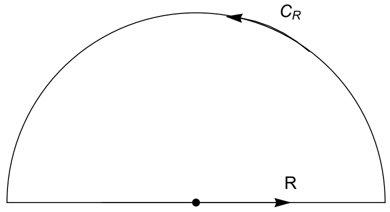
Since $\begin{pmatrix} \left|f\left(R e^{i\theta}\right)\right| \le M_R \\ \left|e^{i a R e^{i\theta}}\right| \le e^{-a R \sin\theta} \end{pmatrix}$,

$$\begin{aligned} \int_{\theta=0}^{\pi} f\left(R e^{i\theta}\right) e^{i a R e^{i\theta}}\, i R e^{i\theta}\, d\theta \le & \left|\int_{\theta=0}^{\pi} f\left(R e^{i\theta}\right) e^{i a R e^{i\theta}}\, i R e^{i\theta}\, d\theta\right| \\ \le & \int_{\theta=0}^{\pi} d\theta\, M_R\, e^{-a R \sin\theta} R = M_R R \int_{\theta=0}^{\pi} e^{-a R \sin\theta}\, d\theta \\ \le & M_R R \int_{\theta=0}^{\pi} e^{-a R \sin\theta}\, d\theta \le M_R R \frac{\pi}{a R} \end{aligned}$$

(Jordan's inequality)

$$\therefore \int_{C_R} f(z)\, e^{i a z} < M_R \frac{\pi}{a} \quad \text{(from the assumption, } M_R \text{ goes to zero as } R \text{ tends to } \infty)$$

Finally we have proved $\lim_{R\to\infty} \int_{C_R} f(z)\, e^{i a z}\, dz = 0$



Therefore, contour integral of $f(z)\, e^{i a z}$ about above contour will be;

$$\oint f(z)\, e^{i a z}\, dz = \int_{C_R} f(z)\, e^{i a z}\, dz + \int_{-\infty}^{\infty} f(x)\, e^{i a x}\, dx = 2\pi i \left(\sum \text{residues of } f(z)\, e^{i a z}\right)$$

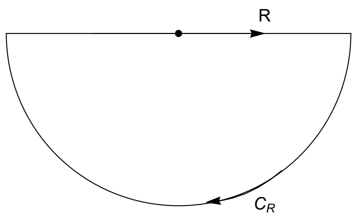
By Jordan's lemma, $\int_{C_R}$ tends to zero as R goes to infinity.

$$\therefore\ 0 + \int_{-\infty}^{\infty} f(x)\, e^{i a x}\, dx = 2\pi i \left(\sum \text{residues of } f(z)\, e^{i a z}\right)$$

(and it seems to have some relation to fourier transform..)

Exercise.
Extend the Jordan's lemma by considering the case $a < 0$

In the lower half plane, the exponent will diverge (since $\sin(\theta) \le 2\,\theta/\pi$) therefore, for the lower half plane case, we have to consider the case $a < 0$.



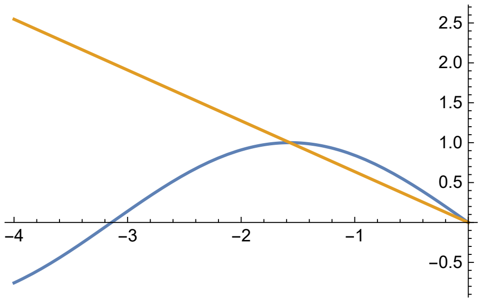
For the case $a < 0$, $\oint = \int_{C_R} + \int_{-\infty}^{\infty} f(x)\, e^{i a x}\, dx = (-2\pi i)\left(\sum \text{residues at lower half plane}\right)$ (negative sign for clockwise contour integral).

$$\int_{C_R} f(z)\, e^{i a z}\, dz = \begin{pmatrix} z \to R e^{i\theta} \\ dz \to i R e^{i\theta} d\theta \end{pmatrix} \Rightarrow \int_{\theta=0}^{-\pi} f\left(R e^{i\theta}\right) e^{i a R e^{i\theta}}\, i R e^{i\theta}\, d\theta$$

Since $\begin{pmatrix} \left|f\left(R e^{i\theta}\right)\right| \le M_R \\ \left|e^{i a R e^{i\theta}}\right| \le e^{-a R \sin\theta} \end{pmatrix}$,

$$\begin{aligned}\int_{\theta=0}^{-\pi} f\left(R e^{i\theta}\right) e^{i a R e^{i\theta}} i R e^{i\theta} d\theta \;\le\;& \left|\int_{\theta=0}^{-\pi} f\left(R e^{i\theta}\right) e^{i a R e^{i\theta}} i R e^{i\theta} d\theta\right| \\ \le\;& \int_{\theta=0}^{-\pi} d\theta\, M_R\, e^{-a R \sin\theta} R = M_R R \int_{\theta=0}^{-\pi} e^{a R(-\sin\theta)} d\theta\end{aligned}$$

```
Plot[{-Sin[u], -2/π u}, {u, 0, -4}]
```



$$-\sin\theta \ge -\frac{2}{\pi}\theta \quad \text{at} \quad -\frac{\pi}{2} \le \theta \le 0 \;(\text{Jordan's inequality})$$

By Jordan's inequality, $a R(-\sin\theta) \le a R(-2\theta/\pi)$ (since a is negative)

Hence,

$$M_R R \int_{\theta=0}^{-\pi} e^{a R(-\sin\theta)} d\theta \le M_R R \int_{\theta=0}^{-\pi} e^{a R(-2\theta/\pi)} d\theta = M_R R \frac{\pi}{-2 a R}\left[e^{a R(-2\theta/\pi)}\right]_{\theta=0}^{-\pi} = \frac{\pi M_R}{-2 a}\left(e^{2 a R} - 1\right)$$

As R tends to ∞, $\int_{C_R} \le \frac{\pi M_R}{2 a}$(don't forget that a is a negative) $= 0$ (by the assumption)

In short,

$$\begin{aligned}\oint f(z) e^{i a z} dz = \int_{C_R} f(z) e^{i a z} dz + \int_{-\infty}^{\infty} f(x) e^{i a x} dx &= (-2\pi i)\,(\textstyle\sum \text{residues at lower half plane}) \\ = 0 + \int_{-\infty}^{\infty} f(x) e^{i a x} dx &= (-2\pi i)\,(\textstyle\sum \text{residues at lower half plane})\end{aligned}$$

< Remark 4.4 > Integrals with complex exponentials

$$\int_{-\infty}^{\infty} f(x) e^{i a x} dx = \begin{cases} (2\pi i)\,(\sum \text{residues at upper half plane}) & a > 0 \\ \int_{-\infty}^{\infty} f(x)\, dx & a = 0 \\ (-2\pi i)\,(\sum \text{residues at lower half plane}) & a < 0 \end{cases}$$

Exercise.

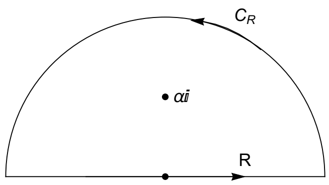
Evaluate the Fourier transform of $f(t) = \sqrt{\frac{1}{2\pi}\frac{2\alpha}{\alpha^2+t^2}}$

Perhaps you have learned Fourier transform at spring ;

$$\begin{cases} \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i t \omega} dt & \text{Fourier transform of } f(t) \\ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega t} d\omega & \text{Inverse Fourier transform of } \hat{f}(x) \end{cases}$$

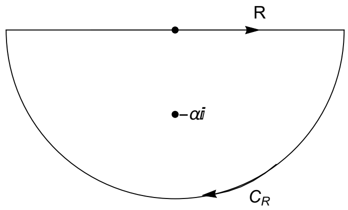
With help of <Remark 4.4>, Fourier transform of $f(t)$ is

$$\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{2\alpha}{\alpha^2+t^2} e^{i\omega t} dt = \begin{cases} (2\pi i)\,(\sum \text{residues at upper half plane}) & \omega > 0 \\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2+t^2} dt & \omega = 0 \\ (-2\pi i)\,(\sum \text{residues at lower half plane}) & \omega < 0 \end{cases}$$



For upper half plane, $\frac{1}{2\pi}\frac{2\alpha}{\alpha^2+t^2}e^{i\omega t}$ has simple pole at αi whose residue is

$$\left[\frac{1}{2\pi}\frac{2\alpha}{t+\alpha i}e^{i\omega t}\right]_{t=\alpha i} = \frac{1}{2\pi i}e^{-\alpha\omega}$$



For lower half plane, $\frac{1}{2\pi}\frac{2\alpha}{\alpha^2+t^2}e^{i\omega t}$ has simple pole at $-\alpha i$ whose residue is

$$\left[\frac{1}{2\pi}\frac{2\alpha}{t-\alpha i}e^{i\omega t}\right]_{t=-\alpha i} = -\frac{1}{2\pi i}e^{\alpha\omega}$$

For ω=0 ,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\alpha}{\alpha^2+t^2}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\alpha}{\alpha^2\left(1+(t/\alpha)^2\right)}dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{2}{\alpha\left(1+\tan^2\theta\right)}\alpha\sec^2\theta\, d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}2\, d\theta = 1$$

$$\mathbf{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2+t^2}dt}$$

```
ConditionalExpression[√(1/a²) a, Re[a²] ≥ 0 || a² ∉ ℝ]
```

Hence,

$$\hat{f}(\omega) = \begin{cases} (2\pi i)\left(\frac{1}{2\pi i}e^{-\alpha\omega}\right) = e^{-\alpha\omega} & \omega > 0 \\ 1 & \omega = 0 \\ (-2\pi i)\left(-\frac{1}{2\pi i}e^{\alpha\omega}\right) = e^{\alpha\omega} & \omega < 0 \end{cases}$$

In short, the Fourier transform of $\sqrt{\frac{1}{2\pi}}\frac{2\alpha}{\alpha^2+t^2}$, $\hat{f}(\omega)$ is $e^{-\alpha|\omega|}$.

```
FourierTransform[2 α / (√(2 π) (α² + t²)), t, ω,
  Assumptions → {α ∈ Reals && α > 0 && ω ∈ Reals}] // TraditionalForm
```

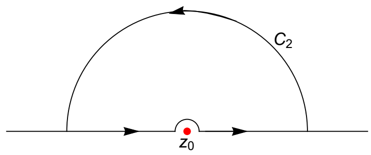
$e^{-\alpha\omega}\left(e^{2\alpha\omega}\theta(-\omega)+\theta(\omega)\right)$

Which means that $e^{\alpha\omega}\theta(-\omega)+e^{-\alpha\omega}\theta(\omega) = e^{-\alpha|\omega|}$

Exercise.

$\int_{-\infty}^{\infty}\frac{\cos bx-\cos ax}{x^2}dx$, for $a > b > 0$

$$\int_{-\infty}^{\infty}\frac{\cos bx-\cos ax}{x^2}dx = \int_{-\infty}^{\infty}\frac{e^{ibx}+e^{-ibx}-e^{iax}-e^{-iax}}{2x^2}dx$$
$$= \int_{-\infty}^{\infty}\frac{e^{ibx}}{2x^2}+\frac{e^{-ibx}}{2x^2}-\frac{e^{iax}}{2x^2}-\frac{e^{-iax}}{2x^2}dx$$



Please remember that above figure and
$P\int f(z)\,dz = -I_{\text{over}} - \int_{C_2} f(z)\,dz + 2\pi i \sum \text{residues (other than at } z = z_0) \text{ with } I_{\text{over}} = -i\pi a_{-1}$

- $\int_{-\infty}^{\infty} \frac{e^{ibx}}{2x^2}\,dx = \int_{-\infty}^{\infty} \frac{1+ibx+(ibx)^2/2!+\cdots}{2x^2}\,dx \longrightarrow \int_{-\infty}^{\infty} \frac{e^{ibx}}{2x^2}\,dx - \pi i\left(\frac{ib}{2}\right) + 0 = 0 \;\therefore \int_{-\infty}^{\infty} \frac{e^{ibx}}{2x^2}\,dx = -\frac{b\pi}{2}$
- $\int_{-\infty}^{\infty} \frac{e^{-ibx}}{2x^2}\,dx - \begin{pmatrix} x \to -y \\ dx \to -dy \end{pmatrix} \to -\int_{\infty}^{-\infty} \frac{e^{iby}}{2y^2}\,dy = \int_{-\infty}^{\infty} \frac{e^{iby}}{2y^2}\,dy = -\frac{b\pi}{2}$
- $\int_{-\infty}^{\infty} \frac{e^{iax}}{2x^2}\,dx = \int_{-\infty}^{\infty} \frac{1+iax+(iax)^2/2!+\cdots}{2x^2}\,dx \longrightarrow \int_{-\infty}^{\infty} \frac{e^{iax}}{2x^2}\,dx - \pi i\left(\frac{ia}{2}\right) + 0 = 0 \;\therefore \int_{-\infty}^{\infty} \frac{e^{iax}}{2x^2}\,dx = -\frac{a\pi}{2}$
- $\int_{-\infty}^{\infty} \frac{e^{-iax}}{2x^2}\,dx - \begin{pmatrix} x \to -y \\ dx \to -dy \end{pmatrix} \to -\int_{\infty}^{-\infty} \frac{e^{iay}}{2y^2}\,dy = \int_{-\infty}^{\infty} \frac{e^{iay}}{2y^2}\,dy = -\frac{a\pi}{2}$

Hence,

$$\int_{-\infty}^{\infty} \frac{e^{ibx}}{2x^2} + \frac{e^{-ibx}}{2x^2} - \frac{e^{iax}}{2x^2} - \frac{e^{-iax}}{2x^2}\,dx = 2\left(-\frac{b\pi}{2} + \frac{a\pi}{2}\right) = \pi(a-b)$$

In *Mathematica,*

$$\int_{-\infty}^{\infty} \frac{\mathbf{Cos[b\,x] - Cos[a\,x]}}{\mathbf{x^2}}\,d\mathbf{x}$$

```
ConditionalExpression[π (Abs[a] - Abs[b]), a ∈ ℝ && b ∈ ℝ]
```

Exercise.

Show that $\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$

Since $\frac{\sin^2 x}{x^2}$ is an even function, $\int_0^{\infty} \frac{\sin^2 x}{x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}$

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \int_{-\infty}^{\infty} \frac{(1-\cos 2x)/2}{x^2}\,dx = \int_{-\infty}^{\infty} \frac{(1-\cos 2x)}{2x^2}\,dx$$
$$= \int_{-\infty}^{\infty} \frac{1-(e^{i2x}+e^{-i2x})/2}{2x^2}\,dx = \int_{-\infty}^{\infty} \frac{2-(e^{i2x}+e^{-i2x})}{4x^2}\,dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{2x^2} - \frac{e^{i2x}}{4x^2} - \frac{e^{-i2x}}{4x^2}\,dx$$

- $\int_{-\infty}^{\infty} \frac{e^{i2x}}{4x^2}\,dx = -\frac{2\pi}{4} = -\frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{e^{-i2x}}{4x^2}\,dx$ (from previous exercise)

Therefore,

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = 0 - \left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) = \pi,\quad \int_0^{\infty} \frac{\sin^2 x}{x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} = \frac{\pi}{2}$$

Mathematica returns answer immediately.

$$\int_0^{\infty} \mathbf{Sin[x]^2 / x^2}\,d\mathbf{x}$$

$$\frac{\pi}{2}$$

Exercise.

Show that $\int_0^{\infty} \frac{x\sin x}{x^2+1}\,dx = \frac{\pi}{2e}$

$$\int_0^\infty \frac{x\sin x}{x^2+1}\,dx = \int_0^\infty \frac{x\left(e^{ix}-e^{-ix}\right)/(2i)}{x^2+1}\,dx = \frac{1}{2i}\int_0^\infty \frac{x\left(e^{ix}-e^{-ix}\right)}{x^2+1}\,dx$$
$$= \frac{1}{2i}\left(\int_0^\infty \frac{x e^{ix}}{x^2+1}\,dx - \int_0^\infty \frac{x e^{-ix}}{x^2+1}\,dx\right)$$

• $-\int_0^\infty \frac{x e^{-ix}}{x^2+1}\,dx - \begin{pmatrix} x \to -y \\ dx \to -dy \end{pmatrix} \to -\int_0^{-\infty} \frac{-y e^{iy}}{y^2+1} - dy = -\int_0^{-\infty} \frac{y e^{iy}}{y^2+1}\,dy = \int_{-\infty}^0 \frac{y e^{iy}}{y^2+1}\,dy$

$$\therefore \frac{1}{2i}\left(\int_0^\infty \frac{x e^{ix}}{x^2+1}\,dx - \int_0^\infty \frac{x e^{-ix}}{x^2+1}\,dx\right) = \frac{1}{2i}\left(\int_0^\infty \frac{x e^{ix}}{x^2+1}\,dx + \int_{-\infty}^0 \frac{y e^{iy}}{y^2+1}\,dy\right) = \frac{1}{2i}\left(\int_{-\infty}^\infty \frac{x e^{ix}}{x^2+1}\,dx\right)$$
$$= \frac{1}{2i}\left(2\pi i * \text{Residue at } x = i\right)$$

$\texttt{Residue}\left[\frac{x\, e^{i x}}{x^2+1}, \{x, i\}\right]$

$\frac{1}{2\,e}$

$$\frac{1}{2i}\left(2\pi i * \text{Residue at } x = i\right) = \frac{\pi}{2e}$$

$\int_0^\infty x\, \texttt{Sin}[x] \,/\, \left(x^2+1\right) dx$

$\frac{\pi}{2\,e}$

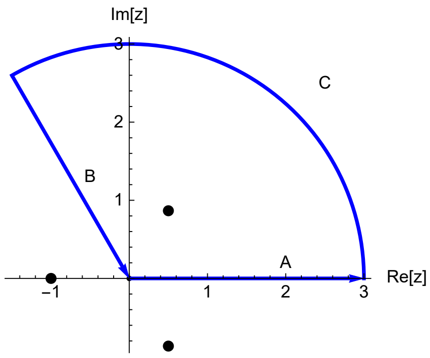
■ Another Integration Technique

Exercise.

Evaluate $\int_0^\infty \frac{dx}{x^3+1}$

Since $\frac{1}{x^3+1}$ is not even, we cannot convert into range $(-\infty, \infty)$. Instead, setting $z = r e^{i\theta}$, what argument θ makes $z^3 = r^3$?; $\theta = 2\pi/3$. Then, consider following contour and corresponding contour integral.

$$\oint \frac{dz}{z^3+1} = \int_0^\infty \frac{dz}{z^3+1} + \int_B \frac{dz}{z^3+1} + \int_C \frac{dz}{z^3+1} \quad \text{where } \int_C \text{ goes to zero}$$



• Contour A : $I = \int_0^\infty \frac{dz}{z^3+1}$

• Contour B : $\int_B \frac{dz}{z^3+1} = \begin{pmatrix} z \to r e^{2\pi i/3} \\ dz \to e^{2\pi i/3} dr \end{pmatrix} \Rightarrow \int_\infty^0 \frac{e^{2\pi i/3} dr}{r^3+1} = -e^{2\pi i/3}\int_0^\infty \frac{dr}{r^3+1} = -e^{2\pi i/3}\, I$

• Contour A→C→B→A : $\oint \frac{dz}{z^3+1} = 2\pi i\left(\text{Residue at } z = e^{i\pi/3}\right)$

```
Residue[1/(z^3 + 1), {z, e^(i π/3)}]
```

$-\frac{1}{3}(-1)^{1/3}$

Residue at $z = e^{i\pi/3} \to -\frac{1}{3}e^{i\pi/3}$

$$\oint \frac{dz}{z^3+1} = 2\pi i\left(-\tfrac{1}{3}e^{i\pi/3}\right) = \int_0^\infty \frac{dz}{z^3+1} + \int_B \frac{dz}{z^3+1} + \int_C \frac{dz}{z^3+1}$$

$$= \boldsymbol{I} + \left(-e^{2\pi i/3}\right)\boldsymbol{I} + 0 = \left(1 - e^{2\pi i/3}\right)\boldsymbol{I}$$

$$2\pi i\left(-\tfrac{1}{3}e^{i\pi/3}\right) = \left(1 - e^{2\pi i/3}\right)\boldsymbol{I}$$

$$2\pi i\left(-\tfrac{1}{3}\right) = \left(e^{-i\pi/3} - e^{i\pi/3}\right)\boldsymbol{I} = -2 i \sin(\pi/3)\boldsymbol{I}$$

$$2\pi i\left(-\tfrac{1}{3}\right) = -2 i\left(\frac{\sqrt{3}}{2}\right)\boldsymbol{I}$$

$$\therefore \boldsymbol{I} = \frac{2\pi}{3\sqrt{3}}$$

With help of *Mathematica*,

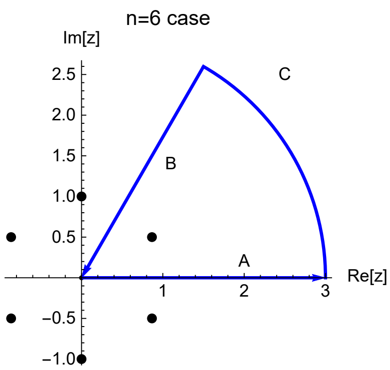
```
∫_0^∞ 1/(x^3 + 1) dx
```

$\frac{2\pi}{3\sqrt{3}}$

Exercise.

Evaluate $\int_0^\infty \frac{dx}{x^n+1}$

Pole at $z = e^{i\pi/n}\left(z^n = -1 = e^{i\pi}\right)$ The argument makes $\left(r e^{i\theta}\right)^n = r^n$ is $2\pi / n$



• Contour A : $\boldsymbol{I} = \int_0^\infty \frac{dz}{z^n+1}$

• Contour B : $\int_B \frac{dz}{z^n+1} = \left(\begin{array}{c} z \to r e^{2\pi i/n} \\ dz \to e^{2\pi i/n} dr \end{array}\right) \Rightarrow \int_\infty^0 \frac{e^{2\pi i/n} dr}{r^n+1} = -e^{2\pi i/n}\int_0^\infty \frac{dr}{r^n+1} = -e^{2\pi i/n}\boldsymbol{I}$

• Contour A→C→B→A : $\oint \frac{dz}{z^n+1} = 2\pi i\left(\text{Residue at } z = e^{i\pi/n}\right)$

```
Residue[1/(z^n + 1), {z, e^(i π/n)}]
```

```
0
```

It doesn’t know..

$$\frac{\mathrm{D}\left[-\mathbb{e}^{\frac{\mathrm{i}\pi}{n}}+z,\,z\right]}{\mathrm{D}[1+z^n,\,z]}\;/.\;z\to\mathbb{e}^{\mathrm{i}\pi/n}$$

$$\frac{\left(\mathbb{e}^{\frac{\mathrm{i}\pi}{n}}\right)^{1-n}}{n}$$

Residue at $e^{i\pi/n} \to \dfrac{e^{i(1-n)\pi/n}}{n}$

$$\oint \frac{dz}{z^n+1} = 2\pi i\left(\frac{e^{i(1-n)\pi/n}}{n}\right) = \int_0^\infty \frac{dz}{z^n+1} + \int_B \frac{dz}{z^n+1} + \int_C \frac{dz}{z^n+1}$$
$$= I + \left(-e^{2\pi i/n}\right)I + 0 = \left(1-e^{2\pi i/n}\right)I$$
$$2\pi i\left(\tfrac{1}{n}e^{i(1-n)\pi/n}\right) = \left(1-e^{2\pi i/n}\right)I$$
$$2\pi i\left(-\tfrac{1}{n}e^{i\pi/n}\right) = \left(1-e^{2i\pi/n}\right)I$$
$$2\pi i\left(-\tfrac{1}{n}\right) = \left(e^{-i\pi/n}-e^{i\pi/n}\right)I$$
$$2\pi i\left(-\tfrac{1}{n}\right) = -2i\sin(\pi/n)\,I$$
$$\therefore I = \frac{\pi/n}{\sin(\pi/n)}$$

Mathematica says...

$$\int_0^\infty \frac{1}{1+x^n}\,dx$$

$$\text{ConditionalExpression}\left[\frac{\pi\,\text{Csc}\left[\frac{\pi}{n}\right]}{n},\ \left(\text{Re}\left[(-1)^{\frac{1}{n}}\right]\le 0\ ||\ (-1)^{\frac{1}{n}}\notin\mathbb{R}\right)\ \&\&\ \text{Re}[n]>1\right]$$

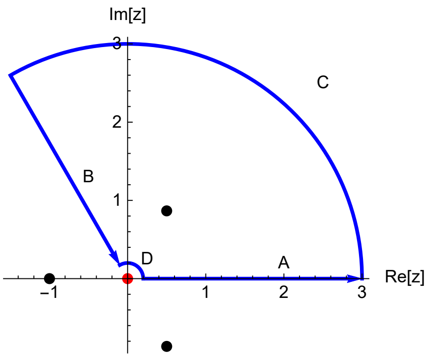
It describes appropriate condition too. How kind!

■ Avoidance of Branch Point

Exercise.

Evaluate $\int_0^\infty \frac{\ln x\,dx}{x^3+1}$

Consider similar contour and corresponding contour integral



• Contour A : $I = \int_0^\infty \frac{\ln z\,dz}{z^3+1}$

• Contour B :

$$\int_B \frac{\ln z\, dz}{z^3+1} = \begin{pmatrix} z \to r\, e^{2\pi i/3} \\ dz \to e^{2\pi i/3}\, dr \end{pmatrix} \Rightarrow \int_\infty^0 \frac{\ln(r\, e^{2\pi i/3})\, e^{2\pi i/3}\, dr}{r^3+1} = -e^{2\pi i/3}\left(\int_0^\infty \frac{\ln r\, dr}{r^3+1} + \frac{2\pi i}{3}\int_0^\infty \frac{dr}{r^3+1}\right)$$
$$= -e^{2\pi i/3}\left(I + \frac{2\pi i}{3}\, \frac{2\pi}{3\sqrt{3}}\right)$$

- Contour C : goes to zero
- Contour D : $\int_D \frac{\ln z\, dz}{z^3+1} = \begin{pmatrix} z \to \delta\, e^{i\theta} \\ dz \to i\,\delta\, e^{i\theta}\, d\theta \end{pmatrix} \Rightarrow \lim_{\delta\to 0} \int_{\theta=2\pi/3}^{0} \frac{\ln(\delta\, e^{i\theta})}{\delta^3\, e^{3i\theta}+1}\, i\,\delta\, e^{i\theta}\, d\theta = 0$
- Contour A→C→B→D→A : $\oint \frac{\ln z\, dz}{z^3+1} = 2\pi i\,(\text{Residue at } z = e^{i\pi/3})$

```
Residue[Log[z]/(z^3 + 1), {z, ⅇ^(ⅈ π/3)}]
```

$$-\frac{1}{9}\,(-1)^{5/6}\,\pi$$

Residue at $z = e^{i\pi/3} \to -\frac{\pi}{9}\, e^{5\pi i/6}$

$$\oint \frac{dz}{z^3+1} = \int_A + \int_B + \int_C + \int_D = I - e^{2\pi i/3}\left(I + \frac{2\pi i}{3}\, \frac{2\pi}{3\sqrt{3}}\right) + 0 + 0$$

$$2\pi i\left(-\frac{\pi}{9}\, e^{5\pi i/6}\right) = \left(1 - e^{2\pi i/3}\right) I - e^{2\pi i/3}\, \frac{4 i\pi^2}{9\sqrt{3}}$$

$$2\pi i\left(-\frac{\pi}{9}\, e^{3\pi i/6}\right) = \left(e^{-\pi i/3} - e^{\pi i/3}\right) I - e^{\pi i/3}\, \frac{4 i\pi^2}{9\sqrt{3}} \quad (\text{Divide both sides by } e^{\pi i/3})$$

$$2\pi i\left(-\frac{\pi}{9}\, i\right) = -2 i \sin(\pi/3)\, I - [\cos(\pi/3) + i\sin(\pi/3)]\, \frac{4 i\pi^2}{9\sqrt{3}}$$

$$2\pi i\left(-\frac{\pi}{9}\, i\right) = -i\sqrt{3}\, I - \left[\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right] \frac{4 i\pi^2}{9\sqrt{3}}$$

$$i\sqrt{3}\, I = -\frac{1}{2}\, \frac{4 i\pi^2}{9\sqrt{3}} \qquad \therefore I = -\frac{2\pi^2}{27}$$

Mathematica says,

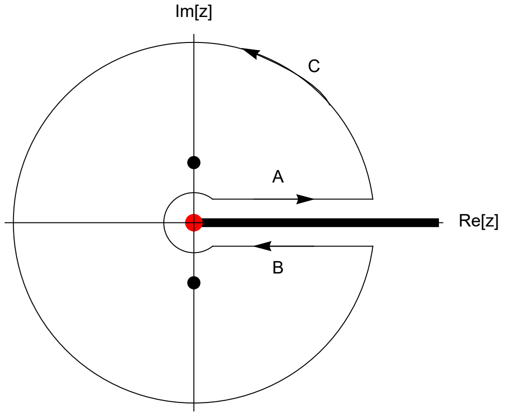
```
∫_0^∞ Log[x]/(x^3 + 1) ⅆx
```

$$-\frac{2\pi^2}{27}$$

■ Exploiting Branch Cuts

Exercise.
Evaluate $\int_0^\infty \frac{x^p\, dx}{x^2+1}$ $0 < p < 1$

Integrand has branch point at $z = 0$ Now consider the following contour



Then $\oint \frac{z^p\, dz}{z^2+1} = \int_A \frac{z^p\, dz}{z^2+1} + \int_B \frac{z^p\, dz}{z^2+1} + \int_C \frac{z^p\, dz}{z^2+1}$

• Contour A : $I = \int_0^\infty \frac{x^p\, dx}{x^2+1}$

• Contour B : $\int_B \frac{z^p\, dz}{z^2+1} = \begin{pmatrix} z \to r\, e^{2\pi i} \\ dz \to e^{2\pi i}\, dr = dr \end{pmatrix} \Rightarrow \int_\infty^0 \frac{(r\, e^{2\pi i})^p\, dr}{r^2+1} = -e^{2p\pi i}\left(\int_0^\infty \frac{r^p\, dr}{r^2+1}\right) = -e^{2p\pi i}\, I$

• Contour C : goes to zero

• Contour A→C→B→A : $\oint \frac{z^p\, dz}{z^2+1} = 2\pi i\left(\text{Residue at } z = e^{i\pi/2},\ z = e^{3\pi i/2}\right)$

$$\text{Residue at} \begin{cases} z = e^{i\pi/2} & \left[\frac{z^p}{z - e^{3\pi i/2}}\right]_{z=e^{i\pi/2}} = 1/2\, e^{3\pi i/2}\, e^{p\pi i/2} \\ z = e^{3\pi i/2} & \left[\frac{z^p}{z - e^{i\pi/2}}\right]_{z=e^{3\pi i/2}} = 1/2\, e^{\pi i/2}\, e^{3p\pi i/2} \end{cases}$$

```
Table[Residue[z^p/(z^2 + 1), {z, n}], {n, {E^(I π/2), E^(3 π I/2)}}]
```

$$\left\{-\frac{i\, i^p}{2},\ \frac{1}{2}\, i\, (-i)^p\right\}$$

Evaluating contour integral;

$$\oint \frac{z^p\, dz}{z^2+1} = \int_A + \int_B + \int_C = I + \left(-e^{2p\pi i}\right) I + 0 = 2\pi i\left(\frac{e^{(3+p)\pi i/2} + e^{(3p+1)\pi i/2}}{2}\right)$$

$$2\pi i\left(e^{(3+p)\pi i/2} + e^{(3p+1)\pi i/2}\right)/2 = \left(1 - e^{2p\pi i}\right) I$$

$$2\pi i\left(e^{(3-p)\pi i/2} + e^{(p+1)\pi i/2}\right)/2 = \left(e^{-p\pi i} - e^{p\pi i}\right) I = -2 i \sin(p\pi)\, I$$

$$2\pi i\left(-i\, e^{-p\pi i/2} + i\, e^{p\pi i/2}\right)/2 = -2 i \sin(p\pi)\, I$$

$$2\pi i\left(-\sin(p\pi/2)\right) = -2 i \sin(p\pi)\, I$$

$$I = \frac{\pi \sin(p\pi/2)}{\sin(p\pi)} = \frac{\pi \sin(p\pi/2)}{2\sin(p\pi/2)\cos(p\pi/2)} = \frac{\pi}{2\cos(p\pi/2)} = \frac{\pi}{2}\sec(p\pi/2)$$

$$\int_0^\infty \frac{x^p}{x^2+1}\, dx$$

```
ConditionalExpression[1/2 π Sec[p π/2], -1 < Re[p] < 1]
```

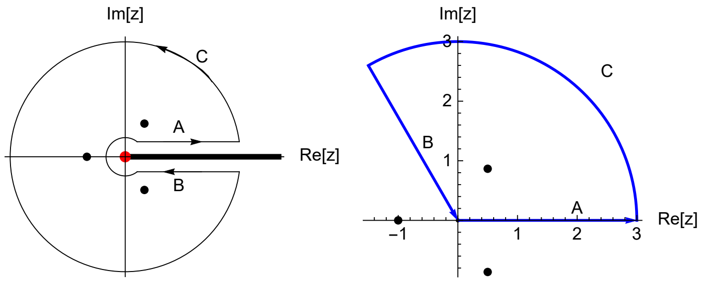
Exercise.

Evaluate $\int_0^\infty \frac{dx}{x^3+1}$ (we have already met this problem!)

Instead of evaluating circular sector(see the second figure), consider the following integral

$$\oint \frac{\ln z\, dz}{z^3+1}$$

with given contour(first figure).



$$\oint \frac{\ln z\, dz}{z^3+1} = \int_A \frac{\ln z\, dz}{z^3+1} + \int_B \frac{\ln z\, dz}{z^3+1} + \int_C \frac{\ln z\, dz}{z^3+1}$$

- Contour A : $\int_0^\infty \frac{\ln x\, dx}{x^3+1}$
- Contour B : $\int_B \frac{\ln z\, dz}{z^3+1} = \begin{pmatrix} z \to r\, e^{2\pi i} \\ dz \to e^{2\pi i}\, dr = dr \end{pmatrix} \Rightarrow \int_\infty^0 \frac{\ln\left(r\, e^{2\pi i}\right) dr}{r^3+1} = \int_\infty^0 \frac{2\pi i + \ln r\, dr}{r^3+1} = -\int_0^\infty \frac{2\pi i + \ln r\, dr}{r^3+1}$
- Contour C : goes to zero
- Contour A→C→B→A : $\oint \frac{\ln z\, dz}{z^3+1} = 2\pi i \left(\text{Residue at } z = e^{i\pi/3},\ z = e^{\pi i},\ z = e^{5\pi i/3}\right)$

$$\text{Residue at} \begin{cases} z = e^{i\pi/3} & \left[\frac{d/dz\left[\left(z-e^{i\pi/3}\right)\ln z\right]}{d/dz\left[z^3+1\right]}\right]_{z=e^{i\pi/3}} = 1/9\, \pi i\, e^{-2\pi i/3} \\ z = e^{\pi i} & \left[\frac{d/dz\left[\left(z-e^{\pi i}\right)\ln z\right]}{d/dz\left[z^3+1\right]}\right]_{z=e^{\pi i}} = 1/3\, \pi i\, e^{-2\pi i} \\ z = e^{5\pi i/3} & \left[\frac{d/dz\left[\left(z-e^{5\pi i/3}\right)\ln z\right]}{d/dz\left[z^3+1\right]}\right]_{z=e^{5\pi i/3}} 5/9\, \pi i\, e^{-10\pi i/3} \end{cases}$$

$$\frac{\mathrm{Log}[z]}{3\, z^2} \;/.\; z \to \left\{e^{i\pi/3},\ e^{\pi i},\ e^{5 i\pi/3}\right\}$$

$$\left\{\frac{1}{9}\, i\, e^{-\frac{2 i\pi}{3}}\pi,\ \frac{i\pi}{3},\ -\frac{1}{9}\, i\, e^{\frac{2 i\pi}{3}}\pi\right\}$$

In this case, **Residue[]** does not give enough information.

$$\mathrm{Table}\left[\mathrm{Residue}\left[\frac{\mathrm{Log}[z]}{z^3+1},\ \{z, n\}\right],\ \left\{n,\ \left\{e^{i\pi/3},\ e^{i\pi},\ e^{5\pi i/3}\right\}\right\}\right]$$

$$\left\{-\frac{1}{9}(-1)^{5/6}\pi,\ \mathrm{Residue}\left[\frac{\mathrm{Log}[z]}{1+z^3},\ \{z, -1\}\right],\ \frac{1}{9}(-1)^{1/6}\pi\right\}$$

Evaluating contour integral;

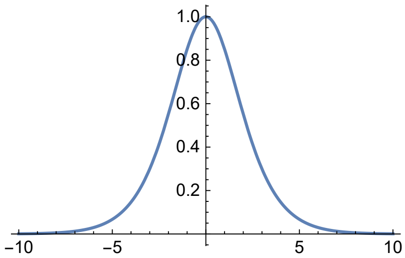
$$\oint \frac{\ln z\, dz}{z^3+1} = \int_A + \int_B + \int_C = \int_0^\infty \frac{\ln x\, dx}{x^3+1} - \int_0^\infty \frac{2\pi i + \ln r\, dr}{r^3+1} + 0 = -2\pi i \int_0^\infty \frac{dr}{r^3+1}$$
$$= 2\pi i \left(\frac{e^{-2\pi i/3} + 3\, e^{-2\pi i} + 5\, e^{-10\pi i/3}}{9}\, \pi i\right)$$

$$\int_0^\infty \frac{dr}{r^3+1} = \frac{e^{-2\pi i/3} + 3\, e^{-2\pi i} + 5\, e^{-10\pi i/3}}{-9}\pi i = \frac{3 + e^{-2\pi i/3} + 5\, e^{-12\pi i/3}\, e^{2\pi i/3}}{-9}\pi i = \frac{3 + e^{-2\pi i/3} + 5\, e^{2\pi i/3}}{-9}\pi i$$
$$= \frac{3 - 3 + 2 i\sqrt{3}}{-9}\pi i = \frac{2\pi}{3\sqrt{3}} \quad \blacksquare$$

■ Exploiting Periodicity

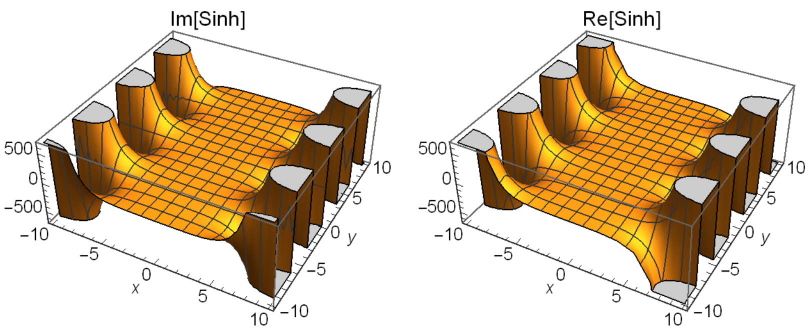
Exercise.
Evaluate $I = \int_0^\infty \frac{x\,dx}{\sinh x}$

```
Plot[x / Sinh[x], {x, -10, 10}]
```



Sinh[z] has the sinusoidal behavior. Take a deep look at below figure .

```
GraphicsGrid[
 {{Plot3D[Im[Sinh[x + ⅈ y]], {x, -10, 10}, {y, -10, 10}, AxesLabel → Automatic,
    PlotLabel → "Im[Sinh]"], Plot3D[Re[Sinh[x + ⅈ y]], {x, -10, 10},
    {y, -10, 10}, AxesLabel → Automatic, PlotLabel → "Re[Sinh]"]}}]
```



Since

$\sinh(x + i\,y) = \frac{e^{(x+i\,y)} - e^{-(x+i\,y)}}{2} = \frac{e^x(\cos y + i \sin y) - e^{-x}(\cos y - i \sin y)}{2} = \cos y\left[\frac{e^x - e^{-x}}{2}\right] + i \sin y\left[\frac{e^x + e^{-x}}{2}\right]$

```
TrigExpand[Sinh[x + ⅈ y]]
```

```
ⅈ Cosh[x] Sin[y] + Cos[y] Sinh[x]
```

and for all x, $\cosh(x) \geq 1$ so that the imaginary part of $\sinh(z)$ is zero only when $y = n\,\pi$. Therefore, $z/\sinh(z)$ has poles at $z = 0 + n\,\pi\,i$ for any integer n. Moreover, $\lim_{z\to 0} \frac{z}{\sinh z} = 1$ so it does not have pole at $z = 0$.

```
Solve[Sinh[z] == 0, z]
```

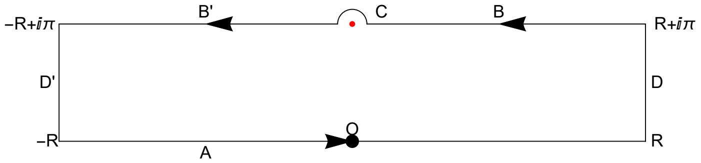
```
{{z → ConditionalExpression[2 ⅈ π C[1], C[1] ∈ ℤ]},
 {z → ConditionalExpression[ⅈ π + 2 ⅈ π C[1], C[1] ∈ ℤ]}}
```

```
Series[z / Sinh[z], {z, 0, 4}]
```

$1 - \frac{z^2}{6} + \frac{7 z^4}{360} + O[z]^5$

In short , $z/\sinh(z)$ has poles at $n\,\pi\,i$ for any integer but 0 . For this reason we can consider following contour integral with given contour $\{R,\ R+i\pi,\ -R+i\pi,\ -R,\ R\}$ where R goes to infinity.

$$\oint \frac{z\,dz}{\sinh z} = \int_D + \int_B + \int_C + \int_{B'} + \int_{D'} + \int_A$$



• Contour A : $\int_{-\infty}^{\infty} \frac{x\,dx}{\sinh x} = 2\int_{0}^{\infty} \frac{x\,dx}{\sinh x} = 2\,I$ (since $x/\sinh(x)$ is an even function see above figure!)

• Contour $B + B'$

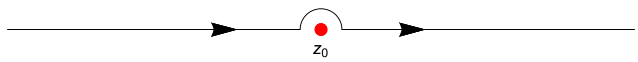
:

$$P\int_{\infty}^{-\infty} \frac{z\,dz}{\sinh z} = \left(\begin{array}{c} z \to x + i\pi \\ dz \to dx \end{array}\right) \Rightarrow P\int_{\infty}^{-\infty} \frac{(x+i\pi)\,dx}{\sinh(x+i\pi)} = P\int_{\infty}^{-\infty} \frac{(x+i\pi)\,dx}{-\sinh(x)}$$

(since $\sinh(x+iy) = i\cosh(x)\sin(y) + \cos(y)\sinh(x)$)

$$= P\int_{-\infty}^{\infty} \frac{(x+i\pi)\,dx}{\sinh(x)} = P\int_{-\infty}^{\infty} \frac{(x)\,dx}{\sinh(x)} + P\int_{-\infty}^{\infty} \frac{(i\pi)\,dx}{\sinh(x)} = 2\,I + 0 \ \left(\text{since } 1/\sinh(x) \text{ is an odd function}\right)$$

• Contour C : We have dealt with that topic at previous section. You have to remember;



$$I_{\text{over}} = -i\pi\,a_{-1}$$

In this case, since we have opposite direction, $\int_C = i\pi$ (Residue at $z = i\pi$)

• Contour D and D' : goes to zero

• Contour $D \to B \to C \to B' \to D' \to A \to D$: $\oint \frac{z\,dz}{\sinh z} = 2\pi i$ (Residue at $z = i\pi$)

```
Residue[z / Sinh[z], {z, ⅈ π}]
```

```
- ⅈ π
```

Evaluating contour integral;

$$\oint \frac{z\,dz}{\sinh z} = \int_D + \int_B + \int_C + \int_{B'} + \int_{D'} + \int_A = \left(\int_D + \int_{D'}\right) + \left(\int_B + \int_{B'} + \int_A\right) + \left(\int_C\right)$$

$$(2\pi i\,(-i\pi) = 2\pi^2) = (0) + (2I + 2I) + (i\pi\,(-i\pi))$$

$$\pi^2 = 4\,I \qquad \therefore\ I = \frac{\pi^2}{4} \quad \blacksquare$$

$$\int_0^{\infty} \frac{x}{\text{Sinh}[x]}\, dx$$

$$\frac{\pi^2}{4}$$

Verified!

4.7 Bromwich Integral

Laplace transform is a special case of Fourier transform. Let’s see!

$$\begin{cases} \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i t \omega}\, dt & \text{Fourier transform of } f(t) \\ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-i \omega t}\, d\omega & \text{Inverse Fourier transform of } \hat{f}(x) \end{cases}$$

$$\{ \mathcal{L} f(s) = \int_0^{\infty} f(t)\, e^{-ts}\, dt \quad \text{Laplace transform of } f(t)$$

For the inverse Laplace transform $\mathcal{L}^{-1}$, let, $F(t) = \mathcal{L}^{-1}[f](s)$. It seems that $F(t)$ diverge exponentially , so let us extract an exponential factor $e^{\beta t}$ from $F(t)$. Then it will be $F(t) = e^{\beta t}\, G(t)$.

If $F(t)$ diverges as $e^{\alpha t}$, we require β to be greater than α so that $G(t)$ will be convergent.

Let $G(t) = \begin{cases} G(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$ then,

its inverse Fourier transformation : $\mathcal{F}^{-1}[G](v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(v)\, e^{-i v u}\, dv = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} G(v)\, e^{-i v u}\, dv$

its FT of IFT(inverse Fourier transform) : $\mathcal{F}\mathcal{F}^{-1}[G](v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i u t}\, du \left(\frac{1}{\sqrt{2\pi}} \int_0^{\infty} G(v)\, e^{-i v u}\, dv \right)$

$$\mathcal{F}\mathcal{F}^{-1}[G](v) = G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i u t}\, du \int_0^{\infty} G(v)\, e^{-i v u}\, dv$$

Since $F(t) = e^{\beta t}\, G(t)$,

$$F(t) = \frac{e^{\beta t}}{2\pi} \int_{-\infty}^{\infty} e^{i u t}\, du \int_0^{\infty} G(v)\, e^{-i v u}\, dv = \frac{e^{\beta t}}{2\pi} \int_{-\infty}^{\infty} e^{i u t}\, du \int_0^{\infty} F(v)\, e^{-\beta v}\, e^{-i v u}\, dv$$

Let s to be a complex number $s = \beta + i\, u$ (β is a certain constant) then,

$$F(t) = \frac{e^{\beta t}}{2\pi} \int_{-\infty}^{\infty} e^{i u t}\, du \int_0^{\infty} F(v)\, e^{-s v}\, dv$$

where $\int_0^{\infty} F(v)\, e^{-s v}\, dv$ is $\mathcal{L}[F](v) = f(s)$ and by transfrom $\begin{Bmatrix} s \to \beta + i\, u \\ ds \to i\, du \end{Bmatrix}$,

$$F(t) = \frac{e^{\beta t}}{2\pi} \int_{\beta - i\infty}^{\beta + i\infty} \frac{ds}{i}\, e^{(s-\beta)t}\, f(s) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} ds\, [e^{s t}\, f(s)] \quad \text{(Bromwich integral)}$$

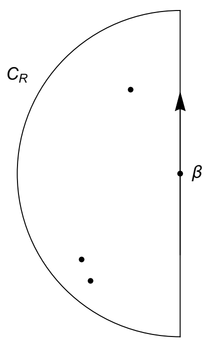
This is the inverse Laplace transform which is called “Bromwich integral”

Note that at the Laplace transform of $f(s) = \int_0^{\infty} F(v)\, e^{-s v}\, dv = \int_0^{\infty} G(v)\, e^{\beta v}\, e^{-s v}\, dv$,

to guarantee the convergence of infinite integral, real part of $\beta - s$, that is, $\beta - \mathrm{Re}[s]$ must be less than or equal to zero ($\beta - \mathrm{Re}[s] \leq 0$). In term of Fourier transform we have extended the Laplace transform onto the complex plane ($\mathrm{Re}[s] \geq \beta$).

To evaluate the inverse Laplace transform, you may need the contour integration.

If $t > 0$ and $f(s)$ is analytic except for isolated singularities, also small at large $|s|$,

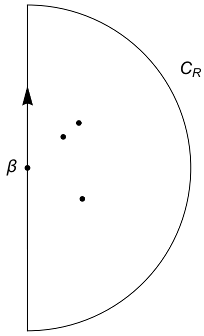


$\int_{C_R}$ goes to zero as s tends to infinity so that the

$$\oint = \int_{C_R} + \int_{\beta - i\infty}^{\beta + i\infty} ds\, [e^{s t}\, f(s)] = 2\pi i \sum (\text{residues in the region } \mathrm{Re}[s] < \beta)$$

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} ds\,[e^{st} f(s)] = F(t) = \sum(\text{residues in the region } \mathrm{Re}[s] < \beta)$$

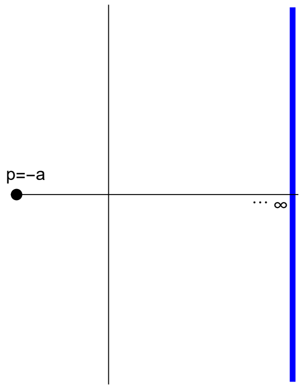
For $t < 0$,



$$\oint = \int_{C_R} + \int_{\beta-i\infty}^{\beta+i\infty} ds\,[e^{st} f(s)] = (-2\pi i)\sum(\text{residues in the region } \mathrm{Re}[s] < \beta)$$

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} ds\,[e^{st} f(s)] = F(t) = -\sum(\text{residues in the region } \mathrm{Re}[s] < \beta)$$

Exercise.
Find the inverse Laplace transform of $\frac{1}{s+a}$



Since there is only one pole at $p = -a$, the residue of $s = -a$ will be the inverse Laplace transform of $\frac{1}{s+a}$.

Residue of $\frac{e^{st}}{s+a}$ at $s = -a$: $\left[\frac{(s+a)\,e^{st}}{s+a}\right]_{s=-a} = e^{-at}$

```
Residue[e^(s t)/(s + a), {s, -a}]
```

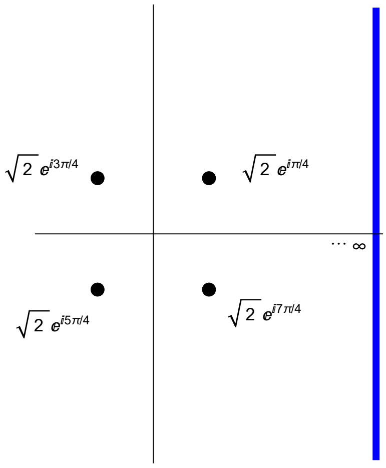
$e^{-a t}$

Hence, the inverse Laplace transform of the $\frac{1}{s+a}$ is e^{-at} ■

```
InverseLaplaceTransform[1/(s + a), s, t]
```

$e^{-a t}$

Exercise.
Find the inverse Laplace transform of $\frac{s^3}{s^4+4}$



sum of residue at $\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$ and $\begin{pmatrix} -1+i \\ -1-i \end{pmatrix}$

By the Schwarz reflection theorem, you don't have to compute the case for complex conjugate of $(1+i)$ and $(-1+i)$

<Theorem> Schwarz Reflection Theorem

- If a function $f(z)$ is $\begin{cases} a. & \text{analytic over some region including a portion of the real axis.} \\ b. & \text{real when } z \text{ is real.} \end{cases}$,

$$\text{then} \quad f^*(z) = f(z^*)$$

Proof can be derived from the theorem of Laurent series ■

$$\text{Residue at} \begin{cases} s = 1+i & \Rightarrow \left[\frac{(s-(1+i))\, s^3\, e^{s t}}{s^4+4}\right]_{s=1+i} = \left[\frac{\left(s-\sqrt{2}\ e^{i\pi/4}\right) s^3\, e^{s t}}{s^4+4}\right]_{s=\sqrt{2}\ e^{i\pi/4}} \\ & = \left[\frac{\left(4 s^3-3\sqrt{2}\ e^{i\pi/4}\, s^2\right) e^{s t}}{4 s^3}\right]_{s=\sqrt{2}\ e^{i\pi/4}} = \left[\frac{\left(2\sqrt{2}\ e^{i 3\pi/4}\right) e^{\sqrt{2}\ t \operatorname{Exp}[i\pi/4]}}{8\sqrt{2}\ e^{i 3\pi/4}}\right] = \frac{e^{t(1+i)}}{4} \\ s = -1+i & \Rightarrow \left[\frac{(s-(-1+i))\, s^3\, e^{s t}}{s^4+4}\right]_{s=-1+i} = \left[\frac{\left(s-\sqrt{2}\ e^{i 3\pi/4}\right) s^3\, e^{s t}}{s^4+4}\right]_{s=\sqrt{2}\ e^{i 3\pi/4}} \\ & = \left[\frac{\left(4 s^3-3\sqrt{2}\ e^{i 3\pi/4}\, s^2\right) e^{s t}}{4 s^3}\right]_{s=\sqrt{2}\ e^{i 3\pi/4}} = \left[\frac{\left(2\sqrt{2}\ e^{i 9\pi/4}\right) e^{\sqrt{2}\ t \operatorname{Exp}[i 3\pi/4]}}{8\sqrt{2}\ e^{i 9\pi/4}}\right] = \frac{e^{t(-1+i)}}{4} \end{cases}$$

$$\text{where} \begin{cases} \frac{e^{t(1+i)}}{4} = \frac{e^{t}(\cos(t)+i\sin(t))}{4} & \frac{e^{t(1-i)}}{4} = \frac{e^{t}(\cos(t)-i\sin(t))}{4} \\ \frac{e^{t(-1+i)}}{4} = \frac{e^{-t}(\cos(t)+i\sin(t))}{4} & \frac{e^{t(-1-i)}}{4} = \frac{e^{-t}(\cos(t)-i\sin(t))}{4} \end{cases}$$

$$\text{sum of residues}: \frac{e^{t}(\cos(t)+i\sin(t))}{4} + \frac{e^{t}(\cos(t)-i\sin(t))}{4} + \frac{e^{-t}(\cos(t)+i\sin(t))}{4} + \frac{e^{-t}(\cos(t)-i\sin(t))}{4}$$

$$= \frac{1}{2} e^{t}(\cos(t)) + \frac{1}{2} e^{-t}(\cos(t)) = \cos(t)\cosh(t) \ ■$$

```
InverseLaplaceTransform[s^3/(s^4 + 4), s, t]
```

$$\frac{1}{4} e^{(-1-i)\, t} \left(1+e^{2 i t}\right) \left(1+e^{2 t}\right)$$

```
ExpToTrig[1/4 e^((-1-i) t) (1 + e^(2 i t)) (1 + e^(2 t))]
```

$\frac{1}{4}\left(1+\text{Cos}[2\,t]+i\,\text{Sin}[2\,t]\right)\left(\text{Cosh}[(1+i)\,t]-\text{Sinh}[(1+i)\,t]\right)\left(1+\text{Cosh}[2\,t]+\text{Sinh}[2\,t]\right)$

```
Simplify[
 1/4 (1 + Cos[2 t] + i Sin[2 t]) (Cosh[(1 + i) t] - Sinh[(1 + i) t]) (1 + Cosh[2 t] + Sinh[2 t])]
```

Cos[t] Cosh[t]

Exercise.
Find the inverse Laplace transform of $\frac{a}{s^2+a^2}$

Compute residue at $s = a\,i$ and $-a\,i$. (Thanks to Schwarz reflection principle, you can do just one.)

Residue at $s = a\,i \Rightarrow \{\left[\frac{(s-a\,i)\,a\,e^{s\,t}}{s^2+a^2}\right]_{s=-a\,i} = \left[\frac{a\,e^{s\,t}}{2\,s}\right]_{s=-a\,i} = \frac{i}{2}\,e^{-i\,a\,t} = \frac{i}{2}(\cos(a\,t) - i\sin(a\,t))$

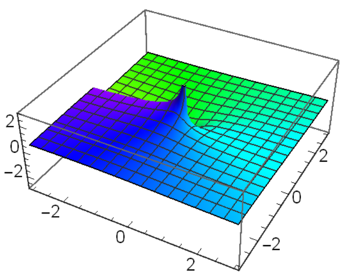
By Schwarz reflection principle, residue at $s = -a\,i \to \frac{-i}{2}(\cos(a\,t) + i\sin(a\,t))$

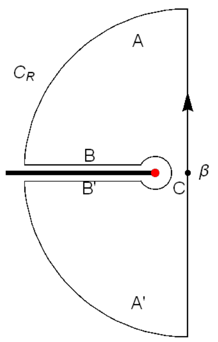
$$\text{sum of residues}: \frac{i}{2}(\cos(a\,t) - i\sin(a\,t)) + \frac{-i}{2}(\cos(a\,t) + i\sin(a\,t)) = \sin(a\,t) \blacksquare$$

```
InverseLaplaceTransform[a/(s^2 + a^2), s, t]
```

Sin[a t]

Exercise.
Find the inverse Laplace transform of $s^{-1/2}$





$f(s) = s^{-1/2}$ has a branch point at $s = 0$ and negative x – axis is its branch cut.

Since interior of the contour does not contain poles, whole contour integral is zero.

$$\oint e^{s\,t}\,f(s)\,ds = \int_A + \int_{A'} + \int_B + \int_{B'} + \int_C + \int_{\beta-i\infty}^{\beta+i\infty} ds\,[e^{s\,t}\,f(s)] = 0$$

- $\int_A + \int_{A'} = \int_{C_R}$ goes to zero as R tends to infinity
- $\int_B e^{s\,t}\,f(s)\,ds = \begin{pmatrix} s \to R\,e^{i\,\pi} \\ ds \to -dR \end{pmatrix}$

$$= \int_\infty^0 e^{R\,t\,e^{i\,\pi}}\left(R\,e^{i\,\pi}\right)^{-1/2}(-dR) = \int_\infty^0 e^{-R\,t}\left(R^{-1/2}\,e^{-i\,\pi/2}\right)(-dR) = \int_\infty^0 \frac{e^{-R\,t}}{\sqrt{R}}\,i\,dR$$

$$\bullet \int_{B'} e^{st} f(s)\, ds = \begin{pmatrix} s \to R e^{-i\pi} \\ ds \to -dR \end{pmatrix}$$

$$= \int_0^\infty e^{R t e^{-i\pi}} \left(R e^{-i\pi}\right)^{-1/2} (-dR) = \int_0^\infty e^{-Rt} \left(R^{-1/2} e^{i\pi/2}\right) (-dR) = -\int_0^\infty \frac{e^{-Rt}}{\sqrt{R}}\, i\, dR$$

$$\therefore \int_B + \int_{B'} = 2i \int_\infty^0 \frac{e^{-Rt}}{\sqrt{R}}\, dR$$

$$\bullet \int_C e^{st} f(s)\, ds = \begin{pmatrix} s \to \epsilon e^{i\theta} \\ ds \to i \epsilon e^{i\theta} d\theta \end{pmatrix}$$

$$= \int_\pi^{-\pi} e^{\epsilon t e^{i\theta}} \left(\epsilon e^{i\theta}\right)^{-1/2} i \epsilon e^{i\theta} d\theta = \int_\pi^{-\pi} e^{\epsilon t e^{i\theta}} \left(\epsilon e^{i\theta}\right)^{1/2} i\, d\theta \text{ tends to zero as } \epsilon \to 0$$

Hence,

$$\oint e^{st} f(s)\, ds = \int_A + \int_{A'} + \int_B + \int_{B'} + \int_C + \int_{\beta - i\infty}^{\beta + i\infty} ds \left[e^{st} s^{-1/2}\right] = 0$$

$$= 0 + 2i \int_\infty^0 \frac{e^{-Rt}}{\sqrt{R}}\, dR + 0 + \int_{\beta - i\infty}^{\beta + i\infty} ds \left[e^{st} s^{-1/2}\right] = 0$$

$$\therefore \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} ds \left[e^{st} s^{-1/2}\right] = \frac{1}{\pi} \int_0^\infty \frac{e^{-Rt}}{\sqrt{R}}\, dR$$

$$\mathbf{\frac{1}{\pi} \int_0^\infty e^{-R\, t}\, R^{-1/2}\, dR}$$

$$\text{ConditionalExpression}\left[\frac{1}{\sqrt{\pi}\,\sqrt{t}},\ \text{Re}[t] > 0\right]$$

Or apply Gamma function; $\Gamma(a) = \int_0^\infty e^{-z} z^{a-1}\, dz$

$$\Gamma(a) = \int_0^\infty e^{-z} z^{a-1}\, dz \quad \begin{pmatrix} z \to R t \\ dz \to t\, dR \end{pmatrix} \Rightarrow \int_0^\infty e^{-Rt} (R t)^{a-1}\, t\, dR$$

$$\Gamma(1/2) = \int_0^\infty e^{-Rt} (R t)^{-1/2}\, t\, dR = \sqrt{t} \int_0^\infty \frac{e^{-Rt}}{\sqrt{R}}\, dR$$

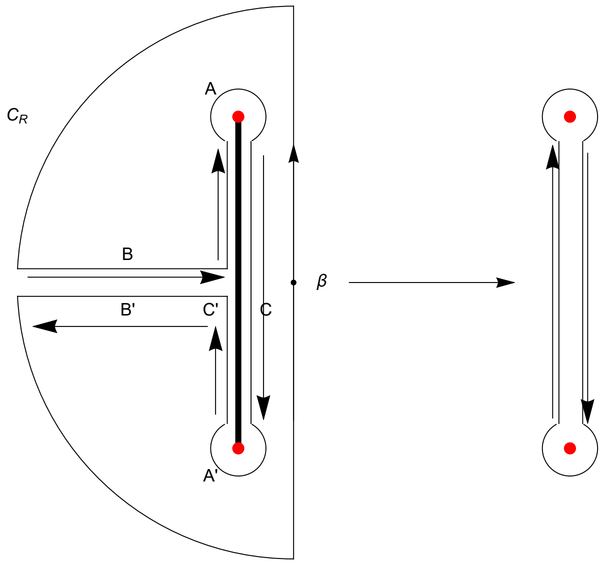
$$\therefore \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} ds \left[e^{st} s^{-1/2}\right] = \frac{1}{\pi} \int_0^\infty \frac{e^{-Rt}}{\sqrt{R}}\, dR = \frac{1}{\pi \sqrt{t}} \Gamma(1/2) = \frac{1}{\sqrt{\pi t}}$$

By *Mathematica*,

```
InverseLaplaceTransform[1 / √s , s, t]
```

$$\frac{1}{\sqrt{\pi}\,\sqrt{t}}$$

Exercise.
Find the inverse Laplace transform of $f(s) = \left(1+s^2\right)^{-1/2}$



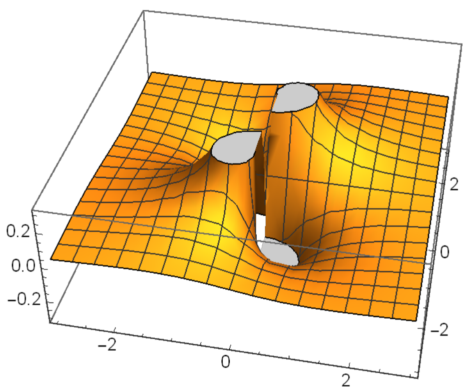
$$\oint = 0 = \int_{C_R} + \int_A + \int_{A'} + \int_B + \int_{B'} + \int_C + \int_{C'} + 2\pi i\, F(t) \quad \text{where } F(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} ds\,[e^{st} f(s)]$$

We already know that $\int_{C_R} = 0,\ \int_A = 0,\ \int_{A'} = 0,\ \int_B + \int_{B'} = 0$. Thus,

$$\oint = \int_C + \int_{C'} + 2\pi i\, F(t) = 0$$

$$F(t) = \frac{1}{2\pi i}\left(-\int_C - \int_{C'}\right) = \frac{1}{2\pi i}\left(\int_{-i}^{i} ds\left[\frac{e^{st}}{(1+s^2)^{1/2}}\right]_{\text{RHS of } b.c} + \int_{i}^{-i} ds\left[\frac{e^{st}}{(1+s^2)^{1/2}}\right]_{\text{LHS of } b.c}\right)$$

```
Plot3D[Im[(1 + (x + ⅈ y)^-2)^-1/2], {x, -3, 3}, {y, -3, 3}]
```



As you can see in that above figure, right hand side of branch cut(RHS of b.c) and left hand side of branch cut is not the same. Let RHS of b.c. be positive then LHS of b.c. will be negative. Then,

$$\frac{1}{2\pi i}\left(\int_{-i}^{i} ds\left[\frac{e^{st}}{(1+s^2)^{1/2}}\right]_{\text{RHS of } b.c} + \int_{i}^{-i} ds\left[\frac{e^{st}}{(1+s^2)^{1/2}}\right]_{\text{LHS of } b.c}\right) = \frac{1}{2\pi i}\left(\int_{-i}^{i}\frac{e^{st}}{(1+s^2)^{1/2}}\,ds + \int_{i}^{-i} ds\ -\left[\frac{e^{st}}{(1+s^2)^{1/2}}\right]\right)$$

$$= \frac{1}{2\pi i}\left(\int_{-i}^{i}\frac{e^{st}}{(1+s^2)^{1/2}}\,ds + \int_{-i}^{i} ds\,\frac{e^{st}}{(1+s^2)^{1/2}}\right) = \frac{1}{\pi i}\int_{-i}^{i}\frac{e^{st}}{(1+s^2)^{1/2}}\,ds = \left(\begin{matrix} s \to u\, i \\ ds \to i\, du \end{matrix}\right) \Rightarrow \frac{1}{\pi}\int_{-1}^{1}\frac{e^{iut}}{(1-u^2)^{1/2}}\,du$$

Where $e^{iut} = \cos(u\,t) + i\sin(u\,t)$, then the odd part of integrand $i\sin(u\,t)/(1-u^2)^{1/2}$ killed.

$$\therefore\ F(t) = \frac{2}{\pi}\int_0^1 \frac{\cos(u\,t)}{(1-u^2)^{1/2}}\,du$$

The integral is an Bessel function, $J_0(t)$

$$\frac{2}{\pi}\int_0^1 \frac{\text{Cos}[u\,t]}{\left(1-u^2\right)^{1/2}}\,du$$

```
ConditionalExpression[BesselJ[0, Abs[t]], t ∈ ℝ]
```

```
Integrate[2 Cos[u t] / (π (1 - u^2)^(1/2)), {u, 0, 1}, Assumptions → t ∈ Reals]
```

```
BesselJ[0, Abs[t]]
```

Applying InverseLaplaceTransform directly,

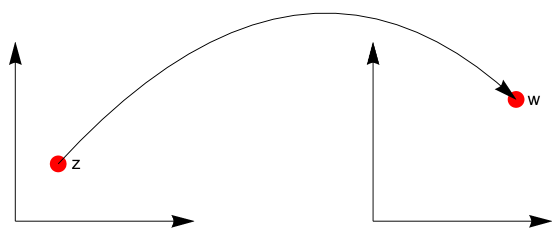
```
InverseLaplaceTransform[(1 + s^2)^(-1/2), s, t] // TraditionalForm
```

$J_0(t)$

5. Mapping

5.1 Before Start

Mapping is a complex function which means that if you entered a 2-dimensional value, then the mapping outputs 2-dimensional value consist of real part and imaginary part. Following figure is describes the mapping : $w = f(z)$



The functions we've been handling in the meantime takes n-dimensional information and returns a one-dimensional value; $f(x) = x^2$, $f(x, y) = x \sin y$ $f(x, y, z) = x^y \sin z$, however, in this section, we will encounter slightly different guys.

Consider the mapping $w(z) = u + i\, v$ where $z = x + i\, y$. By this mapping, the point (x, y) transformed to $(u(x, y), v(x, y))$ as I mentioned above. For example, we can talk about some linear transforms that you have learned at this spring.

Exercise.
Linear Transformations

Consider the $w(z) = A\,z$ with $A = a\, e^{i\,\alpha}$. Then,

$$w = a\, e^{i\,\alpha} z = a\,(\cos\alpha + i \sin\alpha)\,(x + i\,y)\;\begin{cases} u(x, y) = a(x\cos\alpha - y\sin\alpha) \\ v(x, y) = a(x\sin\alpha + y\cos\alpha) \end{cases}$$

or,

$$\begin{pmatrix} u \\ v \end{pmatrix} = a \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which shows rotation! In this case, polar forms make you feel better. Let $z(r,\ \theta) = r\,e^{i\,\theta}$

$$\text{then,}\quad w(r,\ \theta) = a\,e^{i\,\alpha}\,r\,e^{i\,\theta} = a\,r\,e^{i(\alpha+\theta)}$$

which provides some insights to why this mapping represents rotation.

Next, take one more step. Add $B = b + i\,\beta$ to original, then,

$$\begin{pmatrix} u \\ v \end{pmatrix} = a \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ \beta \end{pmatrix} \quad \text{or} \quad a\,r\,e^{i(\alpha+\theta)} + b + i\,\beta$$

where B represents translation.

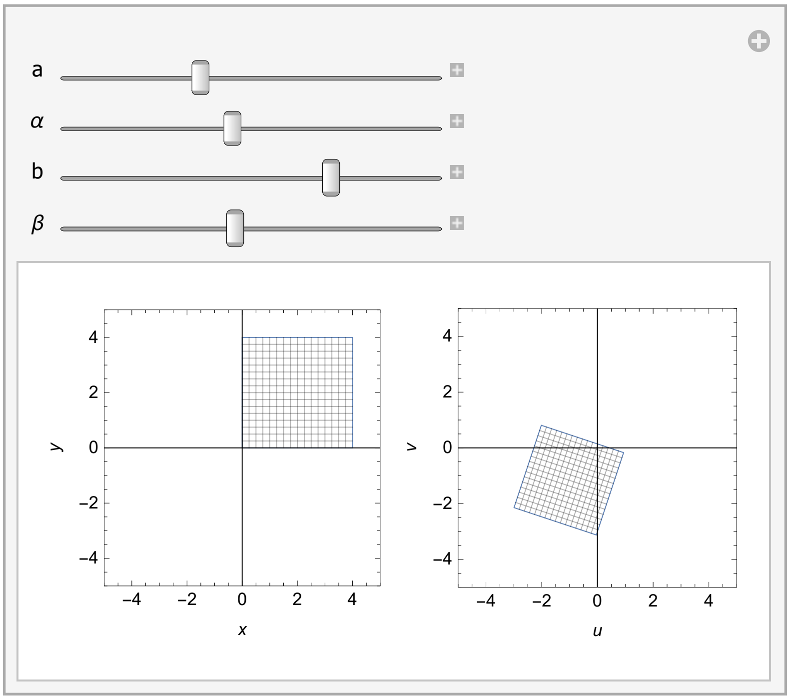
With **ParametricPlot[]** we can graphically construct the mappings. Make your own **Conformal[]** function.

In[17]:=
```
Conformal =
 GraphicsGrid[{{ParametricPlot[{Re[#1], Im[#1]}, #3, #4, Mesh → Automatic, PlotStyle →
        None, #5, FrameLabel → {x, y}], ParametricPlot[{Re[#2], Im[#2]}, #3,
      #4, Mesh → Automatic, PlotStyle → None, #5, FrameLabel → {u, v}]}}] &
```

Out[17]=
```
GraphicsGrid[
  {{ParametricPlot[{Re[#1], Im[#1]}, #3, #4, Mesh → Automatic, PlotStyle → None,
     #5, FrameLabel → {x, y}], ParametricPlot[{Re[#2], Im[#2]}, #3, #4,
     Mesh → Automatic, PlotStyle → None, #5, FrameLabel → {u, v}]}}] &
```

Let's see our mapping ; $w(z) = A\,z + B$.

```
Manipulate[Conformal[x + ⅈ y, a ⅇ^(ⅈ α) (x + ⅈ y) + (b + ⅈ β), {x, 0, 4}, {y, 0, 4},
  PlotRange → {{-5, 5}, {-5, 5}}], {a, 0.1, 2}, {α, 0, 2 π}, {b, -2, 2}, {β, -2, 2}]
```



Manipulating each value, it will enhance your understanding.

Exercise.

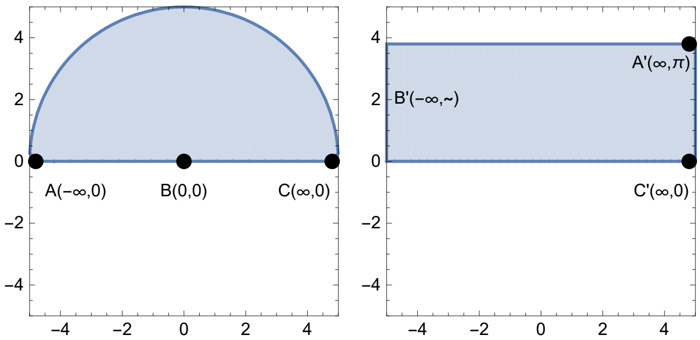
Mappings by ln z

Let $z = r\,e^{i\theta}$ then $w = \ln\left(r\,e^{i\theta}\right) = \ln r + i\,\theta$ so $\begin{cases} u(r,\theta) = \ln r \\ v(r,\theta) = \theta \end{cases}$

Then how is the upper half plane mapped? We have to think about 3 points;

$A = \lim_{R\to\infty} R\,e^{i\pi}$, $B = 0\begin{cases} \epsilon e^{i0} \\ \epsilon e^{i\pi} \end{cases}$, $C = \lim_{R\to\infty} R\,e^{i0}$ goes to

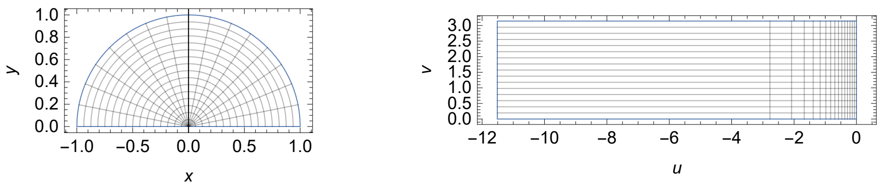
$A' = (\ln R, \pi)$, $B' = \begin{cases} (-\infty, 0) \\ (-\infty, \pi) \end{cases}$, $C' = (\ln R, 0)$



By mapping, hence, line AB and line BC mapped into line $A'B'$ and line $B'C'$

For $R = 1$ case,

```
Conformal[(r ⅇ^ⅈu), Log[(r ⅇ^ⅈu)], {r, 0.00001, 1},
 {u, 0, π}, PlotRange → {All, {{-3.5, 3.5}, {-3.5 , 3.5}}}]
```



The mesh shown in the figure is streamline and isotherms or equipotentials. You will see them again later.

5.2 Conformal Mapping

In many physical situations, such as electrostatics or heat conduction, can be mathematically formulated in terms of Laplace's equation;

$$\nabla^2 \Phi(x, y) = 0 \implies \Phi_{xx} + \Phi_{yy} = 0$$

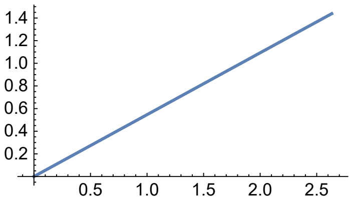
Cauchy-Riemann condition says that real part and imaginary part of analytic function are also satisfying Laplace's equation. Then we can reduce the problems of complicated physical conditions to the problems of finding analytic function in given domain and boundary condition. Moreover, by method of **mapping**(generally conformal mapping), we can simplifying the variables in the situation. Few theorems are ready to help you to be friendly with conformal mapping. After meet some theorems let's see how mapping can utilized to solve some messy problems.

<Theorem 7>

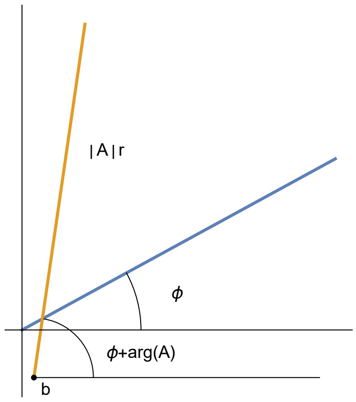
- Assume that $f(z)$ is analytic and not constant in a domain D of the complex z plane. For any point $z \in D$ for which $f'(z) \neq 0$, this mapping is **conformal**, that is, it preserves the angle between two differentiable arcs.

Before prove this theorem, consider a simple case: transformation $w(z) = A\,z + b$ of line segment.

Consider following line segment of curve $z(r) = r\,e^{i\phi}$, note that ϕ is some constant.



The line segment mapped as following by the transformation $w(z) = A\,z + b$ with A and b are some complex numbers.



$$\begin{aligned} w &= A\,z + b \\ &= A\left(r\,e^{i\phi}\right) + b \\ &= |A|\,e^{i\arg(A)}\left(r\,e^{i\phi}\right) + b \\ &= |A|\,r\,e^{i(\arg(A)+\phi)} + b \end{aligned}$$

So, the mapped curve is $\arg(A) = \arg(f'(z))$ rotation of original curve

Now let's consider for general case.

Let C is a differentiable arc given by $C: z(r) = x(r) + i\,y(r)$ $(x, y \in \mathbb{R})$ and write the mapping $w(z)$ as $w = u(x, y) + i\,v(x, y)$ $(u, v \in \mathbb{R})$. Then, the image of the curve C is the C^*: $w(r) = u(x(r), y(r)) + i\,v(x(r), y(r))$. Note that image of differentiable curve is a differentiable curve.

Defining $\frac{dz(r)}{dr} = \frac{dx(r)}{dr} + i\,\frac{dy(r)}{dr}$, the image of differentiation $\frac{dw(r)}{dr}$ will be

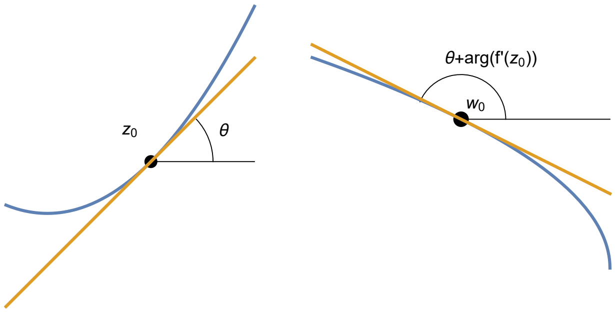
$$\left[\frac{dw(r)}{dr}\right]_{r=r_0} = f'(z_0)\left[\frac{dz(r)}{dr}\right]_{r=r_0} \quad \left(\begin{array}{c}\text{the image of } C \text{ is } w(r) = f(z(r)) \\ z_0 \equiv z(r_0)\end{array}\right)$$

If $f'(z_0) \neq 0$ and $z'(r_0) \neq 0$, it follows that $w'(r_0) \neq 0$ and

$$\begin{aligned} |w'(r_0)|\,e^{i\arg(w'(r_0))} &= |f'(z_0)|\,e^{i\arg(f'(z_0))}\;|z'(r_0)|\,e^{i\arg(z'(r_0))} \\ &= |f'(z_0)|\;|z'(r_0)|\,e^{i\arg(f'(z_0))}\,e^{i\arg(z'(r_0))} \\ &= |f'(z_0)|\;|z'(r_0)|\,e^{i[\arg(f'(z_0))+\arg(z'(r_0))]} \end{aligned}$$

$$\text{Then, } \begin{cases} \arg(w'(r_0)) = \arg(f'(z_0)) + \arg(z'(r_0)) \\ \text{Or, } \arg(dw) = \arg(dz) + \arg(f'(z_0)) \end{cases}$$

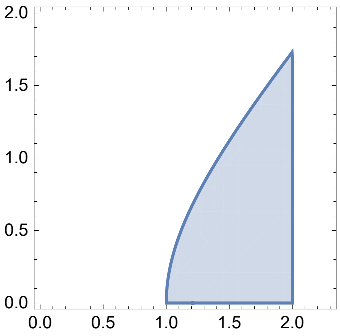
In short, under the analytic transformation $f(z)$, the directed tangent to any curve through z_0 is rotated by an angle $\arg(f'(z_0))$.



Exercise.

Find the image of the region R_z bounded by $y = 0$, $x = 2$, and $x^2 - y^2 = 1$ for $x \geq 0$ and $y \geq 0$, under the transformation $w = z^2$

```
RegionPlot[x^2 - y^2 ≥ 1 && 0 ≤ x < 2 && y > 0, {x, 0, 2.3}, {y, 0, 2}]
```



By mapping $w = z^2$,

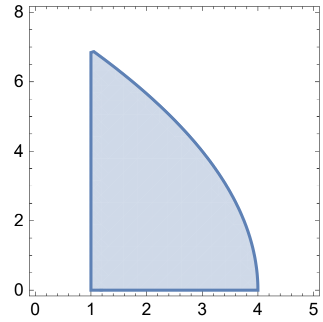
$$w(x, y) = (x + i\, y)^2 = (x^2 - y^2) + i\,(2\,x\,y) = u(x, y) + i\, v(x, y)$$

$$\begin{cases} u(x, y) = x^2 - y^2 \\ v(x, y) = 2\,x\,y \end{cases}$$

Hence,

$$\begin{cases} y = 0 & \to u = x^2\ ,\ v = 0 \\ x = 2 & \to u = 4 - y^2,\ v = 4\,y\ \left(y : 0 \sim \sqrt{3}\right) \\ x^2 - y^2 = 1 & \to u = 1,\ v = 2\,x\,y\ \left(x : 1 \sim 2,\ y : 0 \sim \sqrt{3}\right) \end{cases}$$

```
RegionPlot[u > 1 && v > 0 && u + v^2 / 16 < 4, {u, 0, 5}, {v, 0, 8}]
```

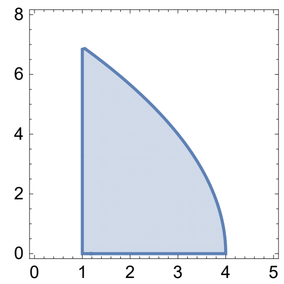


Or,

```
W[z_] := z^2;
z /. Solve[W[z] == w, z]
```

$\{-\sqrt{w}, \sqrt{w}\}$

```
RegionPlot[Re[#]^2 - Im[#]^2 ≥ 1 && 0 ≤ Re[#] < 2 && Im[#] > 0, {u, 0, 5}, {v, 0, 8}] &[
  Sqrt[u + ⅈ v]]
```



Exercise.

Show the mapping $w = \sqrt{1 - z^2}$ maps the hyperbola $2\,x^2 - 2\,y^2 = 1$ onto itself.

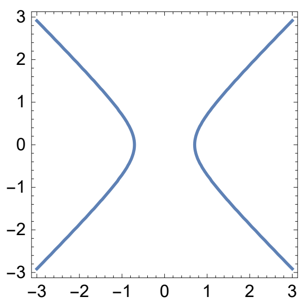
Do it yourself!

```
Clear[W]
```

```
W[z_] := Sqrt[1 - z^2];
z /. Solve[W[z] == w, z]
```

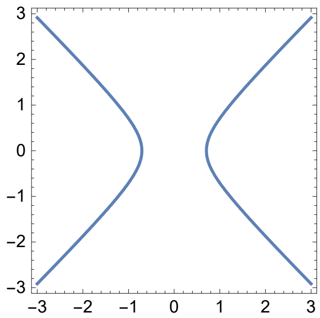
$\{-\sqrt{1 - w^2}, \sqrt{1 - w^2}\}$

```
ContourPlot[2 Re[#]^2 - 2 Im[#]^2 == 1, {x, -3, 3}, {y, -3, 3}] &[x + ⅈ y]
```



transforms to

```
ContourPlot[2 Re[#]^2 - 2 Im[#]^2 == 1, {u, -3, 3}, {v, -3, 3}] &[√(1 - (u + ⅈ v)^2)]
```



<Remark 5.1>

- How to display conformal mapping in *Mathematica* ?

1. There is some region(or curve) in z plane and given mapping $w(z)$
2. Get the inverse mapping of $w(z)$
3. Insert inverse mapping to initial region (or curve)

This remark will helps you much while studying this chapter.

5.3 Critical Points and Inverse Mapping

For any point $z \in D$ for which $f'(z) \neq 0$, we call this mapping is **conformal**. But what about if $f'(z) = 0$?

In this case, the point z_0 such that $f'(z_0) = 0$ is called a critical point of f. At point $z = z_0$, the mapping f ceases to be conformal. Then let's see what happens at critical point.

Let $\delta z = z - z_0$ then, $\delta w = f(z) - f(z_0)$ where $f(z)$ around the $z = z_0$ is

$$f(z) = f(z_0) + \frac{1}{1!} f^{(1)}(z_0)(z - z_0)^1 + \frac{1}{2!} f^{(2)}(z_0)(z - z_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n + \cdots$$

According to this,

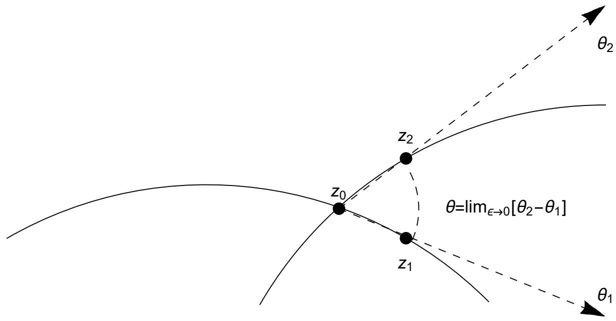
$$\delta w = f(z) - f(z_0) = \frac{1}{1!} f^{(1)}(z_0)(z - z_0)^1 + \frac{1}{2!} f^{(2)}(z_0)(z - z_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n + \cdots$$

<Theorem 8>

- Assume that $f(z)$ is analytic and not constant in a domain D of the complex z plane. Suppose that $f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0) = 0$, while $f^{(n)}(z_0) \neq 0$, $z_0 \in D$. Then the mapping $z \to f(z)$ magnifies n times the angle between two intersection differentiable arcs that meet at z_0.

The proof is easy!

Let $z_1(r)$ and $z_2(r)$ be the equations of the two curves in the domain D



If z_1 and z_2 are points on this arcs that have a distance ϵ from z_0, then z_1 and z_2 are

$$\begin{cases} z_1 = z_0 + \epsilon e^{i\theta_1} \\ z_2 = z_0 + \epsilon e^{i\theta_2} \end{cases}, \quad e^{i(\theta_2-\theta_1)} = \frac{z_2-z_0}{z_1-z_0}$$

As the ϵ tends to zero, θ tends to the angle formed by the two intersecting arcs in the complex z plane.

Let θ be the angle in the complex z plane and ϕ be the angle in the complex w plane.
By mapping $w = f(z)$,

$$\theta = \lim_{\epsilon\to 0} \arg\left(\frac{z_2-z_0}{z_1-z_0}\right), \qquad \phi = \lim_{\epsilon\to 0} \arg\left(\frac{f(z_2)-f(z_0)}{f(z_1)-f(z_0)}\right)$$

$$\phi = \lim_{\epsilon\to 0} \arg\left(\frac{f(z_2)-f(z_0)}{f(z_1)-f(z_0)}\right) = \lim_{\epsilon\to 0} \arg\left[\left(\frac{\frac{f(z_2)-f(z_0)}{(z_2-z_0)^n}}{\frac{f(z_1)-f(z_0)}{(z_1-z_0)^n}}\right)\left(\frac{z_2-z_0}{z_1-z_0}\right)^n\right]$$

By assumption$(f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0) = 0,\ \text{while } f^{(n)} \neq 0)$,

$$\delta w = f(z) - f(z_0) = \frac{1}{1!} f^{(1)}(z_0)(z-z_0)^1 + \frac{1}{2!} f^{(2)}(z_0)(z-z_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n + \cdots$$

$$= 0 + 0 + \cdots + \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n + \cdots$$

$$\text{Hence, } \begin{cases} f(z_2) - f(z_0) = \frac{1}{n!} f^{(n)}(z_0)(z_2-z_0)^n + \cdots \\ f(z_1) - f(z_0) = \frac{1}{n!} f^{(n)}(z_0)(z_1-z_0)^n + \cdots \end{cases}$$

It follows that

$$\lim_{\epsilon\to 0} \frac{f(z_2)-f(z_0)}{(z_2-z_0)^n} = \lim_{\epsilon\to 0} \frac{f(z_1)-f(z_0)}{(z_1-z_0)^n} = \frac{f^{(n)}(z_0)}{n!}$$

Therefore,

$$\phi = \lim_{\epsilon\to 0} \arg\left[\left(\frac{\frac{f(z_2)-f(z_0)}{(z_2-z_0)^n}}{\frac{f(z_1)-f(z_0)}{(z_1-z_0)^n}}\right)\left(\frac{z_2-z_0}{z_1-z_0}\right)^n\right] = \lim_{\epsilon\to 0} \arg\left[\left(\frac{z_2-z_0}{z_1-z_0}\right)^n\right] = \lim_{\epsilon\to 0} \arg\left[e^{i n(\theta_2-\theta_1)}\right] = n\,\theta \quad \blacksquare$$

<Theorem 9>

- Assume that $f(z)$ is analytic at z_0 and that $f'(z_0) \neq 0$. Then $f(z)$ is single-valued in the neighborhood of z_0. That is, $w = f(z)$ has a unique analytic inverse map $z = F(w)$.
- Assume that $f(z)$ is analytic at z_0 and that it has a zero of order n. That is, the first nonvanishing derivative of $f(z)$ at $z = z_0$ is $f^{(n)}(z_0)$. Then to each w around the $w_0 = f(z_0)$, there correspond n distinct point z in the neighborhood of z_0, each of which has w as its image under the mapping $w = f(z)$.

By Taylor expansion, $w = f(z) = f(z_0) + \frac{1}{1!} f^{(1)}(z_0)(z-z_0) + \frac{1}{2!} f^{(2)}(z_0)(z-z_0)^2 + \cdots$

If $f'(z_0) \neq 0$, $w - w_0 = f(z) - f(z_0)$ is approximately $f'(z_0)(z - z_0)$ and there exists unique inverse map.

$$w - w_0 \approx f'(z_0)(z - z_0), \quad z \approx \frac{w - w_0}{f'(z_0)} + z_0$$

However if $f'(z_0) = 0$ and the first nonvanishing derivative of $f(z)$ at $z = z_0$ is $f^{(n)}(z_0)$, then

$$w - w_0 \approx \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n, \quad z \approx \left[\frac{n!(w - w_0)}{f^{(n)}(z_0)}\right]^{1/n} + z_0$$

Is $z(w)$ is single-valued? NO! At the critical point, the inverse mapping of f is a multi-valued function. More precisely, w_0 would be a branch point of order n. We already know that the function who has branch point of order n lives in n sheeted Riemann surface.(Remember $\ln z$; a infinite whirlpool potato!)

<Theorem 10>

Let C be a simple closed contour enclosing a domain D , and let $f(z)$ be analytic on C and in D. Suppose $f(z)$ takes no value more than once on C. Then,

- the map $w = f(z)$ takes C enclosing D to a simple closed contour C^* enclosing a domain D^* in the w plane.
- $w = f(z)$ is a one-to-one map from D to D^*
- if z traverses C in the positive direction, then $w = f(z)$ traverses C^* in the positive direction.

Exercise.

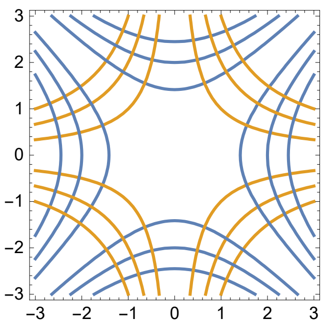
Find the families of curves on which $\text{Re}[z^2] = C_1$ for constant C_1, and $\text{Im}[z^2] = C_2$ for constant C_2. Show that is two families are orthogonal to each other.

In the z plane, $\text{Re}[z^2] = \text{Re}[(x + i y)^2] = x^2 - y^2$ and $\text{Im}[z^2] = 2xy$

It is obvious that $\nabla\text{Re}[z^2] \perp \nabla\text{Im}[z^2]$ since,

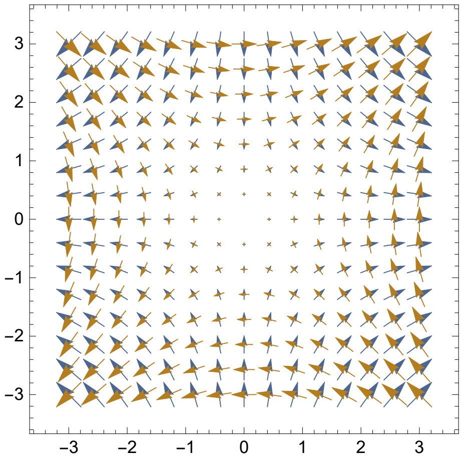
$$\left(2x\hat{i} - 2y\hat{j}\right)\cdot\left(2y\hat{i} + 2x\hat{j}\right) = 0$$

```
Show[ContourPlot[{(x² - y² == #), (2 x y == #)}, {x, -3, 3},
    {y, -3, 3}, ContourLabels → True] & /@ Cases[Range[-6, 6, 2], Except[0]]]
```



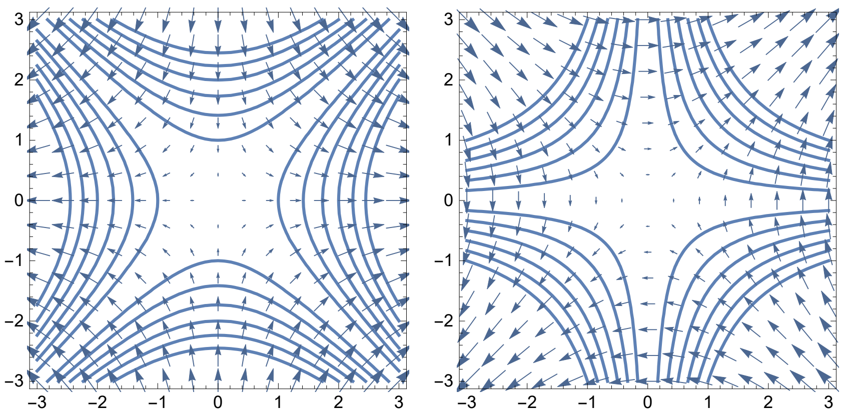
Blue one is $\text{Re}[z^2] = C_1$ and yellow one is $\text{Im}[z^2] = C_2$ and their gradients shape like this;

```
VectorPlot[{{2 x, -2 y}, {2 y, 2 x}}, {x, -3, 3}, {y, -3, 3}]
```



Note that their gradients are perpendicular to its mother contour.

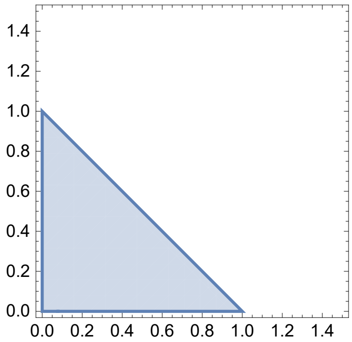
```
GraphicsGrid[
 {{Show[ContourPlot[{(x² - y² == #)}, {x, -3, 3}, {y, -3, 3}, ContourLabels → True] & /@
      Cases[Range[-6, 6, 1], Except[0]],
    {VectorPlot[{2 x, -2 y}, {x, -3, 3}, {y, -3, 3}]}],
   Show[ContourPlot[{(2 x y == #)}, {x, -3, 3}, {y, -3, 3}, ContourLabels → True] & /@
      Cases[Range[-6, 6, 1], Except[0]],
    {VectorPlot[{2 y, 2 x}, {x, -3, 3}, {y, -3, 3}]}]}}]
```



This exercise is important to understand streamline and equipotential.

Exercise.
Find the image of *D* (see below figure!) under the mapping $w = z^3$

```
RegionPlot[x > 0 && y > 0 && x + y < 1, {x, 0, 1.5}, {y, 0, 1.5}]
```



$$w(x, y) = z^3 = (x + i y)^3 = (x^3 - 3xy^2) + i(3x^2 y - y^3) = u(x, y) + i v(x, y)$$

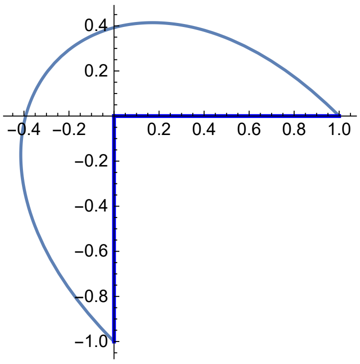
$$\begin{cases} u(x, y) = x^3 - 3xy^2 \\ v(x, y) = 3x^2 y - y^3 \end{cases}$$

By <Theorem 9>,

$$\begin{cases} x = 0 \to & u = 0,\ v = -y^3\ (y : 0 \sim 1) \\ y = 0 \to & u = x^3,\ v = 0\ (x : 0 \sim 1) \\ x + y = 1 \to & u = 1 - 3y + 2y^3 \\ & v = 3y - 6y^2 + 2y^3 \end{cases}$$

Display it to the complex *w* plane.

```
Show[{ParametricPlot[{1 - 3 y + 2 y^3, 3 y - 6 y^2 + 2 y^3}, {y, 1, 0}],
  Graphics[{Blue, Thick, Line[{{0, 0}, {0, -1}}], Line[{{0, 0}, {1, 0}}]}]}]
```

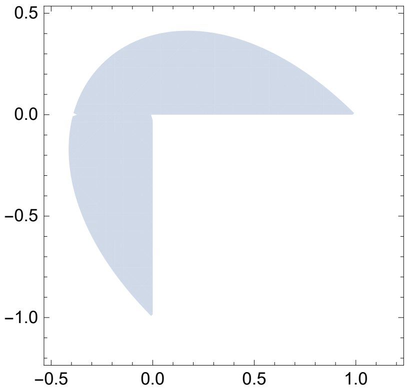


Or use *Mathematica* directly! (See Remark 5.1!)

```
W[z_] := z^3;
z /. Solve[W[z] == w, z]
```

$\{w^{1/3}, -(-1)^{1/3} w^{1/3}, (-1)^{2/3} w^{1/3}\}$

```
Show[
 RegionPlot[(Re[#] > 0 && Im[#] > 0 && Re[#] + Im[#] < 1), {u, -0.5, 1.2}, {v, -1.2, 0.5},
   BoundaryStyle → None] & /@ {(u + ⅈ v)^(1/3), -(-1)^(1/3) (u + ⅈ v)^(1/3), (-1)^(2/3) (u + ⅈ v)^(1/3)}]
```



Exercise.

Express the transformations $\begin{cases} (a) & u = 4x^2 - 8y,\ \ v = 8x - 4y^2 \\ (b) & u = x^3 - 3xy^2,\ \ v = 3x^2 y - y^3 \end{cases}$

in the form $w = F(z, \bar{z})$, $z = x + i\, y$, $\bar{z} = x - i\, y$.

Then $x = (z + \bar{z})/2$ and $y = -i(z - \bar{z})/2$

$$\begin{cases} (a) & u = (z + \bar{z})^2 + i\, 4\,(z - \bar{z}),\ \ v = 4\,(z + \bar{z}) + (z - \bar{z})^2 \\ (b) & u = \left(z^3 + \bar{z}^3\right)/2,\ \ v = -i\left(z^3 - \bar{z}^3\right)/2 \end{cases}$$

Hence,

$$\begin{cases} (a) & w = u + i\, v = F(z, \bar{z}) = (z + \bar{z})^2 + i\,((z - \bar{z})^2 + 8\,z) \\ (b) & w = u + i\, v = F\,(z, \bar{z}) = \left(z^3 + \bar{z}^3\right)/2 + i\left(-i\left(z^3 - \bar{z}^3\right)/2\right) = z^3 \end{cases}$$

Then, which is conformal mapping? Conformal mapping has to satisfy (1)analytic and not constant, (2)nonzero 1st derivative.

$$\text{For } (a),\ \begin{cases} \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y} & (8\,x \neq -8\,y) \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} & (-8 = -8) \end{cases}\ \text{ this mapping is not analytic}$$

or just glancing at the $F(z, \bar{z})$, we can find impurity; $\bar{z}$ neither analytic nor differentiable

$$\text{For } (b),\ \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} & (3\,x^2 - 3\,y^2 = 3\,x^2 - 3\,y^2) \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} & (-6\,x\,y = -6\,x\,y) \end{cases}\ \text{ this mapping is analytic}$$

Moreover, (b) is differentiable and its 1st derivative is $3\,z^2$ that is, (b) is the conformal mapping in the entire region but $z = 0$ in complex z plane.

5.4 Applications of Conformal Mapping

In this section, we're going to utilize the mapping to solve the problems in the physical world. That is, the problems of Laplace's equation; $\nabla^2 \Phi = \Phi_{xx} + \Phi_{yy} = 0$

When the twice differentiable function $\Phi(x, y)$ satisfying Laplace's equation in a region R, it is called harmonic function in region R. For $z = x + i\, y$, let $V(z)$ be analytic in R. If $V(z) = u(x, y) + i\, v(x, y)$, where u, $v \in \mathbb{R}$ and are twice differentiable, then,

By Cauchy-Riemann condition: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Rightarrow \nabla^2 u(x, y) = 0$ and $\nabla^2 v(x, y) = 0$

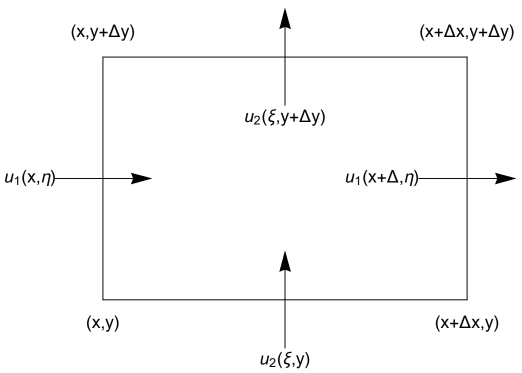
both $u(x, y)$ and $v(x, y)$ are harmonic in region R. In this case they are called conjugate each other.

Let u_1 and u_2 be the components of some vector $\vec{u}$ $\left(\vec{u} = (u_1, u_2)\right)$ If the components of the vector $\vec{u}$ satisfy the equation $\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = \nabla \cdot \vec{u} = 0$, then there exists a scalar function $\Phi(x, y)$ such that

$$u_1 = \frac{\partial \Phi}{\partial x},\ u_2 = \frac{\partial \Phi}{\partial y} \quad \text{that is,} \quad \nabla \Phi = (u_1, u_2)$$

Then the $\Phi(x, y)$ is harmonic. We can think $\Phi(x, y)$ as a potential and u_1 and u_2 as force along the x and y axis.

Exercise.
Two dimensional ideal fluid flow.



Let ρ be the function of x, y and t, then the accumulation rate of fluid in above rectangle is given by

$$\frac{d}{dt}\int_x^{x+\Delta x}\int_y^{y+\Delta y} \rho \, dx \, dy$$

By the law of conservation of mass,

$$\frac{d}{dt}\int_x^{x+\Delta x}\int_y^{y+\Delta y} \rho \, dx \, dy = \int_y^{y+\Delta y}[(\rho u_1)(x, \eta) - (\rho u_1)(x+\Delta x, \eta)]\, d\eta$$
$$+\int_x^{x+\Delta x}[(\rho u_2)(\xi, y) - (\rho u_2)(\xi, y+\Delta y)]\, d\xi$$

Divide the equation by $\Delta x \Delta y$ and take the limit as Δx, Δy tends to zero, then,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x} + \frac{\partial(\rho u_2)}{\partial y} = 0$$

$\frac{\partial \rho}{\partial t}$ is zero since the flow is steady and if the fluid is incompressible, the ρ is constant. Hence,

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = (\nabla \cdot \vec{u}) = 0$$

It follows that there exists a harmonic scalar function $\Phi(x, y)$ such that $u_1 = \partial \Phi / \partial x$ and $u_2 = \partial \Phi / \partial y$

If the flow is irrotational, its curl would be zero; $\nabla \times \vec{u} = \left(0, 0, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) = (0, 0, 0)$ that is,

$$\frac{\partial u_2}{\partial x} = \frac{\partial u_1}{\partial y}$$

which also follows the existence of a harmonic scalar function $\Phi(x, y)$.

Because the function $\Phi(x, y)$ is harmonic, there exists its conjugate function $\Psi(x, y)$ such that

$$\Omega(x, y) = \Phi(x, y) + i\,\Psi(x, y)$$

with $\Omega(x, y)$ analytic. Differentiating $\Omega(x, y)$,

$$\frac{d\Omega}{dz} = \frac{d\Phi}{dx} + i\frac{d\Psi}{dx} = (\text{by Cauchy} - \text{Riemann condition,}) \Rightarrow \frac{d\Phi}{dx} - i\frac{d\Phi}{dy}$$

That is first derivative of Ω, $d\Omega/dz$ is $u_1 - i\,u_2 = u^*$ where $u = u_1 + i\,u_2$ is called complex velocity.

$$\text{Complex velocity } (u) = \left(\frac{d\Omega}{dz}\right)^*$$

The function $\Psi(x, y)$ is called the stream function while $\Omega(x, y)$ is called the complex velocity potential. The families of the curves $\Psi(x, y) = \text{const}$ are called streamlines of the flow. We know that if C is the curve of the path, then the tangent to C is the gradient of $\Phi(x, y)$. By Cauchy-Riemann condition,

$$\nabla\Phi \cdot \nabla\Psi = 0 \text{ ... since } \begin{cases} \Phi_x = \Psi_y \\ -\Phi_y = \Psi_x \end{cases}, \quad \Phi_x \Psi_x + \Phi_y \Psi_y = 0$$

That is, the vector perpendicular to curve C is $\nabla\Psi$. Therefore, from knowledge of vector calculus, the curve C is given by $\Psi = \text{const}$.

Exercise.
Electrostatics

Maybe you're not friendly with Maxwell's equation; $\begin{cases} \nabla \cdot \vec{E} = \rho/\epsilon_0 & (\rho: \text{charge density}) \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{cases}$

But someday, it will annoying you. two-dimension, if there is no source charge, that is, at electrostatic situation, ρ and $\vec{B}$ goes to zero; $\nabla \cdot \vec{E} = 0$, $\nabla \times \vec{E} = 0$

$$\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} = 0, \quad \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} = 0, \quad \vec{E} = (E_1, E_2)$$

Therefore, we have $\Phi(x, y)$ such that

$$E_1 = -\frac{\partial\Phi}{\partial x}, \quad E_2 = -\frac{\partial\Phi}{\partial y} \quad (\text{- signs are convention})$$

Since $\Phi(x, y)$ is a harmonic function, there exists conjugate function $\Psi(x, y)$. Then we have complex electrostatic potential Ω

$$\Omega(z) = \Phi(x, y) + i\,\Psi(x, y)$$

which is analytic in any region does not contain source charge.

Differentiating $\Omega(z)$ by z, we obtain the complex electric field E^*

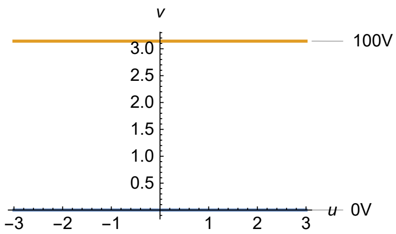
$$\frac{d\Omega}{dz} = \frac{d\Phi}{dx} + i\frac{d\Psi}{dx} = \frac{d\Phi}{dx} - i\frac{d\Phi}{dy} = -E^*$$

where $E = E_1 + i\,E_2$ and $E^* = E_1 - i\,E_2$ the curves of the families $\Phi(x, y) = \text{const}$ and $\Psi(x, y) = \text{const}$ are called equipotential and flux lines.

Exercise.
Steady Temperatures or Electrostatic Potentials

Consider two infinite parallel plates, one at potential $V = 0\,V$ and one at potential $V = 100\,V$

```
Plot[{0, π}, {u, -3, 3}, PlotLabels → {"0V", "100V"}, AxesLabel → {u, v}]
```



We already know that the potential between parallel plates ; $\Phi(u, v) = \frac{100}{\pi} v$ which is the solution of Laplace equation with given boundary condition. Since $\Phi(u, v)$ is harmonic function there must be exists the conjugate $\Psi(u, v)$. Then the function $\Omega(w) = \Phi(u, v) + i\,\Psi(u, v)$ is analytic in any region.

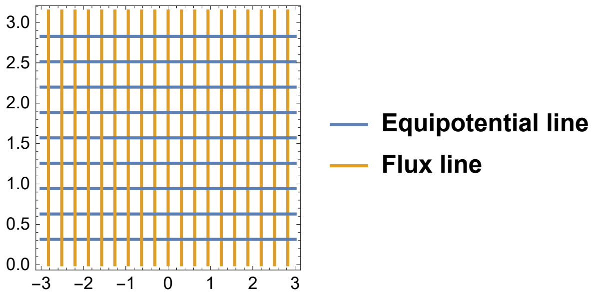
$$\partial_u \Phi = \partial_v \Psi = 0, \quad -\partial_v \Phi = \partial_u \Psi = -100/\pi \Longrightarrow \Psi(u, v) = -\frac{100}{\pi} u$$

or, using ‘complex velocity’(in this case, complex electric field)

$$\left(\frac{d\Omega}{dw}\right)^* = 0 + i\,\frac{100}{\pi} \Longrightarrow \Omega(w) = -i\,\frac{100}{\pi}(w) = -i\,\frac{100}{\pi}(u + i\,v) = \frac{100}{\pi} v - i\,\frac{100}{\pi} u$$

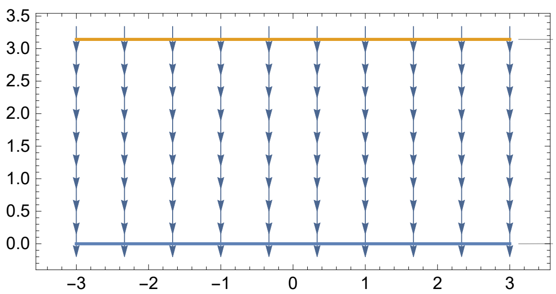
$$\begin{pmatrix} \Phi(u, v) = C_1 : \text{Equipotential line} \\ \Psi(u, v) = C_2 : \text{Flux line or streamline} \end{pmatrix}$$

```
Show[ContourPlot[{(100/π v == #), (-100/π u == #)}, {u, -3, 3}, {v, 0, π},
    PlotLegends → {"Equipotential line", "Flux line"}] & /@ Range[-90, 90, 10]]
```



and gradient of equipotential line(electric field) is $E_1 = -\frac{\partial \Phi}{\partial u}$, $E_2 = -\frac{\partial \Phi}{\partial v}$, $(E_1, E_2) = \left(0, -\frac{100}{\pi}\right)$

```
Show[{VectorPlot[{0, -100/π}, {u, -3, 3}, {v, 0, π},
   AspectRatio → 1/2, VectorPoints → 10, VectorScale → Small],
  Plot[{0, π}, {u, -3, 3}, PlotLabels → {"0V", "100V"}, AxesLabel → {u, v}]}]
```



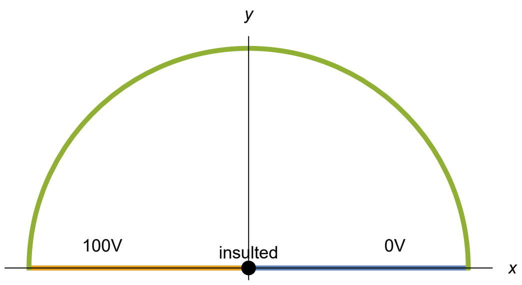
However, above $u\,v$ plane is the image of mapping $w = \ln z$ of following $x\,y$ plane.(we have shown at previous section.) Note that $w = u + i\,v = \ln r + i\,\theta$ with $r = (x^2 + y^2)^{1/2}$, $\theta = \arctan(y/x)$

$$\Longrightarrow \begin{pmatrix} u = \ln(x^2 + y^2)^{1/2} \\ v = \arctan(y/x) \end{pmatrix} \text{ or } \Omega(w) = -i\,\frac{100}{\pi}(w) \Longrightarrow \Omega'(z) = -i\,\frac{100}{\pi} \ln z$$

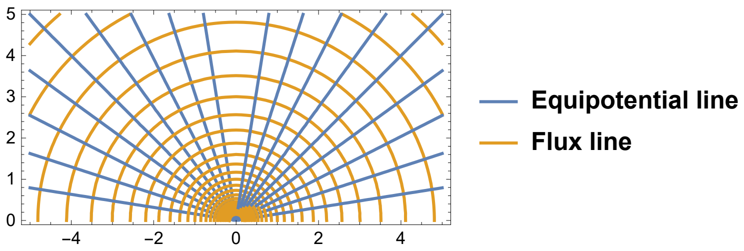
By inverse mapping, $\Omega(w)$ transformed into $\Omega'(z)$ where

$$\Omega'(z) = \Phi'(x, y) + i\,\Psi'(x, y) = \frac{100}{\pi}[\arctan(y/x)] - i\,\frac{100}{\pi}\left[\ln(x^2 + y^2)^{1/2}\right]$$

Therefore, by the theory of conformal mapping , the physical states of the above parallel infinite plates are the image of below left-half and right-half plate, that is to say, the physics of both two planes are identical. So, we can a transform which took the boundaries of a simple region (ex. a rectangle) for which we knew the solution, into the boundaries of a more complicated region for which we wanted the solution.




```
Show[ContourPlot[{(100/π ArcTan[y / x] == #), (- 100/π Log[Sqrt[x^2 + y^2]] == #)},
    {x, -5, 5}, {y, 0, 5}, AspectRatio → 1 / 2,
    PlotLegends → {"Equipotential line", "Flux line"}] & /@ Range[-60, 60, 5]]
```



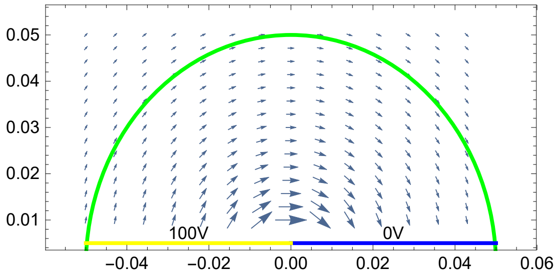
Dose it make sense?

Take a look at electric field; $-\nabla\Phi$

```
Grad[- 100/π ArcTan[y / x], {x, y}]
```

$$\left\{\frac{100\,y}{\pi\,x^2\left(1 + \frac{y^2}{x^2}\right)}, -\frac{100}{\pi\,x\left(1 + \frac{y^2}{x^2}\right)}\right\}$$

```
Show[{VectorPlot[{100 y/(π x^2 (1 + y^2/x^2)), -100/(π x (1 + y^2/x^2))},
   {x, -0.05, 0.05}, {y, 0.01, 0.05}, AspectRatio → 1/2],
  Graphics[{Thick, Green, Circle[{0, 0}, 0.05, {0, π}], Blue,
    Line[{{0, 0.005}, {0.05, 0.005}}], Yellow, Line[{{0, 0.005}, {-0.05, 0.005}}],
    {Black, Text["100V", {-0.025, 0.007}], Text["0V", {0.025, 0.007}]}}]}]
```



How about exchanging the two of them?;

$\Psi'(x, y) = \frac{100}{\pi}[\arctan(y/x)]$, $\Phi'(x, y) = -\frac{100}{\pi}\left[\ln(x^2+y^2)^{1/2}\right]$

$\left(\text{or } \Omega'(z) \to \tilde{\Omega}(z) = \frac{100}{\pi}\ln z\right)$

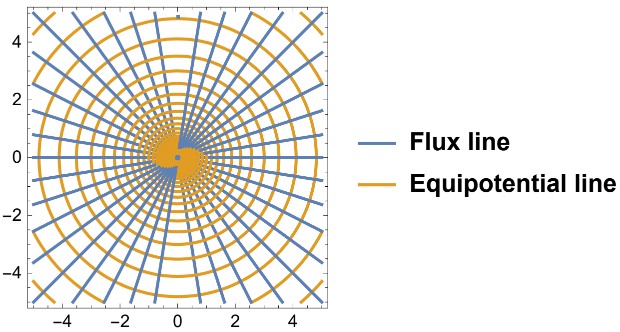
Therefore, $(E'_1, E'_2) = -\nabla\Phi'$

```
Grad[100/π Log[Sqrt[x^2 + y^2]], {x, y}]
```

$$\left\{\frac{100\,x}{\pi\,(x^2+y^2)}, \frac{100\,y}{\pi\,(x^2+y^2)}\right\}$$

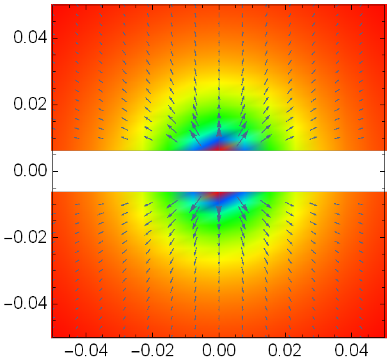
Notice that the length of the gradient is $\frac{100}{\pi(x^2+y^2)} = \frac{100}{\pi}\frac{1}{r^2}$, where electric field of a point charge is $\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$

In this case, z_∞ 0V



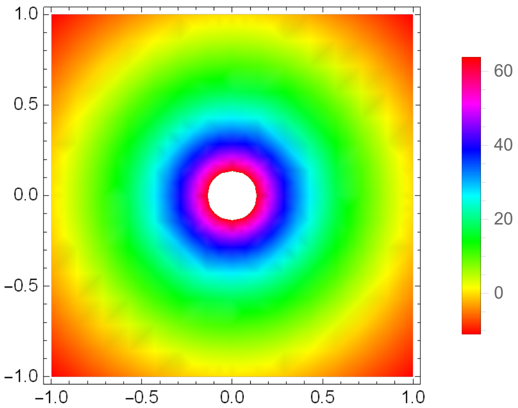
The electric field is formed as follows.

```
Show[{VectorDensityPlot[{100 x/(π (x^2 + y^2)), 100 y/(π (x^2 + y^2))}, {x, -0.05, 0.05}, {y, 0.01, 0.05},
  PlotRange → {{-0.05, 0.05}, {-0.05, 0.05}}, ColorFunction → Hue], VectorDensityPlot[
  {100 x/(π (x^2 + y^2)), 100 y/(π (x^2 + y^2))}, {x, -0.05, 0.05}, {y, -0.05, -0.01}, ColorFunction → Hue]}]
```



Equipotential line is,

```
DensityPlot[-100/π Log[Sqrt[x^2 + y^2]], {x, -1, 1},
 {y, -1, 1}, ColorFunction → Hue, PlotLegends → Automatic]
```



It is a equipotential of single source at $z = 0$

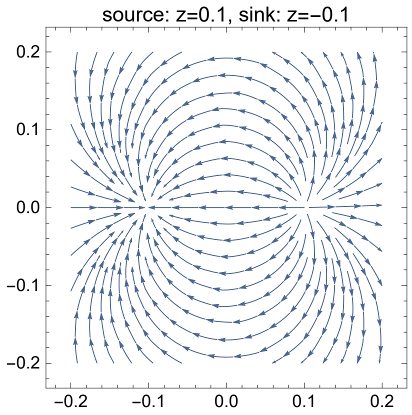
<Remark 5.2>

- **VectorPlot[{Grad[**potential**]}]** shows you a vector field and **VectorDensityPlot[]** shows you its density.

- **DensityPlot[]** or **ContourPlot[]** shows density of the potential.

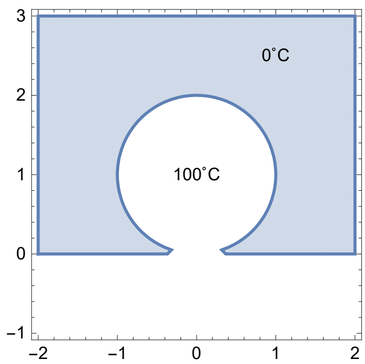
- The complex potential of source at point $z = a$: $\Omega(z) = k \ln(z - a)$, sink at point $z = a$: $\Omega(z) = -k \ln(z - a)$

```
field = Grad[ComplexExpand[Re[Log[((x + ⅈ y) - 0.1)/((x + ⅈ y) + 0.1)]]], {x, y}];
StreamPlot[field, {x, -.2, .2}, {y, -.2, .2},
 PlotLabel → "source: z=0.1, sink: z=-0.1"]
```



Exercise.
Find the temperature in the shaded region. (↓below figure)\ mapping $w = 1/z$ will help you.

```
Show[{RegionPlot[x^2 + (y - 1)^2 > 1 && y > 0, {x, -2, 2}, {y, -1, 3}],
  Graphics[{Text["100°C", {0, 1}], Text["0°C", {1, 2.5}]}]}]
```



In polar coordinate, shaded area is $\text{Im}\left[i + r\, e^{i\theta}\right] \geq 0 \;\&\&\; r \geq 1$

$$w = u + i\,v = 1/z = 1\Big/\left(i + r\, e^{i\theta}\right) = \frac{r\cos\theta}{r^2\cos^2\theta + (1 + r\sin\theta)^2} + i\left(-\frac{1 + r\sin\theta}{r^2\cos^2\theta + (1 + r\sin\theta)^2}\right)$$

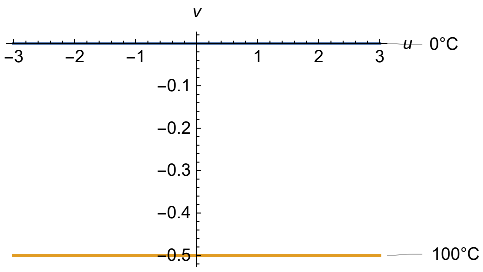
$$\begin{cases} u = \dfrac{r\cos\theta}{r^2\cos^2\theta + (1 + r\sin\theta)^2} = \dfrac{r\cos\theta}{(r^2 + 1) + 2\,r\sin\theta} = \dfrac{x}{x^2 + (y)^2} \\[2ex] v = -\dfrac{1 + r\sin\theta}{r^2\cos^2\theta + (1 + r\sin\theta)^2} = -\dfrac{1 + r\sin\theta}{(r^2 + 1) + 2\,r\sin\theta} = -\dfrac{(y)}{x^2 + (y)^2} \end{cases}$$

then, $r = 1$ goes to $\left(\dfrac{\cos\theta}{\cos^2\theta + (1 + \sin\theta)^2}, -\dfrac{1 + \sin\theta}{\cos^2\theta + (1 + \sin\theta)^2}\right) = \left(\dfrac{\cos\theta}{2\,(1 + \sin\theta)}, -\dfrac{1}{2}\right) \Rightarrow v = -1/2$

and $r = \infty$ goes to $v = 0$

Therefore, this problem on the complex w plane is solving heat flux between the two infinite parallel heat plate

```
Plot[{0, -1/2}, {u, -3, 3}, PlotLabels → {"0°C", "100°C"}, AxesLabel → {u, v}]
```

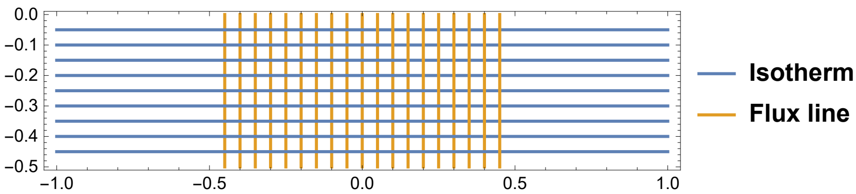


that has $T(u, v) = -200\, v$ as a solution(check it!). So we have complex temperature $\Omega(w)$ as follows

$$\Omega(w) = T(u, v) + i\,\Psi(u, v) = -200\, v + i\, 200\, u \quad (\text{or,}\ \ i\, 200\, w)$$

note that $T(u, v) = C_1$ is isotherm

```
Show[ContourPlot[{(-200 v == #), (200 u == #)},
    {u, -1, 1}, {v, -0.5, 0}, AspectRatio → 1/4] & /@ Range[-90, 90, 10]]
```

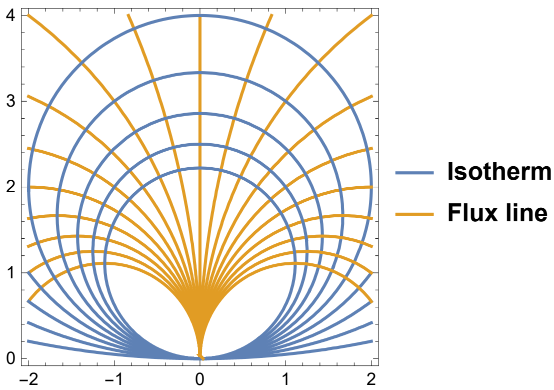


By mapping $w = 1/z$, $\Omega(w)$ transforms to $\Omega'(z)$

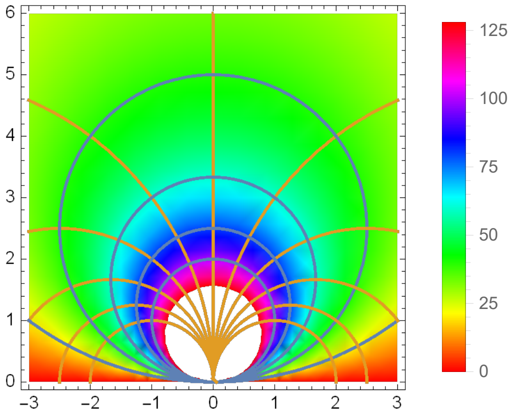
$$\Omega'(z) = T'(x, y) + i\,\Psi'(x, y) = 200\left(\frac{(y)}{x^2+(y)^2}\right) + i\, 200\left(\frac{x}{x^2+(y)^2}\right) \quad (\text{or,}\ \ i\, 200/z)$$

Isotherm and flux line in complex z plane;

```
Show[ContourPlot[{(200 (y/(x^2 + y^2)) == #), (200 (x/(x^2 + y^2)) == #)}, {x, -2, 2}, {y, 0, 4}] & /@
  Range[-90, 90, 10]]
```

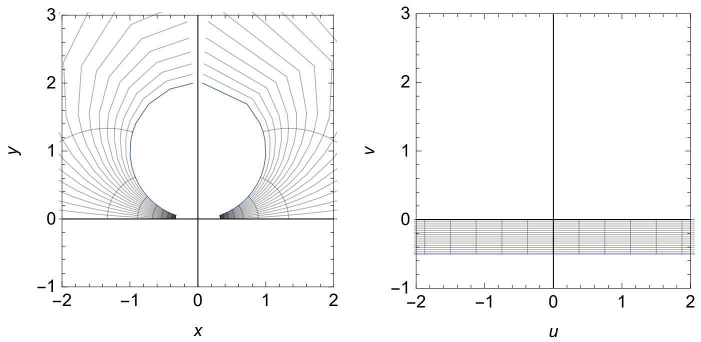


```
Show[DensityPlot[{(200 (y/(x^2 + y^2)))}, {x, -3, 3},
  {y, 0, 6}, ColorFunction → Hue, PlotLegends → Automatic],
 ContourPlot[{(200 (y/(x^2 + y^2)) == #), (200 (x/(x^2 + y^2)) == #)}, {x, -3, 3}, {y, 0, 6}] & /@
  Range[-100, 100, 20]]
```



Using our own **Conformal[]** function,

```
Conformal[(x + ⅈ y)^-1, (x + ⅈ y), {x, -3, 3}, {y, 0, -1/2}, PlotRange → {{-2, 2}, {-1, 3}}]
```



Exercise.
Simulate fringing effect by using conformal mapping ($z = w + e^{w}$).
→ See Griffith book 218±2p!

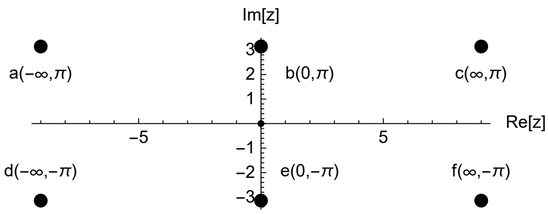
$z = w + e^{w} = (u + i\,v) + e^{(u + i\,v)} =$

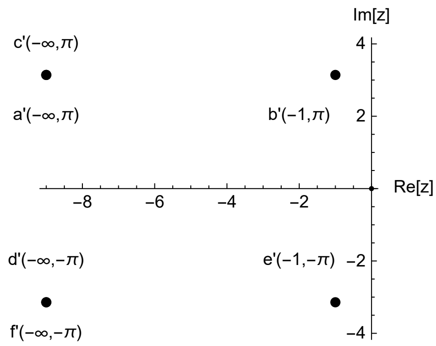
$$(u + i\,v) + e^{u}\,e^{i\,v} = (u + i\,v) + e^{u}(\cos v\ +\ i \sin v) = (u + e^{u}\cos v) + i\,(v + e^{u}\sin v)$$

$$x + i\,y = (u + e^{u}\cos v) + i\,(v + e^{u}\sin v) \to \begin{cases} x = u + e^{u}\cos v \\ y = v + e^{u}\sin v \end{cases}$$

So,

$$\begin{pmatrix} \text{complex } w \text{ plane} & \Rightarrow & \text{complex } z \text{ plane} \\ (-\infty, \pi), (-\infty, -\pi) & \Rightarrow & (-\infty, \pi), (-\infty, -\pi) \\ (\infty, \pi), (\infty, -\pi) & \Rightarrow & (-\infty, \pi), (-\infty, -\pi) \\ (0, \pi), (0, -\pi) & \Rightarrow & (-1, \pi), (-1, -\pi) \end{pmatrix}$$





(It seems like we fold $\overline{a\,b\,c}$ into $\overline{a'\,b'}$ and $\overline{d\,e\,f}$ into $\overline{d'\,e'}$)

As we did at above, let $\overline{a\,b\,c}$ be the infinite parallel plate 100V and $\overline{d\,e\,f}$ be the -100V . Then we have

$$\Omega(w) = \Phi(u, v) + i\,\Psi(u, v) = \frac{100}{\pi} v - i\,\frac{100}{\pi} u \quad \text{(note that } \Phi \text{ is an electric potential)}$$

```
Reduce[w + ⅇ^w == z, {z, w}]
```

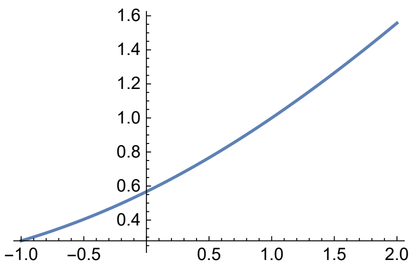
(C[1] ∈ ℤ && w == z − ProductLog[C[1], ⅇ^z]) || (z == 1 && w == 0)

∗ProductLog[z] gives the solution for w in $z = we^w$ moreover, ProductLog[k, z] gives k th solution

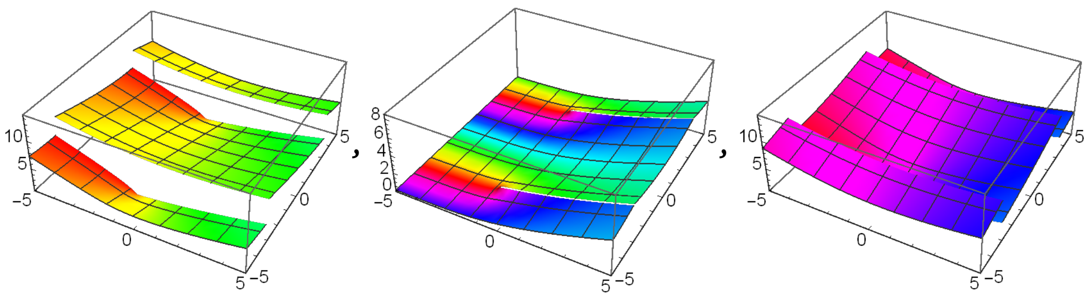
```
ProductLog[ⅇ^z] // TraditionalForm
```

$W(e^z)$

```
Plot[ProductLog[0, ⅇ^x], {x, -1, 2}]
```



```
{ComplexPlot3D[ProductLog[-1, ⅇ^(x+ⅈ y)],
  {x, -10, 10}, {y, -10, 10}, PlotRange → {{-5, 5}, {-5, 5}}],
 ComplexPlot3D[ProductLog[0, ⅇ^(x+ⅈ y)], {x, -10, 10}, {y, -10, 10},
  PlotRange → {{-5, 5}, {-5, 5}}], ComplexPlot3D[ProductLog[1, ⅇ^(x+ⅈ y)],
  {x, -10, 10}, {y, -10, 10}, PlotRange → {{-5, 5}, {-5, 5}}]}
```



Anyway,

```
W[z_] := z - ProductLog[ⅇ^z]
```

```
ComplexExpand[W[x + i y]]
```

```
x + i (y - Im[ProductLog[e^(x+i y)]]) - Re[ProductLog[e^(x+i y)]]
```

Thus, $\begin{cases} u = x - \mathrm{Re}\left[W\left(e^{x+i y}\right)\right] \\ v = y - \mathrm{Im}\left[W\left(e^{x+i y}\right)\right] \end{cases}$ and therefore, Ω in complex z plane would be

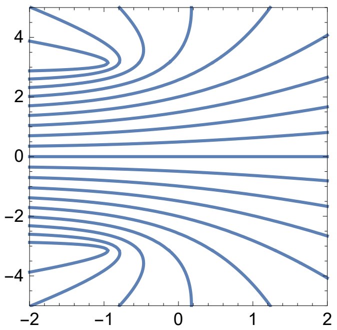
$$\Omega'(z) = \Phi'(x, y) + i\,\Psi'(x, y) = \frac{100}{\pi}\left(y - \mathrm{Im}\left[W\left(e^{x+i y}\right)\right]\right) - i\,\frac{100}{\pi}\left(x - \mathrm{Re}\left[W\left(e^{x+i y}\right)\right]\right)$$

Hence, the electric field near the fringing is

$$(E_x, E_y) = \nabla\Phi(x, y) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\left(\frac{100}{\pi}\left(y - \mathrm{Im}\left[W_{-1,0,1}\left(e^{x+i y}\right)\right]\right)\right)$$

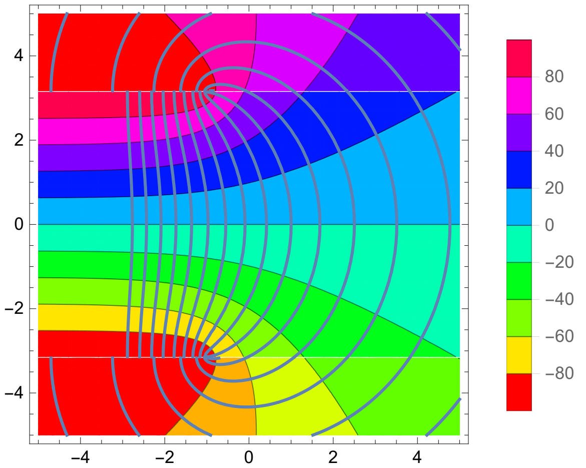
and the equipotential line appears as follows (W_{-1} for $y < -\pi$, W_0 for $-\pi < y < \pi$, W_1 for $y > \pi$)

```
Show[ContourPlot[{(100/π (y - Im[ProductLog[0, e^(x+i y)]]) == #)}, {x, -2, 2},
    {y, -π, π}, PlotRange → {{-2, 2}, {-5, 5}}] & /@ Range[-90, 90, 10],
  ContourPlot[{(100/π (y - Im[ProductLog[1, e^(x+i y)]]) == #)}, {x, -2, 2}, {y, π, 5}] & /@
   Range[-90, 90, 10],
  ContourPlot[{(100/π (y - Im[ProductLog[-1, e^(x+i y)]]) == #)}, {x, -2, 2}, {y, -5, -π}] & /@
   Range[-90, 90, 10]]
```



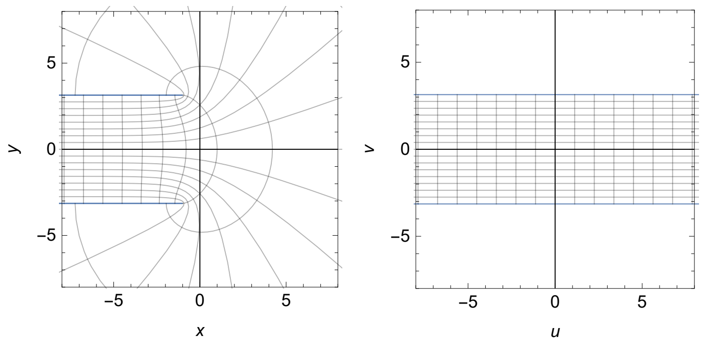
with flux line ($\Psi'(x, y)$),

```
Show[{ContourPlot[100/π (y - Im[ProductLog[1, e^(x+i y)]]), {x, -5, 5}, {y, π, 5},
   ColorFunction → Hue], ContourPlot[100/π (y - Im[ProductLog[0, e^(x+i y)]]),
   {x, -5, 5}, {y, -π, π}, ColorFunction → Hue, PlotLegends → Automatic],
  ContourPlot[100/π (y - Im[ProductLog[-1, e^(x+i y)]]),
   {x, -5, 5}, {y, -5, -π}, ColorFunction → Hue],
  ContourPlot[{(100/π (-x + Re[ProductLog[0, e^(x+i y)]]) == #)}, {x, -5, 5}, {y, -π, π}] &/@
   Range[-90, 90, 10],
  ContourPlot[{(100/π (-x + Re[ProductLog[1, e^(x+i y)]]) == #)}, {x, -5, 5}, {y, π, 5}] &/@
   Range[-90, 90, 10],
  ContourPlot[{(100/π (-x + Re[ProductLog[-1, e^(x+i y)]]) == #)}, {x, -5, 5}, {y, -5, -π}] &/@
   Range[-90, 90, 10]}, PlotRange → {{-5, 5}, {-5, 5}}]
```



Using Conformal[],

```
Conformal[(u + i v) + e^(u+i v), (u + i v),
 {u, -9, 9}, {v, -π, π}, PlotRange → {{-8, 8}, {-8, 8}}]
```



Finally we solve fringing effect problem! See also E. R. Dietz, Am. J. Phys. 72, 1499 (2004).

Exercise.

Find the streamlines for the flows of water around the inside of a right-angle

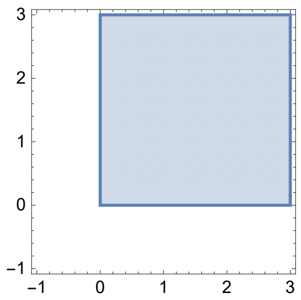
boundary.

By mapping $w(z) = z^2$ situation in straight channel transforms to situation around the inside of a right-angle boundary.

$$w(z) = u + i\,v = z^2 = \left(r\, e^{i\,\theta}\right)^2 = r^2\, e^{i\,2\,\theta} = r^2(\cos(2\,\theta) + i\sin(2\,\theta))$$

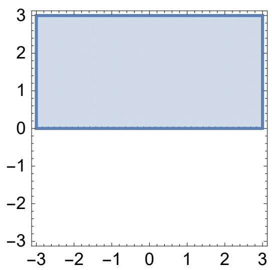
$$\begin{cases} u = r^2\cos(2\,\theta) = r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2 \\ v = r^2\sin(2\,\theta) = r^2\, 2\cos\theta\sin\theta = 2\,x\,y \end{cases}$$

```
RegionPlot[x > 0 && y > 0, {x, -1, 3}, {y, -1, 3}]
```



Inverse mapping $Z(w) = \sqrt{w}$

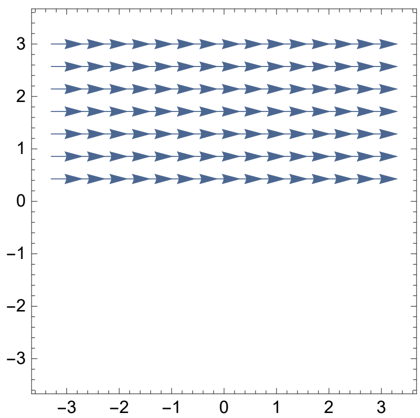
```
RegionPlot[Re[#] > 0 && Im[#] > 0, {u, -3, 3}, {v, -3, 3}] &[√(u + ⅈ v)]
```



Check it!

In the complex w plane, velocity of fluid is $v\,\hat{u}$, that is, potential is $\Phi(u, v) = v\,u$. It follows that is conjugate function is $\Psi(u, v) = v\,v$

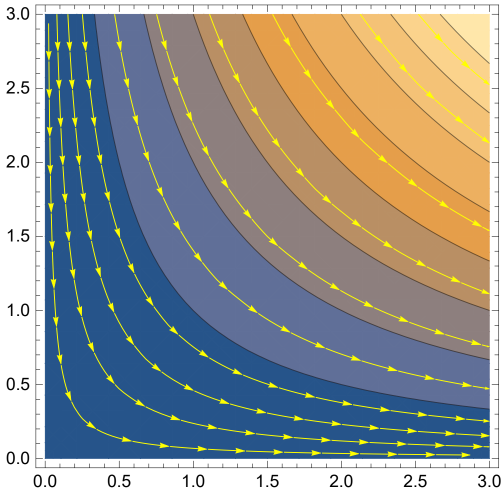
```
VectorPlot[{(HeavisideTheta[v]), 0}, {u, -3, 3}, {v, -3, 3}]
```



Hence, the complex velocity potential is
$[\Omega(w) = \Phi(u, v) + i\,\Psi(u, v) = v\,u + i\,v\,v] \Longrightarrow \left[\tilde{\Omega}(z) = \tilde{\Phi}(x, y) + i\,\tilde{\Psi}(x, y) = v\,(x^2 - y^2) + i\,v\,2\,x\,y\right]$

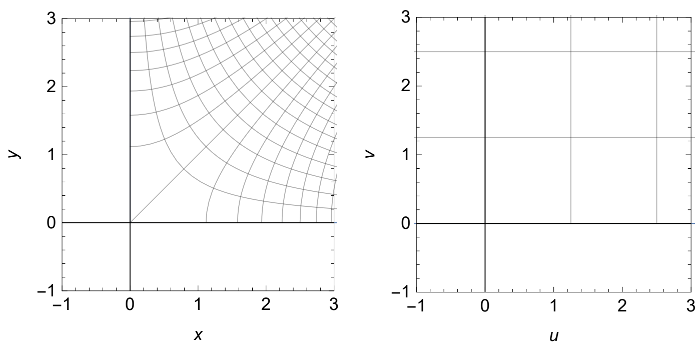
We obtain $\begin{cases} \tilde{\Phi}(x,y) = \nu\,(x^2 - y^2) \\ \tilde{\Psi}(x,y) = \nu\,2\,x\,y \end{cases}$ then, $\begin{cases} \text{gradient of } \tilde{\Phi} = \nu\left(2\,x\,\hat{i} - 2\,y\,\hat{j}\right) \\ \text{gradient of } \tilde{\Psi} = \nu\left(2\,y\,\hat{i} + 2\,x\,\hat{j}\right) \end{cases}$ Note that $\nabla\,\tilde{\Phi}$ is a 'velocity field' and $\tilde{\Psi}$ is a streamline

```
Show[{ContourPlot[2 x y, {x, 0, 3}, {y, 0, 3}],
  StreamPlot[{2 x, -2 y}, {x, 0, 3}, {y, 0, 3}, StreamStyle → Yellow, StreamPoints → 10]}]
```



Using **Conformal[],**

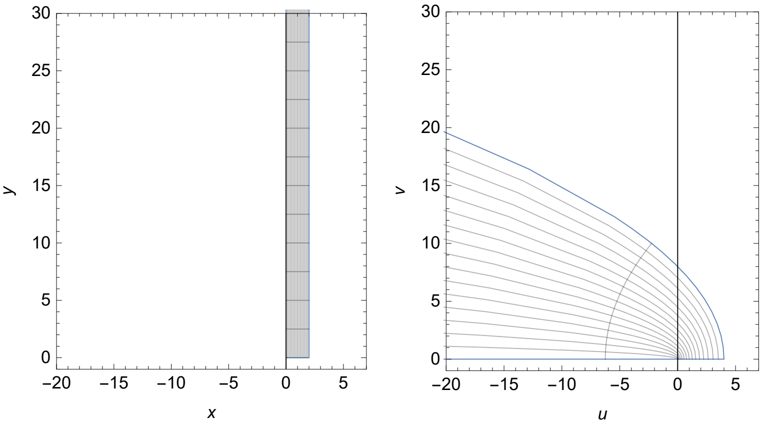
```
Conformal[(x + ⅈ y)^(1/2), (x + ⅈ y), {x, -10, 10}, {y, 0, 20}, PlotRange → {{-1, 3}, {-1, 3}}]
```



5.4 Appendix: Table of Transformations of Regions

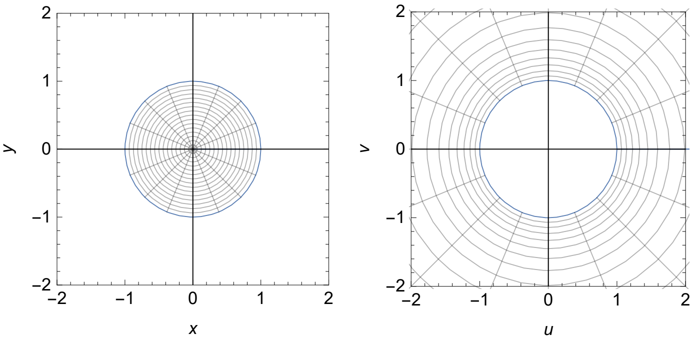
- ***w* = *z*²**

```
Conformal[(x + ⅈ y), (x + ⅈ y)^2, {x, 0, 2}, {y, 0, 40}, PlotRange → {{-20, 7}, {-1, 30}}]
```



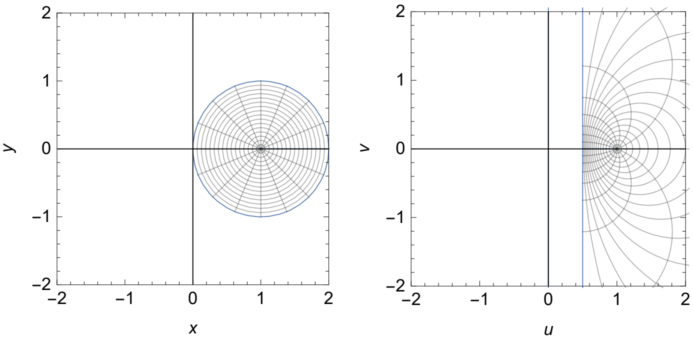
■ ***w* = 1/*z***

```
Conformal[r ⅇ^(ⅈ u), 1/(r ⅇ^(ⅈ u)), {r, 0, 1}, {u, 0, 2 π}, PlotRange → {{-2, 2}, {-2, 2}}]
```



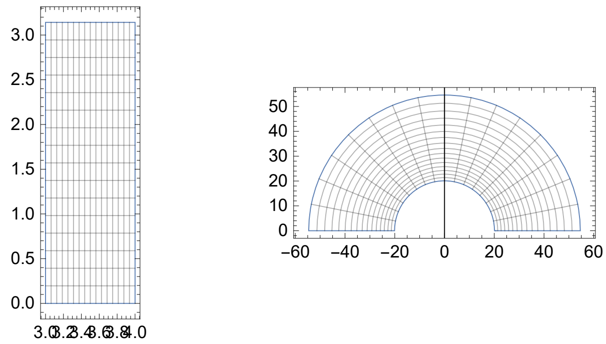
■ ***w* = 1/*z***

```
Conformal[1 + r ⅇ^(ⅈ u), 1/(1 + r ⅇ^(ⅈ u)), {r, 0, 1}, {u, 0, 2 π}, PlotRange → {{-2, 2}, {-2, 2}}]
```



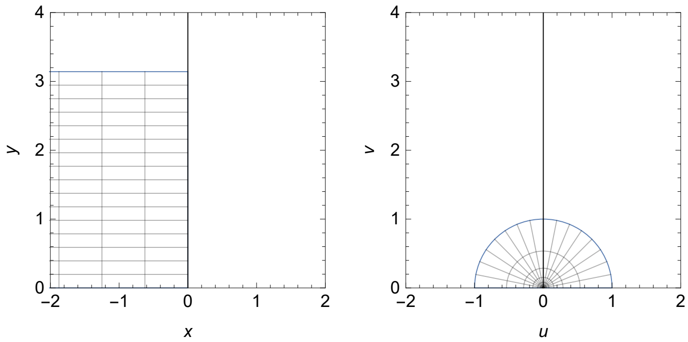
■ ***w* = exp *z***

```
Conformal[(x + ⅈ y), Exp[x + ⅈ y], {x, 3, 4}, {y, 0, π}, PlotRange → All]
```



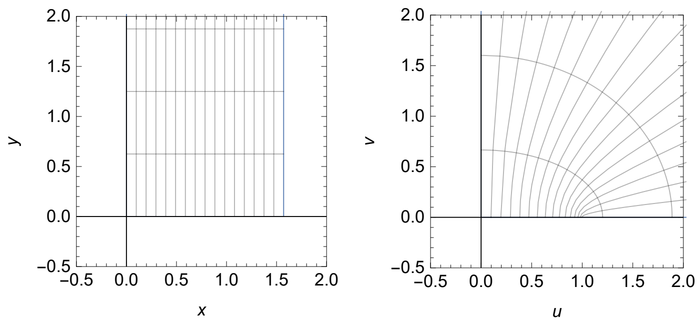
▪ $w = \exp z$

```
Conformal[(x + ⅈ y), Exp[x + ⅈ y], {x, -10, 0}, {y, 0, π}, PlotRange → {{-2, 2}, {0, 4}}]
```



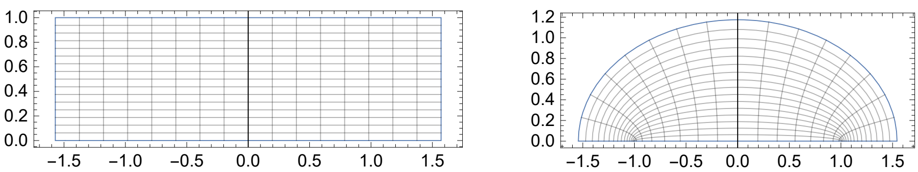
▪ $w = \sin z$

```
Conformal[(x + ⅈ y), Sin[x + ⅈ y], {x, 0, π/2},
 {y, 0, 10}, PlotRange → {{-0.5, 2}, {-0.5, 2}}]
```



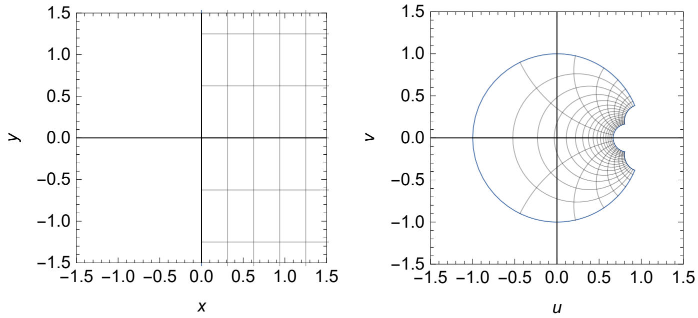
▪ $w = \sin z$

```
Conformal[(x + ⅈ y), Sin[x + ⅈ y], {x, -π/2, π/2}, {y, 0, 1}, PlotRange → All]
```



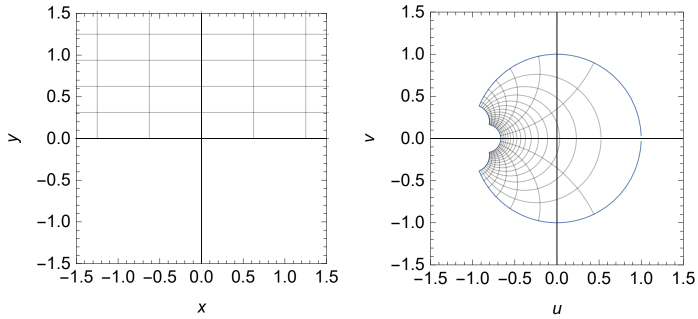
▪ $w = \frac{z-1}{z+1}$

```
Conformal[(x + ⅈ y), ((x + ⅈ y) - 1)/((x + ⅈ y) + 1), {x, 0, 5},
 {y, -5, 5}, PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}}]
```



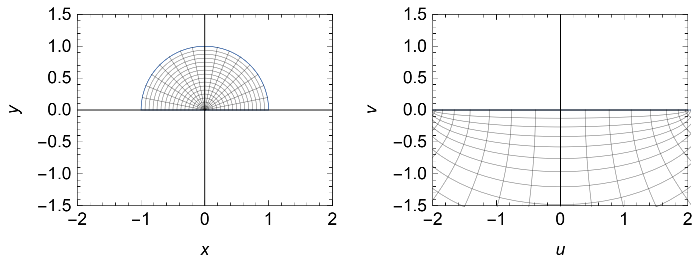
- $w = \frac{i - z}{i + z}$

```
Conformal[(x + ⅈ y), (ⅈ - (x + ⅈ y))/(ⅈ + (x + ⅈ y)), {x, -5, 5},
 {y, 0, 5}, PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}}]
```



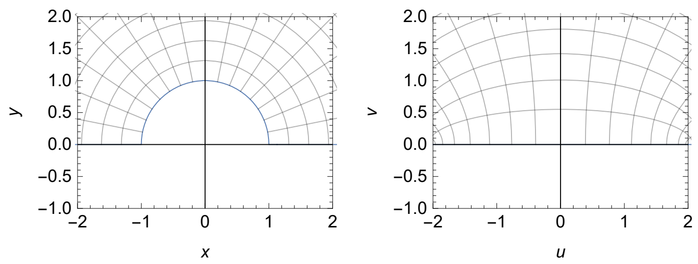
- $w = z + \frac{1}{z}$

```
Conformal[r ⅇ^(ⅈ u), (r ⅇ^(ⅈ u)) + 1/(r ⅇ^(ⅈ u)), {r, 0, 1},
 {u, 0, π}, PlotRange → {{-2, 2}, {-1.5, 1.5}}]
```



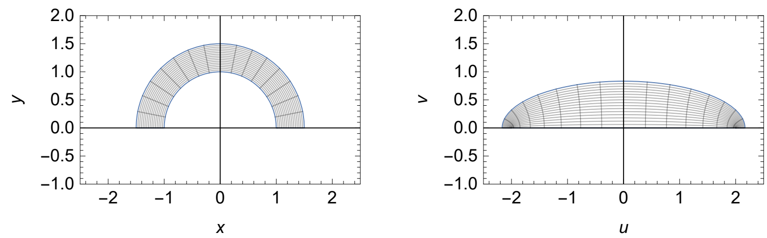
- $w = z + \frac{1}{z}$

```
Conformal[r ⅇ^(ⅈ u), (r ⅇ^(ⅈ u)) + 1/(r ⅇ^(ⅈ u)), {r, 1, 6}, {u, 0, π}, PlotRange → {{-2, 2}, {-1, 2}}]
```



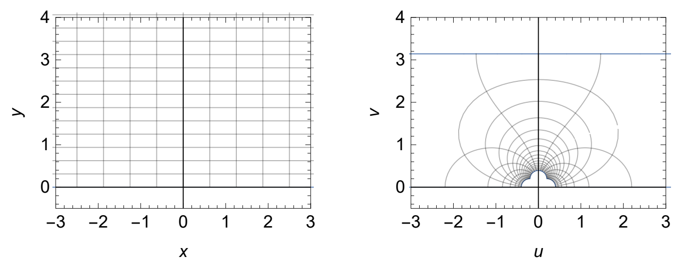
- $w = z + \frac{1}{z}$

```
Conformal[r ⅇ^(ⅈ u), (r ⅇ^(ⅈ u)) + 1/(r ⅇ^(ⅈ u)), {r, 1, 1.5},
 {u, 0, π}, PlotRange → {{-2.5, 2.5}, {-1, 2}}]
```



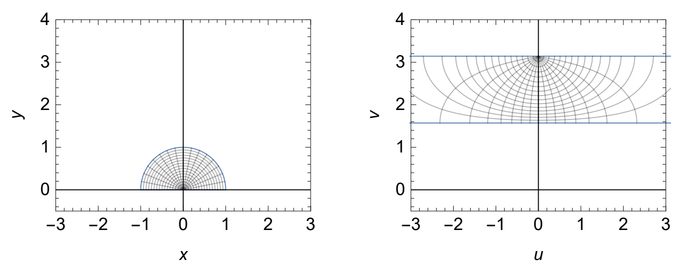
- $w = \ln \frac{z-1}{z+1}$

```
Conformal[(x + ⅈ y), Log[((x + ⅈ y) - 1)/((x + ⅈ y) + 1)],
 {x, -5, 5}, {y, 0, 5}, PlotRange → {{-3, 3}, {-0.5, 4}}]
```



- $w = \ln \frac{z-1}{z+1}$

```
Conformal[(r ⅇ^(ⅈ u)), Log[((r ⅇ^(ⅈ u)) - 1)/((r ⅇ^(ⅈ u)) + 1)],
 {r, 0, 1}, {u, 0, π}, PlotRange → {{-3, 3}, {-0.5, 4}}]
```

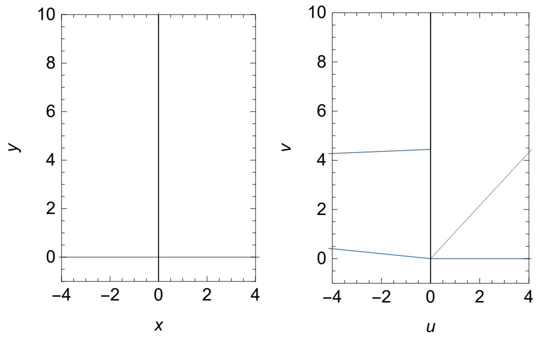


left hand side circle is $x^2 + (y + \cot h)^2 = \csc^2 h$, in above case, $h = \pi/2 = 1.5708$

The upper limit of right hand side band is π and the lower limit is $(\pi - h)$

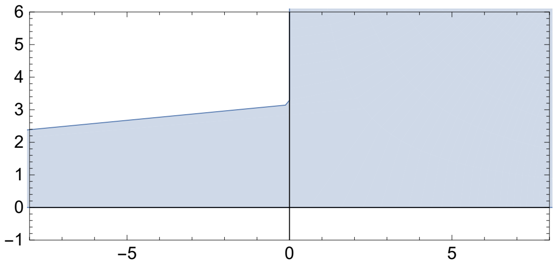
- $\boldsymbol{w = 2\,(z+1)^{1/2} + \ln \frac{(z+1)^{1/2}-1}{(z+1)^{1/2}+1}}$

```
Conformal[(x + ⅈ y), (2 Sqrt[(1 + x + ⅈ y)] + Log[(Sqrt[(1 + x + ⅈ y)] - 1)/(Sqrt[(1 + x + ⅈ y)] + 1)]),
 {x, -300, 300}, {y, 0, 2000}, PlotRange → {{-4, 4}, {-1, 10}}]
```



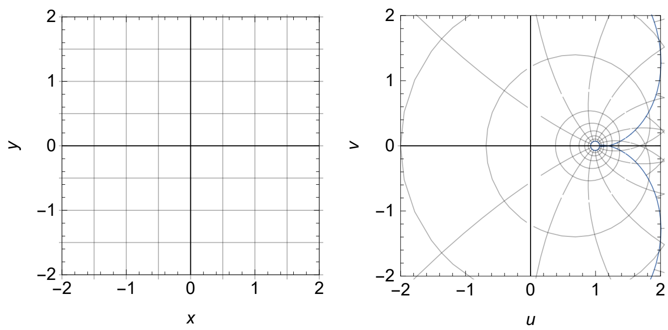

```
w1[x_, y_] := (2 Sqrt[(1 + x + ⅈ y)] + Log[(Sqrt[(1 + x + ⅈ y)] - 1)/(Sqrt[(1 + x + ⅈ y)] + 1)]);
```

```
ParametricPlot[{Re[w1[x, y]], Im[w1[x, y]]},
 {x, -20, 20}, {y, 0, 200}, PlotRange → {{-8, 8}, {-1, 6}}]
```

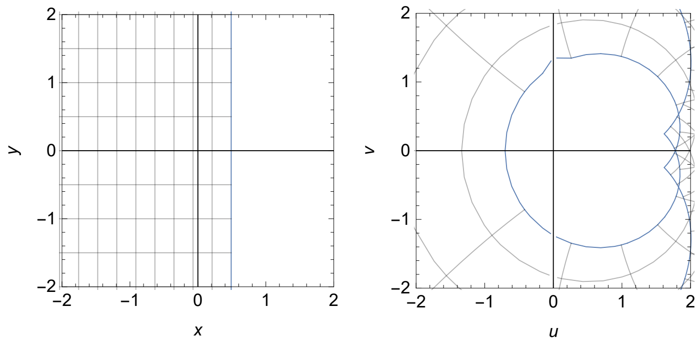


- $\boldsymbol{w = 1/\zeta(z)}$

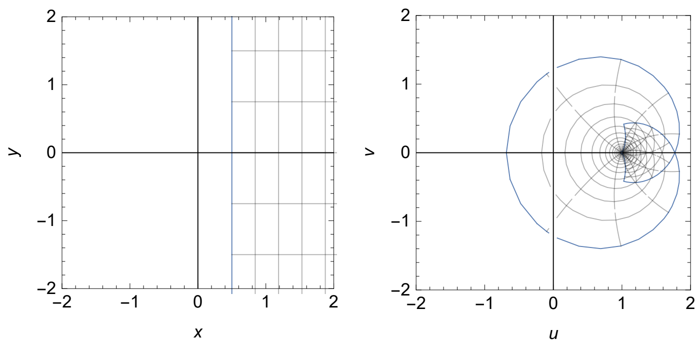
```
Conformal[(x + ⅈ y), 1/Zeta[x + ⅈ y],
 {x, -4, 4}, {y, -4, 4}, PlotRange → {{-2, 2}, {-2, 2}}]
```



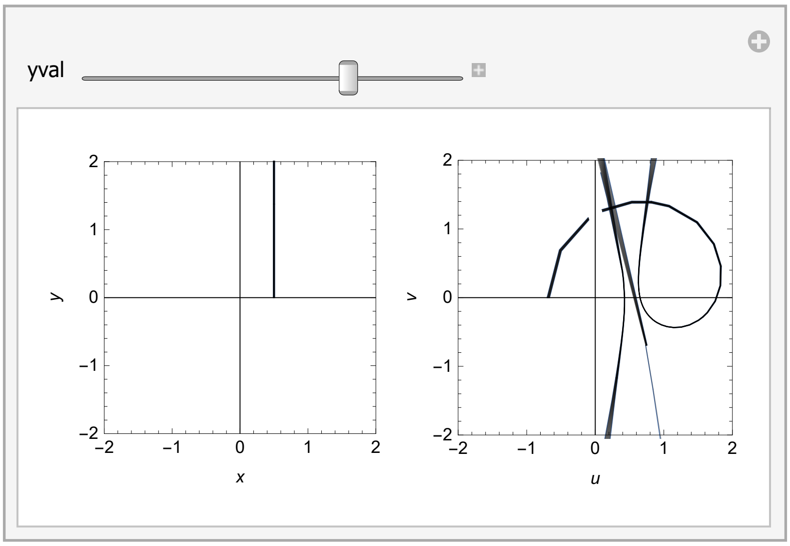
```
Conformal[(x + ⅈ y), 1 / Zeta[x + ⅈ y],
 {x, -4, 0.49}, {y, -4, 4}, PlotRange → {{-2, 2}, {-2, 2}}]
```



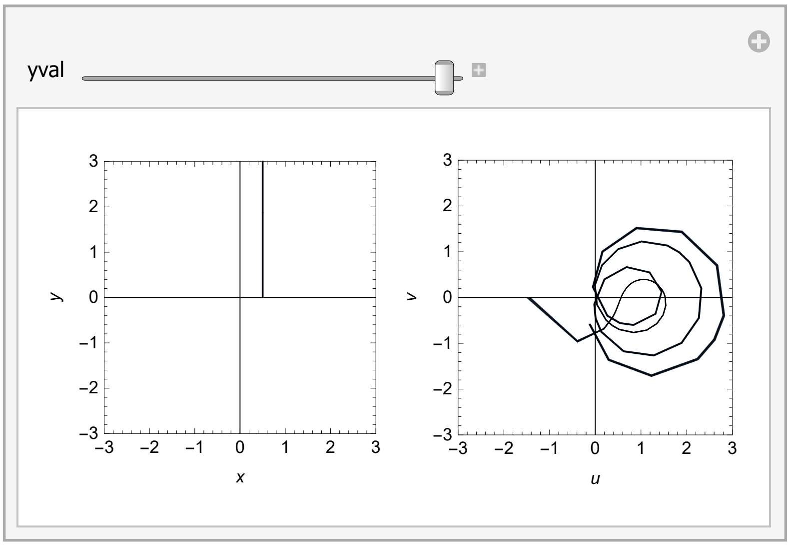
```
Conformal[(x + ⅈ y), 1 / Zeta[x + ⅈ y],
 {x, 1 / 2, 6}, {y, -6, 6}, PlotRange → {{-2, 2}, {-2, 2}}]
```



```
Manipulate[Conformal[(x + ⅈ y), 1 / Zeta[x + ⅈ y], {x, 0.49, 0.51},
  {y, 0, yval}, PlotRange → {{-2, 2}, {-2, 2}}], {yval, 1, 30, 1}]
```



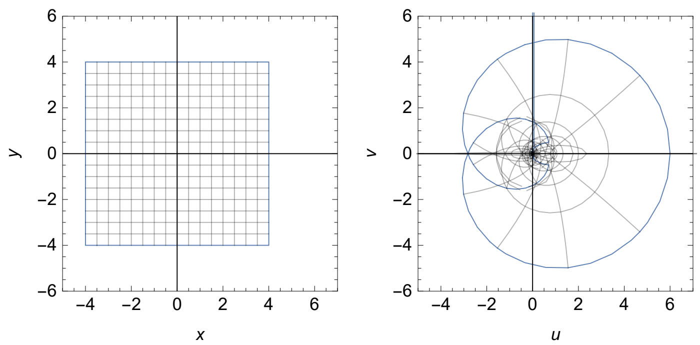
```
Manipulate[Conformal[(x + ⅈ y), Zeta[x + ⅈ y], {x, 0.49, 0.51},
  {y, 0, yval}, PlotRange → {{-3, 3}, {-3, 3}}], {yval, 1, 30, 1}]
```



Note that when it reaches non-trivial zero its mapping goes to origin of complex w plane.

- $w = \Gamma(z)$

```
Conformal[(x + ⅈ y), Gamma[x + ⅈ y], {x, -4, 4}, {y, -4, 4}, PlotRange → {{-5, 7}, {-6, 6}}]
```



6. Asymptotic Evaluation of Integrals

6.1 Asymptotic Series

(If you are 2nd grader)Maybe next year, at quantum mechanics class, you will see this guy;

$$H\Psi = \mathrm{E}\Psi \Longrightarrow i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi = E\Psi$$

For free particle, there is no potential as a barrier ($V(x) = 0$). Setting all constants to 1,

$$i\Psi_t + \Psi_{xx} = 0$$

which has the solution as follows;

$$\Psi(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(k)\,e^{ikx - ik^2 t}\,dk$$

(Don’t be nervous! you will learn this next spring.) It is useful to study their behavior for large x and t (just accept it!). But how can we evaluate such integrals? ; We will see three friends. Perhaps they will be Sherpa to help you climb the mt. Quantum Mechanics.

Before start, let's take a deep look at **asymptotic properties.** (The word asymptote is derived from Greek 'asumptotos' which means not falling together.) Suppose we want to find the value of the integral

$$I(\epsilon) = \int_0^\infty \frac{e^{-t}}{1+\epsilon t} dt, \quad \epsilon > 0$$

for sufficiently small real positive value of ϵ. By integration by parts,

$$I(\epsilon) = 1 - \epsilon \int_0^\infty \frac{e^{-t}}{(1+\epsilon t)^2} dt$$

Repeating,

$$I(\epsilon) = 1 - \epsilon + 2!\,\epsilon^2 - 3!\,\epsilon^3 + \cdots + (-1)^N N!\,\epsilon^N + (-1)^{N+1}(N+1)!\,\epsilon^{N+1} \int_0^\infty \frac{e^{-t}}{(1+\epsilon t)^{N+2}} dt$$

Let's interpret the above equation as a given definition.

Definitions.

- The notation $\boxed{f(k) = O(g(k)), \quad k \to k_0}$ which is read "$f(k)$ is of order $g(k)$ as k goes to k_0" means that there is a finite constant M and a neighborhood of k_0 where $|f| \leq M\,|g|$
- The notation $\boxed{f(k) << g(k), \quad k \to k_0}$ which is read "$f(k)$ is much smaller than $g(k)$ as k goes to k_0" means $\boxed{\lim_{k \to k_0} \left| \frac{f(k)}{g(k)} \right| = 0}$
- We shall say that $\boxed{f(k) \text{ is an approximation to } I(k) \text{ valid to order } \delta(k), \text{ as } k \to k_0}$ if $\lim_{k \to k_0} \frac{I(k) - f(k)}{\delta(k)} = 0$

(1) $-\epsilon = O(\epsilon)$ is of order ϵ, while $2!\,\epsilon^2 = O(\epsilon^2)$ is of order ϵ^2

(2) $2!\,\epsilon^2 << \epsilon$ as ϵ tends to zero.

(3) $f(\epsilon) = 1 - \epsilon + 2!\,\epsilon^2$ is an approximation to $I(\epsilon)$ valid to order ϵ^2;
$\lim_{\epsilon \to 0} \frac{I(\epsilon) - f(\epsilon)}{\epsilon^2} = \lim_{\epsilon \to 0} \frac{-3!\,\epsilon^3 + \cdots}{\epsilon^2} = \lim_{\epsilon \to 0} -3!\,\epsilon + \cdots = 0$

Definitions.

- The ordered sequence of functions $\{\delta_j(k)\}$, $j = 1, 2, \cdots$ is called an asymptotic sequence as $k \to k_0$ if $\delta_{j+1}(k) << \delta_j(k), \quad k \to k_0$
- Let $I(k)$ be continuous and let $\{\delta_j(k)\}$ be an asymptotic sequence as $k \to k_0$. The formal series $\sum_{j=1}^N a_j\,\delta_j$ is called an asymptotic expansion of $I(k)$, as k tends to k_0, valid to order $\delta_N(k)$ if

$$I(k) = \sum_{j=1}^m a_j\,\delta_j(k) + O(\delta_{m+1}(k)), \quad k \to k_0, \quad m = 1, 2, \cdots, N$$

then, $I(k) \sim \sum_{j=1}^N a_j\,\delta_j(k)$, $\quad k \to k_0$

(1) As $\epsilon \to 0$, $\{1, \epsilon, \epsilon^2, \epsilon^3, \cdots\}$ is an asymptotic sequence since $\epsilon^{j+1} << \epsilon^j$, $\epsilon \to 0$

(2) $1 - \epsilon + 2!\,\epsilon^2$ is an asymptotic expansion of $I(\epsilon)$ as $\epsilon \to 0$ valid to order ϵ^2

Exercise.

Find an asymptotic expansion for $J(k) = \int_0^\infty \frac{e^{-kt}}{1+t}\,dt$ as real $k \to \infty$

Let $t' = k\,t$ and $\epsilon = 1/k$ then,

$$\boxed{J(k) = \int_0^\infty \frac{e^{-kt}}{1+t}\,dt \text{ as real } k \to \infty} \text{ turns to } \boxed{J(\epsilon) = \epsilon \int_0^\infty \frac{e^{-t'}}{1+\epsilon t'}\,dt' \text{ as real } \epsilon \to 0}$$

We already know it ;

$J(\epsilon) =$

$$\epsilon \int_0^\infty \frac{e^{-t'}}{1+\epsilon t'}\,dt' = \epsilon\Big[1 - \epsilon + 2!\,\epsilon^2 - 3!\,\epsilon^3 + \cdots + (-1)^{N-1}(N-1)!\,\epsilon^{N-1} + (-1)^N N!\,\epsilon^N \int_0^\infty \frac{e^{-t}}{(1+\epsilon t)^{N+1}}\,dt\Big]$$

$$= \epsilon - \epsilon^2 + 2!\,\epsilon^3 - 3!\,\epsilon^4 + \cdots + (-1)^{N-1}(N-1)!\,\epsilon^N + (-1)^N N!\,\epsilon^{N+1} \int_0^\infty \frac{e^{-t}}{(1+\epsilon t)^{N+1}}\,dt$$

$$= \frac{1}{k} - \left(\frac{1}{k}\right)^2 + 2!\left(\frac{1}{k}\right)^3 - 3!\left(\frac{1}{k}\right)^4 + \cdots + (-1)^{N-1}(N-1)!\left(\frac{1}{k}\right)^N + R_N(k)$$

where $R_N(k) = (-1)^N N!\left(\frac{1}{k}\right)^{N+1} \int_0^\infty \frac{e^{-t}}{(1+t/k)^{N+1}}\,dt$

Note that $|R_N(k)| = N!\left(\frac{1}{k}\right)^{N+1} \int_0^\infty \frac{e^{-t}}{(1+t/k)^{N+1}}\,dt \le N!\left(\frac{1}{k}\right)^{N+1} \int_0^\infty e^{-t}\,dt = N!\left(\frac{1}{k}\right)^{N+1} << \left(\frac{1}{k}\right)^N$ (since $1 + t/k \ge 1$) and as k goes to infinity, $\left\{\frac{1}{k}, \frac{1}{k^2}, \cdots\right\}$ forms an asymptotic sequence.

Series[] does the job.

```
Series[∫_0^∞ e^(-k t)/(1 + t) ⅆt, {k, ∞, 5}, Assumptions → {k ∈ Reals && k > 0}]
```

$$\frac{1}{k} - \left(\frac{1}{k}\right)^2 + \frac{2}{k^3} - \frac{6}{k^4} + \frac{24}{k^5} + O\left[\frac{1}{k}\right]^6$$

Exercise.

Find an asymptotic expansion for $I(k) = \int_k^\infty \frac{e^{-t}}{t}\,dt$ as real $k \to \infty$

$$\int_k^\infty \frac{e^{-t}}{t}\,dt \xrightarrow{\text{Integration by parts}} \left[-\frac{e^{-t}}{t}\right]_k^\infty - \int_k^\infty \frac{e^{-t}}{t^2}\,dt =$$

$$\frac{e^{-k}}{k} - \int_k^\infty \frac{e^{-t}}{t^2}\,dt \;-\cdots\to e^{-k}\left[\frac{0!}{k} - \frac{1!}{k^2} + \frac{2!}{K^3} - \cdots + (-1)^{N-1}\frac{(N-1)!}{k^N}\right] + (-1)^N N! \int_k^\infty \frac{e^{-t}}{t^{N+1}}\,dt$$

where $R_N(k) = (-1)^N N! \int_k^\infty \frac{e^{-t}}{t^{N+1}}\,dt$. As $k \to \infty$, $\left\{\frac{e^{-k}}{k}, \frac{e^{-k}}{k^2}, \cdots\right\}$ forms an asymptotic sequence and

$$|R_N(k)| < N! \int_k^\infty \frac{e^{-t}}{k^{N+1}}\,dt = N!\,\frac{e^{-k}}{k^{N+1}} << \left(\frac{e^{-k}}{k^N}\right)$$

With **Series[],**

```
Series[∫_k^∞ e^(-t)/t ⅆt, {k, ∞, 5}, Assumptions → {k ∈ Reals && k > 0}]
```

$$e^{-k+O\left[\frac{1}{k}\right]^6}\left(\frac{1}{k} - \left(\frac{1}{k}\right)^2 + \frac{2}{k^3} - \frac{6}{k^4} + \frac{24}{k^5} + O\left[\frac{1}{k}\right]^6\right)$$

Moreover, asymptotic series give remarkably good approximations. In this case, let $k = 10$ and $N = 2$. Then $|R_2(10)| < 2!\,\frac{e^{-10}}{10^3}$ is almost zero. But we cannot take too many terms because of '$N!$' things. In most cases taking the first few terms of the asymptotic expansion is sufficient.

Previous definitions about real k can be extended to complex value k. Consider a function $f(z)$ that is analytic everywhere outside a circle $|z| = R$. Then we know that $f(z)$ has a convergent Taylor series at $z \to \infty$; $f(z) = a_0 + a_1/z + a_2/z^2 + \cdots$. In this case the convergent Taylor series is equivalent to a convergent asymptotic series with respect to the asymptotic sequence $\{1/z^j\}_{j=0}^{\infty}$. If $f(z)$ is not analytic at infinity,(ex. $f(z) = e^z$)it cannot have an asymptotic expansion valid for all $\arg(z)$ as $z \to \infty$,that is, the expansion is constrained by some bounds on $\arg(z)$

$f(\epsilon) = e^{\epsilon}$ for instance, there exists an asymptotic expansion as ϵ goes to zero; $f(\epsilon) = 1 + \epsilon + \frac{1}{2!}\epsilon^2 + \cdots$ for all $\arg(\epsilon)$, however, for the function $f(\epsilon) = e^{-1/\epsilon}$, as ϵ goes to zero, $-1/\epsilon$ tends to $+\infty$ or $-\infty$ that has asymptotic expansion only at $\mathrm{Re}[\epsilon] > 0$; $f(\epsilon) \sim 0$. If $\mathrm{Re}[\epsilon] < 0$, then $-1/\epsilon$ goes to $+\infty$ and cannot have asymptotic expansion.

Definition.

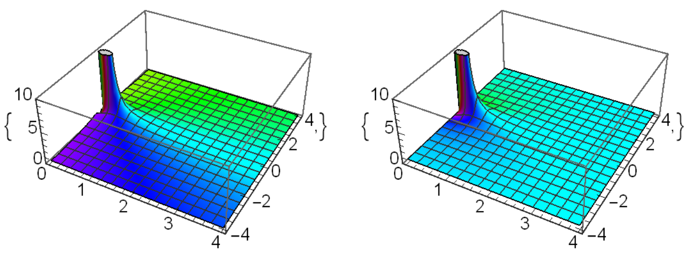
- A function f is said to have an asymptotic power series in a sector of the z plane as $z \to \infty$ if $f(z) \sim a_0 + a_1/z + a_2/z^2 + \cdots$
- Let another function $g(z)$ have an asymptotic power series representation in the same sector of the form; $g(z) \sim b_0 + b_1/z + \cdots$
- Then $f + g$ (sum) and $f\,g$ (product) also have asymptotic power series representations that are obtained by adding or multiplying the series.

Exercise.

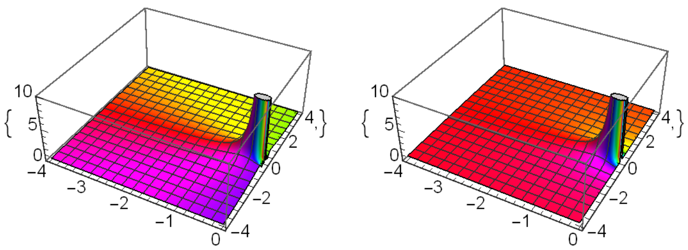
Discuss the asymptotic behavior of $I(z) = \sinh(z^{-1}),\ z \to 0,\ z \in \mathbb{C}$

Note that $\sinh(z^{-1}) = \frac{1}{2}e^{z^{-1}} - \frac{1}{2}e^{-z^{-1}}$
$$\begin{cases} \frac{1}{2}e^{z^{-1}} \text{ is the dominant term} & (\text{in } \mathrm{Re}[z] > 0,\ z^{-1} \to +\infty \text{ and } -z^{-1} \to -\infty \text{ as } z \to 0) \\ -\frac{1}{2}e^{-z^{-1}} \text{ is the dominant term} & (\text{in } \mathrm{Re}[z] < 0,\ z^{-1} \to -\infty \text{ and } -z^{-1} \to +\infty \text{ as } z \to 0) \end{cases}$$

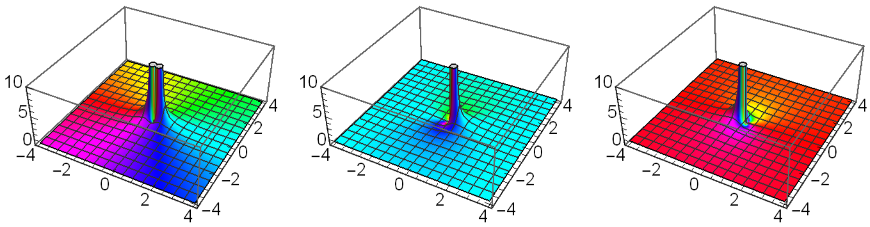
```
GraphicsGrid[
 {{ComplexPlot3D[Sinh[(x + ⅈ y)^-1], {x, 0, 4}, {y, -4, 4}, PlotRange → {0, 10}],
   ComplexPlot3D[1/2 Exp[(x + ⅈ y)^-1], {x, 0, 4}, {y, -4, 4}, PlotRange → {0, 10}]}}]
```



```
GraphicsGrid[
 {{ComplexPlot3D[Sinh[(x + ⅈ y)^-1], {x, -4, 0}, {y, -4, 4}, PlotRange → {0, 10}],
   ComplexPlot3D[-1 / 2 Exp[- (x + ⅈ y)^-1], {x, -4, 0}, {y, -4, 4}, PlotRange → {0, 10}]}}]
```



quiz! match the function and its appropriate graphical representation.



$-1/2e^{-z^{-1}}$ $\quad$ $\sinh(z^{-1})$ $\quad$ $1/2e^{z^{-1}}$

Right! the answer is . Thus, the asymptotic expansion of $\sinh(z^{-1})$ changes discontinuously across the $\mathrm{Re}[z]=0$ line (or $\theta=\pi/2$ ray). This is referred to as the **Stokes phenomenon,** which arises frequently in applications. ■

Now we are going to meet some elementary examples. Before journey, I'll give you some useful tips. Keep it cherished!

<Remark 6.1>

tips:

$$\begin{cases} \int_a^b f(k,t)\,dt, \ \ k\to k_0, \ \text{where } f(k,t)\sim f_0(t) \text{ as } k\to k_0 \quad \Longrightarrow \int_a^b f(k,t)\,dt \sim \int_a^b f_0(t)\,dt, \ \ k\to k_0 \\ \int_k^b f(t)\,dt, \ \ k\to\infty \ \ (\text{use integration by parts!}) \end{cases}$$

Exercise.

Find the first two nonzero terms of the asymptotic expansion of

$I(k) = \int_0^1 \frac{\sin t k}{t} \, dt$ as $k \to 0$.

As k tends to zero, $\frac{\sin t k}{t}$ goes to $\frac{1}{t}\left(t k - \frac{1}{3!} t^3 k^3 + \cdots\right) = \left(k - \frac{1}{3!} t^2 k^3 + \cdots\right)$. Therefore, by tip 1,
$\int_0^1 \frac{\sin t k}{t} \, dt \sim \int_0^1 \left(k - \frac{1}{3!} t^2 k^3 + \cdots\right) dt, \quad k \to 0$

$$\int_0^1 \left(k - \frac{1}{3!} t^2 k^3 + \cdots\right) dt = \left[k t - \frac{1}{3 \cdot 3!} t^3 k^3 + \cdots\right]_{t=0}^1 = k - \frac{1}{3 \cdot 3!} k^3 + \cdots$$

```
Series[∫_0^1 Sin[t k]/t dt, {k, 0, 5}]
```

$k - \frac{k^3}{18} + \frac{k^5}{600} + O[k]^6$

Exercise.
Evaluate $I(k) = \int_k^\infty e^{-t^2} \, dt$ as $k \to 0$ Hint: Gamma function is defined by $\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} \, du$

As k tends to zero, the main contribution of integral is $\int_0^\infty e^{-t^2} \, dt$. Note that
$\Gamma(1/2) = \int_0^\infty u^{-1/2} e^{-u} \, du = 2 \int_0^\infty e^{-t^2} \, dt$

$$I(k) = \int_k^\infty e^{-t^2} \, dt = \int_0^\infty e^{-t^2} \, dt - \int_0^k e^{-t^2} \, dt \text{ (near zero} \to \text{Taylor expansion!)}$$
$$= \frac{1}{2}\Gamma(1/2) - \int_0^k 1 - t^2 + \frac{1}{2!} t^4 - \cdots \, dt = \frac{\sqrt{\pi}}{2} - \left[t - \frac{1}{3} t^3 + \frac{1}{5 \cdot 2!} t^5 - \cdots\right]_0^k$$
$$= \frac{\sqrt{\pi}}{2} - k + \frac{1}{3} k^3 - \frac{1}{5 \cdot 2!} k^5 + \cdots$$

```
Series[∫_k^∞ e^(-t^2) dt, {k, 0, 5}]
```

$\frac{\sqrt{\pi}}{2} - k + \frac{k^3}{3} - \frac{k^5}{10} + O[k]^6$

Exercise.
Evaluate $E_1(k) = \int_k^\infty \frac{e^{-t}}{t} \, dt, \quad k \to 0^+$

Since the integrand has a singularity at $t = 0$, we have to cancel the singular point by ‘add and subtract’ skill. Let us add and subtract $\frac{1}{t(t+1)}$.

$$E_1(k) = \int_k^\infty \frac{e^{-t}}{t} + \frac{1}{t(t+1)} - \frac{1}{t(t+1)} \, dt$$
$$= \int_k^\infty \frac{e^{-t}}{t} - \frac{1}{t(t+1)} \, dt + \int_k^\infty \frac{1}{t(t+1)} \, dt$$
$$= \int_k^\infty \frac{1}{t(t+1)} \, dt + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right) \frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1}\right) \frac{dt}{t}$$
$$= \left[\ln\left(\frac{t}{t+1}\right)\right]_k^\infty + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right) \frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1}\right) \frac{dt}{t}$$

The first integral is $0 - \ln \frac{k}{k+1} = \ln(k+1) - \ln(k) = -\ln k + k - \frac{k^2}{2} + \frac{k^3}{3} - \cdots$

The second integral is a constant that we call $-\gamma$. γ is called Euler constant, while *Mathematica* calls it **EulerGamma.**

```
∫₀^∞ (e^-t - 1/(t+1)) (1/t) dt
```

```
-EulerGamma
```

```
N[-EulerGamma]
```

```
-0.577216
```

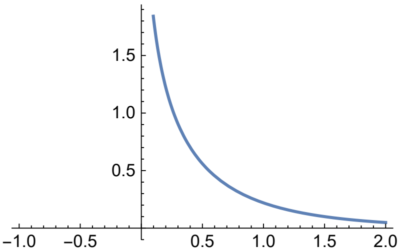
The last integral tends to $-\int_0^k \left(1 - t + \frac{1}{2!}t^2 - \cdots - (1 - t + t^2 - \cdots)\right) \frac{dt}{t}$ as $k \to 0$

$$-\int_0^k \left(1 - t + \frac{1}{2!}t^2 - \cdots - (1 - t + t^2 - \cdots)\right) \frac{dt}{t} = -\int_0^k \left(-\frac{1}{2!}t + \cdots\right) dt = \frac{k^2}{4} - \cdots$$

Hence, $E_1(k) = -\gamma - \ln k + k - \frac{k^2}{4} + \cdots$

Mathematica has exponential integral function, $E_n(z) = \int_1^\infty e^{-zt}/t^n\, dt$, calls it as **ExpIntegralE[n,z]**

```
Plot[ExpIntegralE[1, k], {k, -1, 2}]
```



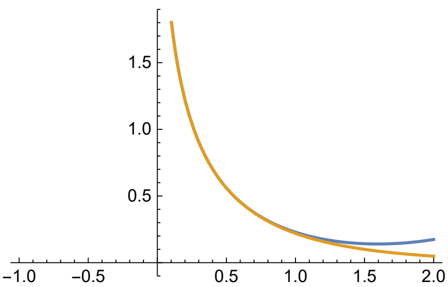
```
Series[∫_k^∞ e^-t/t dt, {k, 0, 5}, Assumptions → {k > 0}]
```

$$(-\text{EulerGamma} - \text{Log}[k]) + k - \frac{k^2}{4} + \frac{k^3}{18} - \frac{k^4}{96} + \frac{k^5}{600} + O[k]^6$$

```
Series[ExpIntegralE[1, k], {k, 0, 5}, Assumptions → {k > 0}]
```

$$(-\text{EulerGamma} - \text{Log}[k]) + k - \frac{k^2}{4} + \frac{k^3}{18} - \frac{k^4}{96} + \frac{k^5}{600} + O[k]^6$$

```
Plot[{(-EulerGamma - Log[k]) + k - k^2/4 + k^3/18, ExpIntegralE[1, k]}, {k, -1, 2}]
```



Exercise.
Evaluate $I(k) = \int_k^\infty e^{-t^2}\, dt,\ k \to \infty$

Now it's time to utilize the tip 2; Integration by part!

$$I(k) = \int_k^\infty e^{-t^2}\, dt = \int_k^\infty -2\,t\,e^{-t^2} \left(\frac{1}{-2\,t}\right) dt = \left[\frac{e^{-t^2}}{-2\,t}\right]_k^\infty - \int_k^\infty \left(\frac{e^{-t^2}}{2\,t^2}\right) dt = \cdots = \frac{e^{-k^2}}{2\,k} - \frac{e^{-k^2}}{4\,k^3} + O\left(\frac{e^{-k^2}}{k^5}\right)$$

$$\left(\int_k^\infty \left(\frac{e^{-t^2}}{2t^2}\right) dt = \int_k^\infty -2\,t\,e^{-t^2}\left(\frac{1}{-4t^3}\right) dt = \left[\frac{e^{-t^2}}{-4t^3}\right]_k^\infty - \int_k^\infty \left(\frac{3\,e^{-t^2}}{4t^4}\right) dt = \frac{e^{-k^2}}{4k^3} + O\left(\frac{e^{-k^2}}{k^5}\right)\right)$$

Note that $\left\{\frac{e^{-k^2}}{k^{2j-1}}\right\}_{j=1}^\infty$ is an asymptotic sequence as k goes to infinity.

```
Series[∫_k^∞ e^(-t^2) dt, {k, ∞, 4}]
```

$$e^{-k^2+O\left[\frac{1}{k}\right]^5}\left(\frac{1}{2k} - \frac{1}{4k^3} + O\left[\frac{1}{k}\right]^5\right)$$

Exercise.
Evaluate $I(k) = \int_0^k t^{-1/2} e^{-t} dt,\ k \to \infty$

As k tends to infinity, the main contribution of the integral is $\int_0^\infty t^{-1/2} e^{-t} dt$. Thus,

$$\begin{aligned} I(k) &= \int_0^k t^{-1/2} e^{-t} dt = \int_0^\infty t^{-1/2} e^{-t} dt - \int_k^\infty t^{-1/2} e^{-t} dt \\ &= \Gamma(1/2) - \int_k^\infty t^{-1/2} e^{-t} dt \end{aligned}$$

where

$$\begin{aligned} \int_k^\infty t^{-1/2} e^{-t} dt &= \left[-t^{-1/2} e^{-t}\right]_k^\infty - \int_k^\infty (1/2)\, t^{-3/2} e^{-t} dt \\ &= \left(0 + \frac{e^{-k}}{k^{1/2}}\right) + O\left(\frac{e^{-k}}{k^{3/2}}\right) = \frac{e^{-k}}{k^{1/2}} + O\left(\frac{e^{-k}}{k^{3/2}}\right) \end{aligned}$$

Hence,

$$I(k) = \Gamma(1/2) - \int_k^\infty t^{-1/2} e^{-t} dt = \sqrt{\pi} - \frac{e^{-k}}{k^{1/2}} + O\left(\frac{e^{-k}}{k^{3/2}}\right)$$

```
Series[∫_0^k t^(-1/2) e^(-t) dt, {k, ∞, 3}]
```

$$\sqrt{\pi} + e^{-k+O\left[\frac{1}{k}\right]^4}\left(-\sqrt{\frac{1}{k}} + \frac{1}{2}\left(\frac{1}{k}\right)^{3/2} - \frac{3}{4}\left(\frac{1}{k}\right)^{5/2} + O\left[\frac{1}{k}\right]^{7/2}\right)$$

Exercise.
Find the asymptotic expansions of the following integrals:
(a) $\int_0^1 \frac{\sin kt}{t} dt,\ k \to 0$, (b) $\int_0^k t^{-1/4} e^{-t} dt,\ k \to 0^+$, (c) $\int_k^\infty e^{-t^4} dt,\ k \to 0$

(a) As k goes to zero, $\frac{\sin kt}{t} = \left(k - \frac{1}{3!} k^3 t^2 + \cdots\right)$. Hence,

$$\begin{aligned} \int_0^1 \frac{\sin kt}{t} dt &= \int_0^1 \left(k - \frac{1}{3!} k^3 t^2 + \cdots\right) dt = \left[\left(k\,t - \frac{1}{3\cdot 3!} k^3 t^3 + \cdots\right)\right]_0^1 \\ &= \left(k - \frac{1}{18} k^3 + \cdots\right) \end{aligned}$$

```
Series[∫_0^1 Sin[k t]/t dt, {k, 0, 3}]
```

$$k - \frac{k^3}{18} + O[k]^4$$

(b) Let $s = t^{1/4}$ then, $\int_0^k t^{-1/4} e^{-t} dt = \int_0^{k^{1/4}} s^{-1} e^{-s^4} 4 s^3 ds = \int_0^{k^{1/4}} 4 s^2 e^{-s^4} ds$. As $s \to 0$,

$$\int_0^{k^{1/4}} 4 s^2 e^{-s^4} ds = \int_0^{k^{1/4}} 4 s^2 \left(1 - s^4 + \frac{1}{2!} s^8 - \cdots\right) ds = \int_0^{k^{1/4}} \left(4 s^2 - 4 s^6 + \frac{1}{2!} 4 s^{10} - \cdots\right) ds$$

$$= \left[\tfrac{4}{3}s^3 - \tfrac{4}{7}s^7 + \tfrac{1}{22}4\,s^{11} - \cdots\right]_0^{k^{1/4}} = \tfrac{4}{3}k^{3/4} - \tfrac{4}{7}k^{7/4} + \tfrac{2}{11}k^{11/4} - \cdots$$

$$\texttt{Series}\left[\int_0^k t^{-1/4}\, e^{-t}\, dt,\ \{k,\ 0,\ 3\}\right]$$

$$\frac{4\,k^{3/4}}{3} - \frac{4\,k^{7/4}}{7} + \frac{2\,k^{11/4}}{11} + O[k]^{13/4}$$

(c) Main contribution of the integral : $\int_0^\infty e^{-t^4}\, dt = \int_0^\infty 1/4\; s^{-3/4}\, e^{-s}\, ds = \frac{1}{4}\Gamma(1/4)$

(note that $\Gamma(z) = \int_0^\infty u^{z-1}\, e^{-u}\, du$)

Thus, $\int_k^\infty e^{-t^4}\, dt = \int_0^\infty e^{-t^4}\, dt - \int_0^k e^{-t^4}\, dt = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) - \int_0^k e^{-t^4}\, dt$

As $k \to 0$, $-\int_0^k e^{-t^4}\, dt = -\int_0^k \left(1 - t^4 + \frac{1}{2!}t^8 - \cdots\right) dt = -\left[t - \frac{1}{5}t^5 + \frac{1}{18}t^9 - \cdots\right]_0^k = -k + \frac{1}{5}k^5 - \frac{1}{18}k^9 - \cdots$

Hence, $\int_k^\infty e^{-t^4}\, dt = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) - \int_0^k e^{-t^4}\, dt = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) - k + \frac{1}{5}k^5 - \frac{1}{18}k^9 - \cdots$

$$\texttt{Series}\left[\int_k^\infty e^{-t^4}\, dt,\ \{k,\ 0,\ 9\},\ \texttt{Assumptions} \to \{k \in \texttt{Reals}\ \&\&\ k > 0\}\right]$$

$$\frac{1}{4}\,\texttt{Gamma}\left[\frac{1}{4}\right] - k + \frac{k^5}{5} - \frac{k^9}{18} + O[k]^{10}$$

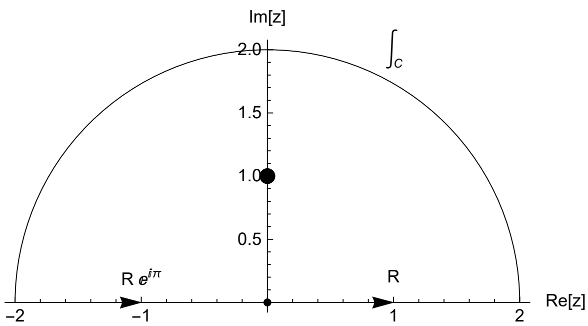
Exercise.
Find the asymptotic expansion of the integral $I(z) = \int_a^b \frac{u(x)}{x-z}\, dx$ $\mathrm{Im}(z) \neq 0$ as $z \to \infty$

$$I(z) = \int_a^b \frac{u(x)}{x-z}\, dx = \frac{1}{-z}\int_a^b \frac{u(x)}{1-x/z}\, dx = \frac{1}{-z}\int_a^b u(x)\left(1 + \frac{x}{z} + \frac{x^2}{z^2} + \cdots\right) dx \sim -\sum_{n=0}^\infty \frac{\int_a^b u(x)\, x^n\, dx}{z^{n+1}}$$

Exercise.
Find the asymptotic expansion of $\int_0^k \frac{e^{-t}}{1+t^2}\, dt$ as $k \to \infty$

The main contribution of the integral is $\int_0^\infty \frac{e^{-t}}{1+t^2}\, dt$

I tried to evaluate this integral by my pen, but it was too hard



But *Mathematica* is ready to help me;

$$\int_0^\infty \frac{e^{-t}}{1 + t^2}\, dt$$

$$\frac{1}{2}\pi\,\texttt{Cos}[1] + \texttt{CosIntegral}[1]\ \texttt{Sin}[1] - \texttt{Cos}[1]\ \texttt{SinIntegral}[1]$$

```
N[1/2 π Cos[1] + CosIntegral[1] Sin[1] - Cos[1] SinIntegral[1]]
```

```
0.62145
```

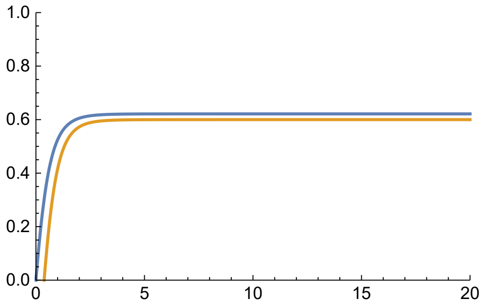
let's call it α. Then it follows

$$\int_0^k \frac{e^{-t}}{1+t^2} dt = \int_0^\infty \frac{e^{-t}}{1+t^2} dt - \int_k^\infty \frac{e^{-t}}{1+t^2} dt = \alpha - \int_k^\infty \frac{e^{-t}}{1+t^2} dt$$

By integration by parts,

$$\int_k^\infty \frac{e^{-t}}{1+t^2} dt = \left[-\frac{e^{-t}}{1+t^2}\right]_k^\infty - \int_k^\infty \frac{2\,t\,e^{-t}}{(1+t^2)^2} dt = \frac{e^{-k}}{1+k^2} + O\left(\frac{k\,e^{-k}}{(1+k^2)^2}\right)$$

```
Plot[{NIntegrate[e^-t/(1 + t^2), {t, 0, k}], 0.6 - e^-k/(1 + k^2)},
 {k, 0, 20}, PlotRange → {{0, 20}, {0, 1}}]
```



6.2 Laplace Type Integrals

In this section we will study the asymptotic behavior as $k \to +\infty$ of integrals of the form $I(k) = \int_a^b f(t)\, e^{-k\,\phi(t)}\, dt$. Such integrals are called Laplace type integrals. A special case of Laplace type integral is Laplace transform; when $\phi(t) = t$, $a = 0$, $b = \infty$

To evaluate the integral, $I(k) = \int_a^b f(t)\, e^{-k\,\phi(t)}\, dt$, we have to consider the dominant contribution of the integral. There are two cases;

$$\begin{cases} \phi(t) \text{ is monotonic in the region } [a, b] \\ \phi(t) \text{ has a local minimum } c \text{ in } [a, b] \quad (f(c) \neq 0) \end{cases}$$

When k goes to infinity and $\phi(t)$ has a local minimum c in region $[a, b]$, (while $f(c)$ is nonzero) the dominant contribution is given by the neighborhood of $t = c$. On the other hand, when $\phi(t)$ is monotonic, the minimum occurs at boundary that the dominant contribution comes from boundary. First, we will discuss about monotonic $\phi(t)$. Integration by parts is always useful tool.

Exercise.
Evaluate $\int_0^\infty (1 + t^2)^{-2}\, e^{-k\,t}\, dt$ as $k \to \infty$

$\phi(t) = t$ is monotonic. We are secure! Using integration by parts,

$$\left[\frac{(1+t^2)^{-2}}{-k}e^{-kt}\right]_0^\infty + \frac{1}{k}\int_0^\infty -2\,(1+t^2)^{-3}\,(2\,t)\,e^{-kt}\,dt = \frac{1}{k} - \frac{4}{k}\int_0^\infty t(1+t^2)^{-3}\,e^{-kt}\,dt$$
$$= \frac{1}{k} - \frac{4}{k}\left(\left[\frac{t\,(1+t^2)^{-3}}{-k}\,e^{-kt}\right]_0^\infty - \frac{1}{k}\int_0^\infty(\ldots)\,e^{-kt}\,dt\right)$$
$$= \frac{1}{k} + \frac{4}{k^2}\int_0^\infty(\ldots)\,e^{-kt}\,dt$$
$$= \frac{1}{k} + O\left(\frac{1}{k^3}\right)$$

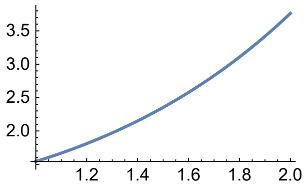
```
Series[∫_0^∞ (1 + t²)^-2 e^-k t dt, {k, ∞, 3}]
```

$$\frac{1}{k} - \frac{4}{k^3} + O\left[\frac{1}{k}\right]^4$$

Exercise.
Evaluate $I(k) = \int_1^2 e^{k\cosh t}\,dt$ as $k \to \infty$

Since $(\cosh t)' = \sinh t > 0$ in the interval $[1, 2]$, $\cosh t$ is monotonic.

```
Plot[Cosh[t], {t, 1, 2}]
```



Using integration by parts,

$$I(k) = \int_1^2 e^{k\cosh t}\,dt = \int_1^2 (k\sinh t)\,e^{k\cosh t}\,\frac{1}{k\sinh t}\,dt = \left[\frac{e^{k\cosh t}}{k\sinh t}\right]_1^2 - \int_1^2(\ldots)\,e^{k\cosh t}\,dt$$

As k goes to infinity, $\frac{e^{k\cosh 2}}{k\sinh 2}$ tends to dominant term. Hence, $I(k) \sim \frac{e^{k\cosh 2}}{k\sinh 2}$, $k \to \infty$

From above exercises, we have deal with such Laplace type integrals by using integration by parts.

If you want more rigorous justification of the integration by parts, see the below theorem.

<Theorem 11> Integration by parts

- Consider the integral $I(k) = \int_a^b f(t)\,e^{-kt}\,dt$ where the interval $[a, b]$ is a finite segment of the real axis. Let $f^{(m)}(t)$ denote the m th derivative of $f(t)$. Suppose that $f(t)$ has $N+1$ continuous derivatives while $f^{(N+2)}(t)$ is piecewise continuous on $a \le t \le b$. Then,

$$I(k) \sim \sum_{n=0}^{N} \frac{e^{-ka}}{k^{n+1}}\,f^{(n)}(a),\quad k \to +\infty$$

To prove this lemma, repeat integration by parts m times.

$$I(k) = \int_a^b f(t)\,e^{-kt}\,dt = \left[\frac{e^{-kt}}{-k}\,f(t)\right]_a^b - \int_a^b \frac{e^{-kt}}{-k}\,f(t)\,dt = \left[\frac{e^{-kt}}{-k}\,f(t)\right]_a^b + \frac{1}{k}\int_a^b f^{(1)}(t)\,e^{-kt}\,dt$$
$$= \left[\frac{e^{-kt}}{-k}\,f(t)\right]_a^b + \left[\frac{e^{-kt}}{-k^2}\,f^{(1)}(t)\right]_a^b + \frac{1}{k^2}\int_a^b f^{(2)}(t)\,e^{-kt}\,dt = \cdots$$
$$= \left[\frac{e^{-kt}}{-k}\,f(t)\right]_a^b + \left[\frac{e^{-kt}}{-k^2}\,f^{(1)}(t)\right]_a^b + \cdots + \left[\frac{e^{-kt}}{-k^m}\,f^{(m-1)}(t)\right]_a^b + \frac{1}{k^m}\int_a^b f^{(m)}(t)\,e^{-kt}\,dt$$
$$= \sum_{n=0}^{m-1}\frac{e^{-ka}}{k^{n+1}}\,f^{(n)}(a) - \sum_{n=0}^{m-1}\frac{e^{-kb}}{k^{n+1}}\,f^{(n)}(b) + \frac{1}{k^m}\int_a^b f^{(m)}(t)\,e^{-kt}\,dt,\quad m = 1, 2, \cdots, N$$

Since $a \le b$ and k is near ∞, $\frac{e^{-kb}}{-k^{n+1}}$ is much smaller than $\frac{e^{-ka}}{-k^{n+1}}$ we can asymptotically neglect the upper endpoint $t = b$.

Furthermore,

$$\begin{aligned} R_{m-1}(k) &= \tfrac{1}{k^m} \int_a^b f^{(m)}(t)\, e^{-k\,t}\, dt \\ &= \left[\tfrac{e^{-k\,t}}{-k^{m+1}} f^{(m)}(t)\right]_a^b + \tfrac{1}{k^{m+1}} \int_a^b f^{(m+1)}(t)\, e^{-k\,t}\, dt \\ &\simeq \tfrac{e^{-k\,a}}{k^{m+1}} f^{(m)}(a) + \tfrac{1}{k^{m+1}} \int_a^b f^{(m+1)}(t)\, e^{-k\,t}\, dt \end{aligned}$$

Therefore, as $k \to \infty$, $R_{m-1}(k) = O\left(\frac{e^{-k\,a}}{k^{m+1}}\right)$. For $m = N$, we can decompose the interval $[a, b]$ into subintervals in each of which $f^{(N+2)}(t)$ is continuous. Then a final integration by parts for R_N completes the proof. ■

Two generalization of above result:

- (a) $b = \infty$
  Laplace transform ($a = 0$)
  $\mathcal{L}[f] = \int_0^\infty f(t)\, e^{-k\,t}\, dt \sim \sum_{n=0}^N \frac{1}{k^{n+1}} f^{(n)}(0), \quad k \to +\infty$
- (b) $\phi(t)$ is monotonic in $[a, b]$
  By changing variables $\tau = \phi(t)$,
  $\int_a^b f(t)\, e^{-k\,\phi(t)}\, dt$ can be transformed to $\int_a^b \tilde{f}(t)\, e^{-k\,t}\, dt$

Exercise.
Evaluate $I(\epsilon) = \int_0^\infty (1 + \epsilon\, t)^{-1}\, e^{-t}\, dt$ as $\epsilon \to 0$

Nice to see you again, $\int_0^\infty (1 + \epsilon\, t)^{-1}\, e^{-t}\, dt$! We have seen that $I(\epsilon)$ has asymptotic series

$$I(\epsilon) = 1 - \epsilon + 2!\,\epsilon^2 - 3!\,\epsilon^3 + \cdots + (-1)^N N!\,\epsilon^N + (-1)^{N+1} (N+1)!\,\epsilon^{N+1} \int_0^\infty \frac{e^{-t}}{(1+\epsilon\, t)^{N+2}}\, dt \ \text{ as } \epsilon \to 0$$

It can be transformed to Laplace type integral by letting $\epsilon\, t = \tau$

$$\begin{aligned} \int_0^\infty (1 + \epsilon\, t)^{-1}\, e^{-t}\, dt &= \int_0^\infty (1 + \tau)^{-1}\, e^{-\tau/\epsilon}\, \tfrac{1}{\epsilon}\, d\tau, \quad \epsilon \to 0 \\ &= k \int_0^\infty (1 + \tau)^{-1}\, e^{-k\,\tau}\, d\tau, \quad k \to \infty \quad (1/\epsilon = k) \end{aligned}$$

By the theorem,

$$\tilde{I}(k) \sim k \sum_{n=0}^N \frac{1}{k^{n+1}} f^{(n)}(0) = \sum_{n=0}^N \frac{1}{k^n} f^{(n)}(0), \quad k \to +\infty$$

$$\text{where } f(\tau) = (1 + \tau)^{-1}$$

Note that $\frac{d^n}{d\tau^n}[(1 + \tau)^{-1}] = (-1)^n\, n!\,(1 + \tau)^{-(n+1)}$, $f^{(n)}(0) = (-1)^n\, n!$. Hence,

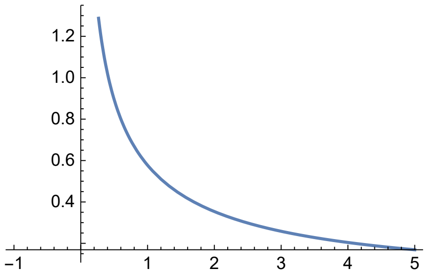
$$\begin{aligned} I(\epsilon) &\sim \sum_{n=0}^N (-1)^n\, n!\,\epsilon^n, \quad \epsilon \to 0 \\ &= 1 - \epsilon + 2!\,\epsilon^2 - 3!\,\epsilon^3 + \cdots \end{aligned}$$

Sometimes $f(t)$ is not sufficiently smooth at $t = a$. In this case above theorem may not work.

Exercise.
Evaluate $I(k) = \int_0^5 (t^2 + 2\, t)^{-1/2}\, e^{-k\,t}\, dt$ as $k \to \infty$

Unfortunately, $f(t) = (t^2 + 2\, t)^{-1/2}$ is not smooth at $t = 0$

```
Plot[(t^2 + 2 t)^(-1/2), {t, -1, 5}]
```



By integration by parts,

$$\left[\frac{(t^2+2t)^{-1/2}}{-k} e^{-kt}\right]_0^\infty + \frac{1}{k}\int_0^5 e^{-kt}\frac{d}{dt}\left[(t^2+2t)^{-1/2}\right] dt$$

the first term is singular at $t = 0$; a straightforward integration by parts fails.

Intuitively, we expected that the main contribution to $I(k)$ for large k will come near $t = 0$. Thus it would be desirable to expand $(t^2 + 2t)^{-1/2}$ in the neighborhood of the origin.

$$(t^2+2t)^{-1/2} = (2t)^{-1/2}\left(1+\frac{t}{2}\right)^{-1/2} = (2t)^{-1/2}\left(1-\frac{t}{4}+\cdots\right)$$

above is only valid for region $|t/2| < 1$ (in this case, $0 < t < 2$)

However, since the rapid decay of e^{-kt}, $I(k)$ should be <u>asymptotically equivalent to</u>

$\boxed{\int_0^R (t^2+2t)^{-1/2} e^{-kt}\, dt,}$ where R is sufficiently small but finite.

If $R < 2$, we can expand $(t^2+2t)^{-1/2}$ as above and $I(k)$ is asymptotically equivalent to $\int_0^R (t^2+2t)^{-1/2} e^{-kt}\, dt$

$$I(k) = \int_0^R (2t)^{-1/2}\left(1-\frac{t}{4}+\cdots\right) e^{-kt}\, dt = \int_0^R e^{-kt}(2t)^{-1/2}\, dt - \frac{1}{8}\int_0^R e^{-kt}(2t)^{1/2}\, dt + \cdots$$

$$I(k) \sim \int_0^R e^{-kt}(2t)^{-1/2}\, dt - \frac{1}{8}\int_0^R e^{-kt}(2t)^{1/2}\, dt$$

Even if you change the R into ∞ there will be exponentially small error as k tends to infinity

$$I(k) \sim \int_0^R e^{-kt}(2t)^{-1/2}\, dt - \frac{1}{8}\int_0^R e^{-kt}(2t)^{1/2}\, dt \sim \int_0^\infty e^{-kt}(2t)^{-1/2}\, dt - \frac{1}{8}\int_0^\infty e^{-kt}(2t)^{1/2}\, dt$$

and it looks similar to gamma function $\left(\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt\right)$

$$\int_0^\infty e^{-kt}(2t)^{-1/2}\, dt - \frac{1}{8}\int_0^\infty e^{-kt}(2t)^{1/2}\, dt = \int_0^\infty e^{-\tau}(2\tau/k)^{-1/2}\frac{d\tau}{k} - \frac{1}{8}\int_0^\infty e^{-\tau}(2\tau/k)^{1/2}\frac{d\tau}{k}$$

$$= \frac{1}{(2k)^{1/2}}\int_0^\infty (\tau)^{-1/2} e^{-\tau}\, d\tau - \frac{1}{2(2t)^{3/2}}\int_0^\infty (\tau)^{1/2} e^{-\tau}\, d\tau = \frac{\Gamma(1/2)}{(2k)^{1/2}} - \frac{\Gamma(3/2)}{2(2t)^{3/2}}$$

Now the following lemma gives you a rigorous justification for the above approach.

<Theorem 12> Watson’s lemma

- Consider the integral $I(k) = \int_0^b f(t)\, e^{-kt}\, dt$, $b > 0$. Suppose that $f(t)$ is integrable in $(0, b)$ and that it has the asymptotic series expansion; $f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n}$, $t \to 0^+$ $(\alpha > -1,\ \beta > 0)$ Then

$$I(k) \sim \sum_{n=0}^\infty a_n \frac{\Gamma(\alpha+\beta n+1)}{k^{\alpha+\beta n+1}},\ k \to \infty$$

If b is finite, we require that for $t > 0$, $|f(t)| \le A$, where A is a constant

if $b = \infty$, we need only require that $|f(t)| \le M e^{ct}$, where c and M are constants.

proof

First, break the integral in two parts, $I(k) = I_1(k) + I_2(k)$, where

$$I_1(k) = \int_0^R f(t)\, e^{-k t}\, dt,\quad I_2(k) = \int_R^b f(t)\, e^{-k t}\, dt,\quad 0 < R < b$$

The integral $I_2(k)$ is exponentially small as k tends to infinity.
For finite b, because $f(t)$ is bounded for $t > 0$, there exists a positive constant A such that $|f(t)| \le A$ for $t \ge R$. Thus,

$$|I_2(k)| \le A \int_R^b e^{-k t}\, dt = \frac{A}{k}\left(e^{-R k} - e^{-b k}\right)$$

As $k \to \infty$, $e^{-k b}$ is much smaller than $e^{-k R}$, that is we can neglect the above second term.

$$I_2(k) \sim O\left(\frac{e^{-k R}}{k}\right)$$

By the way, $f(t) = t^\alpha \left(\sum_{n=0}^N a_n\, t^{\beta n} + O\left(t^{\beta(N+1)}\right)\right)$, $t \to 0^+$ $(\alpha > -1,\ \beta > 0)$ for each positive integer N, which implies

$$\begin{aligned} I_1(k) &= \int_0^R f(t)\, e^{-k t}\, dt = \int_0^R t^\alpha \left(\sum_{n=0}^N a_n\, t^{\beta n} + O\left(t^{\beta(N+1)}\right)\right) e^{-k t}\, dt \\ &= \int_0^R \left[\sum_{n=0}^N a_n\, t^{\alpha+\beta n} + O\left(t^{\alpha+\beta(N+1)}\right)\right] e^{-k t}\, dt,\quad k \to \infty \\ &= \int_0^R \left[\sum_{n=0}^N a_n\, t^{\alpha+\beta n}\right] e^{-k t}\, dt + \int_0^R O\left(t^{\alpha+\beta(N+1)}\right) e^{-k t}\, dt,\quad k \to \infty \end{aligned}$$

However, $\left(\text{do not forget } \Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt\, !\right)$

$$\begin{aligned} \int_0^R t^{\alpha+\beta n}\, e^{-k t}\, dt &= \int_0^\infty t^{\alpha+\beta n}\, e^{-k t}\, dt - \int_R^\infty t^{\alpha+\beta n}\, e^{-k t}\, dt \\ &= \int_0^\infty (\tau/k)^{\alpha+\beta n}\, e^{-\tau}\, \frac{1}{k}\, d\tau + O\left(\frac{e^{-k R}}{k}\right) \quad (\tau = k t) \\ &= \left(\frac{1}{k}\right)^{\alpha+\beta n+1} \int_0^\infty \tau^{\alpha+\beta n}\, e^{-\tau}\, d\tau + O\left(\frac{e^{-k R}}{k}\right) \\ &= \frac{\Gamma(\alpha+\beta n+1)}{k^{\alpha+\beta n+1}} + O\left(\frac{e^{-k R}}{k}\right),\quad k \to \infty \end{aligned}$$

Moreover, using the definition we have met above,

- The notation $\boxed{f(k) = O(g(k)),\ k \to k_0}$ which is read "$f(k)$ is of order $g(k)$ as k goes to k_0" means that there is a finite constant M and a neighborhood of k_0 where $|f| \le M\, |g|$

$$\begin{aligned} &\int_0^R O\left(t^{\alpha+\beta(N+1)}\right) e^{-k t}\, dt \le A_N \int_0^R t^{\alpha+\beta(N+1)}\, e^{-k t}\, dt \\ &\le A_N \int_0^\infty t^{\alpha+\beta(N+1)}\, e^{-k t}\, dt = A_N\, \frac{\Gamma(\alpha+\beta(N+1)+1)}{k^{\alpha+\beta(N+1)+1}} \quad \text{for some constant } A_N \end{aligned}$$

Therefore,

$$I(k) = I_1(k) + I_2(k) = \sum_{n=0}^N a_n\, \frac{\Gamma(\alpha+\beta n+1)}{k^{\alpha+\beta n+1}} + O\left(\frac{1}{k^{\alpha+\beta(N+1)+1}}\right)$$

Note that the assumption $\alpha > -1$, $\beta > 0$ are necessary for convergence at $t = 0$.
Also, if $b = \infty$, it is necessary that $|f(t)| \le M\, e^{c t}$ for some constants M and c, in order to have convergence at $t \to +\infty$

In this case, $I_2(k) \le \int_R^\infty M\, e^{c t}\, e^{-k t}\, dt = M\, \frac{e^{-(k-c) R}}{k-c} = O\left(\frac{e^{-k R}}{k}\right)$ as $k \to \infty$ ■

Get back to the exercise. First, expand the $f(t)$

$$f(t) =$$

$$\left(t^2 + 2\, t\right)^{-1/2} = (2\, t)^{-1/2} \left(1 + \frac{t}{2}\right)^{-1/2} = (2\, t)^{-1/2} \left(1 + \frac{\left(-\frac{1}{2}\right)}{1!}\, \frac{t}{2} + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!} \left(\frac{t}{2}\right)^2 + \ldots + \frac{\Gamma(-1/2+1)}{\Gamma(n+1)\, \Gamma(-1/2-n+1)} \left(\frac{t}{2}\right)^n + \ldots\right)$$

$$\therefore f(t) = (2\, t)^{-1/2} \sum_{n=0}^\infty \left(\frac{t}{2}\right)^n \frac{\Gamma(1/2)}{\Gamma(n+1)\, \Gamma(1/2-n)} = (2\, t)^{-1/2} \sum_{n=0}^\infty \left(\frac{t}{2}\right)^n \tilde{a}_n$$

where $\alpha = -1/2$, $\beta = 1$, and $a_n = \frac{\tilde{a}_n}{2^{n+1/2}}$

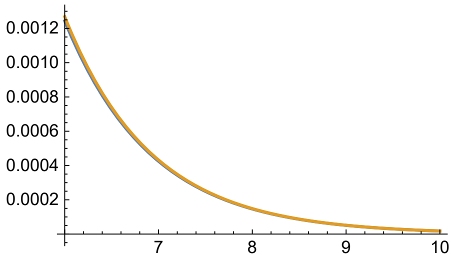
By lemma,

$$I(k) \sim \sum_{n=0}^{\infty} \frac{\tilde{a}_n}{2^{n+1/2}} \frac{\Gamma(-1/2+n+1)}{k^{-1/2+n+1}} = \sum_{n=0}^{\infty} \frac{\tilde{a}_n}{2^{n+1/2}} \frac{\Gamma(1/2+n)}{k^{1/2+n}},\ k \to \infty$$

For very very large k, $\sum_{n=0}^{\infty} \frac{\tilde{a}_n}{2^{n+1/2}} \frac{\Gamma(1/2+n)}{k^{1/2+n}} \sim \frac{\tilde{a}_0}{2^{1/2}} \frac{\Gamma(1/2)}{k^{1/2}} = \frac{1}{\sqrt{2k/\pi}}$, $I(k) \sim \frac{1}{\sqrt{2k/\pi}}$

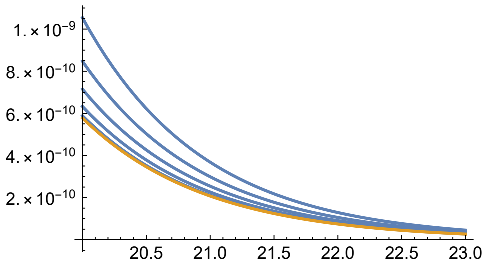
$I(k)$ is related to the modified Bessel function of the second kind, $K_0(k)$

```
Plot[{BesselK[0, k], e^-k/Sqrt[2 k/π]}, {k, 6, 10}]
```



As $k \to \infty$, $K_p(k) \sim \left(\frac{e^{-k}}{\sqrt{2k/\pi}}\right)$ (it is shown that $K_0(k) = e^{-k} I(k)$)

```
Plot[{BesselK[#, k] & /@ Range[5], e^-k/Sqrt[2 k/π]}, {k, 20, 23}]
```



Now then, let's talk about not monotonic case. Suppose that function $\phi(t)$ has a local minimum at $t = c$ such that $a < c < b$, $\phi'(c) = 0$ and $\phi''(c) > 0$. Furthermore, we assume that $\phi'(t) \neq 0$ in the interval $[a, b]$ except at $t = c$.

Then the Laplace type integral $I(k) = \int_a^b f(t)\, e^{-k\,\phi(t)}$ is asymptotic to

$$\int_{c-R}^{c+R} f(c) \exp\left[-k\left[\phi(c) + \frac{(t-c)^2}{2}\phi''(c)\right]\right] dt \text{ as } k \to \infty$$

(since $\phi(t) = \phi(c) + \frac{(t-c)}{1!}\phi'(c) + \frac{(t-c)^2}{2!}\phi''(c) + \cdots$ in the neighborhood of $t = c$, and $\phi'(c) = 0$)

where R is small but finite. Let's evaluate above integral.

$$\int_{c-R}^{c+R} f(c) \exp\left[-k\left[\phi(c) + \frac{(t-c)^2}{2}\phi''(c)\right]\right] dt = f(c)\, e^{-k\,\phi(c)} \int_{c-R}^{c+R} e^{-\frac{k\phi''(c)}{2}(t-c)^2} dt$$

letting $\tau = \sqrt{\frac{k\phi''(c)}{2}}\,(t-c)$,

$$f(c)\, e^{-k\,\phi(c)} \int_{c-R}^{c+R} e^{-\frac{k\phi''(c)}{2}(t-c)^2} dt = f(c)\, e^{-k\,\phi(c)} \int_{c-R}^{c+R} e^{-\tau^2} dt$$

$$= \frac{f(c)\, e^{-k\,\phi(c)}}{\sqrt{k\,\phi''(c)/2}} \int_{-R\sqrt{k\,\phi''(c)/2}}^{R\sqrt{k\,\phi''(c)/2}} e^{-\tau^2}\, d\tau$$

As $k \to \infty$, $\int_{-R\sqrt{k\phi''(c)/2}}^{R\sqrt{k\phi''(c)/2}} e^{-\tau^2} d\tau \simeq \int_{-\infty}^{\infty} e^{-\tau^2} d\tau = 2\int_0^{\infty} e^{-\tau^2} d\tau = 2\int_0^{\infty} \frac{1}{2} k^{-1/2} e^{-k} dk = \Gamma(1/2) = \sqrt{\pi}$

Hence,

$$I(k) = \int_a^b f(t)\, e^{-k\phi(t)} \sim f(c)\, e^{-k\phi(c)} \sqrt{\frac{2\pi}{k\phi''(c)}}, \quad k \to \infty$$

If you want the rigorous derivation, see following lemma. It's must be tough, but don't let it get to you.
(the notation $f(t) \in C^n[a, b]$ means that $f(t)$ has n derivatives and that $f^{(n)}(t)$ is continuous in the interval $[a, b]$)

<Theorem 13> Laplace's method

- Consider the Laplace type integral $I(k) = \int_a^b f(t)\, e^{-k\phi(t)} dt$ and assume that $\phi'(c) = 0$, $\phi''(c) > 0$ for some point c in the interval $[a, b]$. Further, assume that $\phi'(t) \neq 0$ in $[a, b]$ except at $t = c$, $\phi \in C^4[a, b]$, and $f \in C^2[a, b]$. Then if c is an interior point,

$$I(k) = \int_a^b f(t)\, e^{-k\phi(t)} dt \sim f(c)\, e^{-k\phi(c)} \sqrt{\frac{2\pi}{k\phi''(c)}}, \quad k \to \infty$$

expresses the leading term of an asymptotic expansion of $I(k)$ as $k \to \infty$, with an error $O\left(e^{-k\phi(c)} / k^{3/2}\right)$.
If c is an endpoint, then the leading term is half that obtained when c is an interior point, and the error is $O\left(e^{-k\phi(c)} / k\right)$

proof

Splitting the $[a, b]$ into two half-open intervals $[a, c)$, $(c, b]$, in each of which $\phi(t)$ is monotonic. Then we can apply Watson's lemma to each monotonic $\phi(t)$. Let $I(k) = I_a(k) + I_b(k)$ where $I_a = \int_a^c f(t)\, e^{-k\phi(t)} dt$ and $I_b = \int_c^b f(t)\, e^{-k\phi(t)} dt$.
Let's consider I_b. Since ϕ is monotonic in $[c, b]$, let $\phi(t) - \phi(c) = \tau$ then,

$$I_b = \int_c^b f(t)\, e^{-k\phi(t)} dt = \int_0^{\phi(b)-\phi(c)} f(t)\, e^{-k(\phi(c)+\tau)} \frac{d\tau}{\phi'(t)} = e^{-k\phi(c)} \int_0^{\phi(b)-\phi(c)} \left[\frac{f(t)}{\phi'(t)}\right]_{t=t(\tau)} e^{-k\tau} d\tau$$

and because ϕ is monotonic, we can use the inversion of $\phi(t) - \phi(c) = \tau$ to determine the $t = t(\tau)$

$$t(\tau) = \phi^{-1}(\tau + \phi(c))$$

To apply Watson's lemma, we need to know the behavior of $\left[\frac{f(t)}{\phi'(t)}\right]_{t=t(\tau)}$ as $\tau \to 0^+$

From $\phi(t) = \phi(c) + \frac{(t-c)^2}{2!}\phi''(c) + \frac{(t-c)^3}{3!}\phi'''(c) + O((t-c)^4)$,

$$\tau = \frac{(t-c)^2}{2!}\phi''(c) + \frac{(t-c)^3}{3!}\phi'''(c) + O((t-c)^4)$$
$$2\tau = \phi''(c)(t-c)^2 + \frac{\phi'''(c)}{3}(t-c)^3 + O((t-c)^4)$$

To find $t = t(\tau)$ near $\tau = 0$, solve above equation recursively.

$$(t-c) = \sqrt{\frac{2}{\phi''(c)}}\, \tau^{1/2} + A\,\tau^{\alpha} + \cdots \text{ and substitute to the above equation. } (\alpha > 1/2)$$

$$\textbf{Expand}\left[\left(\sqrt{\frac{2}{\phi''}}\ \tau^{1/2}+A\,\tau^{\alpha}\right)^2 * \phi'' + \left(\sqrt{\frac{2}{\phi''}}\ \tau^{1/2}+A\,\tau^{\alpha}\right)^3 * \phi'''/3\right]$$

$$2\,\tau+\frac{2\sqrt{2}\ A\,\tau^{\frac{1}{2}+\alpha}}{\sqrt{\frac{1}{\phi''}}}+A^2\,\tau^{2\alpha}\,\phi''+\frac{1}{3}A^3\,\tau^{3\alpha}\,\phi^{(3)}+$$

$$\sqrt{2}\ A^2\,\tau^{\frac{1}{2}+2\alpha}\sqrt{\frac{1}{\phi''}}\ \phi^{(3)}+\frac{2}{3}\sqrt{2}\ \tau^{3/2}\left(\frac{1}{\phi''}\right)^{3/2}\phi^{(3)}+\frac{2\,A\,\tau^{1+\alpha}\,\phi^{(3)}}{\phi''}$$

we find $\alpha = 1$ and $A = -\left(\frac{\phi'''(c)}{3\,(\phi''(c))^2}\right)$,

$$\textbf{Expand}\left[\left(\sqrt{\frac{2}{\phi''}}\ \tau^{1/2}+A\,\tau^{\alpha}\right)^2 * \phi'' + \left(\sqrt{\frac{2}{\phi''}}\ \tau^{1/2}+A\,\tau^{\alpha}\right)^3 * \phi'''/3\right]\ /.\ \left\{A\to\frac{-\phi'''}{3\,(\phi'')^2},\ \alpha\to 1\right\}$$

$$2\,\tau-\frac{5\,\tau^2\left(\phi^{(3)}\right)^2}{9\,(\phi'')^3}+\frac{1}{9}\sqrt{2}\ \tau^{5/2}\left(\frac{1}{\phi''}\right)^{9/2}\left(\phi^{(3)}\right)^3-\frac{\tau^3\left(\phi^{(3)}\right)^4}{81\,(\phi'')^6}$$

Thus,

$$\begin{aligned}(t-c) &= \sqrt{\frac{2}{\phi''(c)}}\ \tau^{1/2}-\left(\frac{\phi'''(c)}{3\,(\phi''(c))^2}\right)\tau^1+O\left(\tau^{3/2}\right)\\ &= \sqrt{\frac{2}{\phi''(c)}}\ \tau^{1/2}\left[1-\frac{\phi'''(c)}{3\sqrt{2}\ (\phi''(c))^{3/2}}\ \tau^{1/2}+O(\tau)\right]\end{aligned}$$

Using

$$\begin{cases} f(t)=f(c)+(t-c)\,f'(c)+O\left((t-c)^2\right)\\ \phi'(t)=\phi'(c)+\phi''(c)\,(t-c)+\frac{\phi'''(c)}{2}(t-c)^2+O\left((t-c)^3\right)\\ \qquad\quad =\phi''(c)\,(t-c)+\frac{\phi'''(c)}{2}(t-c)^2+O\left((t-c)^3\right)\end{cases},$$

$$\frac{f(c)+(t-c)\,f'(c)}{\phi''(c)\,(t-c)+\frac{\phi'''(c)}{2}(t-c)^2}+O((t-c))=\frac{f(c)+(t-c)\,f'(c)}{\phi''(c)\,(t-c)\left[1+\frac{\phi'''(c)}{2\,\phi''(c)}(t-c)\right]}+O((t-c))=\frac{\left[f(c)+(t-c)\,f'(c)\right]}{\phi''(c)\,(t-c)}\left[1-\frac{\phi'''(c)}{2\,\phi''(c)}(t-\right.$$

$$\frac{\left[f(c)+(t-c)\,f'(c)\right]}{\phi''(c)\,(t-c)}-\frac{\phi'''(c)\left[f(c)+(t-c)\,f'(c)\right]}{2\,(\phi''(c))^2}+\cdots$$

$$\frac{f(c)}{\phi''(c)\,(t-c)}+\left(\frac{f'(c)}{\phi''(c)}-\frac{\phi'''(c)\,f(c)}{2\,(\phi''(c))^2}\right)+O(t-c)\quad\left((t-c)=\sqrt{\frac{2}{\phi''(c)}}\ \tau^{1/2}\left[1-\frac{\phi'''(c)}{3\sqrt{2}\ (\phi''(c))^{3/2}}\ \tau^{1/2}+O(\tau)\right]\right)$$

Beep! *Mathematica* chance!

$$\textbf{Expand}\left[\frac{f}{\phi''}\left(\sqrt{\frac{2}{\phi''}}\ t^{1/2}\left(1-\frac{\phi'''}{3\sqrt{2}\ (\phi'')^{3/2}}\ t^{1/2}\right)\right)^{-1}\right]$$

$$\frac{f\sqrt{\frac{1}{\phi''}}}{\sqrt{2}\ \sqrt{t}\ \left(1-\frac{\sqrt{t}\ \phi^{(3)}}{3\sqrt{2}\ (\phi'')^{3/2}}\right)}$$

$$= \frac{f(c)}{\sqrt{2\,\phi''(c)}}\,\tau^{-1/2}\left(1 - \frac{\phi'''(c)}{3\sqrt{2}\,(\phi''(c))^{3/2}}\,\tau^{1/2}\right)^{-1} + \left(\frac{f'(c)}{\phi''(c)} - \frac{\phi'''(c)\,f(c)}{2\,(\phi''(c))^2}\right) + O\left(\tau^{1/2}\right)$$

$$= \frac{f(c)}{\sqrt{2\,\phi''(c)}}\,\tau^{-1/2}\left(1 + \frac{\phi'''(c)}{3\sqrt{2}\,(\phi''(c))^{3/2}}\,\tau^{1/2} + \cdots\right) + \left(\frac{f'(c)}{\phi''(c)} - \frac{\phi'''(c)\,f(c)}{2\,(\phi''(c))^2}\right) + O\left(\tau^{1/2}\right)$$

$$= \frac{f(c)}{\sqrt{2\,\phi''(c)}}\,\tau^{-1/2} + \frac{f(c)\,\phi'''(c)}{6\,(\phi''(c))^2} + \left(\frac{f'(c)}{\phi''(c)} - \frac{\phi'''(c)\,f(c)}{2\,(\phi''(c))^2}\right) + O\left(\tau^{1/2}\right)$$

$$= \frac{f(c)}{\sqrt{2\,\phi''(c)}}\,\tau^{-1/2} + \left(\frac{f'(c)}{\phi''(c)} - \frac{\phi'''(c)\,f(c)}{3\,(\phi''(c))^2}\right) + O\left(\tau^{1/2}\right)$$

$$\equiv a_0\,\tau^{-1/2} + a_1 + O\left(\tau^{1/2}\right)$$

Finally,

$$\left[\frac{f(t)}{\phi'(t)}\right]_{t=t(\tau)} = a_0\,\tau^{-1/2} + a_1 + O\left(\tau^{1/2}\right) = \tau^{-1/2}\left(a_0 + a_1\,\tau^{1/2} + O(\tau)\right)$$

$$I_b = e^{-k\,\phi(c)} \int_0^{\phi(b)-\phi(c)} \tau^{-1/2}\left(a_0 + a_1\,\tau^{1/2} + O(\tau)\right) e^{-k\,\tau}\,d\tau$$

It’s time to recall Watson’s lemma;

- Consider the integral $I(k) = \int_0^b f(t)\,e^{-k\,t}\,dt$, $b > 0$. Suppose that $f(t)$ is integrable in $(0, b)$ and that it has the asymptotic series expansion; $f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n\,t^{\beta\,n}$, $t \to 0^+$ $(\alpha > -1,\ \beta > 0)$ Then

  $\boxed{I(k) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta\,n + 1)}{k^{\alpha+\beta\,n+1}},\ k \to \infty}$

In this case, $\alpha = -1/2$ and $\beta = 1/2$ then, $e^{k\,\phi(c)} I_b(k) = a_0 \frac{\Gamma(-1/2+1)}{k^{-1/2+1}} + a_1 \frac{\Gamma(-1/2+1/2+1)}{k^{-1/2+1/2+1}} + O\left(\frac{1}{k^{3/2}}\right)$

$$e^{k\,\phi(c)}\,I_b(k) = a_0\,\frac{\Gamma(1/2)}{k^{1/2}} + a_1\,\frac{\Gamma(1)}{k} + O\left(\frac{1}{k^{3/2}}\right)$$

$$I_b(k) = \frac{\sqrt{\pi}\,f(c)}{\sqrt{2\,k\,\phi''(c)}}\,e^{-k\,\phi(c)} + \frac{1}{k}\left(\frac{f'(c)}{\phi''(c)} - \frac{\phi'''(c)\,f(c)}{3\,(\phi''(c))^2}\right) e^{-k\,\phi(c)} + O\left(\frac{e^{-k\,\phi(c)}}{k^{3/2}}\right),\ k \to \infty$$

Similarly, for the interval $[a.c)$,

$$I_a = \int_a^c f(t)\,e^{-k\,\phi(t)}\,dt = e^{-k\,\phi(c)} \int_{\phi(a)-\phi(c)}^{0} \left[\frac{f(t)}{\phi'(t)}\right]_{t=t(\tau)} e^{-k\,\tau}\,d\tau$$

$$= -e^{-k\,\phi(c)} \int_0^{\phi(a)-\phi(c)} \left[\frac{f(t)}{\phi'(t)}\right]_{t=t(\tau)} e^{-k\,\tau}\,d\tau$$

Moreover, in this interval,

$$(t - c) = -\sqrt{\frac{2}{\phi''(c)}}\;\tau^{1/2}\left[1 - \frac{\phi'''(c)}{3\sqrt{2}\,(\phi''(c))^{3/2}}\,\tau^{1/2} + O(\tau)\right]$$

Hence,

$$-e^{k\,\phi(c)}\,I_a(k) = -\frac{f(c)}{\sqrt{2\,\phi''(c)}}\,\frac{\Gamma(1/2)}{k^{1/2}} + \left(\frac{f'(c)}{\phi''(c)} - \frac{\phi'''(c)\,f(c)}{3\,(\phi''(c))^2}\right)\frac{\Gamma(1)}{k} + O\left(\frac{1}{k^{3/2}}\right)$$

$$I_a(k) = \frac{\sqrt{\pi}\,f(c)}{\sqrt{2\,k\,\phi''(c)}}\,e^{-k\,\phi(c)} - \frac{1}{k}\left(\frac{f'(c)}{\phi''(c)} - \frac{\phi'''(c)\,f(c)}{3\,(\phi''(c))^2}\right) e^{-k\,\phi(c)} + O\left(\frac{e^{-k\,\phi(c)}}{k^{3/2}}\right),\ k \to \infty$$

So, we have

$$\begin{cases} I_b(k) = \dfrac{\sqrt{\pi}\,f(c)}{\sqrt{2\,k\,\phi''(c)}}\,e^{-k\,\phi(c)} + \left(\dfrac{f'(c)}{\phi''(c)} - \dfrac{\phi'''(c)\,f(c)}{3\,(\phi''(c))^2}\right)\dfrac{e^{-k\,\phi(c)}}{k} + O\left(\dfrac{e^{-k\,\phi(c)}}{k^{3/2}}\right), & k \to \infty \\ I_a(k) = \dfrac{\sqrt{\pi}\,f(c)}{\sqrt{2\,k\,\phi''(c)}}\,e^{-k\,\phi(c)} - \left(\dfrac{f'(c)}{\phi''(c)} - \dfrac{\phi'''(c)\,f(c)}{3\,(\phi''(c))^2}\right)\dfrac{e^{-k\,\phi(c)}}{k} + O\left(\dfrac{e^{-k\,\phi(c)}}{k^{3/2}}\right), & k \to \infty \end{cases}$$

• If c is an interior point, $I(k) = I_a(k) + I_b(k)$ with error term $O\left(\frac{e^{-k\,\phi(c)}}{k^{3/2}}\right)$

• If $c = b$, $I(k) = I_b(k)$ with error term $O\left(\frac{e^{-k\,\phi(c)}}{k}\right)$

• If $c = a$, $I(k) = I_a(k)$ with error term $O\left(\frac{e^{-k\,\phi(c)}}{k}\right)$ ∎

Exercise.
Evaluate $I(k) = \int_a^b f(t)\, e^{k\,\phi(t)}\, dt,\ k \to \infty,$ when $\phi(t)$ has a unique maximum at an interior point $t = c$.

As we did at above, (before Laplace's method) consider the main dominant contribution of the integral. As k goes to ∞, the integration around the neighborhood of $t = c$ tends to be dominant exponentially. Use the same idea as those motivating Laplace's method

$$I(k) \sim \int_{c-R}^{c+R} f(c)\, e^{k\left[\phi(c) + \frac{(t-c)^2}{2}\phi''(c)\right]}\, dt,$$

letting $\tau = \sqrt{-k\,\phi''(c)/2}\ (t-c)$, (note that $\phi''(c) < 0$ since $t = c$ is a maximum)

$$= f(c)\, e^{k\,\phi(c)} \int_{-R\sqrt{-k\,\phi''(c)/2}}^{R\sqrt{-k\,\phi''(c)/2}} e^{-\tau^2} \frac{d\tau}{\sqrt{-k\,\phi''(c)/2}}$$

$$\sim \frac{f(c)\, e^{k\,\phi(c)}}{\sqrt{-k\,\phi''(c)/2}} \int_{-\infty}^{\infty} e^{-\tau^2}\, d\tau = \frac{f(c)\, e^{k\,\phi(c)}}{\sqrt{-k\,\phi''(c)/2}}\, 2 \int_0^{\infty} \frac{1}{2} s^{-1/2}\, e^{-s}\, ds$$

$$= \frac{f(c)\, e^{k\,\phi(c)}}{\sqrt{-k\,\phi''(c)/2}}\, \Gamma(1/2) = \sqrt{\frac{2\pi}{-k\,\phi''(C)}}\ f(c)\, e^{k\,\phi(c)} = \sqrt{\frac{2\pi}{k\,|\phi''(C)|}}\ f(c)\, e^{k\,\phi(c)}$$

Exercise.
Use Laplace's method to show that for an appropriate class of functions the 'L_p norm' converges to the 'maximum' norm, as $p \to \infty$.

The L_p norm of a function g is given by $\|g\|_p = (I(p))^{1/p}$, where $I(p) = \int_a^b |g(t)|^p\, dt$. We assume that $|g(t)| \in C^4$ and that it has a unique maximum with $g(c) \neq 0$ at $t = c$ inside $[a, b]$.
Using Laplace's method, $\phi(t) = \ln|g(t)|$, $\phi'(c) = 0$, $\phi''(c) = g''(c)/g(c)$,

$$I(p) = \int_a^b |g(t)|^p\, dt = \int_a^b e^{p \ln|g(t)|}\, dt$$

$$= \sqrt{\frac{2\pi}{p\,|\phi''(c)|}}\ 1\, e^{p \ln|g(c)|} \text{(result from above exercise)}$$

$$= \sqrt{\frac{2\pi\,|g(c)|}{|g''(c)|}}\ \frac{1}{\sqrt{p}}\ |g(c)|^p \left(\text{let } A = \sqrt{\frac{2\pi\,|g(c)|}{|g''(c)|}}\right)$$

Thus,

$$\|g\|_p \sim A^{1/p}\, p^{-1/(2p)}\, |g(c)| = |g(c)| \left\{1 - \frac{\ln p}{2p} + O\left(\frac{1}{p}\right)\right\}$$

because $A^{1/p} = e^{\frac{1}{p}\ln A} \sim 1 + \frac{\ln A}{p}$, $p^{-\frac{1}{2p}} = e^{-\frac{1}{2p}\ln p} \sim 1 - \frac{\ln p}{2p}$

Hence the L_p norm of $g(t)$ tends to $|g(c)|$ as $p \to \infty$, which is referred to as the maximum norm.

Exercise.
Evaluate $I(k) = \int_a^b f(t)\, e^{k\,\phi(t)}\, dt,\ k \to \infty,$ with $\phi'(c) = \phi''(c) = \cdots = \phi^{(p-1)}(c) = 0$ and $\phi^{(p)}(c) < 0$, where ϕ is maximum at the interior point $t = c$.

Expanding $f(t)$ and $\phi(t)$ near $t = c$,

$$I(k) \sim \int_{c-R}^{c+R} f(c)\, e^{k\left[\phi(c) + \frac{(t-c)^p}{p!}\phi^{(p)}(c)\right]}\, dt$$

letting $\tau = \left(-\frac{k\,\phi^{(p)}(c)}{p!}\right)^{1/p} (t-c)$,

$$\int_{c-R}^{c+R} f(c)\, e^{k\left[\phi(c)+\frac{(t-c)^p}{p!}\phi^{(p)}(c)\right]} dt = f(c)\, e^{k\,\phi(c)} \int_{-R\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}}^{+R\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}} e^{-\tau^p} \frac{d\tau}{\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}}$$

$$= \frac{f(c)\, e^{k\,\phi(c)}}{\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}} \int_{-R\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}}^{+R\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}} e^{-\tau^p}\, d\tau \sim \frac{f(c)\, e^{k\,\phi(c)}}{\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}} \int_{-\infty}^{+\infty} e^{-\tau^p}\, d\tau$$

$$= \frac{2\, f(c)\, e^{k\,\phi(c)}}{\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}} \int_0^{\infty} e^{-\tau^p}\, d\tau = \frac{2\, f(c)\, e^{k\,\phi(c)}}{\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}} \int_0^{\infty} \frac{1}{p}\, e^{-s} (s)^{\frac{1}{p}-1}\, ds$$

$$= \frac{f(c)\, e^{k\,\phi(c)}}{\left(-k\,\phi^{(p)}(c)/p!\right)^{1/p}} \left(\frac{2\,\Gamma\left(\frac{1}{p}\right)}{p}\right)$$

Exercise.
Evaluate $I = \int_0^5 \sin s\, e^{-k \sinh^4 s}\, ds$ as $k \to \infty$

Note that $\phi(s) = \sinh^4 s$ has the minimum at $s = 0$. (in the neighborhood of $s = 0$, $\sin s$ has its Taylor series; $s - \frac{s^3}{3!} + \cdots \sim s$ and $\sinh s$ has its Taylor series, $s + \frac{s^3}{3!} + \cdots \sim s$)

$$I \sim \int_0^R s\, e^{-k\,s^4}\, ds$$

letting $s^2 = t$, $2\,s\,ds \to dt$

$$\int_0^R s\, e^{-k\,s^4}\, ds = \frac{1}{2} \int_0^{\sqrt{R}} e^{-k\,t^2}\, dt$$

letting $k\,t^2 = \tau$, $dt \to \frac{1}{2\sqrt{k}}\, \tau^{-1/2}\, d\tau$

$$\frac{1}{2} \int_0^{\sqrt{R}} e^{-k\,t^2}\, dt = \frac{1}{4\sqrt{k}} \int_0^{k\,R} \tau^{-1/2}\, e^{-\tau}\, d\tau\,, \quad \text{where } k \to \infty$$

$$\sim \frac{1}{4\sqrt{k}} \int_0^{\infty} \tau^{-1/2}\, e^{-\tau}\, d\tau = \frac{1}{4\sqrt{k}} \Gamma(1/2)$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{k}}$$

Exercise.
Evaluate $I = \int_0^{\infty} \frac{e^{-k\,t^2}}{\sqrt{\sinh t}}\, dt$ as $k \to \infty$

$\phi(t) = t^2$ has minimum at $t = 0 \to t = 0$ is the dominant contribution of the integral

```
Series[1/√Sinh[x], {x, 0, 4}]
```

$$\frac{1}{\sqrt{x}} - \frac{x^{3/2}}{12} + \frac{x^{7/2}}{160} + O[x]^{9/2}$$

As you can see, $1/\sqrt{\sinh t}$ has Taylor expansion $\frac{1}{\sqrt{t}} - \cdots$ in the neighborhood of $t = 0$. Therefore,
$\int_0^{\infty} \frac{e^{-k\,t^2}\, dt}{\sqrt{\sinh t}} \sim \int_0^R \frac{e^{-k\,t^2}\, dt}{\sqrt{t}}$

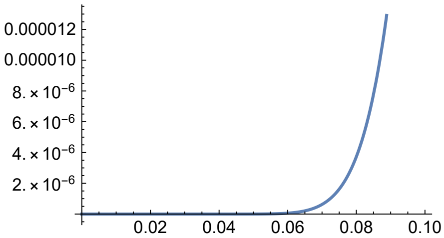
$$\int_0^R \frac{e^{-k\,t^2}\, dt}{\sqrt{t}} = \frac{1}{2\,k^{1/4}} \int_0^{k\,R^2} e^{-\tau}\, \tau^{-3/4}\, d\tau \sim \frac{1}{2\,k^{1/4}} \int_0^{\infty} e^{-\tau}\, \tau^{-3/4}\, d\tau$$

$$= \frac{1}{2\,k^{1/4}} \Gamma(1/4)$$

Exercise.
Evaluate $I(k) = \int_0^\infty e^{-k t - \frac{1}{t}}\, dt$ as a $k \to \infty$

$\int_0^\infty e^{-k t-\frac{1}{t}}\, dt = \int_0^\infty e^{-1/t}\, e^{-k t}\, dt$, $\begin{cases} f(t) = e^{-1/t} \\ \phi(t) = t \end{cases}$ Note that $f(t)$ vanishes exponentially fast at $t = 0$

```
Plot[e^-1/t, {t, 0, 0.1}]
```



Then the main contribution of the integral is the neighborhood of t such that $k t + \frac{1}{t}$ has its minimum.

$$\frac{d}{dt}\left(k t + \frac{1}{t}\right) = k - t^{-2} = 0\,,\ \ t = \frac{1}{\sqrt{k}}$$

that is, the point $t = 1/\sqrt{k}$ is the dominant contribution of the integral which is movable. Letting $t = \frac{s}{\sqrt{k}}$, we can fix the point.

$$I(k) = \int_0^\infty e^{-k t-\frac{1}{t}}\, dt = I(k) = \int_0^\infty e^{-\sqrt{k}\left(s+\frac{1}{s}\right)} \frac{ds}{\sqrt{k}} = \frac{1}{\sqrt{k}} \int_0^\infty e^{-\sqrt{k}\left(s+\frac{1}{s}\right)}\, ds$$
$$= \frac{1}{\sqrt{k}} \int_0^\infty e^{-\sqrt{k}\ \phi(s)}\, ds, \quad \text{where } \phi(s) = s + \frac{1}{s}$$

Since $(s + 1/s)' = 1 - 1/s^2$, $\phi(s)$ has a minimum at $s = 1$, that is, the main contribution of $\frac{1}{\sqrt{k}} \int_0^\infty e^{-\sqrt{k}\ \phi(s)}\, ds$ is neighborhood of $s = 1$ point.

• $\phi(s)$ neighborhood of $s = 1 \to \phi(s) \sim \phi(1) + \phi'(1)(s-1) + \frac{\phi''(1)}{2!}(s-1)^2 = 2 + (s-1)^2$

$$\frac{1}{\sqrt{k}} \int_0^\infty e^{-\sqrt{k}\ \phi(s)}\, ds \sim \frac{1}{\sqrt{k}} \int_{1-R}^{1+R} e^{-\sqrt{k}\ (2+(s-1)^2)}\, ds = \frac{e^{-2\sqrt{k}}}{\sqrt{k}} \int_{1-R}^{1+R} e^{-\sqrt{k}\ (s-1)^2}\, ds$$
$$= \frac{e^{-2\sqrt{k}}}{\sqrt{k}} \int_0^R 2\, e^{-\sqrt{k}\ t^2}\, dt = \frac{e^{-2\sqrt{k}}}{\sqrt{k}} \int_0^\infty 2\, e^{-\tau}\, \frac{1}{2} k^{-1/4}\, \tau^{-1/2}\, d\tau\ \left(\text{letting } k^{1/2} t^2 = \tau\right)$$
$$= \frac{e^{-2\sqrt{k}}}{k^{3/4}} \int_0^\infty e^{-\tau}\, \tau^{-1/2}\, d\tau = \frac{e^{-2\sqrt{k}}}{k^{3/4}}\, \Gamma(1/2) = \frac{\sqrt{\pi}\, e^{-2\sqrt{k}}}{k^{3/4}},\ \ k \to \infty$$

Exercise.
Asymptotic expansion of the gamma function (Stirling’s formula)

Consider $\Gamma(k+1) = \int_0^\infty e^{-t}\, t^k\, dt$ and note that $t^k = e^{k \ln t}$

$$\Gamma(k+1) = \int_0^\infty e^{-t}\, t^k\, dt = \int_0^\infty e^{-t}\, e^{k \ln t}\, dt$$

has maximum value at t such that $-t + k \ln t$ has maximum value

$$\frac{d}{dt}(-t + k \ln t) = -1 + \frac{k}{t} = 0,\ \ t = k \text{ is the maximum point}$$

Since $t = k$ is movable, fix it by changing variables $t \to s\, k$

$$\Gamma(k+1) = \int_0^\infty e^{-t}\, t^k\, dt = \int_0^\infty e^{-s\,k} (s\,k)^k\, k\, ds = k^{k+1} \int_0^\infty e^{-s\,k}\, s^k\, ds$$
$$= k^{k+1} \int_0^\infty e^{-s\,k}\, e^{k \ln s}\, ds = k^{k+1} \int_0^\infty e^{-k(-\ln s + s)}\, ds = k^{k+1} \int_0^\infty e^{-k\,\phi(s)}\, ds$$

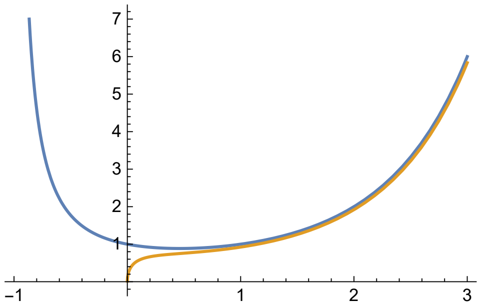
where $\phi(s) = -\ln s + s$ that has a minimum at $s = 1$

• Taylor series of $\phi(s)$ in the neighborhood of $s = 1$: $\phi(s) \sim \phi(1) + \frac{\phi''(1)}{2!}(s-1)^2 = 1 + \frac{1}{2}(s-1)^2$

Hence,

$$k^{k+1} \int_0^\infty e^{-k\,\phi(s)}\, ds \sim k^{k+1} \int_{1-R}^{1+R} e^{-k\left[1+\frac{1}{2}(s-1)^2\right]}\, ds = k^{k+1}\, e^{-k} \int_{1-R}^{1+R} e^{-k((s-1)^2/2)}\, ds$$
$$= k^{k+1}\, e^{-k} \int_0^{1+R} 2\, e^{-k((s-1)^2/2)}\, ds = k^{k+1}\, e^{-k} \int_0^R 2\, e^{-k(t^2/2)}\, dt \sim k^{k+1}\, e^{-k} \int_0^\infty 2\, e^{-k(t^2/2)}\, dt$$
$$(\text{letting } k\,t^2/2 \to \tau) \;=\; k^{k+1}\, e^{-k} \int_0^\infty 2\, e^{-\tau}\, k^{-1/2}\, \frac{1}{\sqrt{2}}\, \tau^{-1/2}\, d\tau$$
$$= k^{k+1/2}\, e^{-k} \int_0^\infty \sqrt{2}\; \tau^{-1/2}\, e^{-\tau}\, d\tau = \sqrt{2\,k} \left(\frac{k}{e}\right)^k \Gamma(1/2) = \sqrt{2\,k\,\pi} \left(\frac{k}{e}\right)^k$$
$$\Gamma(k+1) \sim \sqrt{2\,k\,\pi} \left(\frac{k}{e}\right)^k,\ k \to \infty$$

```
Plot[{Gamma[k + 1], √(2 k π) (k / e)^k}, {k, -1, 3}]
```



Exercise.
Use Laplace's method to determine the leading behavior of $I(k) = \int_{-1/2}^{1/2} e^{-k \sin^4 t}\, dt$

$\phi(t) = \sin^4 t$ has a local minimum at $t = 0$ and its Taylor expansion around the neighborhood of $t = 0$ is,

```
Series[Sin[t]^4, {t, 0, 6}]
```

$$t^4 - \frac{2\,t^6}{3} + O[t]^7$$

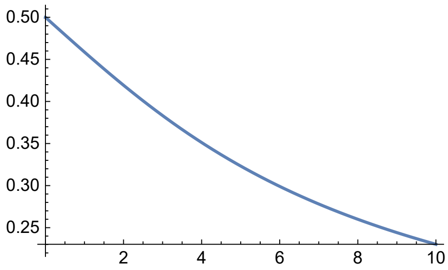
$$I(k) = \int_{-1/2}^{1/2} e^{-k \sin^4 t}\, dt \sim \int_{0-R}^{0+R} e^{-k\left(t^4 + O(t^6)\right)}\, dt = 2 \int_0^\infty e^{-k\,t^4}\, dt$$

letting $k\,t^4 = \tau$,

$$2 \int_0^\infty e^{-k\,t^4}\, dt = 2 \int_0^\infty e^{-\tau}\, \frac{1}{4\,k}\, \tau^{-3/4}\, k^{3/4}\, d\tau = \frac{1}{2\,k^{1/4}}\, \Gamma(1/4)$$

Exercise.
Show that $\int_0^\infty \ln\left(\frac{u}{1-e^{-u}}\right)\frac{e^{-ku}}{u}\,du \sim \frac{1}{2k},\ \ k\to\infty$

```
Plot[Log[u/(1 - e^-u)]/u, {u, 0, 10}]
```



$\phi(u)=u$ has minimum at $u=0$ and in the neighborhood of $u=0$, $\frac{1}{u}\ln\left(\frac{u}{1-e^{-u}}\right)$ has Taylor expansion;

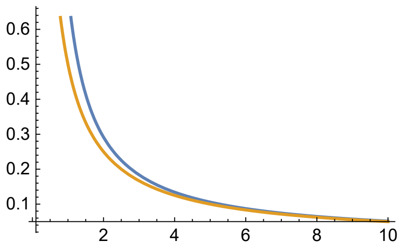
$$1-e^{-u}=u-\frac{1}{2}u^2+O(u^3),\quad \frac{u}{1-e^{-u}}=\frac{1}{1-\frac{1}{2}u+O(u^2)}\sim\left(1-\frac{1}{2}u\right)^{-1}$$

$$\frac{1}{u}\ln\left(\frac{u}{1-e^{-u}}\right)\sim-\frac{1}{u}\ln\left(1-\frac{1}{2}u\right)=-\frac{1}{u}\left(-\frac{u}{2}-\frac{u^2}{8}-\frac{u^3}{24}+O(u^4)\right)=\frac{1}{2}+\frac{u}{8}+\frac{u^2}{24}+O(u^3)$$

Hence,

$$\int_0^\infty \ln\left(\frac{u}{1-e^{-u}}\right)\frac{e^{-ku}}{u}\,du\sim\int_0^\infty\left[\frac{1}{2}+\frac{u}{8}+\frac{u^2}{24}+O(u^3)\right]e^{-ku}\,du$$
$$\sim\int_0^\infty\left(\frac{1}{2}e^{-ku}+\frac{u}{8}e^{-ku}+\frac{u^2}{24}e^{-ku}\right)du=\int_0^\infty\left(\frac{1}{2}e^{-\tau}+\frac{\tau}{8k}e^{-\tau}+\frac{\tau^2}{24k^2}e^{-\tau}\right)\frac{d\tau}{k}$$
$$=\frac{1}{2k}\Gamma(1)+\frac{1}{8k^2}\Gamma(2)+\frac{1}{24k^3}\Gamma(3)+O\left(\frac{1}{k^4}\right)$$
$$=\frac{1}{2k}+\frac{1}{8k^2}+\frac{1}{12k^3}+O\left(\frac{1}{k^4}\right)$$

```
Plot[{1/(2 k) + 1/(8 k^2) + 1/(12 k^3), 1/(2 k)}, {k, 0.1, 10}]
```



6.3 Fourier Type Integrals

Fourier type integrals are

$$I(k)=\int_a^b f(t)\,e^{i k\phi(t)}\,dt$$

where $f(t)$ and $\phi(t)$ are real continuous functions. For $\phi(t)\in\mathbb{C}$ case, we need the steepest descent method which we will meet next section. Just as we deal with Laplace type integral, first, we consider the monotonic $\phi(t)$ case and next, we consider the case that $\phi'(t)$ vanishes in the interval $[a,b]$

If $\phi(t)$ is monotonic and $\phi(t)$ and $f(t)$ are sufficiently smooth, an asymptotic expansion of $I(k)=\int_a^b f(t)\,e^{ik\phi(t)}\,dt$ can be obtained by the integration by parts procedure.

<Theorem 14> Integration by parts

- Consider the integral $I(k) = \int_a^b f(t)\, e^{i k t}\, dt$ where the interval $[a, b]$ is a finite segment of the real axis. Let $f^{(m)}(t)$ denote the m th derivative of $f(t)$. Assume that $f(t)$ has $N+1$ continuous derivatives and that $f^{(N+2)}$ is piecewise continuous on $[a, b]$. Then

$$I(k) \sim \sum_{n=0}^{N} \frac{(-1)^n}{(i k)^{n+1}} \left[f^{(n)}(b)\, e^{i k b} - f^{(n)}(a)\, e^{i k a} \right], \quad k \to \infty$$

proof

For $m \le N$, integration by parts yields

$$I(k) = \int_a^b f(t)\, e^{i k t}\, dt = \left[\frac{e^{i k t}}{i k} f(t)\right]_a^b - \int_a^b \frac{f'(t)}{i k} e^{i k t}\, dt = \left[\frac{e^{i k t}}{i k} f(t)\right]_a^b - \left[\frac{e^{i k t}}{(i k)^2} f'(t)\right]_a^b + \int_a^b \frac{f''(t)}{(i k)^2} e^{i k t}\, dt$$

$$= \cdots = \left[\frac{e^{i k t}}{i k} f^{(0)}(t)\right]_a^b - \left[\frac{e^{i k t}}{(i k)^2} f^{(1)}(t)\right]_a^b + \left[\frac{e^{i k t}}{(i k)^3} f^{(3)}(t)\right]_a^b - \cdots + (-1)^n \left[\frac{e^{i k t}}{(i k)^{n+1}} f^{(n)}(t)\right] + \cdots$$

$$\sim \sum_{n=0}^{N} \frac{(-1)^n}{(i k)^{n+1}} \left[f^{(n)}(b)\, e^{i k b} - f^{(n)}(a)\, e^{i k a} \right], \quad k \to \infty$$

Note that $N = 0$ case,

$$I(k) \sim \frac{1}{i k} \left[f(b)\, e^{i k b} - f(a)\, e^{i k a} \right], \quad k \to \infty \quad \blacksquare$$

Exercise.

Evaluate $I(k) = \int_0^1 \frac{e^{i k t}}{1+t}\, dt$ as $k \to \infty$

Using integration by parts, (note that n th derivative of $(1+t)^{-1}$ is $(-1)^n\, n!\, (1+t)^{-(n+1)}$)

$$I(k) = \int_0^1 \frac{e^{i k t}}{1+t}\, dt = \left[\frac{e^{i k t}}{i k}\left(\frac{1}{1+t}\right)\right]_0^1 - \int_0^1 \frac{e^{i k t}}{(i k)^2}\, \partial_t\left(\frac{1}{1+t}\right) dt$$

$$= \left[\frac{e^{i k t}}{i k}\left(\frac{1}{1+t}\right)\right]_0^1 - \left[\frac{e^{i k t}}{(i k)^2}\, \partial_t\left(\frac{1}{1+t}\right)\right]_0^1 + \int_0^1 \frac{e^{i k t}}{(i k)^2}\, \partial_t^{\,2}\left(\frac{1}{1+t}\right) dt$$

$$= \cdots \sim \sum_{n=0}^{N} \frac{(-1)^n}{(i k)^{n+1}} \left[f^{(n)}(1)\, e^{i k} - f^{(n)}(0) \right] = \sum_{n=0}^{N} \frac{(-1)^n}{(i k)^{n+1}} \left[\frac{(-1)^n\, n!}{2^{n+1}} e^{i k} - (-1)^n\, n! \right]$$

$$= \sum_{n=0}^{N} \frac{1}{(i k)^{n+1}} \left[\frac{n!}{2^{n+1}} e^{i k} - n! \right] = e^{i k}\left[\frac{1}{2\,(i k)} + \frac{1}{2^2\,(i k)^2} + \frac{2!}{2^3\,(i k)^3} + \cdots\right] - \left[\frac{1}{(i k)} + \frac{1}{(i k)^2} + \frac{2!}{(i k)^3} + \cdots\right]$$

```
Series[∫_0^1 (e^(i k t))/(1 + t) dt, {k, ∞, 3}]
```

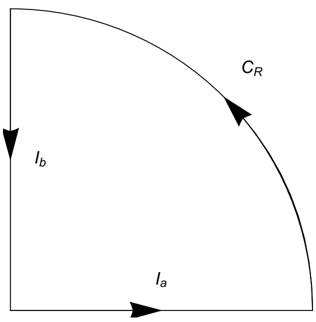
$$e^{i k + O\left[\frac{1}{k}\right]^4} \left(-\frac{i}{2 k} - \frac{1}{4 k^2} + \frac{i}{4 k^3} + O\left[\frac{1}{k}\right]^4 \right) + \left(\frac{i}{k} + \left(\frac{1}{k}\right)^2 - \frac{2 i}{k^3} + O\left[\frac{1}{k}\right]^4 \right)$$

Now it's time to introduce Watson's lemma for Fourier type integrals. To derive the lemma, we will need to compute the integral $\int_0^\infty t^\gamma\, e^{i \mu t}\, dt$ where $\gamma, \mu \in \mathbb{R}$, $\gamma > -1$. See the following exercise.

Exercise.

Show that $I = \int_0^\infty t^{\gamma-1}\, e^{i t}\, dt = e^{\frac{\gamma \pi i}{2}}\, \Gamma(\gamma)$, $\gamma \in \mathbb{R}$, $0 < \gamma < 1$

We need to rotate the contour so that the I can be related to a gamma function.



Consider above contour ; $\oint z^{\gamma-1} e^{iz}\, dz = \int_a + \int_b + \int_{C_R} = 0$ (because there are no singularities in the first quadrant)

Thanks to Jordan's lemma, $\int_{C_R}$ goes to zero, so that $\oint z^{\gamma-1} e^{iz}\, dz = \int_a + \int_b = 0$

$$\text{Changing variables } \begin{cases} \text{Path } a & z = R,\ R: 0 \sim \infty,\ \ dz \to dR \\ \text{Path } b & z = R e^{i\pi/2},\ R: \infty \sim 0,\ dz \to i\, dR \end{cases}$$

That is,

$$\int_0^\infty R^{\gamma-1} e^{iR}\, dR + \int_\infty^0 \left(R e^{i\pi/2}\right)^{\gamma-1} e^{iRe^{i\pi/2}}\, i\, dR = 0$$

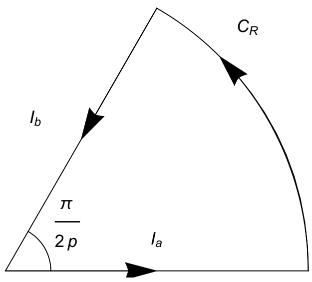
$$\begin{aligned} \int_0^\infty R^{\gamma-1} e^{iR}\, dR &= \int_0^\infty \left(R e^{i\pi/2}\right)^{\gamma-1} e^{-R}\, i\, dR \\ &= \int_0^\infty R^{\gamma-1} e^{i\pi(\gamma-1)/2} e^{-R} e^{i\pi/2}\, dR \\ &= \int_0^\infty R^{\gamma-1} e^{i\pi\gamma/2} e^{-R}\, dR = e^{i\pi\gamma/2} \int_0^\infty R^{\gamma-1} e^{-R}\, dR \\ &= e^{i\pi\gamma/2}\, \Gamma(\gamma) \end{aligned}$$

```
Integrate[t^(γ-1) ⅇ^(ⅈ t), {t, 0, ∞}, Assumptions → {0 < γ < 1 && γ ∈ Reals}]
```

$e^{\frac{i\pi\gamma}{2}}$ Gamma[γ]

Exercise.

Show that $I = \int_0^\infty t^\gamma e^{i v t^p}\, dt = \left(\frac{1}{|v|}\right)^{\frac{\gamma+1}{p}} \frac{\Gamma\left(\frac{\gamma+1}{p}\right)}{p}\, e^{\frac{i\pi}{2p}(\gamma+1)\,\mathrm{sgn}\, v}$, where γ and v are real constants, $\gamma > -1$, and p is a positive integer.



This contour is appropriate for the exercise. It's more general. Let's evaluate the integral step by step as we did at above.

$$\oint z^\gamma e^{i v z^p}\, dz = \int_a + \int_b + \int_{C_R} = 0$$

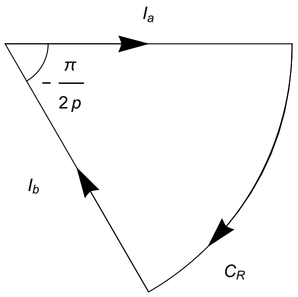
First, we consider the $v > 0$ case.

$$\text{Changing variables } \begin{cases} \text{Path } a & z = R,\ R: 0 \sim \infty,\ \ dz \to dR \\ \text{Path } b & z = R e^{\frac{i\pi}{2p}},\ R: \infty \sim 0,\ dz \to e^{\frac{i\pi}{2p}}\, dR \end{cases}$$

$$\int_0^\infty R^\gamma e^{i v R^p} dR + \int_\infty^0 \left(R e^{\frac{i\pi}{2p}}\right)^\gamma e^{i v R^p e^{\frac{i\pi}{2p} p}} e^{\frac{i\pi}{2p}} dR = 0$$

$$\begin{aligned}
\int_0^\infty R^\gamma e^{i v R^p} dR &= \int_0^\infty \left(R e^{\frac{i\pi}{2p}}\right)^\gamma e^{-v R^p} e^{\frac{i\pi}{2p}} dR \\
&= e^{\frac{i\pi(\gamma+1)}{2p}} \int_0^\infty R^\gamma e^{-v R^p} dR \\
(v R^p \to \tau) &= e^{\frac{i\pi(\gamma+1)}{2p}} \int_0^\infty (\tau/v)^{\gamma/p} e^{-\tau} \frac{1}{v p} (\tau/v)^{-\frac{p-1}{p}} d\tau \\
&= e^{\frac{i\pi(\gamma+1)}{2p}} \int_0^\infty \left(\frac{1}{v}\right)^{\frac{\gamma+1}{p}} \frac{1}{p} \tau^{\frac{1+\gamma}{p}-1} e^{-\tau} d\tau \\
&= \left(\frac{1}{v}\right)^{\frac{\gamma+1}{p}} \frac{\Gamma\left(\frac{1+\gamma}{p}\right)}{p} e^{\frac{i\pi(\gamma+1)}{2p}}
\end{aligned}$$

Similarly, $v < 0$ case is $-\frac{\pi}{2p}$ rotation.



Therefore, the evaluation of the integral is

$$\left(\frac{1}{-v}\right)^{\frac{\gamma+1}{p}} \frac{\Gamma\left(\frac{1+\gamma}{p}\right)}{p} e^{-\frac{i\pi(\gamma+1)}{2p}}$$

Hence, $I = \int_0^\infty t^\gamma e^{i v t^p} dt = \left(\frac{1}{|v|}\right)^{\frac{\gamma+1}{p}} \frac{\Gamma\left(\frac{\gamma+1}{p}\right)}{p} e^{\frac{i\pi}{2p}(\gamma+1)\,\mathrm{sgn}\, v}$

```
Integrate[t^γ ⅇ^(ⅈ v t^p), {t, 0, ∞}, Assumptions → {γ ∈ Reals && v ∈ Reals && γ > -1 && p > 1 + γ}]
```

$$\frac{1}{p} \mathrm{Abs}[v]^{-\frac{1+\gamma}{p}} \mathrm{Gamma}\left[\frac{1+\gamma}{p}\right] \left(\mathrm{Cos}\left[\frac{\pi (1+\gamma)}{2p}\right] + i\, \mathrm{Sign}[v]\, \mathrm{Sin}\left[\frac{\pi(1+\gamma)}{2p}\right]\right)$$

<Remark 6.2>

- $I = \int_0^\infty t^\gamma e^{i v t^p} dt = \left(\frac{1}{|v|}\right)^{\frac{\gamma+1}{p}} \frac{\Gamma\left(\frac{\gamma+1}{p}\right)}{p} e^{\frac{i\pi}{2p}(\gamma+1)\,\mathrm{sgn}\, v}$

From this result, we can derive the following lemma!

<Theorem 15>

- Consider the integral; $I(k) = \int_0^b f(t)\, e^{i k \mu t} dt$, $b > 0$, $\mu \pm 1$, $k > 0$. Suppose that $f(t)$ vanishes infinitely smoothly at $t = b$ and that $f(t)$ and all its derivatives exists in $(0, b]$. Furthermore, assume that $f(t) \sim t^\gamma + o(t^\gamma)$, as $t \to 0^+$, $\gamma \in \mathbb{R}$ && $\gamma > -1$. Then

$$I(k) = \left(\frac{1}{k}\right)^{\gamma+1} \Gamma(\gamma+1)\, e^{\frac{i\pi}{2}(\gamma+1)\mu} + o\left(k^{-(\gamma+1)}\right), \quad k \to \infty$$

In this case, $p = 1$ and $v = k\mu$ ($|v| = k$, $\mu = \mathrm{sgn}\, v$)

Exercise.

Show that $\int_0^1 \sqrt{t}\, e^{i k t} dt \sim \frac{-i e^{i k}}{k} + \frac{\sqrt{\pi}\, e^{3 i \pi/4}}{2 k^{3/2}}$ as $k \to \infty$

By integration by parts,

$$\int_0^1 \sqrt{t}\, e^{i k t} dt = \left[\frac{e^{i k t}}{i k} \sqrt{t}\right]_0^1 - \int_0^1 \frac{e^{i k t}}{i k} \frac{1}{2\sqrt{t}} dt = -\frac{i}{k} e^{i k} + \frac{i}{2k} \int_0^1 \frac{e^{i k t}}{\sqrt{t}} dt$$

By the <Theorem 15>, the leading order term of $\int_0^1 \frac{e^{i k t}}{\sqrt{t}} dt$ is

$$\left(\frac{1}{k}\right)^{1/2} \Gamma(-1/2+1)\, e^{\frac{i\pi}{2}(-1/2+1)} = \sqrt{\frac{\pi}{k}}\, e^{i \pi/4}$$

Therefore,

$$\int_0^1 \sqrt{t}\, e^{i k t} dt = -\frac{i}{k} e^{i k} + \frac{i}{2k} \int_0^1 \frac{e^{i k t}}{\sqrt{t}} dt \sim -\frac{i}{k} e^{i k} + \frac{i}{2k} \sqrt{\frac{\pi}{k}}\, e^{i \pi/4}$$
$$= -\frac{i}{k} e^{i k} + \frac{\sqrt{\pi}\, e^{3 i \pi/4}}{2 k^{3/2}}$$

Note that $(-1) = e^{i \pi}$

```
Series[∫_0^1 √t e^(i k t) dt, {k, ∞, 3}]
```

$$e^{i k + O\left[\frac{1}{k}\right]^4} \left(-\frac{i}{k} + O\left[\frac{1}{k}\right]^4\right) + \left(\frac{1}{2} (-1)^{3/4} \sqrt{\pi} \left(\frac{1}{k}\right)^{3/2} + O\left[\frac{1}{k}\right]^{7/2}\right) + e^{i k + O\left[\frac{1}{k}\right]^4} \left(\frac{1}{2 k^2} - \frac{i}{4 k^3} + O\left[\frac{1}{k}\right]^4\right)$$

Now it’s time to consider the not monotonic case. Recall the derivation of Laplace’s method. A heuristic analysis of the leading term of the asymptotic expansion of Fourier type integrals closely follows Laplace’s method.

Suppose f is continuous, ϕ is twice differentiable, ϕ' vanishes in $[a, b]$ only at the point $t = c$, and $\phi''(c) \neq 0$. Then the large k behavior of the integral $I(k) = \int_a^b f(t)\, e^{i k \phi(t)} dt$ is given by

$$\int_{c-R}^{c+R} f(c)\, e^{i k\left[\phi(c) + \frac{\phi''(c)}{2!}(t-c)^2\right]} dt \quad \text{where } R \text{ is small but finite.}$$

$$\text{Letting} \begin{cases} \mu \tau^2 = (t-c)^2 \frac{\phi''(c)}{2} k & d\tau = \sqrt{\frac{|\phi''(c)|\, k}{2}}\, dt \\ \mu = \operatorname{sgn}(\phi''(c)) \end{cases},$$

$$\int_{c-R}^{c+R} f(c)\, e^{i k\left[\phi(c) + \frac{\phi''(c)}{2!}(t-c)^2\right]} dt = f(c)\, e^{i k \phi(c)} \sqrt{\frac{2}{|\phi''(c)|\, k}} \int_{-R\sqrt{k |\phi''(c)|/2}}^{R\sqrt{k |\phi''(c)|/2}} e^{i \mu \tau^2} d\tau$$

As k goes ∞, the integral reduces to

$$f(c)\, e^{i k \phi(c)} \sqrt{\frac{2}{|\phi''(c)|\, k}} \int_{-\infty}^{\infty} e^{i \mu \tau^2} d\tau$$

$$f(c)\,e^{i\,k\,\phi(c)}\sqrt{\frac{2}{|\phi''(c)|\,k}}\int_{-\infty}^{\infty}e^{i\,\mu\,\tau^2}\,d\tau = f(c)\,e^{i\,k\,\phi(c)}\sqrt{\frac{2}{|\phi''(c)|\,k}}\int_0^{\infty}2\,e^{i\,\mu\,\tau^2}\,d\tau$$

$$= f(c)\,e^{i\,k\,\phi(c)}\sqrt{\frac{2}{|\phi''(c)|\,k}}\int_0^{\infty}2\,\tau^0\,e^{i\,\mu\,\tau^2}\,d\tau$$

$$= f(c)\,e^{i\,k\,\phi(c)}\sqrt{\frac{2}{|\phi''(c)|\,k}}\;2\left[\left(\frac{1}{1}\right)^{\frac{0+1}{2}}\frac{\Gamma\left(\frac{0+1}{2}\right)}{2}\,e^{\frac{i\pi}{4}(0+1)\,\mu}\right]$$

$$= f(c)\,e^{i\,k\,\phi(c)}\sqrt{\frac{2}{|\phi''(c)|\,k}}\;\Gamma(1/2)\,e^{\frac{i\,\pi\,\mu}{4}}$$

$$= f(c)\,e^{i\,k\,\phi(c)}\sqrt{\frac{2\,\pi}{|\phi''(c)|\,k}}\;e^{\frac{i\,\pi\,\mu}{4}}$$

(I used the the result of exercise ahead of Theorem 14.)

Hence,

$$I(k)=\int_a^b f(t)\,e^{i\,k\,\phi(t)}\,dt \sim f(c)\,e^{i\,k\,\phi(c)}\sqrt{\frac{2\,\pi}{|\phi''(c)|\,k}}\;e^{\frac{i\,\pi\,\mu}{4}} \text{ where } \mu=\operatorname{sgn}(\phi''(c)),\; k\to\infty$$

Let's take a deep look at the result. Andiamo!

<Theorem 16> The stationary phase method

- Consider the integral; $I(k)=\int_a^b f(t)\,e^{i\,k\,\phi(t)}\,dt$ and assume that $t=c$ is the only point in $[a, b]$ where $\phi'(t)$ vanishes. Also assume that $f(t)$ vanishes infinitely smoothly at the two end points $t=a$ and $t=b$, and that both f and ϕ are infinitely differentiable on the half-open intervals $[a, c)$ and $(c, b]$. Furthermore assume that
$\phi(t)-\phi(c)\sim\alpha(t-c)^2+o((t-c)^2)$, $f(t)\sim\beta(t-c)^\gamma+o((t-c)^\gamma)$ as t tends to c and $\gamma>-1$. Then
$$\int_a^b f(t)\,e^{i\,k\,\phi(t)}\,dt \sim e^{i\,k\,\phi(c)}\,\beta\,\Gamma\left(\frac{\gamma+1}{2}\right)e^{i\,\pi\frac{(\gamma+1)}{4}\mu}\left(\frac{1}{k\,|\alpha|}\right)^{\frac{\gamma+1}{2}}+o\left(k^{-\frac{(\gamma+1)}{2}}\right),\quad k\to\infty \text{ where } \mu=\operatorname{sgn}\alpha$$

proof

Split the interval $[a, b]$ into $[a, c)$ and $(c, b]$ and let I_a and I_b be the integrals evaluated inside $[a, c)$ and $(c, b]$ (that is , $I(k)=I_a(k)+I_b(k)$) . Consider $I_b(k)$ first.

$$I_b(k)=\int_c^b f(t)\,e^{i\,k\,\phi(t)}\,dt$$

letting $\mu\,u=\phi(t)-\phi(c)$, $\mu=\operatorname{sgn}\alpha$, $(\mu\,du=\phi'(t)\,dt)$

$$I_b(k)=e^{i\,k\,\phi(c)}\int_c^b f(t)\,e^{i\,k\,(\phi(t)-\phi(c))}\,dt = e^{i\,k\,\phi(c)}\int_0^{|\phi(b)-\phi(c)|}f(t)\,e^{i\,k\,\mu\,u}\,\frac{\mu}{\phi'(t)}\,du$$

$$= e^{i\,k\,\phi(c)}\int_0^{|\phi(b)-\phi(c)|}F(u)\,e^{i\,k\,\mu\,u}\,du$$

$$\text{where } F(u)=\frac{\mu\,f(t)}{\phi'(t)}$$

To evaluate this integral, we're going to obtain inversion of $\mu\,u=\phi(t)-\phi(c)$ in the neighborhood of $t=c$.

$$\mu\,u=\phi(t)-\phi(c)\sim\alpha(t-c)^2\;,\quad (t-c)\sim\left(\frac{u}{|\alpha|}\right)^{1/2}$$

Since $\begin{cases} f(t) & \sim\beta(t-c)^\gamma \\ \phi(t) & \sim\phi(c)+\alpha(t-c)^2 \end{cases}$, $\frac{\mu f(t)}{\phi'(t)}\sim\frac{\mu\,\beta(t-c)^\gamma}{2\,\alpha\,(t-c)}$ which follows

$$F(u) = \frac{\mu f(t)}{\phi'(t)} \sim \frac{\mu \beta (t-c)^\gamma}{2\alpha(t-c)} = \frac{\beta}{2|\alpha|}(t-c)^{\gamma-1} \sim \frac{\beta}{2|\alpha|}\left(\left(\frac{u}{|\alpha|}\right)^{1/2}\right)^{\gamma-1} = \frac{\beta}{2|\alpha|^{\frac{\gamma+1}{2}}} u^{\frac{\gamma-1}{2}}, \quad u \to 0^+$$

Hence,

$$\begin{aligned} I_b(k) &= \int_c^b f(t)\, e^{ik\phi(t)}\, dt \sim e^{ik\phi(c)} \int_0^{|\phi(b)-\phi(c)|} \frac{\beta}{2|\alpha|^{\frac{\gamma+1}{2}}} u^{\frac{\gamma-1}{2}} e^{ik\mu u}\, du \\ &= \frac{\beta e^{ik\phi(c)}}{2|\alpha|^{\frac{\gamma+1}{2}}} \int_0^{|\phi(b)-\phi(c)|} u^{\frac{\gamma-1}{2}} e^{ik\mu u}\, du \\ &= \frac{\beta e^{ik\phi(c)}}{2|\alpha|^{\frac{\gamma+1}{2}}} \left(\frac{1}{k}\right)^{\frac{\gamma+1}{2}} \Gamma\left(\frac{\gamma+1}{2}\right) e^{\frac{i\pi}{2}\left(\frac{\gamma+1}{2}\right)\mu} \quad \text{(from Theorem 15.)} \\ &= \frac{1}{2} e^{ik\phi(c)} \beta \Gamma\left(\frac{\gamma+1}{2}\right) e^{\frac{i\pi}{2}\left(\frac{\gamma+1}{2}\right)\mu} \left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}} \end{aligned}$$

Similarly, for $I_a(k)$, (in this case, $\mu u = \phi(t) - \phi(c) \sim \alpha(c-t)^2$ and $f(t) \sim \beta(c-t)^\gamma$)

$$\begin{aligned} I_a(k) &= e^{ik\phi(c)} \int_a^c f(t)\, e^{ik(\phi(t)-\phi(c))}\, dt = e^{ik\phi(c)} \int_{|\phi(a)-\phi(c)|}^0 f(t)\, e^{ik\mu u} \frac{\mu}{\phi'(t)}\, du \\ &= -e^{ik\phi(c)} \int_0^{|\phi(a)-\phi(c)|} F(u)\, e^{ik\mu u}\, du \end{aligned}$$

where $F(u) = \frac{\mu f(t)}{\phi'(t)} \sim \frac{\mu \beta (c-t)^\gamma}{2\alpha(t-c)} = \frac{-\beta}{2|\alpha|}(c-t)^{\gamma-1} \sim \frac{-\beta}{2|\alpha|}\left(\left(\frac{u}{|\alpha|}\right)^{1/2}\right)^{\gamma-1} = \frac{-\beta}{2|\alpha|^{\frac{\gamma+1}{2}}} u^{\frac{\gamma-1}{2}}, \quad u \to 0^+$

Hence,

$$\begin{aligned} I_a(k) &= \int_a^c f(t)\, e^{ik\phi(t)}\, dt \sim e^{ik\phi(c)} \int_{|\phi(a)-\phi(c)|}^0 \frac{-\beta}{2|\alpha|^{\frac{\gamma+1}{2}}} u^{\frac{\gamma-1}{2}} e^{ik\mu u}\, du \\ &= \frac{\beta e^{ik\phi(c)}}{2|\alpha|^{\frac{\gamma+1}{2}}} \int_0^{|\phi(a)-\phi(c)|} u^{\frac{\gamma-1}{2}} e^{ik\mu u}\, du \\ &= \frac{\beta e^{ik\phi(c)}}{2|\alpha|^{\frac{\gamma+1}{2}}} \left(\frac{1}{k}\right)^{\frac{\gamma+1}{2}} \Gamma\left(\frac{\gamma+1}{2}\right) e^{\frac{i\pi}{2}\left(\frac{\gamma+1}{2}\right)\mu} \quad \text{(from Theorem 15.)} \\ &= \frac{1}{2} e^{ik\phi(c)} \beta \Gamma\left(\frac{\gamma+1}{2}\right) e^{\frac{i\pi}{2}\left(\frac{\gamma+1}{2}\right)\mu} \left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}} \end{aligned}$$

Adding $I_a(k)$ and $I_b(k)$, $I(k) = e^{ik\phi(c)} \beta \Gamma\left(\frac{\gamma+1}{2}\right) e^{\frac{i\pi}{2}\left(\frac{\gamma+1}{2}\right)\mu} \left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}}, \quad k \to \infty$ ■

It is possible to describe Fourier type integral even if $\phi(t)$ and $f(t)$ have different asymptotic behaviors as $t \to c^+$ and $t \to c^-$.

If $\begin{cases} \phi(t) - \phi(c) \sim \alpha_+(t-c)^\nu & \phi(t) - \phi(c) \sim a_-(c-t)^\nu \\ f(t) \sim \beta_+(t-c)^\gamma \text{ as } t \to c^+ & f(t) \sim \beta_-(c-t)^\gamma \text{ as } t \to c^- \end{cases}$

$$\text{Then, } \begin{cases} I_a \sim \frac{1}{\nu} e^{ik\phi(c)} \beta_- \Gamma\left(\frac{\gamma+1}{\nu}\right) e^{\frac{i\pi}{2}\left(\frac{\gamma+1}{\nu}\right)\mu_-} \left(\frac{1}{k|\alpha_-|}\right)^{\frac{\gamma+1}{\nu}}, & \mu_- = \operatorname{sgn} \alpha_- \\ I_b \sim \frac{1}{\nu} e^{ik\phi(c)} \beta_+ \Gamma\left(\frac{\gamma+1}{2}\right) e^{\frac{i\pi}{2}\left(\frac{\gamma+1}{\nu}\right)\mu_+} \left(\frac{1}{k|\alpha_+|}\right)^{\frac{\gamma+1}{\nu}}, & \mu_+ = \operatorname{sgn} \alpha_+ \end{cases}$$

Exercise.

Evaluate $I(k) = \int_0^{\pi/2} e^{ik\cos t}\, dt$ as $k \to \infty$

$\phi(t) = \cos t$, $\phi'(t) = -\sin t \to$ vanishes at $t = 0$ and $\phi''(0) = -1$

$\cos t \sim 1 - \frac{t^2}{2!}$, $\cos t - 1 \sim -\frac{t^2}{2!}$

$$I(k) = \int_0^{\pi/2} e^{ik\cos t}\, dt \sim e^{ik} \int_0^{0+R} e^{-ikt^2/2}\, dt$$

let $\tau^2 = \frac{t^2}{2} k \left(\tau = \sqrt{\frac{k}{2}}\, t,\ dt = \sqrt{\frac{2}{k}}\, d\tau\right)$

$$e^{ik}\int_0^{0+R} e^{-ikt^2/2}\,dt = e^{ik}\int_0^{R\sqrt{k/2}} e^{-i\tau^2}\sqrt{\frac{2}{k}}\,d\tau \sim e^{ik}\sqrt{\frac{2}{k}}\int_0^{\infty} e^{-i\tau^2}\,d\tau$$

$$= e^{ik}\sqrt{\frac{2}{k}}\int_0^{\infty} e^{i(-1)\tau^2}\,d\tau = e^{ik}\sqrt{\frac{2}{k}}\left(\frac{1}{1}\right)^{1/2}\frac{\Gamma(1/2)}{2}e^{-\frac{i\pi}{4}} = \sqrt{\frac{\pi}{2k}}\,e^{i\left(k-\frac{\pi}{4}\right)}$$

(recall that $\int_0^{\infty} t^{\gamma} e^{i\nu t^{p}}\,dt = \left(\frac{1}{|\nu|}\right)^{\frac{\gamma+1}{p}}\frac{\Gamma\left(\frac{\gamma+1}{p}\right)}{p}e^{\frac{i\pi}{2p}(\gamma+1)\,\mathrm{sgn}\,\nu}$)

or just apply it to steepest descent method. $\phi''(0) = -1$, $\mu = -1$, $f(t) = 1$, $\phi(0) = 1$

Exercise.

Evaluate the leading behavior of $J_n(n)$ as $n \to \infty$, where the Bessel function $J_n(x)$ is given by $J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin t - n\,t)\,dt$.

$$J_n(n) = \frac{1}{\pi}\int_0^{\pi}\cos(n\sin t - n\,t)\,dt = \frac{1}{\pi}\int_0^{\pi}\mathrm{Re}\left[e^{i(n\sin t - n\,t)}\right]dt$$

$$\sim \frac{1}{\pi}\int_0^{\pi}\mathrm{Re}\left[e^{-i\left(\frac{n\,t^3}{3!}\right)}\right]dt \sim \frac{1}{\pi}\int_0^{0+R}\mathrm{Re}\left[e^{-i\left(\frac{n\,t^3}{3!}\right)}\right]dt$$

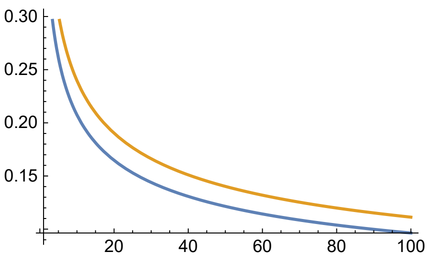
let $n\,t^3/6 = \tau^3$ $\left(\tau = \left(\frac{n}{6}\right)^{1/3} t,\ dt = \left(\frac{6}{n}\right)^{1/3} d\tau\right)$

$$\frac{1}{\pi}\int_0^{0+R}\mathrm{Re}\left[e^{-i\left(\frac{n\,t^3}{3!}\right)}\right]dt = \frac{1}{\pi}\int_0^{R(n/6)^{1/3}}\mathrm{Re}\left[e^{-i\tau^3}\right]\left(\frac{6}{n}\right)^{1/3}d\tau$$

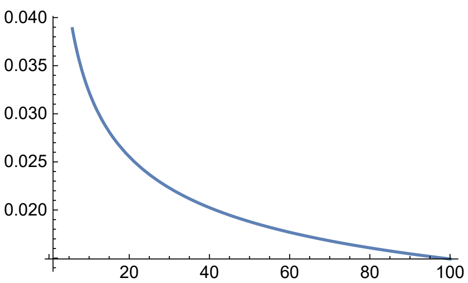
$$\sim \frac{1}{\pi}\left(\frac{6}{n}\right)^{1/3}\int_0^{\infty}\mathrm{Re}\left[e^{-i\tau^3}\right]d\tau = \frac{1}{\pi}\left(\frac{6}{n}\right)^{1/3}\mathrm{Re}\left[\left(\frac{1}{1}\right)^{1/3}\frac{\Gamma(1/3)}{3}e^{-\frac{i\pi}{6}}\right]$$

$$= \frac{1}{\pi}\left(\frac{6}{n}\right)^{1/3}\left(\frac{1}{1}\right)^{1/3}\frac{\Gamma(1/3)}{3}\cos\left(\frac{\pi}{6}\right) = \frac{\Gamma(1/3)}{3\pi}\left(\frac{6}{n}\right)^{1/3}$$

```
Plot[{BesselJ[n, n], Gamma[1/3]/(3 π) 6^(1/3) n^(-1/3)}, {n, 1, 100}]
```




```
Plot[{Abs[BesselJ[n, n] - Gamma[1/3]/(3 π) 6^(1/3) n^(-1/3)]}, {n, 1, 100}]
```



Exercise.

Show that the asymptotic expansion of $I(k) = \int_{-\infty}^{\infty}\frac{e^{ikt^2}}{1+t^2}\,dt$, as $k \to \infty$ is given by $I(k) \sim \sum_{n=0}^{\infty}\left(\frac{1}{k}\right)^{n+1/2}(-1)^n\,\Gamma\left(n+\frac{1}{2}\right)e^{i\frac{\pi}{2}\left(n+\frac{1}{2}\right)}$

$$I(k)=\int_{-\infty}^{\infty}\frac{e^{i k t^2}}{1+t^2}\,dt \sim \int_{0-R}^{0+R}(1-t^2+\cdots)\,e^{i k t^2}\,dt$$

```
Series[1/(1+t^2), {t, 0, 10}]
```

$1-t^2+t^4-t^6+t^8-t^{10}+O[t]^{11}$

letting $\tau^2=k\,t^2$, $\left(\tau=\sqrt{k}\;t,\;\; dt=k^{-1/2}\,d\tau\right)$

$$\begin{aligned}
\int_{0-R}^{0+R}(1-t^2+\cdots)\,e^{i k t^2}\,dt &= \int_{-R\sqrt{k}}^{+R\sqrt{k}}(1-\tau^2/k+\cdots)\,e^{i\tau^2}\,k^{-1/2}\,d\tau \sim \int_{-\infty}^{+\infty}(1-\tau^2/k+\cdots)\,e^{i\tau^2}\,k^{-1/2}\,d\tau\\
&= k^{-1/2}\int_{-\infty}^{+\infty}e^{i\tau^2}\,d\tau-k^{-3/2}\int_{-\infty}^{+\infty}\tau^2\,e^{i\tau^2}\,d\tau+\cdots=2\left[k^{-1/2}\int_0^{\infty}e^{i\tau^2}\,d\tau-k^{-3/2}\int_0^{\infty}\tau^2\,e^{i\tau^2}\,d\tau+\cdots\right]\\
&= 2\left[k^{-1/2}\left(\tfrac{1}{1}\right)^{1/2}\frac{\Gamma(1/2)}{2}\,e^{\frac{i\pi}{4}}-k^{-3/2}\left(\tfrac{1}{1}\right)^{3/2}\frac{\Gamma(3/2)}{2}\,e^{\frac{i3\pi}{4}}+\cdots+(-1)^n\left(\tfrac{1}{1}\right)^{\frac{(2n+1)}{2}}\frac{\Gamma(n+1/2)}{2}\,e^{\frac{i(2n+1)\pi}{4}}+\cdots\right]\\
&= \left(\tfrac{1}{k}\right)^{1/2}\Gamma(1/2)\,e^{\frac{i\pi}{4}}-\left(\tfrac{1}{k}\right)^{3/2}\Gamma(3/2)\,e^{\frac{i3\pi}{4}}+\cdots+(-1)^n\,\Gamma(n+1/2)\,e^{\frac{i(2n+1)\pi}{4}}+\cdots\\
&= \sum_{n=0}^{\infty}(-1)^n\,\Gamma(n+1/2)\,e^{\frac{i(2n+1)\pi}{4}}
\end{aligned}$$

6.4 The Steepest Descents Method

The steepest descent method is a powerful approach for studying the large k asymptotics of integrals of the form

$$I(k)=\int_C f(z)\,e^{k\,\phi(z)}\,dz$$

where C is a contour in the complex z plane and $f(z)$ and $\phi(z)$ ($=u+i\,v$) are analytic function of z. Basic idea of the steepest descent method: Utilize the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $\phi(z)$ has a constant imaginary part.
($\mathrm{Im}[\phi(z)]=v=$ const.)

$$I(k)=\int_C f(z)\,e^{k\,\phi(z)}\,dz=e^{i\,k\,v}\int_{C'}f(z)\,e^{k\,u}\,dz$$

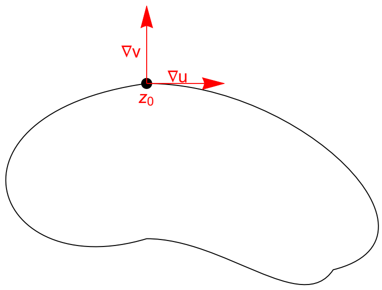
Last integral is a twin brother of Laplace type integral. Note that if $f(z)$ and $\phi(z)$ have singularities(-such as poles or branch points.. I bet that you have almost forgotten these friends), important contribution can arise in the deformation process.

Before start, you need to understand the steepest paths and saddle point.

- **Steepest Paths**

Let $\phi(z)=u(x,y)+i\,v(x,y)$, $z=x+i\,y$. Consider a point $z_0=(x_0+i\,y_0)$ in the complex z plane. A direction away from z_0 in which $u(x,y)$ is decreasing is called a direction of descent. The direction on which the decrease is maximal is called direction of steepest descent. Then the gradient of the differentiable function $u(x,y)$ points in the direction of the most rapid change of $u(x,y)$ at the point (x,y) (direction of steepest ascent is ∇u and that of steepest descent is $-\nabla u$)

The curves of steepest descent associated with a point $z_0=x_0+i\,y_0$ are given by $v(x,y)=v(x_0,y_0)=$ constant. Why? consider the following contour $v(x,y)=$ const. Because $v(x,y)=$ const, its gradient $(\partial v/\partial x,\,\partial v/\partial y)$ is normal to the contour. Recall the Cauchy-Riemann condition; $(\partial_x u=\partial_y v,\; -\partial_y u=\partial_x v)$. Then $(\partial v/\partial x,\,\partial v/\partial y)=(-\partial u/\partial y,\,\partial u/\partial x)$ which is perpendicular to ∇u.



So that the gradient of u is a tangent at point z_0, that is, the rapid change of $u(x, y)$ occurs in the contour of $v(x, y) = \text{const}$.
Or just simply, define $\delta\phi = \phi(z) - \phi(z_0) = \delta u + i\,\delta v$ then δu is maximal only if $\delta v = 0$, $v(x, y) = v(x_0, y_0) = \text{const}$.

- **Saddle Points**

We say that the point z_0 is a saddle point of order N it the first N derivatives vanish, or alternatively, letting $n = N + 1$

$$\left[\frac{d^m\phi}{dz^m}\right]_{z=z_0} = 0, \quad m = 1, 2, \cdots, n-1, \quad \left[\frac{d^n\phi}{dz^n}\right]_{z=z_0} = a\,e^{i\alpha}, \quad a > 0$$

When $N = 1$, we say that z_0 is a saddle point, or a “simple” saddle point, and omit the phrase ‘of order one’.

<Theorem 17>

If the first $n - 1$ derivatives vanish, then there exist n directions of steepest descent and n directions of steepest ascent.
Let $z = z_0 + \rho e^{i\theta}$(neighborhood of z_0) then, these directions are given by

$$\begin{cases} \text{steepest descent directions}: & \theta = -\frac{\alpha}{n} + (2m+1)\frac{\pi}{n}, \quad m = 0, 1, \cdots, n-1 \\ \text{steepest ascent directions}: & \theta = -\frac{\alpha}{n} + 2m\frac{\pi}{n}, \quad m = 0, 1, \cdots, n-1 \end{cases}$$

$$\text{where } \alpha \text{ comes from } \left[\frac{d^m\phi}{dz^m}\right]_{z=z_0} = a\,e^{i\alpha},$$

To derive this, we note that

$$\phi(z) - \phi(z_0) = \frac{\phi^{(1)}(z_0)}{1!}(z-z_0) + \frac{\phi^{(2)}(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{\phi^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$$

because z_0 is a saddle point of order $n - 1$,

$$\begin{aligned} \phi(z) - \phi(z_0) &= 0 + 0 + \cdots + 0 + \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots \\ &= \frac{a\,e^{i\alpha}}{n!}(z-z_0)^n + \cdots \sim \frac{a\,e^{i\alpha}}{n!}(z-z_0)^n \\ &= \frac{a\,e^{i\alpha}}{n!}\left(\rho\,e^{i\theta}\right)^n = \frac{\rho^n a}{n!}e^{i(\alpha+n\theta)} = \frac{\rho^n a}{n!}[\cos(\alpha+n\theta) + i\sin(\alpha+n\theta)] \end{aligned}$$

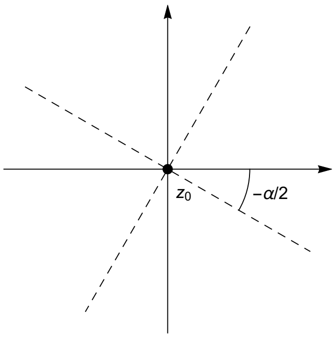
$$\begin{cases} \delta u = \frac{\rho^n a}{n!}\cos(\alpha+n\theta) \\ \delta v = \frac{\rho^n a}{n!}\sin(\alpha+n\theta) \end{cases}$$

Since the directions of steepest descent at $z = z_0$ are defined by $\delta v = 0$, the contour of steepest descent is $\sin(\alpha + n\theta) = 0$ and for u to decrease away from z_0, that is, $\delta u < 0$ $(\cos(\alpha + n\theta) < 0)$.
Similarly, the directions of steepest ascent are given by $\sin(\alpha + n\theta) = 0$ && $\delta u > 0$ $(\cos(\alpha + n\theta) > 0)$. Therefore,

$$\begin{cases} \text{steepest descent directions}: & \theta = -\frac{\alpha}{n} + (2m+1)\frac{\pi}{n}, \quad m = 0, 1, \cdots, n-1 \\ \text{steepest ascent directions}: & \theta = -\frac{\alpha}{n} + 2m\frac{\pi}{n}, \quad m = 0, 1, \cdots, n-1 \end{cases} \quad \blacksquare$$

For $n = 2$, the simple saddle point case the steepest descent and ascent directions are

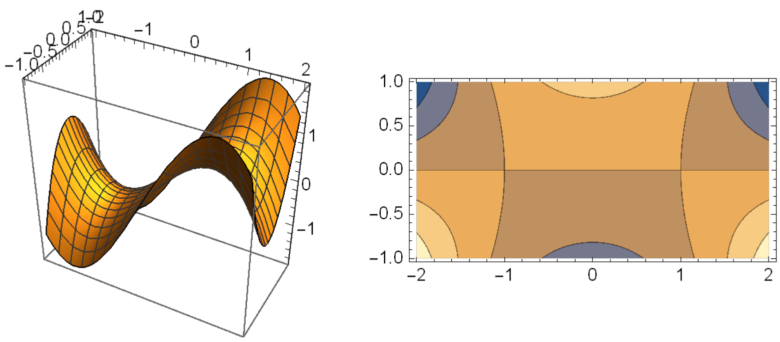
$$\begin{cases} \text{steepest descent directions}: & \theta = -\frac{\alpha}{2} + \frac{\pi}{2}, \ -\frac{\alpha}{2} + \frac{3\pi}{2} \\ \text{steepest ascent directions}: & \theta = -\frac{\alpha}{2}, \ -\frac{\alpha}{2} + \pi \end{cases}$$



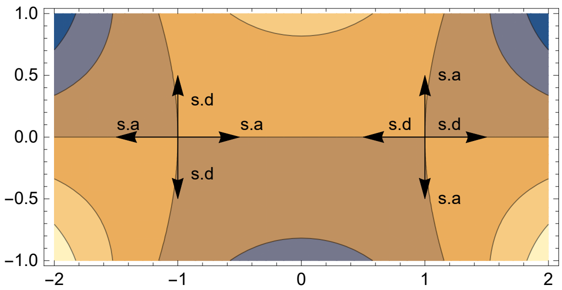
Consider $\phi(z) = z - z^3/3$. Its 1st derivative is $1 - z^2$ and 2nd derivative is $-2z$. So, there exist two simple saddle points at $z_0 = 1$ and $z_0 = -1$

$$\begin{cases} \phi^{(2)}(-1) = 2 = 2\,e^{i\,0} & \alpha = 0, \ \text{steepest descent directions}: \frac{\pi}{2}, \ \frac{3\pi}{2} \\ \phi^{(2)}(1) = -2 = 2\,e^{i\,\pi} & \alpha = \pi, \ \text{steepest descent directions}: 0, \ \pi \end{cases}$$

```
GraphicsGrid[
 {{Plot3D[{Re[(x + i y) - (x + i y)^3 / 3]}, {x, -2, 2}, {y, -1, 1}, BoxRatios → Automatic],
   ContourPlot[Im[(x + i y) - (x + i y)^3 / 3],
    {x, -2, 2}, {y, -1, 1}, AspectRatio → Automatic]}}]
```



Note that the right hand side figure is representing $\text{Im}[\phi(t)] = \text{const}$. Since the directions of steepest descent and steepest ascent are define by $\text{Im}[\phi(t)] = \text{const}$, the contour around the $z = -1$ and $z = 1$ show that the steepest descent and steepest ascent directions.



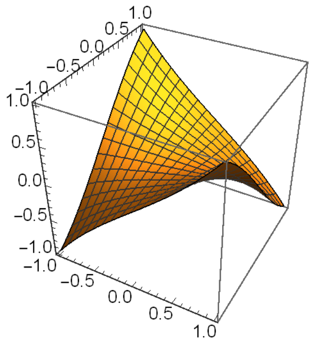
More examples always make you feel better.

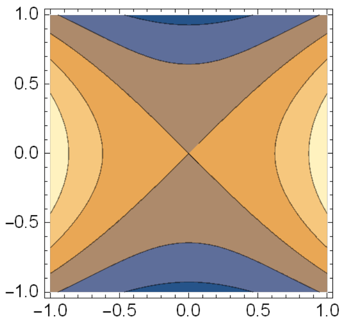
$$\phi(z) = i \cosh z \longrightarrow \begin{cases} \phi^{(1)}(z) = i \sinh z & \phi^{(1)}(0) = 0 \\ \phi^{(2)}(z) = i \cosh z & \phi^{(2)}(0) = i = e^{\frac{i\pi}{2}} \end{cases} \quad i \cosh z \text{ has simple saddle point at } z = 0$$

and $\alpha = \pi/2$

$$\begin{cases} \text{steepest descent directions}: & \theta = \frac{\pi}{4}, \frac{5\pi}{4} \\ \text{steepest ascent directions}: & \theta = -\frac{\pi}{4}, \frac{3\pi}{4} \end{cases}$$

```
GraphicsGrid[
 {{Plot3D[{Re[ⅈ Cosh[x + ⅈ y]]}, {x, -1, 1}, {y, -1, 1}, BoxRatios → Automatic],
   ContourPlot[Im[ⅈ Cosh[x + ⅈ y]], {x, -1, 1}, {y, -1, 1}, AspectRatio → Automatic]}}]
```



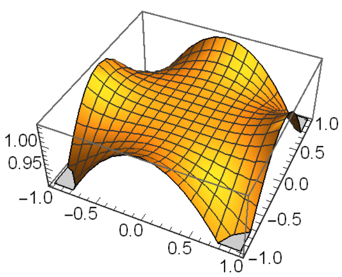


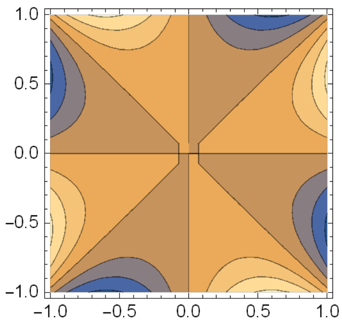
$$\phi(z) = \cosh z - \frac{z^2}{2} \longrightarrow \begin{cases} \phi^{(1)}(z) = \sinh z - z & \phi^{(1)}(0) = 0 \\ \phi^{(2)}(z) = \cosh z - 1 & \phi^{(2)}(0) = 0 \\ \phi^{(3)}(z) = \sinh z & \phi^{(3)}(0) = 0 \\ \phi^{(4)}(z) = \cosh z & \phi^{(4)}(0) = 1 = e^{i\,0} \end{cases} \quad \cosh z - \frac{z^2}{2} \text{ has saddle point of order}$$

3 at $z = 0$ and $\alpha = 0$

$$\begin{cases} \text{steepest descent directions}: & \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \\ \text{steepest ascent directions}: & \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{4} \end{cases}$$

```
GraphicsGrid[
 {{Plot3D[{Re[Cosh[x + ⅈ y] - (x + ⅈ y)^2 / 2]}, {x, -1, 1}, {y, -1, 1}], ContourPlot[
    Im[Cosh[x + ⅈ y] - (x + ⅈ y)^2 / 2], {x, -1, 1}, {y, -1, 1}, AspectRatio → Automatic]}}]
```





Is everything okay? These two are important to construct the steepest descent method.

$$I(k) = \int_C f(z)\, e^{k\,\phi(z)}\, dz = e^{i\,k\,v} \int_{C'} f(z)\, e^{k\,u}\, dz$$

From the basic idea of steepest descent method, notice that because of exponential effect of Laplace integral, its evaluation depends only on the small neighborhood of **critical points**.

$$\text{Critical points}\begin{cases}\text{the points such that } \phi'(z)=0\\ \text{singular points of the integrand}\\ \text{endpoints}\end{cases}$$

By summing all contributions from these points, we can obtain generalized Laplace type integral.

Steepest descent method is called 'steepest descent' method for the reason of using the steepest descent directions for deformed path C'. As you can see in the examples above, the paths of steepest descent go through a saddle point $z = z_0$ such that $\phi'(z_0) = 0$.

<Theorem 18> Laplace's method for complex contour

- Consider the integral $I(k) = \int_C f(z)\, e^{k\,\phi(z)}\, dz$ and suppose that z_0 is a saddle point of order $(n-1)$ and also

$$\begin{cases}\phi(z)-\phi(z_0) \sim \frac{(z-z_0)^n}{n!}\phi^{(n)}(z_0), & \phi^{(n)}(z_0) = \left|\phi^{(n)}(z_0)\right| e^{i\alpha}\\ f(z) \sim f_0\,(z-z_0)^{\beta-1}, & \mathrm{Re}[\beta] > 0\end{cases}$$

Then

$$I(k) \sim \frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n}\,\frac{e^{k\,\phi(z_0)}\,\Gamma(\beta/n)}{\left(k\left|\phi^{(n)}(z_0)\right|\right)^{\beta/n}}$$

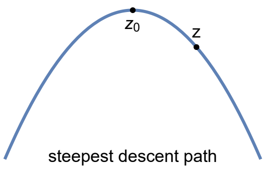
proof

From

$$I(k) = \int_C f(z)\, e^{k\,\phi(z)}\, dz = \int_{C'} f(z)\, e^{k\,\phi(z)}\, dz,$$

since C' is a path of steepest descent, we can make a change of variables ; $-t = \phi(z) - \phi(z_0)$ (note that the imaginary part of $\phi(z)$ on the path of steepest descent is constant and $|\phi(z_0)| > |\phi(z)|$, so that the t is real and positive.). Then

$$\begin{cases}-t = \frac{(z-z_0)^n}{n!}\phi^{(n)}(z_0), & |z-z_0| = t^{1/n}\left(\frac{n!}{\left|\phi^{(n)}(z_0)\right|}\right)^{1/n}\\ -dt = \phi'(z)\, dz & dz = \frac{dt}{-\phi'(z)}\end{cases}$$



On the steepest descent path, the saddle point is a highest point, namely, deformed integration range starts from 0 to $a < \infty$.

$$\begin{aligned}I(k) = \int_{C'} f(z)\, e^{k\,\phi(z)}\, dz &= \int_0^a f(z)\, e^{k\,[\phi(z_0)-t]}\,\frac{dt}{-\phi'(z)}\\ &= e^{k\,\phi(z_0)}\int_0^a \left[\frac{f(z)}{-\phi'(z)}\right] e^{-k\,t}\, dt\\ &\sim e^{k\,\phi(z_0)}\int_0^\infty \left[\frac{f(z)}{-\phi'(z)}\right] e^{-k\,t}\, dt\end{aligned}$$

As $k \to \infty$, the main contribution occurs at neighborhood of $z = 0$. So,
$e^{k\,\phi(z_0)}\int_0^a \left[\frac{f(z)}{-\phi'(z)}\right] e^{-k\,t}\, dt \sim e^{k\,\phi(z_0)}\int_0^\infty \left[\frac{f(z)}{-\phi'(z)}\right] e^{-k\,t}\, dt$

By assumptions,

$$\begin{cases} \phi(z) - \phi(z_0) \sim \frac{(z-z_0)^n}{n!}\,\phi^{(n)}(z_0), & \phi'(z) \sim \frac{(z-z_0)^{n-1}}{(n-1)!}\,\phi^{(n)}(z_0) \\ f(z) \sim f_0\,(z-z_0)^{\beta-1}, & \end{cases}, \left[\frac{f(z)}{-\phi'(z)}\right] \sim \left[\frac{f_0\,(z-z_0)^{\beta-1}}{-\frac{(z-z_0)^{n-1}}{(n-1)!}\,\phi^{(n)}(z_0)}\right]$$

Also,

$$\begin{cases} (z-z_0) = |z-z_0|\, e^{i\theta} & |z-z_0| = t^{1/n}\left(\frac{n!}{|\phi^{(n)}(z_0)|}\right)^{1/n} \\ \phi^{(n)}(z_0) = \left|\phi^{(n)}(z_0)\right| e^{i\alpha} & \end{cases}$$

Hence,

$$\begin{aligned} e^{k\phi(z_0)} \int_0^\infty \left[\frac{f(z)}{-\phi'(z)}\right] e^{-kt}\, dt &= e^{k\phi(z_0)} \int_0^\infty \left[\frac{(n-1)!\, f_0\,(z-z_0)^{\beta-n}}{-\phi^{(n)}(z_0)}\right] e^{-kt}\, dt \\ &= e^{k\phi(z_0)} \int_0^\infty \left[\frac{(n-1)!\, f_0 \left(t^{1/n}\left(\frac{n!}{|\phi^{(n)}(z_0)|}\right)^{1/n} e^{i\theta}\right)^{\beta-n}}{-\left|\phi^{(n)}(z_0)\right| e^{i\alpha}}\right] e^{-kt}\, dt \\ &= e^{k\phi(z_0)} \int_0^\infty \left[\frac{f_0\,(n!)^{\beta/n}\, e^{i[\beta\theta-(\theta n+\alpha)]}}{-n\left|\phi^{(n)}(z_0)\right|^{\beta/n}}\right] t^{\beta/n-1}\, e^{-kt}\, dt \end{aligned}$$

recall that θ is the steepest descent directions.

$$\left\{ \text{steepest descent directions}: \quad \theta = -\frac{\alpha}{n} + (2m+1)\frac{\pi}{n}, \quad m = 0, 1, \cdots, n-1 \right\}$$

$$\theta n + \alpha = (2m+1)\pi = 2m\pi + \pi, \quad e^{i(\theta n+\alpha)} = e^{i(2m\pi+\pi)} = 1\, e^{i\pi} = -1$$

$$\begin{aligned} e^{k\phi(z_0)} \int_0^\infty \left[\frac{f_0\,(n!)^{\beta/n}\, e^{i[\beta\theta-(\theta n+\alpha)]}}{-n\left|\phi^{(n)}(z_0)\right|^{\beta/n}}\right] t^{\beta/n-1}\, e^{-kt}\, dt &= e^{k\phi(z_0)} \int_0^\infty \left[\frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n\left|\phi^{(n)}(z_0)\right|^{\beta/n}}\right] t^{\beta/n-1}\, e^{-kt}\, dt \\ &= \frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n\left|\phi^{(n)}(z_0)\right|^{\beta/n}}\, e^{k\phi(z_0)} \int_0^\infty t^{\beta/n-1}\, e^{-kt}\, dt \\ &= \frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n\left|\phi^{(n)}(z_0)\right|^{\beta/n}}\, e^{k\phi(z_0)} \int_0^\infty \left(\frac{\tau}{k}\right)^{\beta/n-1} e^{-\tau}\, \frac{1}{k}\, d\tau \\ &= \frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n\left(k\left|\phi^{(n)}(z_0)\right|\right)^{\beta/n}}\, e^{k\phi(z_0)} \int_0^\infty \tau^{\beta/n-1}\, e^{-\tau}\, d\tau \\ &= \frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n\left(k\left|\phi^{(n)}(z_0)\right|\right)^{\beta/n}}\, e^{k\phi(z_0)}\, \Gamma(\beta/n) \end{aligned}$$

$$I(k) \sim \frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n} \frac{e^{k\phi(z_0)}\,\Gamma(\beta/n)}{\left(k\left|\phi^{(n)}(z_0)\right|\right)^{\beta/n}} \qquad \blacksquare$$

<Remark 6.3>

■ Basic steps for steepest descent method

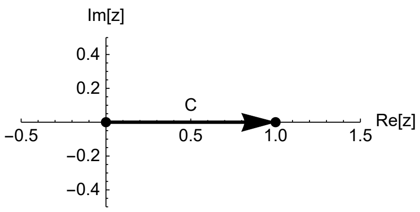
1. Identify all critical points of the integrand
2. Determine the paths of steepest descent, C_s
3. Deform the original contour C onto one or more paths of steepest descent C_s or paths that are asymptotically equivalent to C_s
4. Evaluate the asymptotic expansions by using Laplace's method, Watson's lemma, integration by parts.

Exercise.
Find the complete asymptotic expansion of $I(k) = \int_0^1 \ln t\, e^{ikt}\, dt$ as $k \to \infty$

■ Method of stationary phase → Fail ; $\phi'(t) = [(d/dt)\, t] = 1$ is not vanishes in the interval $[0, 1]$

- Integration by parts → Fail ; $\ln t$ diverges at $t = 0$
- Steepest descent method

$I(k) = \int_0^1 \ln t \, e^{k i t} \, dt \longrightarrow \int_C \ln z \, e^{i k z} \, dz$ where $\phi(z) = i z = i(x + i y) = -y + i x$

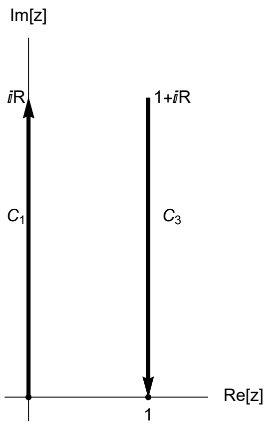


1. Critical points:

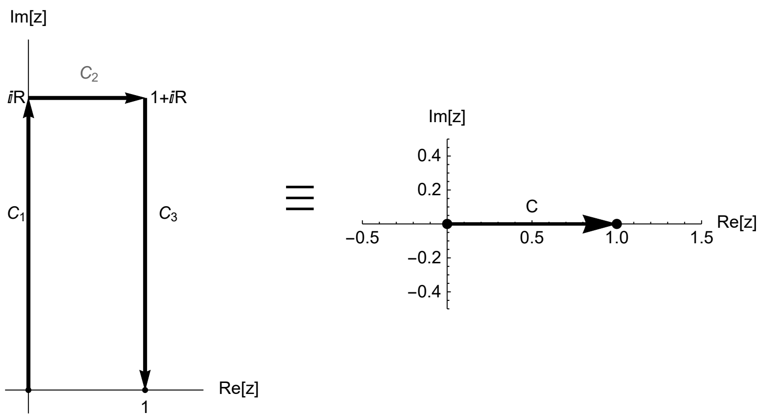
$\phi(z)$ has no saddle point ($\phi'(z) = i$). Endpoints are the dominant contribution of the integral.

2. Path of steepest descent from two endpoints $z = 0$ and $z = 1$:

$\text{Im}[\phi(z)] = x = \text{const}$ (steepest) && $y > 0$ (descent)



3. Deform the contour C onto paths of steepest descent :
By Cauchy’s theorem,



4. Evaluate the integral :

$$
\begin{aligned}
(k) = \int_0^1 \ln t\, e^{ikt}\, dt &= \int_C \ln z\, e^{ikz}\, dz \\
&= \int_{C'} \ln z\, e^{ikz}\, dz \\
&= \int_{C_1+C_2+C_3} \ln z\, e^{ikz}\, dz \\
&= \int_0^R \ln\left(r e^{\frac{i\pi}{2}}\right) e^{-kr}\, i\, dr + \int_0^1 \ln(x + iR)\, e^{ik(x+iR)}\, dx + \int_r^0 \ln\left(1 + r e^{\frac{i\pi}{2}}\right) e^{ik\left(1+re^{\frac{i\pi}{2}}\right)}\, i\, d
\end{aligned}
$$

Thanks to Jordan's lemma, as $R \to \infty$, second integral tend to zero

$$
\begin{aligned}
&= \int_0^\infty \ln(ir)\, e^{-kr}\, i\, dr + 0 + \int_\infty^0 \ln(1 + ir)\, e^{ik(1+ir)}\, i\, dr \\
&= i \int_0^\infty \ln(ir)\, e^{-kr}\, dr - i e^{ik} \int_0^\infty \ln(1 + ir)\, e^{-kr}\, dr
\end{aligned}
$$

letting $s = k\,r$, $(r = s/k,\ ds = k\,dr)$

$$
\left\{
\begin{aligned}
i \int_0^\infty \ln(ir)\, e^{-kr}\, dr \quad &= \frac{i}{k} \int_0^\infty \ln\left(\frac{e^{i\pi/2} s}{k}\right) e^{-s}\, ds = \frac{i}{k} \int_0^\infty \left(\ln\left(e^{i\pi/2}\right) + \ln s - \ln k\right) e^{-s}\, ds \\
&= \frac{i}{k} \int_0^\infty \left(\frac{i\pi}{2} - \ln k + \ln s\right) e^{-s}\, ds = \frac{i}{k}\left(\frac{i\pi}{2} - \ln k\right)(1) + \frac{i}{k} \int_0^\infty \ln s\, e^{-s}\, ds \\
i e^{ik} \int_0^\infty \ln(1 + ir)\, e^{-kr}\, dr \quad &= i e^{ik} \int_0^\infty \left[\frac{ir}{1} - \frac{(ir)^2}{2} + \cdots\right] e^{-kr}\, dr \sim i e^{ik} \int_0^R \left[\frac{ir}{1} - \frac{(ir)^2}{2} + \cdots\right] e^{-kr}\, dr \\
&\sim \frac{i e^{ik}}{k} \int_0^\infty \left[\frac{i\tau}{k} - \frac{(i\tau)^2}{2k^2} + \cdots\right] e^{-\tau}\, dr = \frac{i e^{ik}}{k}\left(\frac{i}{k}\Gamma(2) - \frac{1}{2}\left(\frac{i}{k}\right)^2 \Gamma(3) + \cdots\right) \\
&= i e^{ik} \sum_{n=1}^\infty \frac{(-i)^n\, \Gamma(n+1)}{n\, k^{n+1}} = i e^{ik} \sum_{n=1}^\infty \frac{(-i)^n\, (n-1)!}{k^{n+1}}
\end{aligned}
\right.
$$

```
∫_0^∞ Log[s] ⅇ^-s ⅆs
```

```
-EulerGamma
```

Hence,

$$
\begin{aligned}
&i \int_0^\infty \ln(ir)\, e^{-kr}\, dr - i e^{ik} \int_0^\infty \ln(1 + ir)\, e^{-kr}\, dr \\
&= \frac{i}{k}\left(\frac{i\pi}{2} - \ln k\right) + \frac{i}{k}(-\gamma) + i e^{ik} \sum_{n=1}^\infty \frac{(-i)^n\, (n-1)!}{k^{n+1}} \\
&= -\frac{i \ln k}{k} - \frac{i\gamma + \pi/2}{k} + i e^{ik} \sum_{n=1}^\infty \frac{(-i)^n\, (n-1)!}{k^{n+1}}
\end{aligned}
$$

$$
I(k) = \int_0^1 \ln t\, e^{kit}\, dt \sim -\frac{i \ln k}{k} - \frac{i\gamma + \pi/2}{k} + i e^{ik} \sum_{n=1}^\infty \frac{(-i)^n\, (n-1)!}{k^{n+1}}, \quad k \to \infty
$$

```
Series[∫_0^1 Log[t] ⅇ^(ⅈ k t) ⅆt, {k, ∞, 3}]
```

$$
\mathrm{Cos}\left[k + O\left[\frac{1}{k}\right]^4\right] \left(-\frac{i}{k^3} + O\left[\frac{1}{k}\right]^5\right) + \left(-\frac{i\left(\mathrm{EulerGamma} - \frac{i\pi}{2} + \mathrm{Log}[k]\right)}{k} + O\left[\frac{1}{k}\right]^4\right) + \mathrm{Cos}\left[k + O\left[\frac{1}{k}\right]^4\right] \left(\left(\frac{1}{k}\right)^2 - \frac{2}{k^4} + O\left[\frac{1}{k}\right]^5\right) + \left(\left(\frac{1}{k}\right)^3 + O\left[\frac{1}{k}\right]^5\right) \mathrm{Sin}\left[k + O\left[\frac{1}{k}\right]^4\right] + \left(\frac{i}{k^2} - \frac{2i}{k^4} + O\left[\frac{1}{k}\right]^5\right) \mathrm{Sin}\left[k + O\left[\frac{1}{k}\right]^4\right]
$$

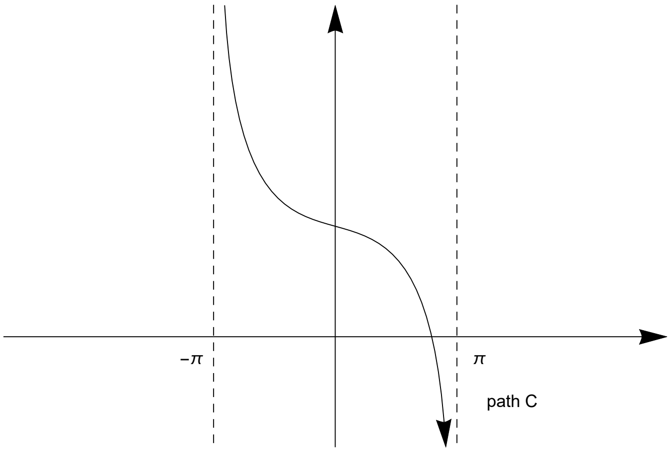
<Remark 6.4>

- $\int_0^\infty \ln t\, e^{-t}\, dt = \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right) \frac{1}{t}\, dt = -\gamma = -0.577216\ldots$

Exercise.

Find the asymptotic behavior of the Hankel function

$H_\nu^{(1)}(k) = \frac{1}{\pi}\int_C e^{i\,k\cos z}\, e^{i\,\nu(z-\pi/2)}\, dz$, as $k \to \infty$

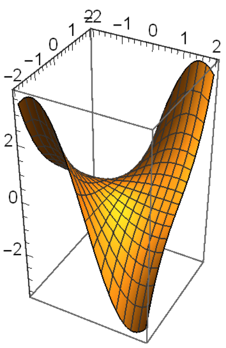


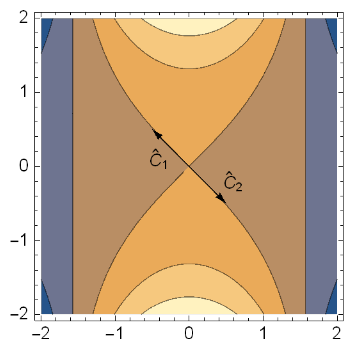
$$H_\nu^{(1)}(k) = \frac{1}{\pi}\int_C e^{i\,k\cos z}\, e^{i\,\nu(z-\pi/2)}\, dz = \frac{1}{\pi}\int_C f(z)\, e^{k\,\phi(z)}\, dz \quad \begin{cases} f(z) = \frac{1}{\pi} e^{i\,\nu(z-\pi/2)} \\ \phi(z) = i\cos z \end{cases}$$

$\phi(z) = i\cos z,\ \ \phi'(z) = -i\sin z,$

$\phi''(z) = -i\cos z\ \left(\phi'(0) = 0,\ \ \phi''(0) = -i = e^{\frac{3\pi i}{2}},\ \text{where } z=0 \text{ is a simple saddle point}\right)$

The directions of steepest descent: $\theta = -\frac{\pi}{4}, \frac{3\pi}{4}$. Let the contour $\hat{C}_1$ has the direction $\theta = \frac{3\pi}{4}$ and $\hat{C}_2$ has the direction $\theta = -\frac{\pi}{4}$.





Then, $\int_C \sim -\int_{\hat{C}_1} + \int_{\hat{C}_2}$

We can utilize the Laplace's method for complex contours: $I(k) \sim \frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n}\ \frac{e^{k\,\phi(z_0)}\,\Gamma(\beta/n)}{\left(k\,|\phi^{(n)}(z_0)|\right)^{\beta/n}}$ where

$$z_0 = 0,\ \ n = 2,\ \ f(z) \sim \frac{e^{-i\,\nu\,\pi/2}}{\pi}(z-0)^{1-1}\ \begin{cases} \beta = 1, \\ f_0 = \frac{e^{-i\,\nu\,\pi/2}}{\pi} \end{cases},\ \ \phi(0) = i,\ \ \left|\phi^{(2)}(0)\right| = 1,\ \ \theta = -\frac{\pi}{4}, \frac{3\pi}{4}$$

- $-\int_{\hat{C}_1} \sim -\left[\frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n}\ \frac{e^{k\,\phi(z_0)}\,\Gamma(\beta/n)}{\left(k\,|\phi^{(n)}(z_0)|\right)^{\beta/n}}\right] =$

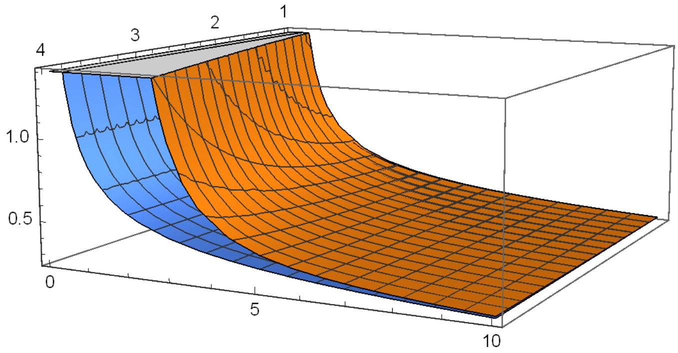
$$-\left[\frac{e^{-i\,\nu\,\pi/2}\sqrt{2}\, e^{3\pi i/4}}{2\pi}\ \frac{e^{i\,k}\,\Gamma(1/2)}{(k)^{1/2}}\right] = e^{-i\,\pi}\left[\sqrt{\frac{1}{2\pi k}}\ e^{i(k-\nu\,\pi/2+3\,\pi/4)}\right] = \sqrt{\frac{1}{2\pi k}}\ e^{i(k-\nu\,\pi/2-\pi/4)}$$

- $\int_{\hat{C}_2} \sim \left[\frac{f_0\,(n!)^{\beta/n}\, e^{i\beta\theta}}{n}\ \frac{e^{k\,\phi(z_0)}\,\Gamma(\beta/n)}{\left(k\,|\phi^{(n)}(z_0)|\right)^{\beta/n}}\right] = \left[\frac{e^{-i\,\nu\,\pi/2}\sqrt{2}\, e^{-\pi i/4}}{2\pi}\ \frac{e^{i\,k}\,\Gamma(1/2)}{(k)^{1/2}}\right] = \sqrt{\frac{1}{2\pi k}}\ e^{i(k-\nu\,\pi/2-\pi/4)}$

Therefore, the leading term is

$$-\int_{\hat{C}_1} + \int_{\hat{C}_2} \sim \sqrt{\frac{2}{\pi k}}\; e^{i(k - v\,\pi/2 - \pi/4)},\;\; k \to \infty$$

```
Plot3D[{Abs[HankelH1[v, k]], Abs[Sqrt[2/(π k)] E^(I (k - v π/2 - π/4))]}, {v, 1, 4}, {k, 0, 10}]
```



As $k \to 10$, two plates(yellow one: Hankel, blue one: Leading term) are getting closer and closer.

Steepest descent curves: $\mathrm{Im}[\phi(z)] = \mathrm{Im}[\phi(0)] = 1 = \cos(x)\cosh(y)$

Note that the saddle point $z = 0$, $\cos(x) \sim \left(1 - \frac{x^2}{2} + \cdots\right)$ and $\cosh(y) \sim \left(1 + \frac{y^2}{2} + \cdots\right)$.

$\cos(x)\cosh(y) \sim \left(1 - \frac{x^2}{2} + \cdots\right)\left(1 + \frac{y^2}{2} + \cdots\right) = 1$ or $\frac{y^2}{2} - \frac{x^2}{2} \approx 0$ (steepest descent one is $y = -x$)

* Want to get higher order terms? See below ;

Let $\phi(z) - \phi(z_0) = -t$, $(t > 0,\; t \in \mathbb{R})$.

$$\begin{aligned}
& i\cos z - i\cos 0 = -t \\
& \Rightarrow i(\cos z - 1) = -t \\
& \Rightarrow \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) = i\,t \\
& \Rightarrow \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \cdots\right) = -i\,t \\
& \Rightarrow \frac{z^2}{2}\left(1 - \frac{z^2}{12} + \cdots\right) = e^{-i\,\pi/2}\,t
\end{aligned}$$

Solving for z as a function of t recursively,

$$\begin{aligned}
z &= \left[2\,e^{-i\,\pi/2}\,t\right]^{1/2}\left(1 - \frac{z^2}{12} + \cdots\right)^{-1/2} \\
&= \sqrt{2}\;e^{-i\,\pi/4}\,t^{1/2}\left(1 + \frac{z^2}{24} + \cdots\right) \\
&= \sqrt{2}\;e^{-i\,\pi/4}\,t^{1/2}\left(1 + \frac{\left(\sqrt{2}\;e^{-i\,\pi/4}\,t^{1/2}\left(1 + \frac{z^2}{24} + \cdots\right)\right)^2}{24} + \cdots\right) \\
&= \sqrt{2}\;e^{-i\,\pi/4}\,t^{1/2} + \frac{\sqrt{2}\;e^{-i\,\pi/4}\,t^{1/2}}{24}\left(2\,e^{-i\,\pi/2}\,t\left(1 + \frac{z^2}{12} + \cdots\right)\right) + \cdots \\
z(t) &= \sqrt{2}\;e^{-i\pi/4}\,t^{1/2} + \frac{\sqrt{2}}{12}\,e^{-i3\pi/4}\,t^{3/2} + \cdots
\end{aligned}$$

The integral

$$H_v^{(1)} = \frac{1}{\pi}\int_C e^{i\,v(z-\pi/2)}\,e^{k\,\phi(z)}\,dz = \frac{1}{\pi}\int_C e^{i\,v(z-\pi/2)}\,e^{k\,\phi(0)}\,e^{k\,[\phi(z)-\phi(0)]}\,dz$$

is asymptotically given by

$$\frac{1}{\pi}\int_{C_s} e^{i\,v(z-\pi/2)}\,e^{i\,k}\,e^{k\,[-t]}\,dz = \frac{e^{i\,k}\,e^{-i\,v\,\pi/2}}{\pi}\int_{C_s} e^{i\,v\,z(t)}\,e^{-k\,t}\,\frac{dz}{dt}\,dt$$

(note that $-\int_{\hat{C}_1} = -\int_{t=\infty}^{0}$, $\int_{\hat{C}_2} = \int_{t=0}^{\infty}$, namely, $\int_{C_s} = -\int_{t=\infty}^{0} + \int_{t=0}^{\infty} = 2\int_{t=0}^{\infty}$)

$$\sim \frac{2\,e^{i k}\,e^{-i \nu \pi/2}}{\pi} \int_0^\infty \left(1 + i\,\nu\,z + \frac{(i\,\nu\,z)^2}{2!} + \cdots\right) e^{-k t}\,\frac{dz}{dt}\,dt$$

$$= \frac{2\,e^{i k}\,e^{-i \nu \pi/2}}{\pi} \int_0^\infty \left(1 + i\,\nu\,\sqrt{2}\;e^{-i\pi/4}\,t^{1/2} + \frac{\left(-\nu^2\,2\,e^{-i\pi/2}\right)t}{2!} + \cdots\right) e^{-k t}\left(\frac{\sqrt{2}}{2}\,e^{-i\pi/4}\,t^{-1/2} + \frac{\sqrt{2}}{8}\,e^{-i 3\pi/4}\,t^{1/2} + \cdots\right) dt$$

$$= \frac{2\,e^{i k}\,e^{-i \nu \pi/2}}{\pi} \int_0^\infty \left(\frac{\sqrt{2}\;e^{-i\pi/4}}{2}\,t^{-1/2} + i\,\nu\,e^{-i\pi/2} + \frac{(1/4-\nu^2)\sqrt{2}\;e^{-i3\pi/4}}{2}\,t^{1/2} + \cdots\right) e^{-k t}\,dt$$

$$= \frac{2\,e^{i k}\,e^{-i \nu \pi/2}}{\pi} \int_0^\infty \left(c_0\,t^{-1/2} + c_1 + c_2\,t^{1/2} + \cdots\right) e^{-k t}\,dt$$

$$= \frac{2\,e^{i k}\,e^{-i \nu \pi/2}}{\pi} \int_0^\infty \left(c_0(\tau/k)^{-1/2} + c_1 + c_2(\tau/k)^{1/2} + \cdots\right) e^{-\tau}\,\frac{d\tau}{k}$$

$$= \frac{2\,e^{i k}\,e^{-i \nu \pi/2}}{\pi} \int_0^\infty \left(\frac{c_0}{k^{1/2}}\,\tau^{-1/2} + \frac{c_1}{k} + \frac{c_2}{k^{3/2}}\,\tau^{1/2} + \cdots\right) e^{-\tau}\,d\tau$$

$$= \frac{2\,e^{i k}\,e^{-i \nu \pi/2}}{\pi} \left(\frac{c_0}{k^{1/2}}\,\Gamma(1/2)\,\tau^{-1/2} + \frac{c_1}{k}\,\Gamma(1) + \frac{c_2}{k^{3/2}}\,\Gamma(3/2)\,\tau^{1/2} + \cdots\right)$$

$$= \frac{2\,e^{i k}\,e^{-i \nu \pi/2}}{\pi} \left(\frac{\sqrt{2}\;e^{-i\pi/4}}{2\,k^{1/2}}\,\sqrt{\pi}\;\tau^{-1/2} + \frac{i\,\nu\,e^{-i\pi/2}}{k} + \frac{(1/4-\nu^2)\sqrt{2}\;e^{-i3\pi/4}}{2\,k^{3/2}}\,\frac{\sqrt{\pi}}{2}\,\tau^{1/2} + \cdots\right)$$

$$= \frac{2\,e^{i k}\,e^{-i \nu \pi/2}}{\pi} \left(\frac{\sqrt{2\,\pi}\;e^{-i\pi/4}}{2\,k^{1/2}}\,\tau^{-1/2} + \frac{\nu}{k} + \frac{(1/4-\nu^2)\sqrt{2\,\pi}\;e^{-i3\pi/4}}{4\,k^{3/2}}\,\tau^{1/2} + \cdots\right) \quad (\Gamma(n+1) = n\,\Gamma(n))$$

Exercise.
Evaluate $I(k) = \int_{-\infty}^{\infty} e^{i k t}(1+t^2)^{-k}\,dt,\ \ k \to \infty$

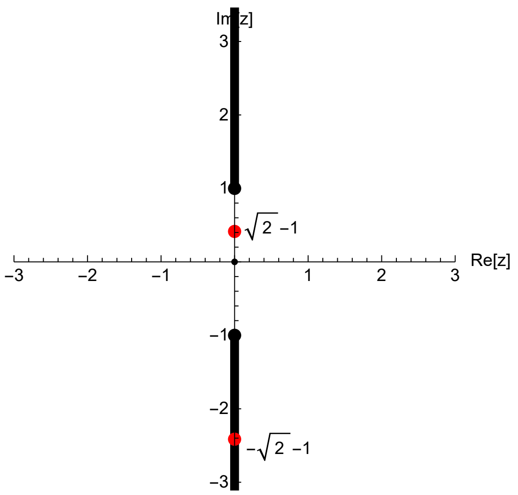
$$I(k) = \int_{-\infty}^{\infty} e^{i k t}(1+t^2)^{-k}\,dt = \int_{-\infty}^{\infty} e^{i k t}\,e^{-k \ln(1+t^2)}\,dt = \int_{-\infty}^{\infty} e^{k\left(i t - \ln(1+t^2)\right)}\,dt = \int_{-\infty}^{\infty} f(t)\,e^{k\,\phi(t)}\,dt$$

(where $\phi(t) = i\,t - \ln(1+t^2)$, $f(t) = 1$)

Then, $\phi(z) = i\,z - \ln(1+z^2)$, $\phi'(z) = i - \frac{2z}{1+z^2}$, $\phi''(z) = -\frac{2}{1+z^2} + \frac{4z^2}{(1+z^2)^2} = \frac{2z^2-2}{(1+z^2)^2}$

$$\begin{cases} \phi'\left(i\left(\pm\sqrt{2}-1\right)\right) = 0 \\ \phi''\left(i\left(\pm\sqrt{2}-1\right)\right) \neq 0 \end{cases} \quad i\left(\sqrt{2}-1\right),\ i\left(-\sqrt{2}-1\right) \text{ are simple saddle point.}$$

Let's take the branch cut for $\ln(1+z^2)$ to be from $\pm i$ to $\pm i\,\infty$



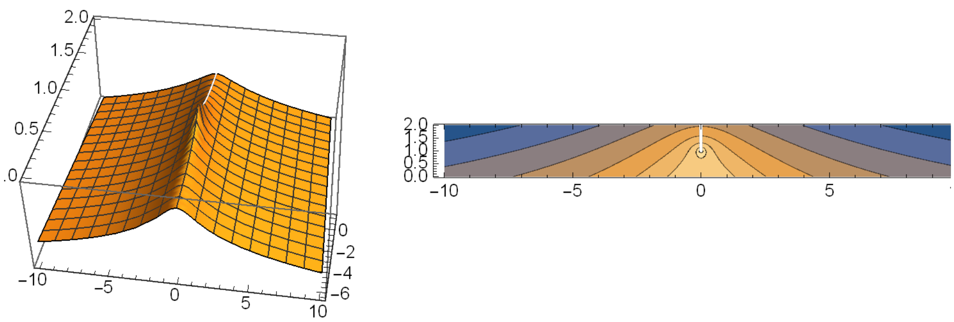
Since $i\left(-\sqrt{2}-1\right)$ is on the branch cut, we cannot use that point. So let's consider the point $z_0 = i\,c$, $c = \sqrt{2}-1$

$$\phi''(i\,c)=\frac{-2c^2}{(1-c^2)^2}=\left(2+\sqrt{2}\right)e^{i\pi}\ \ (\alpha=\pi,\ \text{steepest descent direction}: 0,\ \pi)$$

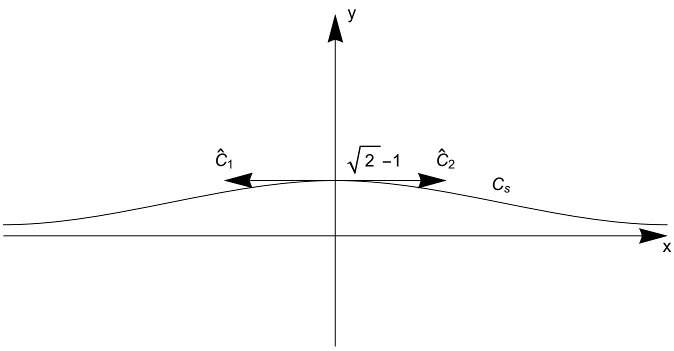
$$\phi(i\,c)=-c-\ln(2\,c)=1-\sqrt{2}-\ln\left(2\sqrt{2}-2\right)$$

Now deform the original contour ($-\infty\sim\infty$) so that it passes through $z_0=i\,c$. The deformed contour is given by $\mathrm{Im}[\phi(z)]=\mathrm{Im}[\phi(i\,c)]=\mathrm{Im}\left[1-\sqrt{2}-\ln\left(2\sqrt{2}-2\right)\right]=0$ or $x-\theta=0$, where $\theta=\arg(1+z^2)$.
$\left(\text{note that } \phi(z)=i\,z-\ln(1+z^2)=i(x+i\,y)-\ln\left(|1+z^2|\,e^{i\arg(1+z^2)}\right),\ \ \mathrm{Im}[\phi(z)]=x-\arg(1+z^2)\right)$

```
In[21]:= GraphicsGrid[{{Plot3D[Re[(ⅈ (x + ⅈ y) - Log[1 + (x + ⅈ y)^2])], {x, -10, 10}, {y, 0, 2}],
    ContourPlot[Re[(ⅈ (x + ⅈ y) - Log[1 + (x + ⅈ y)^2])],
     {x, -10, 10}, {y, 0, 2}, AspectRatio → Automatic]}}]
```



The deformed contour C_s approaches the original contour as $|x|\to\infty$



Use Laplace's method for complex contours.

$$z_0=i\,c,\ \ n=2,\ \ |\phi''(i\,c)|=\left(2+\sqrt{2}\right),\ \ f_0=1,\ \beta=1,\ \ \phi(i\,c)=-c-\ln(2\,c)$$

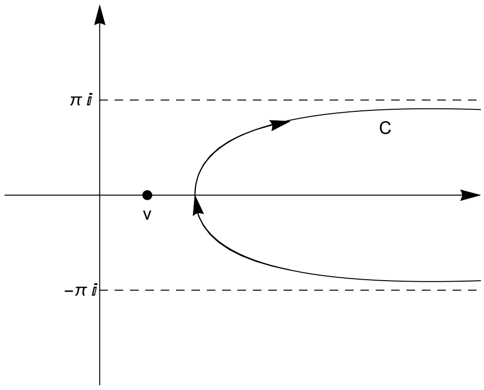
$$\int_{C_s}=-\int_{\hat{C}_1}+\int_{\hat{C}2}$$

$$=-\frac{(2)^{1/2}e^{i\pi}}{2}\,\frac{e^{k(-c-\ln(2c))}\,\Gamma(1/2)}{\left(k\left(2+\sqrt{2}\right)\right)^{1/2}}+\frac{(2)^{1/2}}{2}\,\frac{e^{k(-c-\ln(2c))}\,\Gamma(1/2)}{\left(k\left(2+\sqrt{2}\right)\right)^{1/2}}$$

$$=\frac{\sqrt{2\pi}\,e^{k(-c-\ln(2c))}}{\left(k\left(2+\sqrt{2}\right)\right)^{1/2}}=\sqrt{\frac{2\pi}{\left(2+\sqrt{2}\right)k}}\,e^{-k\left(\sqrt{2}-1\right)}\left(2\sqrt{2}-2\right)^{-k}$$

Exercise.

Evaluate $I(k) = \int_C e^{k(\operatorname{sech} v \sinh z - z)}\, dz,\ k \to \infty,\ v \neq 0,$ where C is so-called Sommerfeld contour illustrated in below and v is a fixed real positive number.



$f(z) = 1,\ \phi(z) = \operatorname{sech} v \sinh z - z,\ \phi'(z) = \operatorname{sech} v \cosh z - 1,$

$\phi''(z) = \operatorname{sech} v \sinh z,\ \ (\phi'(v) = 0,\ \phi''(v) \neq 0)$

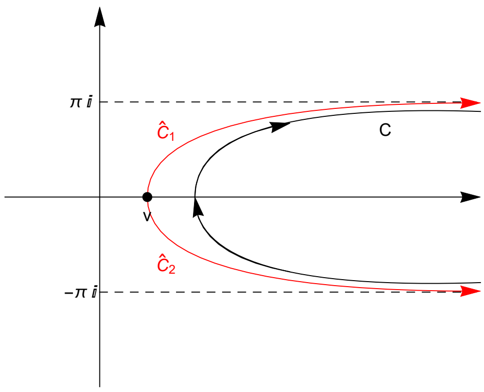
$z = v$ is a simple saddle point and from $\phi''(v) = \tanh v = \tanh v\, e^{i 0}$,

$\alpha = 0$ (steepest descent direction : $\frac{\pi}{2}, \frac{3\pi}{2}$).

$$\phi(x + i y) = \operatorname{sech} v \sinh(x + i y) - (x + i y) = \operatorname{sech} v (\cos y \sinh x + i \sin y \cosh x) - (x + i y)$$
$$= \operatorname{sech} v \cos y \sinh x - x + i (\operatorname{sech} v \sin y \cosh x - y)$$

Steepest descent contour is given by $\operatorname{Im}[\phi(z)] = \operatorname{sech} v \sin y \cosh x - y = \operatorname{Im}[\phi(v)] = 0,\ \frac{\cosh x}{\cosh v} = \frac{y}{\sin y}$;

hence as $y \to \pm\pi$, x tends to ∞

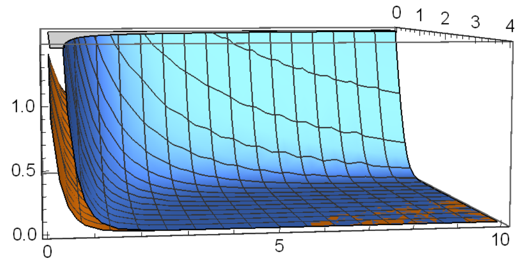


Use Laplace's method for complex contour

$$z_0 = v,\ \phi(v) = \tanh v - v,\ \ |\phi''(v)| = \tanh v,\ n = 2,\ \beta = 1,\ f_0 = 1,\ \int_C = \int_{\hat{C}_1} - \int_{\hat{C}_2}$$

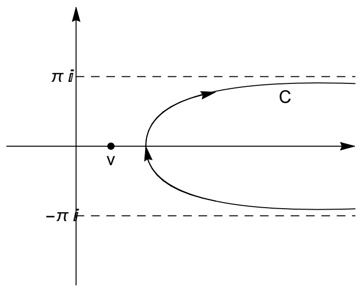
$$\int_C \sim \frac{1\,(2)^{1/2} e^{i\pi/2}}{2} \frac{e^{k(\tanh v - v)}\,\Gamma(1/2)}{(k(\tanh v))^{1/2}} - \frac{1\,(2)^{1/2} e^{i 3\pi/2}}{2} \frac{e^{k(\tanh v - v)}\,\Gamma(1/2)}{(k(\tanh v))^{1/2}}$$
$$= \frac{i\,(2)^{1/2}}{2} \frac{e^{k(\tanh v - v)}\,\Gamma(1/2)}{(k(\tanh v))^{1/2}} + \frac{i(2)^{1/2}}{2} \frac{e^{k(\tanh v - v)}\,\Gamma(1/2)}{(k(\tanh v))^{1/2}}$$
$$= i \sqrt{\frac{2\pi}{k \tanh v}}\, e^{k(\tanh v - v)},\ \ k \to \infty$$

```
Plot3D[{BesselJ[k, k Sech[v]] / 2 π, Sqrt[2 π/(k Tanh[v])] ⅇ^(k (Tanh[v] - v))}, {v, 0, 4}, {k, 0.1, 10}]
```



As $k \to \infty$, $I(k) = i \sqrt{\frac{2\pi}{k \tanh v}}\, e^{k(\tanh v - v)} = J_k(k \operatorname{sech} v)/2\pi i$

Exercise.
Evaluate $I(k) = \int_C e^{k(\sinh z - z)}\, dz$, $k \to \infty$ where C is the Sommerfeld contour.



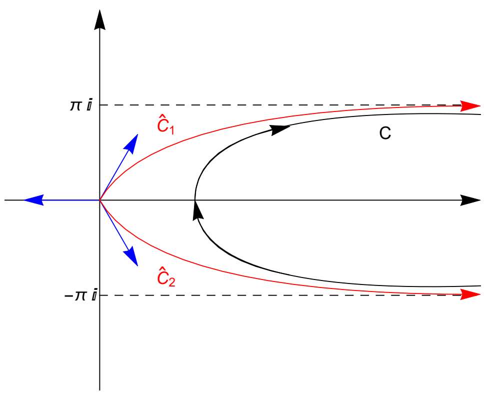
$f(z) = 1$, $\phi(z) = \sinh z - z$, $\phi'(z) = \cosh z - 1$, $\phi''(z) = \sinh z$,
$\phi'''(z) = \cosh z$ $(\phi'(0) = \phi''(0) = 0,\ \phi'''(0) \neq 0)$

$z = 0$ is a saddle point of order 2 $\left(\phi'''(0) = 1 = e^{i0},\ \alpha = 0,\ n = 3\right)$, so that the steepest descent direction is $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

From

$$\begin{aligned}\phi(x + i y) &= \sinh(x + i y) - (x + i y) = \cos y \sinh x + i \sin y \cosh x - (x + i y) \\ &= \cos y \sinh x - x + i(\sin y \cosh x - y),\end{aligned}$$

the steepest descent contour is given by $\mathrm{Im}[\phi(z)] = \sin y \cosh x - y = 0$; as $y \to \pm\pi$, x tends to ∞



Use Laplace' s method for complex contour

$$z_0 = 0,\ \phi(0) = 0,\ \ |\phi'''(0)| = 1,\ n = 3,\ \beta = 1,\ f_0 = 1,\ \ \int_C = \int_{\hat{C}_1} - \int_{\hat{C}_2}$$

$$\int_C \sim \frac{1\,(3!)^{1/3}\,e^{i\pi/3}}{3}\,\frac{\Gamma(1/3)}{(k)^{1/3}} - \frac{1\,(3!)^{1/3}\,e^{i5\pi/3}}{3}\,\frac{\Gamma(1/3)}{(k)^{1/3}}$$

$$= \Gamma\left(\frac{1}{3}\right)\left(\frac{3!}{k}\right)^{1/3}\frac{e^{i\pi/3} - e^{i5\pi/3}}{3}$$

$$= \Gamma\left(\frac{1}{3}\right)\left(\frac{3!}{k}\right)^{1/3}\frac{e^{i\pi/3} - e^{-i\pi/3}}{3} = \Gamma\left(\frac{1}{3}\right)\left(\frac{3!}{k}\right)^{1/3}\frac{2\,i\sin(\pi/3)}{3}$$

$$= \frac{i}{\sqrt{3}}\,\Gamma\left(\frac{1}{3}\right)\left(\frac{6}{k}\right)^{1/3},\ k \to \infty$$

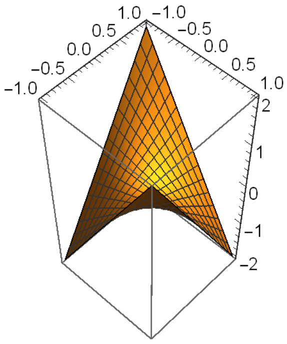
Note that we have met this at Fourier type integral section; $J_n(n) \sim \frac{1}{3\pi}\left(\cos\frac{\pi}{6}\right)\Gamma\left(\frac{1}{3}\right)\left(\frac{6}{n}\right)^{1/3},\ n \to \infty$.

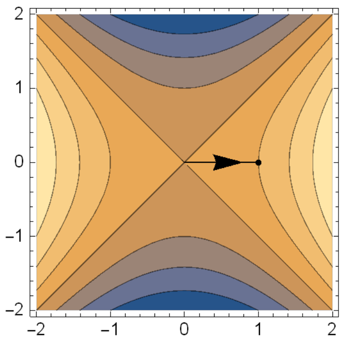
That is, $I(k) \sim \frac{i}{\sqrt{3}}\Gamma\left(\frac{1}{3}\right)\left(\frac{6}{k}\right)^{1/3} \sim \frac{J_k(k)}{2\pi i}$

Exercise.

Find the full asymptotic expansion of $I(k) = \int_0^1 e^{i k t^2}\,dt$, as $k \to \infty$

```
GraphicsGrid[{{Plot3D[Re[i (x + i y)^2], {x, -1, 1}, {y, -1, 1}, BoxRatios → Automatic],
   Show[{ContourPlot[Im[i (x + i y)^2], {x, -2, 2}, {y, -2, 2}], Graphics[{Point[{1, 0}],
      Arrowheads[0.1], Arrow[{{0, 0}, {0.8, 0}}], Line[{{0.5, 0}, {1, 0}}]}]}]}}]
```



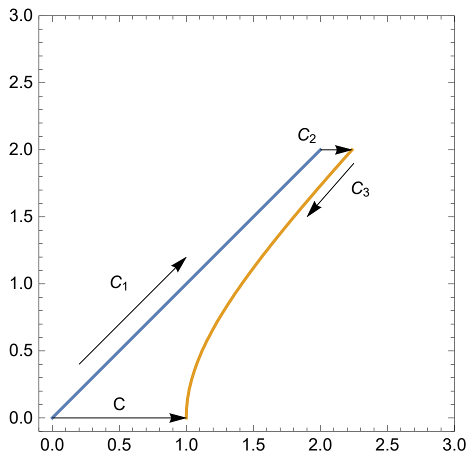


$\phi(z) = i\,z^2,\ \phi'(z) = 2\,i\,z,\ \phi''(z) = 2\,i = 2\,e^{i\pi/2}\ n = 2,\ \alpha = \frac{\pi}{2}$ ($z = 0$ at simple saddle point with steepest descent direction $\frac{\pi}{4}, \frac{5\pi}{4}$)

From

$$\phi(z) = i(x + i\,y)^2 = i(x^2 - y^2 + i\,2\,x\,y) = -2\,x\,y + i\,(x^2 - y^2),\ \ \mathrm{Im}[\phi(z)] = x^2 - y^2,$$

steepest path is $\mathrm{Im}[\phi(z)] = \text{const} = x^2 - y^2$. We need two steepest paths with respect to $z = 0$ and $z = 1$; $x = \pm y,\ x^2 - y^2 = 1$. (see the above contour plot) and its corresponding steepest descent paths are $x = y\ \&\&\ y > 0$ and $x = \sqrt{1 + y^2}$



Connect the two steepest paths, C_1 and C_3, by the contour C_2. Note that by the Jordan's theorem, as $R \to \infty$, $\int_{C_2}$ tends to zero.

$$\text{Therefore,}\quad \int_C = \int_{C_1} + \int_{C_3}$$

- $\int_{C_1}: z = r\, e^{i\pi/4}$,

$$dz = e^{i\pi/4}\, dr \Rightarrow \int_{C_1} e^{i k z^2}\, dz = \int_{r=0}^{\infty} e^{i k r^2 (i)}\, e^{i\pi/4}\, dr = e^{i\pi/4} \int_0^{\infty} e^{-k r^2}\, dr = e^{i\pi/4} \int_0^{\infty} e^{-\tau}\, \frac{1}{2k} (\tau/k)^{-1/2}\, d\tau$$

$$= \frac{1}{2 k^{1/2}} e^{i\pi/4} \int_0^{\infty} \tau^{-1/2}\, e^{-\tau}\, d\tau = \frac{1}{2 k^{1/2}} e^{i\pi/4}\, \Gamma(1/2) = \frac{1}{2} \sqrt{\frac{\pi}{k}}\, e^{i\pi/4}$$

- $\int_{C_3}: z = (x + i\, y) = \left(\sqrt{1 + y^2} + i\, y\right)$,

$$dz = \left(\frac{y}{\sqrt{1+y^2}} + i\right) dy = \int_{C_3} e^{i k z^2}\, dz = \int_{y=\infty}^{0} e^{i k \left(\sqrt{1+y^2} + i y\right)^2} \left(\frac{y}{\sqrt{1+y^2}} + i\right) dy$$

$$= \int_{\infty}^{0} e^{i k \left(1 + i\, 2 y \sqrt{1+y^2}\right)} \left(\frac{y}{\sqrt{1+y^2}} + i\right) dy =$$

$$e^{i k} \int_{\infty}^{0} e^{-2 k y \sqrt{1+y^2}} \left(\frac{y}{\sqrt{1+y^2}} + i\right) dy = -e^{i k} \int_0^{\infty} e^{-2 k y \sqrt{1+y^2}} \left(\frac{y}{\sqrt{1+y^2}} + i\right) dy$$

the dominant contribution is near $y = 0$

$$\sim -e^{i k} \int_0^{\infty} e^{-2 k y \left(1 + \frac{y^2}{2} + \cdots\right)} \left(y \left(1 - \frac{y^2}{2} + \cdots\right) + i\right) dy \sim -e^{i k} \int_0^{\infty} e^{-2 k y} (y + i)\, dy =$$

$$-e^{i k} \int_0^{\infty} y\, e^{-2 k y}\, dy - e^{i k}\, i \int_0^{\infty} e^{-2 k y}\, dy$$

$$= -e^{i k} \int_0^{\infty} \frac{\tau}{2k} e^{-\tau} \frac{d\tau}{2k} - e^{i k}\, i \int_0^{\infty} e^{-2 k y}\, dy = -e^{i k} \left(\frac{\Gamma(2)}{4 k^2} + \frac{i}{2k}\right) = -e^{i k} \left(\frac{1}{4 k^2} + \frac{i}{2k}\right)$$

Finally,

$$I(k) = \int_C \sim \int_{C_1} + \int_{C_3} = \frac{1}{2} \sqrt{\frac{\pi}{k}}\, e^{i\pi/4} - e^{i k} \left(\frac{1}{4 k^2} + \frac{i}{2k}\right), \quad k \to \infty$$

To obtain higher order terms, use the change of variables $s = 2 y \sqrt{1 + y^2}$ when evaluating $\int_{C_3}$

• $\int_{C_3}: z = (x + i\,y) = \left(\sqrt{1+y^2} + i\,y\right),\ z^2 = 1 + 2\,i\,y\sqrt{1+y^2} = 1 + i\,s,$

$z = (1 + i\,s)^{1/2},\ dz = \frac{i}{2\sqrt{1+i\,s}}\,ds$. Hence,

$\int_{C_3} e^{i\,k\,z^2}\,dz =$

$$\int_{s=\infty}^{0} e^{i\,k\,(1+i\,s)}\,\frac{i}{2\sqrt{1+i\,s}}\,ds = -\frac{i}{2}\,e^{i\,k}\int_0^\infty e^{-k\,s}(1+i\,s)^{-1/2}\,ds = -\frac{i}{2}\,e^{i\,k}\int_0^\infty e^{-k\,s}\left(1 - \frac{i\,s}{2} + \frac{3\,(i\,s)^2}{8} - \cdots\right)ds$$

$$= -\frac{i}{2}\,e^{i\,k}\int_0^\infty e^{-k\,s}\left(\sum_{n=0}^\infty \frac{(-1)^n\,(2\,n)!}{2^{2\,n}\,(n!)^2}\,(i\,s)^n\right)ds = -\frac{i}{2}\,e^{i\,k}\sum_{n=0}^\infty \frac{(-1)^n\,(2\,n)!}{2^{2\,n}\,(n!)^2}\,(i)^n\int_0^\infty e^{-k\,s}\,s^n\,ds$$

$$= -\frac{i}{2}\,e^{i\,k}\sum_{n=0}^\infty \frac{(-1)^n\,(2\,n)!}{2^{2\,n}\,(n!)^2}\,(i)^n\int_0^\infty e^{-\tau}(\tau/k)^n\,\frac{d\tau}{k} = -\frac{i}{2}\,e^{i\,k}\sum_{n=0}^\infty \frac{(-1)^n\,(2\,n)!}{2^{2\,n}\,(n!)^2}\,(i)^n\,\frac{1}{k^{n+1}}\int_0^\infty e^{-\tau}\,\tau^n\,d\tau$$

$$= -\frac{i}{2}\,e^{i\,k}\sum_{n=0}^\infty \frac{(-1)^n\,(2\,n)!}{2^{2\,n}\,(n!)^2}\,(i)^n\,\frac{1}{k^{n+1}}\,\Gamma(n+1) = -\frac{i}{2}\,e^{i\,k}\sum_{n=0}^\infty \frac{(-1)^n\,(2\,n)!}{2^{2\,n}\,n!}\,(i)^n\,\frac{1}{k^{n+1}}$$

Therefore,

$$I(k) \sim \int_{C_1} + \int_{C_3} = \frac{1}{2}\sqrt{\frac{\pi}{k}}\,e^{i\pi/4} - \frac{i}{2}\,e^{i\,k}\sum_{n=0}^\infty \frac{(-1)^n\,(2\,n)!}{2^{2\,n}\,n!}\,(i)^n\,\frac{1}{k^{n+1}},\ k \to \infty$$

```
Series[∫_0^1 e^(i k t^2) dt, {k, ∞, 6}]
```

$$\left(\frac{1}{2}\,(-1)^{1/4}\sqrt{\pi}\,\sqrt{\frac{1}{k} + O\left[\frac{1}{k}\right]^{13/2}}\right) + e^{i\,k + O\left[\frac{1}{k}\right]^7}\left(-\frac{i}{2\,k} - \frac{1}{4\,k^2} + \frac{3\,i}{8\,k^3} + \frac{15}{16\,k^4} - \frac{105\,i}{32\,k^5} - \frac{945}{64\,k^6} + O\left[\frac{1}{k}\right]^7\right)$$

Exercise.

Consider the integrals (a) $I_1(k) = \int_{-\infty}^{\infty} \frac{e^{i\,k(t+t^3/3)}}{t^2+t_0{}^2}\,dt,\ \ k > 0$

, (b) $I_\alpha(k) = \int_{-\infty}^{\infty} \frac{e^{i\,k(t+t^3/3)}}{(t^2+1)^\alpha}\,dt,\ \ k > 0,\ 0 < \alpha < 1$ where t_0 is a real positive constant.

Find the leading term of the expansions of these integrals..

Now there are singularities in the region. First, we consider $I_1(k)$.

(a) $I_1(k) = \int_{-\infty}^{\infty} \frac{e^{i\,k(t+t^3/3)}}{t^2+t_0{}^2}\,dt,\ \ k > 0$

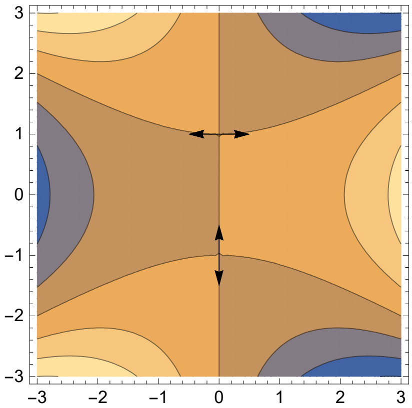
$f(z) = \frac{1}{z^2+t_0{}^2},\ \ \phi(z) = i(z + z^3/3),\ \ \phi'(z) = i(1+z^2),\ \ \phi''(z) = i(2\,z) \Rightarrow z = \pm i$ is simple saddle point

$\phi''(i) = -2 = 2\,e^{i\pi}$ steepest descent direction : $0,\ \pi$;

$\phi''(-i) = 2 = 2\,e^{i0}$ steepest descent direction : $\frac{\pi}{2},\ \frac{3\pi}{2}$

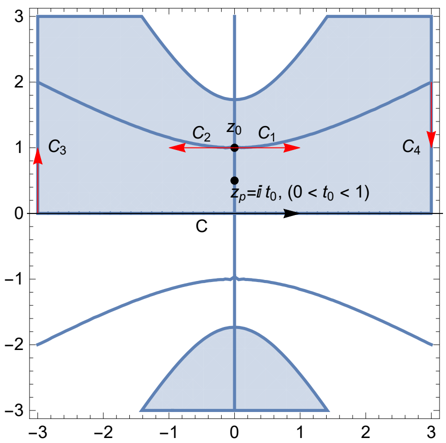
Steepest descent curve is given by $\text{Im}[\phi(z)] = (3\,x + x^3 - 3\,x\,y^2) = \text{Im}[\phi(i)] = 0$

```
Show[{ContourPlot[Im[ⅈ (x + ⅈ y + (x + ⅈ y)^3 / 3)], {x, -3, 3}, {y, -3, 3}],
  Graphics[{Arrow /@ {{{0, 1}, {0.5, 1}}, {{0, -1}, {0, -0.5}},
      {{0, -1}, {0, -1.5}}, {{0, 1}, {-0.5, 1}}}}]}]
```



Moreover, the integral converges in the region where $\text{Re}[\phi(z)] = y(y^2 - 3x^2 - 3) < 0$

```
Show[{RegionPlot[y (y^2 - 3 x^2 - 3) < 0, {x, -3, 3}, {y, -3, 3}],
  ContourPlot[x (3 + x^2 - 3 y^2) == 0, {x, -3, 3}, {y, -3, 3}],
  Graphics[{PointSize[0.02], Point[{0, 0.5}], Point[{0, 1}],
    {Text["z0", {0, 1.3}], Text["zp=ⅈ t0, (0 < t0 < 1)", {1, 0.3}],
     Text["C", {-0.5, -0.2}], Text["C1", {0.5, 1.2}], Text["C2", {-0.5, 1.2}],
     Text["C3", {-2.7, 1}], Text["C4", {2.7, 1}]}, {Red, Arrow /@
      {{{-3, 0}, {-3, 1}}, {{3, 2}, {3, 1}}, {{0, 1}, {1, 1}}, {{0, 1}, {-1, 1}}}},
    Line[{{-4, 0}, {4, 0}}], Arrow[{{-2, 0}, {1, 0}}]}]}]
```



Thus, we use the upper half plane path. We can deform C into $C_3 - C_2 + C_1 + C_4$.
However, if $0 < t_0 < 1$, $\int_C - \left(\int_{C_3} - \int_{C_2} + \int_{C_1} + \int_{C_4}\right) = 2\pi i \left[\text{residue of } f(z)\, e^{k\,\phi(z)} \text{ at } z = z_p\right]$

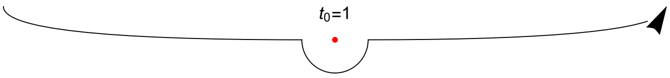
$$2\pi i \left[\frac{e^{i\,k(z + z^3/3)}}{z + i\,t_0}\right]_{z = i\,t_0} = 2\pi i\, \frac{e^{-k(t_0 - t_0{}^3/3)}}{2\,i\,t_0} = \frac{\pi e^{-k(t_0 - t_0{}^3/3)}}{t_0}$$

Use Laplace's method for complex contours to evaluate the integral.

$$\boxed{\begin{array}{c} n = 2,\; z_0 = i,\; f(z) = f(i) + \frac{f'(i)}{1!}(z - i) + \cdots \sim \frac{1}{t_0{}^2 - 1} = f_0 (z - i)^{1-1} \text{ where } f(z) = \frac{1}{z^2 + t_0{}^2} \Rightarrow f_0 = \frac{1}{t_0{}^2 - 1},\; \beta = 1 \\ \phi(i) = -\frac{2}{3},\; \phi''(i) = -2,\; \theta = 0 \text{ for } C_1 \text{ and } \theta = \pi \text{ for } C_2 \end{array}}$$

$$\begin{cases} 0<t_0<1 & I_1(k)=\int_C=\int_{C_3}-\int_{C_2}+\int_{C_1}+\int_{C_4}+\frac{\pi e^{-k(t_0-t_0^3/3)}}{t_0}\ \left(\text{as } R \text{ tends to } \infty,\ \int_{C_3}\text{and}\int_{C_4}\to 0\right) \\ & =\int_{C_1}-\int_{C_2}+\frac{\pi e^{-k(t_0-t_0^3/3)}}{t_0}\sim\frac{(2!)^{1/2}}{2\left(t_0{}^2-1\right)}\frac{e^{-2k/3}\,\Gamma(1/2)}{(2k)^{1/2}}-\frac{(2!)^{1/2}(-1)}{2\left(t_0{}^2-1\right)}\frac{e^{-2k/3}\,\Gamma(1/2)}{(2k)^{1/2}}+\frac{\pi e^{-k(t_0-t_0^3/3)}}{t_0} \\ & =\frac{1}{\left(t_0{}^2-1\right)}\frac{e^{-2k/3}\,\Gamma(1/2)}{(k)^{1/2}}+\frac{\pi e^{-k(t_0-t_0^3/3)}}{t_0}=\frac{\sqrt{\pi/k}}{\left(t_0{}^2-1\right)}e^{-2k/3}+\frac{\pi e^{-k(t_0-t_0^3/3)}}{t_0} \\ t_0>1 & I_1(k)\sim\frac{\sqrt{\pi/k}}{\left(t_0{}^2-1\right)}e^{-2k/3} \end{cases}$$

When $t_0 = 1$, the steepest descent contour can avoid the singular point



which has the effect of treating the saddle point as a Cauchy principal value integral and halving the pole contribution.

$$\int_C-P\int_{-\infty+i}^{\infty+i}\sim\pi i\,[\text{residue at } z=i],\quad\text{therefore,}\quad I_1(k)\sim P\int_{-\infty+i}^{\infty+i}\frac{e^{ik(z+z^3/3)}}{z^2+1}dz+\pi i\left[\frac{e^{ik(z+z^3/3)}}{z+i}\right]_{z=i}$$

Letting $z = s + i$,

$$\begin{aligned} I_1(k)&\sim P\int_{-\infty+i}^{\infty+i}\frac{e^{ik(z+z^3/3)}}{z^2+1}dz+\pi i\left[\frac{e^{-2k/3}}{2i}\right] \\ &=P\int_{-\infty}^{\infty}\frac{e^{ik(s^3/3+i(s^2+2/3))}}{s(s+2i)}ds+\frac{\pi e^{-2k/3}}{2} \\ &=e^{-2k/3}\,P\int_{-\infty}^{\infty}\frac{e^{iks^3/3}\,e^{-ks^2}}{s(s+2i)}ds+\frac{\pi e^{-2k/3}}{2} \end{aligned}$$

let $s\sqrt{k}=u$

$$\begin{aligned} &=e^{-2k/3}\,P\int_{-\infty}^{\infty}\frac{e^{iu^3/3\sqrt{k}}\,e^{-u^2}}{\left(u/\sqrt{k}\right)\left(u/\sqrt{k}+2i\right)}\frac{du}{\sqrt{k}}+\frac{\pi e^{-2k/3}}{2} \\ &=e^{-2k/3}\,P\int_{-\infty}^{\infty}\frac{e^{iu^3/3\sqrt{k}}\,e^{-u^2}}{u\left(u/\sqrt{k}+2i\right)}du+\frac{\pi e^{-2k/3}}{2}\ \text{as } k\to\infty,\ \begin{cases} u^3/3\sqrt{k}\to 0 \\ u/\sqrt{k}\to 0 \end{cases} \\ &\sim e^{-2k/3}\,P\int_{-\infty}^{\infty}\frac{e^0\,e^{-u^2}}{u(0+2i)}du+\frac{\pi e^{-2k/3}}{2}=e^{-2k/3}\,P\int_{-\infty}^{\infty}\frac{e^{-u^2}}{2iu}du+\frac{\pi e^{-2k/3}}{2} \end{aligned}$$

Since the integral is odd, $I_1(k)\sim\frac{\pi e^{-2k/3}}{2}$

In short,

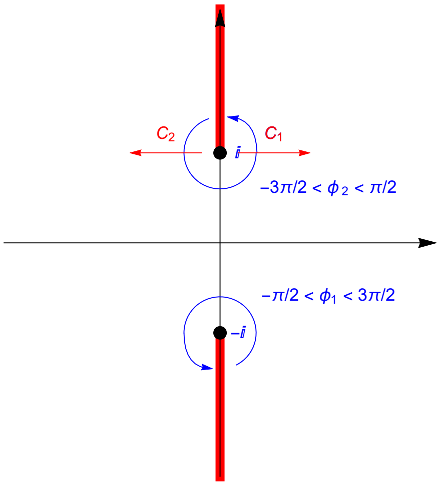
$$I_1(k)\sim\begin{cases}\frac{\sqrt{\pi/k}}{\left(t_0{}^2-1\right)}e^{-2k/3}+\frac{\pi e^{-k(t_0-t_0^3/3)}}{t_0}\mathcal{H}(1-t_0) & t_0\neq 1 \\ \frac{\pi e^{-2k/3}}{2} & t_0=1\end{cases}$$

where $\mathcal{H}$ is the Heaviside step function.

(b) $I_\alpha(k)=\int_{-\infty}^{\infty}\frac{e^{ik(t+t^3/3)}}{(t^2+1)^\alpha}dt,\quad k>0,\ 0<\alpha<1$

$f(z)=(z^2+1)^{-\alpha}$ has branch point at $z=i,\ -i$.

Take the branch cut as $(\pm i,\ \pm i\,\infty)$ with local coordinates $(z^2+1)=(z-i)(z+i)=r_1\,e^{i\phi_1}\,r_2\,e^{i\phi_2}$



Because of the branch cut, C_1 and C_2 are correspond to $\theta = 0$ and $-\pi$. Steepest path is same as above. Now use Laplace's method to evaluate this integral

$$\boxed{\begin{array}{c} n = 2,\ z_0 = i,\ f(z) = \left[\frac{1}{(z+i)^{\alpha}}\right]_{z=i} (z-i)^{-\alpha} = f_0 (z-i)^{\beta-1} \Rightarrow f_0 = \frac{1}{(2i)^{\alpha}},\ \beta = 1-\alpha \\ \phi(i) = -\frac{2}{3},\ \phi''(i) = -2,\ \theta = 0 \text{ for } C_1 \text{ and } \theta = -\pi \text{ for } C_2 \end{array}}$$

note that $f_0 = \frac{1}{(2i)^{\alpha}} = 2^{-\alpha} e^{-i\pi\alpha/2}$. Then,

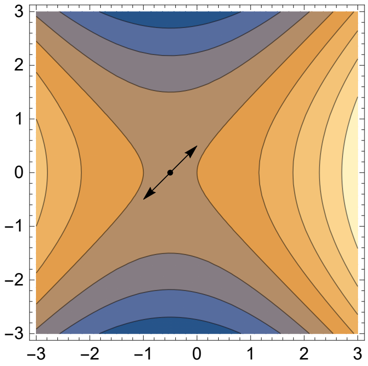
$$\begin{aligned} I_\alpha(k) &\sim -\int_{C_2} + \int_{C_1} = -\frac{e^{-i\pi\alpha/2} (2)^{(1-\alpha)/2} e^{-i\pi(1-\alpha)}}{2^{\alpha+1}} \frac{e^{-2k/3}\, \Gamma\left(\frac{1-\alpha}{2}\right)}{(2k)^{(1-\alpha)/2}} + \frac{e^{-i\pi\alpha/2} (2)^{(1-\alpha)/2}}{2^{\alpha+1}} \frac{e^{-2k/3}\, \Gamma\left(\frac{1-\alpha}{2}\right)}{(2k)^{(1-\alpha)/2}} \\ &= \frac{(2)^{(1-\alpha)/2} e^{i\pi\alpha/2}}{2^{\alpha+1}} \frac{e^{-2k/3}\, \Gamma\left(\frac{1-\alpha}{2}\right)}{(2k)^{(1-\alpha)/2}} + \frac{e^{-i\pi\alpha/2} (2)^{(1-\alpha)/2}}{2^{\alpha+1}} \frac{e^{-2k/3}\, \Gamma\left(\frac{1-\alpha}{2}\right)}{(2k)^{(1-\alpha)/2}} \\ &= \frac{e^{-2k/3}\, \Gamma\left(\frac{1-\alpha}{2}\right)}{(2k)^{(1-\alpha)/2}} \frac{(2)^{(1-\alpha)/2}}{2^{\alpha+1}} \left(e^{i\pi\alpha/2} + e^{-i\pi\alpha/2}\right) = \frac{e^{-2k/3}\, \Gamma\left(\frac{1-\alpha}{2}\right)}{(2k)^{(1-\alpha)/2}} \frac{(2)^{(1-\alpha)/2}}{2^{\alpha+1}} \left(2 \cos \frac{\pi\alpha}{2}\right) \\ &= \frac{e^{-2k/3}\, \Gamma\left(\frac{1-\alpha}{2}\right)}{2^{\alpha} k^{(1-\alpha)/2}} \cos \frac{\pi\alpha}{2} \end{aligned}$$

Exercise.
Evaluate $I(k) = \int_0^1 \frac{e^{i k (t^2+t)}}{\sqrt{t}}\, dt$ as $k \to \infty$

$f(z) = z^{-1/2}$, $\phi(z) = i(z^2 + z)$, $\phi'(z) = i(2z+1)$, $\phi''(z) = 2i = 2e^{i\pi/2}$, $z = -1/2$ is a simple saddle point where $\alpha = \frac{\pi}{2}$.

Direction of steepest descent: $\frac{\pi}{4}, \frac{5\pi}{4}$

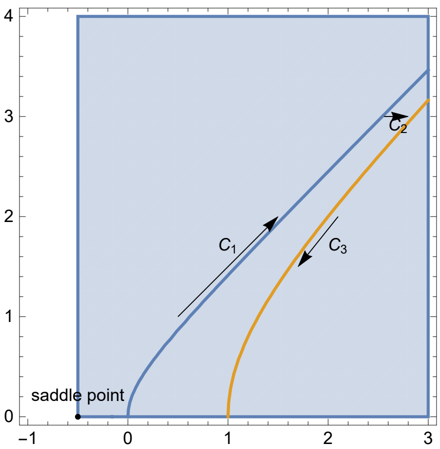
```
Show[{ContourPlot[Im[ⅈ ((x + ⅈ y)^2 + (x + ⅈ y))], {x, -3, 3}, {y, -3, 3}], Graphics[
    {Point[{-0.5, 0}], Arrow /@ {{{-0.5, 0}, {0, 0.5}}, {{-0.5, 0}, {-1, -.5}}}}]}]
```



$\phi(z) = i((x + i\,y)^2 + (x + i\,y)) = y(-1 - 2\,x) + i\,(x + x^2 - y^2)$

Convergence region: $\text{Re}[\phi(z)] = y(-1 - 2\,x) < 0$, Steepest path: $\text{Im}[\phi(z)] = \text{const}$

```
Show[{RegionPlot[y (-1 - 2 x) < 0, {x, -1, 3}, {y, 0, 4}],
  ContourPlot[{x^2 - y^2 + x == 0, x^2 - y^2 + x == 2}, {x, -1, 3}, {y, 0, 4}],
  Graphics[{Point[{-0.5, 0}], Text["saddle point", {-.5, 0.2}], Text["C1", {1, 1.7}],
    Text["C2", {2.7, 2.9}], Text["C3", {2.1, 1.7}], Arrow[{{0.5, 1}, {1.5, 2}}],
    Arrow[{{2.1, 2}, {1.7, 1.5}}], Arrow[{{2.56, 3}, {2.8, 3}}]}]}]
```



$$I(k) \sim \int_{C_1} + \int_{C_2} + \int_{C_3}, \text{ as } R \to \infty, \int_{C_2} \to \infty$$

• $\int_{C_1} : y = \sqrt{x^2 + x}\,,\ z = x + i\sqrt{x^2 + x}\,,$

$\phi(z) = i(z^2 + z) = -(2\,x + 1)\sqrt{x + x^2}\,,\ dz = \left(1 + \frac{i}{2}(x^2 + x)^{-1/2}(2\,x + 1)\right)dx$

$$\Rightarrow \int_{C_1} \frac{e^{i\,k(z^2+z)}}{\sqrt{z}}\,dz = \int_0^\infty \frac{e^{k\left(-(2x+1)\sqrt{x+x^2}\right)}}{\sqrt{x + i\sqrt{x^2+x}}\,,}\left(1 + \frac{i}{2}(x^2 + x)^{-1/2}(2\,x + 1)\right)dx, \text{ letting } s = (2\,x + 1)\sqrt{x^2 + x}$$

Since dominant contribution occurs at the neighborhood of $z = 0$

$s = (2\,x + 1)\,x^{1/2}(1 + x)^{1/2} =$

$$(2\,x + 1)\,x^{1/2}\left(1 + \frac{x}{2} - \frac{x^2}{8} + \cdots\right) = (1 + 2\,x)\left(x^{1/2} + \frac{x^{3/2}}{2} - \frac{x^{5/2}}{8} + \cdots\right) = x^{1/2} + \frac{5\,x^{3/2}}{2} + \frac{7\,x^{5/2}}{8} + \cdots$$

$$x^{1/2} = s - \frac{5\,x^{3/2}}{2} + \cdots = s - \frac{5\left(s - \frac{5\,x^{3/2}}{2} + \cdots\right)^3}{2} + \cdots \sim s - \frac{5}{2}s^3 + \cdots,\quad x = s^2 - 5\,s^4 + \cdots$$

$$\texttt{Series}\left[\frac{1+e^{i\,\pi/2}\,\left(x^2+x\right)^{-1/2}\,\left(2\,x+1\right)/2}{\sqrt{x+e^{i\,\pi/2}\,\sqrt{x^2+x}}},\ \{x,\ 0,\ 3\}\right]$$

$$\frac{\sqrt{i\sqrt{x}}}{2\sqrt{x}\,\sqrt{x}}-\frac{3\,i\,\sqrt{i\sqrt{x}}}{4\sqrt{x}}+\frac{15\,\sqrt{i\sqrt{x}}\,\sqrt{x}}{16\sqrt{x}}+\frac{21\,i\,\sqrt{i\sqrt{x}}\,x}{32\sqrt{x}}-\frac{261\,\sqrt{i\sqrt{x}}\,x^{3/2}}{256\sqrt{x}}-\frac{517\,i\,\sqrt{i\sqrt{x}}\,x^2}{512\sqrt{x}}+\frac{3003\,\sqrt{i\sqrt{x}}\,x^{5/2}}{2048\sqrt{x}}+\frac{6885\,i\,\sqrt{i\sqrt{x}}\,x^3}{4096\sqrt{x}}+O[x]^{7/2}$$

$$\int_0^\infty \frac{e^{k\left(-(2x+1)\sqrt{x+x^2}\right)}}{\sqrt{x+i\sqrt{x^2+x}\,,}}\left(1+\frac{i}{2}(x^2+x)^{-1/2}(2x+1)\right)dx=$$

$$\int_0^\infty e^{-ks}\left(\frac{e^{i\pi/4}}{2}x^{-3/4}-\frac{3\,e^{i3\pi/4}}{4}x^{-1/4}+\frac{15\,e^{i\pi/4}}{16}x^{1/4}+\frac{21\,e^{i3\pi/4}}{32}x^{3/4}+\cdots\right)\left(\frac{dx}{ds}\right)ds$$
$$=\int_0^\infty e^{-ks}\left(\frac{e^{i\pi/4}}{2}s^{-3/2}-\frac{3\,e^{i3\pi/4}}{4}s^{-1/2}+\frac{45\,e^{i\pi/4}}{16}s^{1/2}-\frac{9\,e^{i3\pi/4}}{32}s^{3/2}+\cdots\right)(2s-20s^3+\cdots)\,ds$$
$$=\int_0^\infty e^{-ks}\left(e^{i\pi/4}s^{-1/2}-\frac{3\,e^{i3\pi/4}}{2}s^{1/2}-\frac{35\,e^{i\pi/4}}{8}s^{3/2}+\frac{231\,e^{i3\pi/4}}{16}s^{5/2}+\cdots\right)ds$$
$$=e^{i\pi/4}\int_0^\infty e^{-ks}\left(s^{-1/2}-\frac{3i}{2}s^{1/2}-\frac{35}{8}s^{3/2}+\frac{231\,i}{16}s^{5/2}+\cdots\right)ds$$
$$=e^{i\pi/4}\int_0^\infty e^{-\tau}\left((\tau/k)^{-1/2}-\frac{3i}{2}(\tau/k)^{1/2}-\frac{35}{8}(\tau/k)^{3/2}+\frac{231\,i}{16}(\tau/k)^{5/2}+\cdots\right)\frac{d\tau}{k}$$
$$=e^{i\pi/4}\left(\frac{\Gamma(1/2)}{k^{1/2}}-\frac{3\,i\,\Gamma(3/2)}{2\,k^{3/2}}-\frac{35\,\Gamma(5/2)}{8\,k^{5/2}}+\frac{231\,i\,\Gamma(7/2)}{16\,k^{7/2}}+\cdots\right)=$$
$$e^{i\pi/4}\left(\frac{1}{k^{1/2}}\sqrt{\pi}-\frac{3i}{2\,k^{3/2}}\frac{\sqrt{\pi}}{2}-\frac{35}{8\,k^{5/2}}\frac{1\cdot3\sqrt{\pi}}{2\cdot2}+\frac{231\,i}{16\,k^{7/2}}\frac{1\cdot3\cdot5\sqrt{\pi}}{2\cdot2\cdot2}+\cdots\right)$$
$$=\sqrt{\frac{\pi}{k}}\,e^{i\pi/4}\left(1-\frac{3i}{4k}-\frac{105}{32k^2}+\frac{3465}{128k^3}+\cdots\right)$$

- $\int_{C_3}: y=\sqrt{x^2+x-2}\,,\ z=x+i\sqrt{x^2+x-2}\,,$

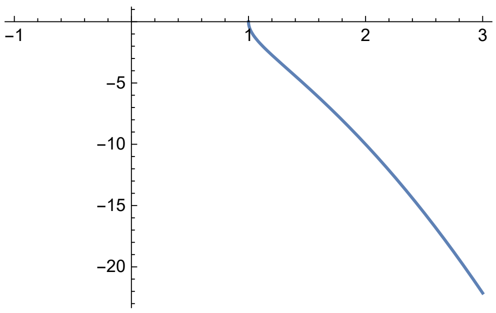
$\phi(z)=i(z^2+z)=-(2x+1)\sqrt{-2+x+x^2}+2i,\ dz=\left(1+\frac{i}{2}(x^2+x-2)^{-1/2}(2x+1)\right)dx$

$$\Rightarrow\int_{C_3}\frac{e^{ik(z^2+z)}}{\sqrt{z}}dz=\int_\infty^0\frac{e^{k\left(2i-(2x+1)\sqrt{-2+x+x^2}\right)}}{\sqrt{x+i\sqrt{x^2+x-2}\,,}}\left(1+\frac{i}{2}(x^2+x-2)^{-1/2}(2x+1)\right)dx,\ \text{letting}$$

$s=(2x+1)\sqrt{x^2+x-2}$

Note that dominant contribution occurs at the neighborhood of $x=1$

$$\texttt{Plot}\left[-\left(2\,x+1\right)\,\sqrt{x^2+x-2}\,,\ \{x,\ -1,\ 3\}\right]$$



$$\texttt{Series}\left[\left(2\,x+1\right)\,\sqrt{x^2+x-2}\,,\ \{x,\ 1,\ 4\}\right]$$

$$3\sqrt{3}\,\sqrt{x-1}+\frac{5}{2}\sqrt{3}\,\left(x-1\right)^{3/2}+\frac{7\,\left(x-1\right)^{5/2}}{8\sqrt{3}}-\frac{\left(x-1\right)^{7/2}}{16\sqrt{3}}+O[x-1]^{9/2}$$

let $(x-1)=t$ then,

$$s = 3\sqrt{3}\, t^{1/2} + \frac{5\sqrt{3}}{2} t^{3/2} + \frac{7}{8\sqrt{3}} t^{5/2} - \cdots,\quad t^{1/2} = \frac{s}{3\sqrt{3}} - \frac{5}{6} t^{3/2} - \cdots = \frac{s}{3\sqrt{3}} - \frac{5}{6}\left(\frac{s}{3\sqrt{3}} - \frac{5\sqrt{3}}{6\sqrt{3}} t^{3/2}\right)^3 + \cdots$$

$$t^{1/2} \sim \frac{s}{3\sqrt{3}}\left(1 - \frac{5s^3}{162}+\cdots\right),\quad t\sim \frac{s^2}{27}\left(1-\frac{5}{81}s^2+\cdots\right)$$

$$\mathbf{Series}\left[\frac{1+\frac{i}{2}\left(x^2+x-2\right)^{-1/2}(2x+1)}{\sqrt{x+i\sqrt{x^2+x-2}}}, \{x,1,4\}\right]$$

$$\frac{i\sqrt3}{2\sqrt{x-1}} + \frac74 - \frac{17}{16}\, i\sqrt3\,\sqrt{x-1} - \frac{117(x-1)}{32} + \frac{8755\, i\, (x-1)^{3/2}}{768\sqrt3} + \frac{6537}{512}(x-1)^2 - \frac{794087\, i\, (x-1)^{5/2}}{18432\sqrt3} - \frac{201997\,(x-1)^3}{4096} + \frac{33577771\, i\, (x-1)^{7/2}}{196608\sqrt3} + \frac{26048125\,(x-1)^4}{131072} + O[x-1]^{9/2}$$

$$-e^{2ik}\int_0^\infty \frac{e^{k\left(-(2x+1)\sqrt{-2+x+x^2}\right)}}{\sqrt{x+i\sqrt{x^2+x-2}\,,}}\left(1+\frac{i}{2}(x^2+x-2)^{-1/2}(2x+1)\right)dx =$$

$$-e^{2ik}\int_0^\infty e^{-ks}\left(\frac{i\sqrt3}{2}t^{-1/2}+\frac74-\frac{17i\sqrt3}{16}t^{1/2}+\cdots\right)\frac{dt}{ds}\,ds$$

$$\sim -e^{2ik}$$

$$\int_0^\infty e^{-ks}\left(\frac{i\sqrt3}{2}\frac{3\sqrt3}{s}\left(1+\frac{5s^3}{162}+\cdots\right)+\frac74-\frac{17i\sqrt3}{16}\frac{s}{3\sqrt3}\left(1-\frac{5s^3}{162}+\cdots\right)+\cdots\right)\left(\frac{2}{27}s-\frac{20}{2187}s^3+\cdots\right)ds$$

$$=-e^{2ik}\int_0^\infty e^{-ks}\left(\frac{i9}{2s}\left(1+\frac{5s^3}{162}+\cdots\right)+\frac74-\frac{17is}{48}\left(1-\frac{5s^3}{162}+\cdots\right)+\cdots\right)\left(\frac2{27}s-\frac{20}{2187}s^3+\cdots\right)ds$$

$$=-e^{2ik}\int_0^\infty e^{-ks}\left(\frac{9i}{2}s^{-1}+\frac74-\frac{17i}{48}s+\frac{45i}{324}s^2\right)\left(\frac2{27}s-\frac{20}{2187}s^3+\cdots\right)ds$$

$$=-e^{2ik}\int_0^\infty e^{-ks}\left(\frac i3+\frac7{54}s-\frac{131i}{1944}s^2-\cdots\right)ds = -e^{2ik}\int_0^\infty e^{-\tau}\left(\frac i3+\frac7{54}(\tau/k)-\frac{131i}{1944}(\tau/k)^2-\cdots\right)\frac{d\tau}{k}$$

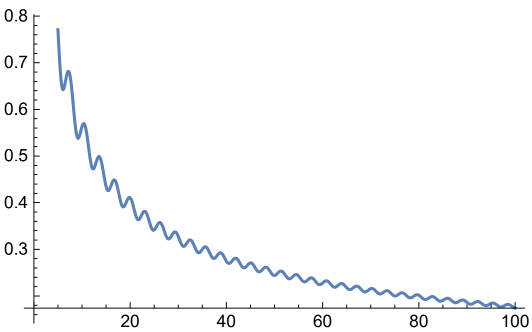
$$=-e^{2ik}\left[\frac i3\frac{\Gamma(1)}{k}+\frac7{54}\frac{\Gamma(2)}{k^2}-\frac{131i}{1944}\frac{\Gamma(3)}{k^3}-\cdots\right]$$

$$=e^{2ik}\left[-\frac{i}{3k}-\frac{7}{54k^2}+\frac{131i}{972k^3}-\cdots\right]$$

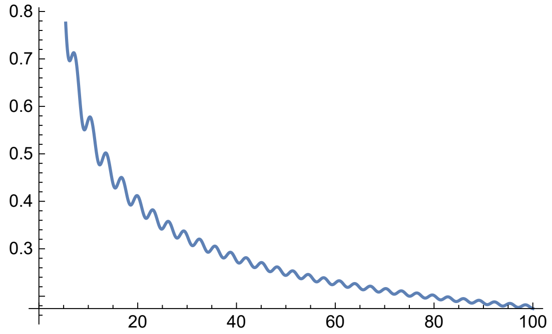
Hence,

$$I(k)=\int_0^1\frac{e^{ik(t^2+t)}}{\sqrt t}\,dt\sim\sqrt{\frac\pi k}\,e^{i\pi/4}\left(1-\frac{3i}{4k}-\frac{105}{32k^2}+\cdots\right)+e^{2ik}\left(-\frac{i}{3k}-\frac{7}{54k^2}+\frac{131i}{972k^3}-\cdots\right)\text{ as } k\to\infty$$

$$\mathbf{Plot}\left[\left\{\mathbf{Abs}\left[\mathbf{NIntegrate}\left[\mathbf{Exp}\left[i\,k\,(t^2+t)\right]\big/\sqrt{t},\ \{t,0,1\}\right]\right]\right\},\ \{k,0,100\}\right]$$



```
Plot[
 {Abs[Sqrt[π/k] e^(i π/4) (1 - 3 i/(4 k) - 105/(32 k^2) + 3465/(128 k^3)) + e^(2 i k) (-i/(3 k) - 7/(54 k^2) + 131 i/(972 k^3))]}, {k, 0, 100}]
```



Appendix

Application of Stationary Phase Method in Quantum Mechanics–Free Particle

(see also Griffith's book) Consider the Schrödinger equation for free particle.

$$i\,\psi_t + \psi_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \quad \psi \to 0 \text{ as } |x| \to \infty$$
$$\psi(x, 0) = \psi_0(x) \text{ with } \int_{-\infty}^{\infty} |\psi_0|^2 \, dx < \infty$$

Using Fourier transform in x,

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b(k, t)\, e^{i k x} \, dk$$

and substituting this integral into the above time -dependent Schrödinger equation.

$$i\,\psi_t + \psi_{xx} = 0 \Longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\,\partial_t b(k, t)\, e^{i k x} \, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -k^2\, b(k, t)\, e^{i k x} \, dk = 0$$
$$\Longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{i k x} (i\,\partial_t - k^2)\, b(k, t) = 0$$
$$\Longrightarrow (i\,\partial_t - k^2)\, b(k, t) = 0, \quad b(k, t) = b(k, 0)\, e^{-i k^2 t}$$

Therefore

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b(k, 0)\, e^{-i k^2 t} e^{i k x} \, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b_0(k)\, e^{i k (x - k t)} \, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b_0(k)\, e^{i t \phi(k)} \, dk$$

where k is a phase velocity, $b_0(k) = b(k, 0)$ and $\phi(k) = k(x/t) - k^2$. Let's consider the large t behavior of $\psi(x, t)$ where (x/t) is fixed.

Since $\phi'(k) = (x/t) - 2k$, $\phi''(k) = -2$, we can apply it to stationary phase method. Note that $\phi'(k)$ vanishes only at $k = (x/2t)$

$$\psi(x,t) \sim \frac{1}{\sqrt{2\pi}} \int_{(x/2t)-R}^{(x/2t)+R} b_0\left(\frac{x}{2t}\right) e^{i t\left((x/2t)^2 + \frac{(k-(x/2t))^2}{2}(-2)\right)} dk = \frac{1}{\sqrt{2\pi}} b_0\left(\frac{x}{2t}\right) e^{i t (x/2t)^2} \int_{(x/2t)-R}^{(x/2t)+R} e^{-i t (k-(x/2t))^2} dk$$

letting $\tau = t^{1/2}(k-(x/2t))$, $dk = t^{-1/2} d\tau$

$$= \frac{1}{\sqrt{2\pi}} b_0\left(\frac{x}{2t}\right) e^{i t (x/2t)^2} t^{-1/2} \int_{-R t^{1/2}}^{+R t^{1/2}} e^{-i \tau^2} d\tau$$

as $t \to \infty$,

$$= \frac{1}{\sqrt{2\pi}} b_0\left(\frac{x}{2t}\right) e^{i t (x/2t)^2} t^{-1/2} \int_{-\infty}^{+\infty} e^{-i \tau^2} d\tau = \frac{1}{\sqrt{2\pi}} b_0\left(\frac{x}{2t}\right) e^{i t (x/2t)^2} t^{-1/2}\, 2 \int_0^{\infty} e^{-i \tau^2} d\tau$$

letting $i\tau^2 = s$, or $e^{i\pi/2} \tau^2 = s$

$$= \frac{1}{\sqrt{2\pi}} b_0\left(\frac{x}{2t}\right) e^{i t (x/2t)^2} t^{-1/2}\, 2 \int_0^{\infty} e^{-s} \frac{1}{2} s^{-1/2} e^{-i\pi/4} ds = \frac{1}{\sqrt{2\pi}} b_0\left(\frac{x}{2t}\right) e^{i t (x/2t)^2} t^{-1/2} e^{-i\pi/4}\, \Gamma(1/2)$$

$$= \frac{e^{i t (x/2t)^2}}{\sqrt{2t}} b_0\left(\frac{x}{2t}\right) e^{-i\pi/4}$$

$\psi(x,t) \sim \frac{e^{i t (x/2t)^2}}{\sqrt{2t}} b_0\left(\frac{x}{2t}\right) e^{-i\pi/4}$ implies that the amplitude decays as $t^{-1/2}$

From $\phi(k) = k(x/t) - k^2$, $\phi'(k) = (x/t) - 2k$, $\phi''(k) = -2$, the important propagation velocity is not the phase velocity k, but the velocity that yields the dominant asymptotic result;
$\phi'(k) = (x/t) - 2k = 0$, $(x/t) = 2k$.

$$\begin{cases} \text{phase velocity } (v_p(k)) : k \\ \text{group velocity}(v_g(k)) : 2k \end{cases}$$

Group velocity is important because, after a sufficiently long time, each wave number k dominates the solution in a region defined by $x \sim v_g(k)\, t$. Namely, after a sufficiently long time, different wave numbers propagate with the group velocity.

In general,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b(k,0)\, e^{-i w(k) t} e^{i k x} dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b_0(k)\, e^{i k\left(x - \frac{w(k)}{k} t\right)} dk$$

then, the phase velocity is given by $v_p(k) = w/k$.
A linear equation in one space, on time dimension, is called dispersive if $w(k)$ is real and $d^2 w/dk^2$ is not identically zero. The group velocity is given by $v_g(k) = dw/dk$ which corresponds to the stationary point

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b_0(k)\, e^{i k\left(x - \frac{w(k)}{k} t\right)} dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b_0(k)\, e^{i t(\phi(k))} dk, \quad \phi(k) = k(x/t) - w$$

$$\phi'(k) = \frac{x}{t} - dw/dk = \frac{x}{t} - v_g(k) = 0$$

So that we can expect the dominant contribution to the solution for large t to be in the neighborhood of $\frac{x}{t} = w'(k) = v_g(k)$.

$$\begin{cases} \text{phase velocity } (v_p(k)) : w/k \\ \text{group velocity}(v_g(k)) : dw/dk \end{cases}$$

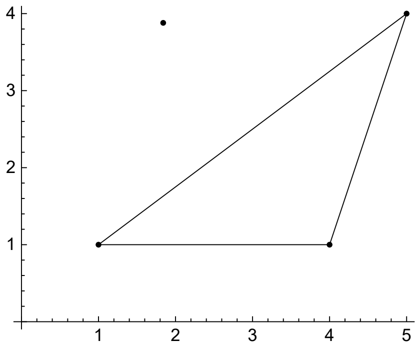
Poincaré Hyperbolic Disk

- Tiling Euclidean plane with triangles

• Reflection in complex z plane $= c' = \frac{(a^* b - b^* a) + (a-b) c^*}{a^* - b^*}$

```
reflection[{r_, p1_, p2_}] :=
 Module[{cr, cp1, cp2, newcr, nr},
  {cr, cp1, cp2} = Map[(First[#] + ⅈ Last[#]) &, {r, p1, p2}];
  newcr = (((Conjugate[cp1] cp2 - Conjugate[cp2] cp1) + (cp1 - cp2) Conjugate[cr]) /
      (Conjugate[cp1] - Conjugate[cp2]));
  nr = N[{Re[newcr], Im[newcr]}]]
```

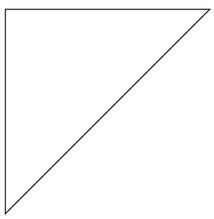
```
Graphics[{Line[{{4, 1}, {5, 4}, {1, 1}, {4, 1}}],
  Point /@ {{4, 1}, {5, 4}, {1, 1}, reflection[{{4, 1}, {5, 4}, {1, 1}}]}},
 Axes -> True, AxesOrigin → {0, 0}]
```



Define triangle

```
triangle[{r_, p1_, p2_}] :=
 {Line[{r, p1, p2, r}]}
```

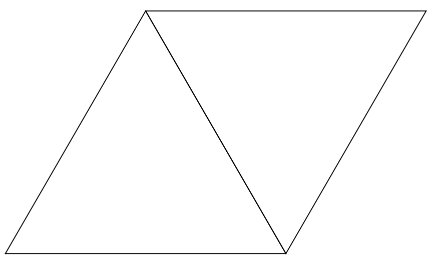
```
Graphics[triangle[{{1, 1}, {1, 2}, {2, 2}}]]
```



Now then, define reflectedTriangle

```
reflectedTriangle[{a_, b_, c_}] :=
 Module[{A}, Show[Graphics[
    {triangle[{a, b, c}],
     triangle[{A = reflection[{a, b, c}], b, c}]}, AspectRatio -> Automatic]]]
```

```
reflectedTriangle[{{0, 0}, {1, 0}, {0.5, √3 / 2}}]
```



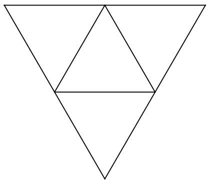
Let's get *n* th reflection!

```
cyclicPermutations[x_List] :=
 NestList[RotateLeft, x, Length[x] - 1]
```

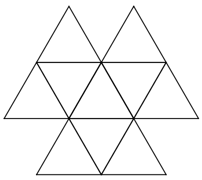
which returns that $\{x, \mathrm{rot}^1(x), \mathrm{rot}^2(x), \ldots, \mathrm{rot}^{(\mathrm{len}(x)-1)}(x)\}$

```
reflectTriangles[triangle[{a_, b_, c_}]] :=
 triangle /@ {ReplacePart[#, 1 → reflection[#]]} & /@ cyclicPermutations[{a, b, c}]
Iteration[{a_, b_, c_}] :=
     ReplacePart[#, 1 → reflection[#]] & /@ cyclicPermutations[{a, b, c}]
```

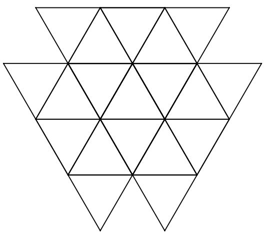
```
Graphics[reflectTriangles[triangle[{{0, 0}, {1, 0}, {0.5, √3 / 2}}]]]
```



```
Graphics[triangle /@ Flatten[Iteration /@
    Append[Iteration[{{0, 0}, {2, 0}, {1, √3}}], {{0, 0}, {2, 0}, {1, √3}}], 1]]
```



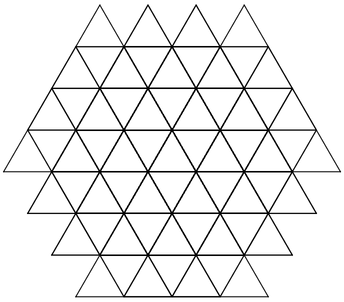
```
Graphics[triangle /@ Flatten[Iteration /@ Flatten[Iteration /@
      Append[Iteration[{{0, 0}, {2, 0}, {1, √3}}], {{0, 0}, {2, 0}, {1, √3}}], 1], 1]]
```



```
sList0 = sList
sList1 = Append[Iteration[sList0], sList0]
sList2 = Flatten[Iteration /@ sList1, 1]
slist3 = Flatten[Iteration /@ sList2, 1]
```

```
nIteration[{a_, b_, c_}, n_] :=
     Module[{sList1 = Append[Iteration[{a, b, c}], {a, b, c}]},
          For[i = 1, i < n, i++, {sList1 = Flatten[Iteration /@ sList1, 1]}];
               sList1]
```

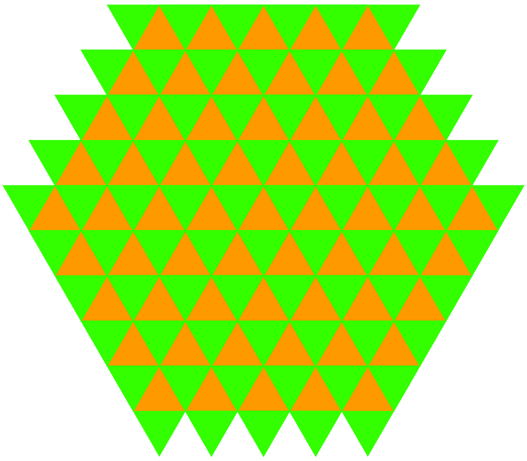
```
Graphics[triangle /@ nIteration[{{0, 0}, {2, 0}, {1, √3}}, 6]]
```



With *Mathematica*'s wonderful color function, you can create your own tessellation!

```
TriangleIteration[Hue[u_], {a_, b_, c_}, n_] :=
     Module[{u0 = u, sList = {Hue[u], Triangle[{a, b, c}]},
   tmp = Iteration[{a, b, c}], ctmp = {Hue[u], Triangle[{a, b, c}]}},
         For[i = 1, i < n, i++,
   {tmp = Flatten[Iteration /@ tmp, 1],
    ctmp = {Hue[u0 + 0.1 * (-1)^i], Triangle /@ tmp}, sList = AppendTo[sList, ctmp]}];
             sList]
```

```
Graphics[TriangleIteration[Hue[0.2], {{0, 0}, {2, 0}, {1, √3}}, 9]]
```

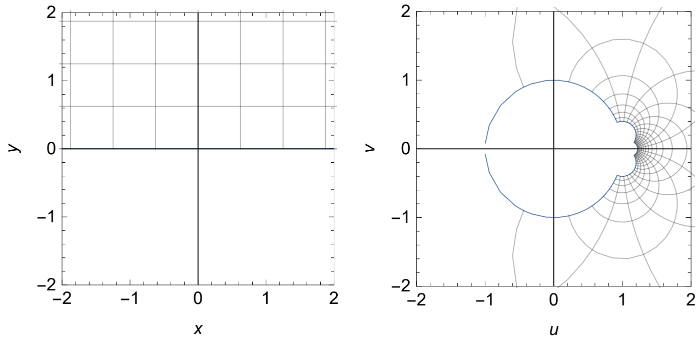


I believe you can think of the better code . If you have better code, then, email me.

■ Tilings of the Poincaré disc

• Hyperbolic transformation: $z \to \frac{\alpha z+\beta}{\gamma z+\delta}$ (whereas Euclidean transformation is $z \to a\,z + b$)

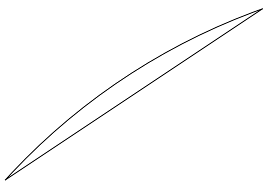
```
Conformal[(x + ⅈ y), ((x + ⅈ y) + ⅈ)/((x + ⅈ y) - ⅈ),
 {x, -5, 5}, {y, 0, 10}, PlotRange → {{-2, 2}, {-2, 2}}]
```



As you can see, a parallel line in hyperbolic plane is an arc in Euclidean plane.
 Following code is from the ‘Wolfram Demonstration Project’. Each codes define the geometry of hyperbolic plane.

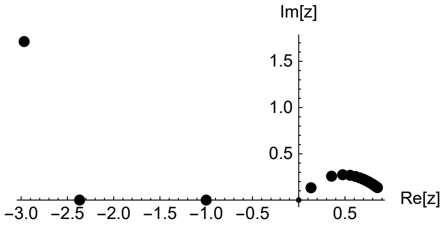
In[1]:=
```
geodesic[{z1_, z2_}] :=
  Module[{x1 = Re[z1], x2 = Re[z2], y1 = Im[z1],
    y2 = Im[z2], det, center, arg1, arg2, argMax, argMin},
   det = 4 (x1 * y2 - y1 * x2);
   If[Abs[det] < 10^(-3), Line[{{x1, y1}, {x2, y2}}],
    center =
     2 / det ((Im[z2 - z1] + Im[z2] Abs[z1]^2 - Im[z1] Abs[z2]^2) +
        I * (Re[z1 - z2] + Re[z1] Abs[z2]^2 - Re[z2] Abs[z1]^2));
    arg1 = Arg[z1 - center];
    arg2 = Arg[z2 - center];
    argMax = Max[arg1, arg2];
    argMin = Min[arg1, arg2];
    If[argMax - argMin ≥ Pi, argMin = argMin + 2 Pi];
    Circle[{Re[center], Im[center]}, Abs[center - z1], {argMin, argMax}]]];
```

```
Graphics[{Line[{{2, 1}, {-1, -1}}], geodesic[{2 + ⅈ, -1 - ⅈ}]}]
```



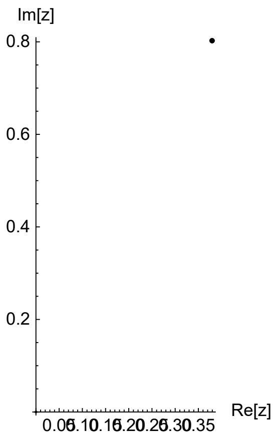
In[2]:=
```
vertex[kot_, n_] := Module[{abs, r =
      (Sin[Pi / n] / Sin[(kot + Pi) / 2])^2 / (1 - (Sin[Pi / n] / Sin[(kot + Pi) / 2])^2)},
   abs = Sqrt[1 + 2 r - 2 Sqrt[r (1 + r)] Cos[Pi - kot / 2 - Pi / 2 - Pi / n]];
   abs * Cos[Pi / n] + I * abs * Sin[Pi / n]];
```

```
ArgandPlot[vertex[1.5, #] & /@ Range[1, 20]]
```




```
In[3]:= rotate[zc_, kot_, z_] :=
          (E^(I * kot) (z - zc) / (1 - z * Conjugate[zc]) + zc) /
           (1 + E^(I * kot) Conjugate[zc] (z - zc) / (1 - z * Conjugate[zc]));
```

```
ArgandPlot[{rotate[1 + i, 1.5, 0]}]
```




```
In[4]:= rotateGeodesic[{zc_, z_}, kot_] :=
          {(E^(I * kot) (z - zc) / (1 - z * Conjugate[zc]) + zc) /
            (1 + E^(I * kot) Conjugate[zc] (z - zc) / (1 - z * Conjugate[zc])), zc};
```

```
In[5]:= reflectnGon[stranica_, n_, m_] :=
          Rest[NestList[rotateGeodesic[#, 2 Pi / m] &, stranica, n - 1]];
```

```
In[6]:= fillAround[nGon_, n_, m_] :=
          Flatten[Map[reflectnGon[Reverse[#], n, m] &, nGon], 1];
```

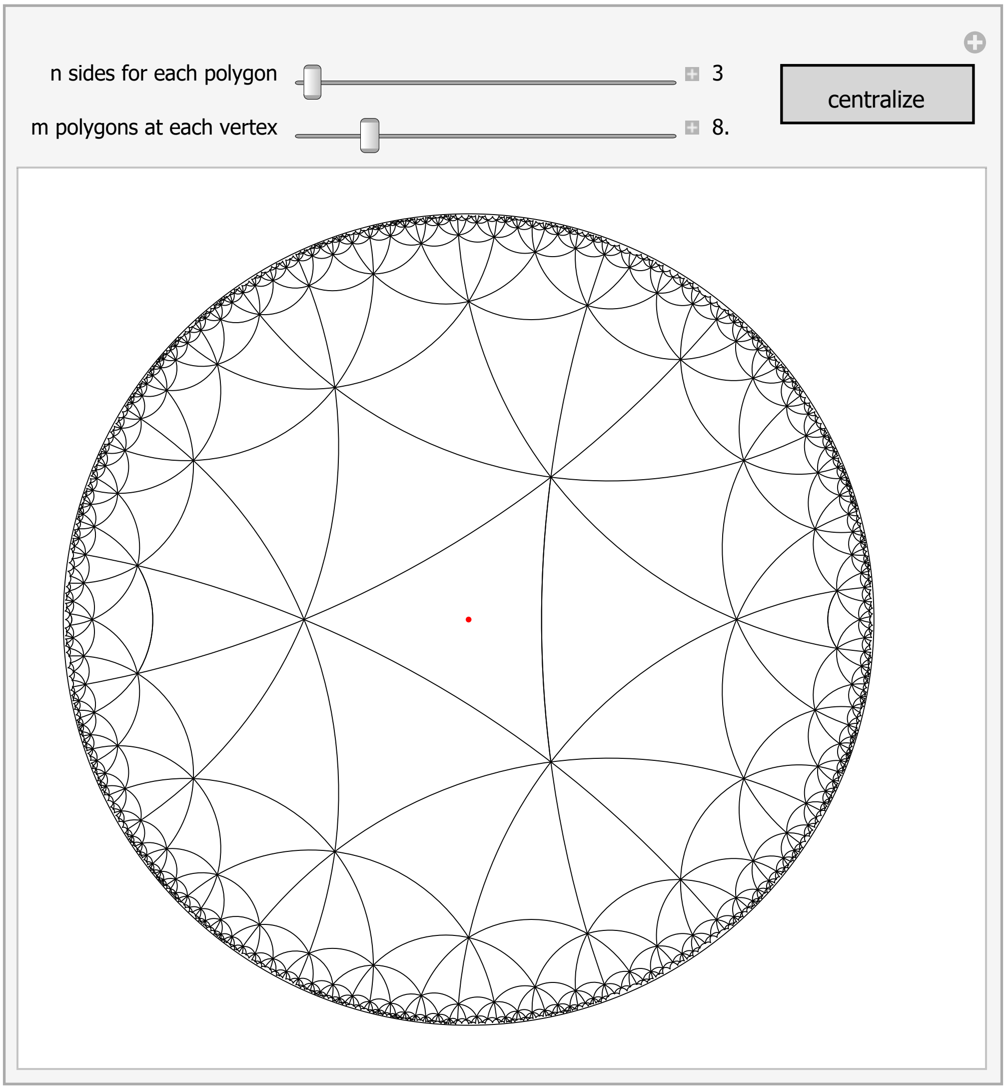
```
In[7]:= geodesicsForTiling[n_, m_, nivo_, c_] := Module[{ogliscaθ, oglisca},
           ogliscaθ = N[Partition[Map[(# + c) / (1 + Conjugate[c] #) &,
               NestList[rotate[0, 2 Pi / n, #] &, vertex[2 Pi / m, n], n]], 2, 1]];
           oglisca = Flatten[NestList[fillAround[#, n, m] &, ogliscaθ, nivo], 2];
           ogliscaθ = Partition[oglisca, 2];
           oglisca = geodesic /@ Union[Round[Map[Sort, ogliscaθ], .001]]
          ];
```

In[8]:=
```
geodesicsForTilingGray[n_, m_, nivo_, c_] := Module[{ogliscaθ, oglisca},
   ogliscaθ = N[Partition[Map[(# + c) / (1 + Conjugate[c] #) &,
       NestList[rotate[θ, 2 Pi / n, #] &, vertex[2 Pi / m, n], n]], 2, 1]];
   oglisca = Round[NestList[fillAround[#, n, m] &, ogliscaθ, nivo], .0001];
   ogliscaθ = Map[geodesic /@ # &, oglisca];
   oglisca = DeleteDuplicates[Flatten[Table[
       Prepend[ogliscaθ[[i]], Opacity[(nivo + 2 - i) / (nivo + 1)]], {i, nivo + 1}], 1]]
   (*oglisca=Union[Round[Map[Sort,ogliscaθ],.0001]]*)
  ];
```

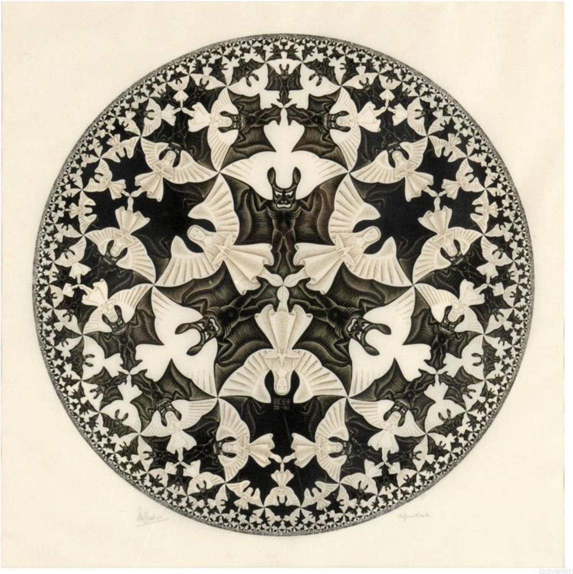
In[9]:=
```
nivo[n_, m_] :=
  If[n > 3, Floor[m / 2],
   If[m == 7, 11,
    If[m == 8, 10,
     If[m ≤ 20, 9, 7
     ]]]
  ];
```

In[10]:=
```
noMin[n_] := If[n == 3, 7, If[n == 4, 5, If[n == 5 || n == 6, 4, 3]]];
noMax[n_] := If[MemberQ[{3, 4, 5}, n], 13, If[n == 6, 10, 8]]
```

```
Manipulate[
 If[n == 3 && m < 7, m = 7];
 If[n == 4 && m < 5, m = 5];
 If[(n == 5 || n == 6) && m < 4, m = 4];
 If[n > 6 && m > 8, m = 8];
 If[n == 6 && m > 12, m = 12];
 EventHandler[Dynamic[Graphics[Join[{Circle[], Red, Point[{Re[c], Im[c]}], Black},
     geodesicsForTiling[n, m, nivo[n, m], c]], ImageSize → 1.1 {400, 400}]],
  "MouseClicked" :→ (Module[{cen = MousePosition["Graphics"]},
      If[Apply[Plus, cen^2] < 1, c = cen[[1]] + I * cen[[2]]]];)],
 {{c, 0}, ControlType → None},
 Row[{Column[{
     Control@{{n, 3, "n sides for each polygon"}, 3, 10, 1, Appearance → "Labeled"},
     Control@{{m, 7, "m polygons at each vertex"},
       noMin[n], noMax[n], 1, Appearance → "Labeled"}}],
   Button["centralize", c = 0, ImageSize → {100, 30}]}],
 TrackedSymbols → {n, m},
 SaveDefinitions → True,
 AutorunSequencing → All]
```



M.C. Esher left a famous painting using hyperbolic tiling $(n, m) = (3, 8)$.



M.C. Esher, ‘ Angels and Demons’, 1941

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