Exercise 5 (Week 4)

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1. Define the bijective binary-counting function $\mathbb{B}^n \to [0, 2^n)$ and show that its inverse equals the function *bin* in Section 1.1.

SOLUTION. The binary-counting function $\mathbb{B}^n \to [0, 2^n)$ is defined as

$$f(b_n b_{n-1} \dots b_1) = \sum_{i=1}^{n} 2^{n-i} b_i.$$

To show that its inverse equals the function *bin*, i.e., $f^{-1}(x) = bin(x)$, we have to prove that

$$f(bin(x)) = x \tag{1}$$

and

$$bin(f(b_n b_{n-1} \dots b_1)) = b_n b_{n-1} \dots b_1. \tag{2}$$

For (1), suppose

$$bin(x) = b_n b_{n-1} \dots b_1.$$

By the conversion rule of binary numbers, we have

$$x = \sum_{i=1}^{n} 2^{n-i} b_i.$$

Then

$$f(bin(x)) = f(b_n b_{n-1} \dots b_1) = \sum_{i=1}^{n} 2^{n-i} b_i = x.$$

For (2), since

$$f(b_n b_{n-1} \dots b_1) = \sum_{i=1}^{n} 2^{n-i} b_i,$$

we have

$$bin(f(b_n b_{n-1} \dots b_1)) = bin(\sum_{i=1}^{n} 2^{n-i} b_i) = b_n b_{n-1} \dots b_1.$$

By (1) and (2), $f^{-1}(x) = bin(x)$ holds. $\square$

2. Check the matrix Λ in the cases $n = 1, 2$.

Solution. We have

$$\begin{aligned}\Lambda(\chi) - E(\chi) &= E(\chi) - \chi \\ \Lambda(\chi) &= 2E(\chi) - \chi \\ \forall x : \mathbb{B}^n \cdot \Lambda(\chi)(x) &= 2E(\chi) - \chi(x)\end{aligned}$$

then for $n = 1$,

$$\forall x : \mathbb{B} \cdot \Lambda(\chi)(x) = 2(\frac{1}{2}\sum_{y:\mathbb{B}} \chi(y)) - \chi(x),$$

$$\Lambda = 2 \times \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

and for $n = 2$,

$$\forall x : \mathbb{B}^2 \cdot \Lambda(\chi)(x) = 2(\frac{1}{4}\sum_{y:\mathbb{B}^2} \chi(y)) - \chi(x),$$

$$\Lambda = 2 \times \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix} - \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1\end{pmatrix}.$$

3. Check that multiplication by matrix (1) achieves rotation in the plane.

Solution. Consider a point represented as $(\cos b, \sin b)$, where b is the angle between the point and the x-axis.

Then after rotation by the matrix (1), we have

$$\begin{aligned}\begin{pmatrix}\cos a & -\sin a\\ \sin a & \cos a\end{pmatrix}\begin{pmatrix}\cos b\\ \sin b\end{pmatrix} &= \begin{pmatrix}\cos a\cos b - \sin a\sin b\\ \sin a\cos b + \cos a\sin b\end{pmatrix}\\ &= \begin{pmatrix}\cos(a+b)\\ \sin(a+b)\end{pmatrix},\end{aligned}$$

which is the point after rotation by angle a.

Therefore, multiplication by matrix (1) achieves rotation in the plane.