HW4

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1. (a). We assume that employees work h hours per day,

So for the optimization problem:

$$\text{maximize} \sum_{i=1}^{5}((\alpha-15h)k_i)-s_i-c_i)$$

$$\text{subject to} \quad k_i-3\geq 0 \quad i=1,\cdots,5$$

$$\sum_{i=1}^{5} s_i \leq 2\sum_{i=1}^{5} 15hk_i$$

(b). This is a LP optimization problem.

(c) First, we need to convert it to a convex optimization problem.

$$\text{minimize} \sum_{i=1}^{5}(s_i+c_i-(\alpha-15h)k_i)$$

$$\text{subject to} \quad 3-k_i\leq 0 \quad i=1,\cdots,5$$

$$\sum_{i=1}^{5}(s_i-30hk_i)\leq 0$$

$$L(k,\lambda_1,\lambda_2)=\sum_{i=1}^{5}(s_i+c_i-(\alpha-15h)k_i)+\sum_{i=1}^{5}\lambda_{1i}(3-k_i)+\lambda_2\left(\sum_{i=1}^{5}(s_i-30hk_i)\right)$$

(d). dual problem:

$$\nabla_k L=\sum_{i=1}^{5}(15h-\alpha)-\sum_{i=1}^{5}\lambda_{1i}-\sum_{i=1}^{5}30\lambda_2 h$$

$$-\sum_{i=1}^{5}\lambda_{1i}-\sum_{i=1}^{5}30\lambda_2 h=0 \quad \text{(complementory slackness)}$$

so for the $15h-\alpha$, if $\alpha>15h$, then $\nabla_k L<0$, it's unbounded below

$$g(\lambda_1,\lambda_2)=-\infty$$

if $\alpha<15h$, then $\nabla_k L>0$,

when $k=0$, $g(\lambda_1,\lambda_2)=\inf_k L=\sum_{i=1}^{5}(s_i+c_i)+3\sum_{i=1}^{5}\lambda_{1i}+\sum_{i=1}^{5}\lambda_2 s_i$

2. (a). When $P_i=0$, it can satisfy the constraints in the context.

(b). when $i=1$, $P_1-M\leq u_1$

For the perturbed problem,

We assume that strong duality holds, and that the dual optimum is attained.

$$p^*(u,v)\geq p^*(0,0)-\lambda^{*T}u-v^{*T}v \qquad v^{*T}=0$$

$$=p^*(0,0)-\lambda^{*T}u$$

$\lambda_1=100$, and it's large, and we loosen the constraint, $u_1>0$,

the optimal value is guaranteed to decrease greatly.

So I should increase the energy budget for store $i=1$.

(c). when $i=2$, $\lambda_2=0.001$. λ_2 is small, and we loosen the constraint, $u_2>0$,

the optimal value will not decrease too much.

so I should not increase the energy budget for store $i=2$.

3. We know that the equality constraint doesn't exist,

then $v_i=0$,

$f_i(x)\leq u_i$, $\quad p^*(0,0)=g(\lambda^*,v^*)\leq f_0(x)+\lambda^{*T}u$

$f_0(x)\geq p^*(0,0)-\lambda^{*T}u$

we can replace $f_0(x)$ with $p^*(u,v)$

then $p^*(u,v)\geq p^*(0,0)-\lambda^{*T}u$.

4. For the KKT conditions:

① $f_i(\tilde{x})\leq 0$
② $h_i(\tilde{x})=0$ primal feasible
③ $\tilde{\lambda}_i\geq 0$ dual feasible
④ $\tilde{\lambda}_i f_i(\tilde{x})=0 \rightarrow$ complementary slackness
⑤ $\frac{\partial L}{\partial x}=0$

$$Ax=\begin{bmatrix}-2&-1&0\\4&3&1\\1&1&1\\4&2&0\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}-3\\6\\1\\6\end{bmatrix}\leq\begin{bmatrix}-3\\6\\1\\7\end{bmatrix}$$

for complementary slackness, $f_4(x)<0$, then $\lambda_4=0$.

$$L(x,\lambda)=c^Tx+\lambda^T(Ax-b)$$

then $\frac{\partial L}{\partial x}=c^T+\lambda^TA=0$

$$[30,4,-8]+[\lambda_1,\lambda_2,\lambda_3,\lambda_4]\begin{bmatrix}-2&-1&0\\4&3&1\\1&1&1\\4&2&0\end{bmatrix}$$

so $\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\end{bmatrix}=\begin{bmatrix}40\\14\\-6\\0\end{bmatrix}$

for complementary slackness, $\lambda_i^*>0$, $f_i(x^*)=0$.

$f_3(x^*)=0$, but $\lambda_3<0$, so x^* isn't an optimal solution.

5. For this problem, we need to prove the objective function is a convex function.

$f(x)=\max(x)-\min(x)$.

We need to prove $f(\theta x_1+(1-\theta)x_2)\leq\theta f(x_1)+(1-\theta)f x_2$ $\quad x_1,x_2\in\mathbb{R}^n$

Because $f(x)\geq 0$,

then $f(\theta x)=\theta f(x)$

$f(\theta x_1+(1-\theta)x_2)=\max(\theta x_1+(1-\theta)x_2)-\min(\theta x_1+(1-\theta)x_2)$

$\theta f(x_1)+(1-\theta)f(x_2)=\theta\max(x_1)-\theta\min(x_1)+(1-\theta)\max(x_2)-(1-\theta)\min(x_2)$

We know that: $\max(\theta x_1+(1-\theta)x_2)\leq\theta\max(x_1)+(1-\theta)\max(x_2)$

$\min(\theta x_1+(1-\theta)x_2)\geq\theta\min(x_1)+(1-\theta)\min(x_2)$

So $f(x)$ is a convex function, this is a convex optimization problem.