Exercise 4

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1. Show that the matrix H_n is unitary and that quantum initialisation (1) is given by $H_n\delta_0$.

SOLUTION. The matrix H_n is defined as

$$H_{2^n} = 2^{-n/2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}^{\otimes n}.$$

We can prove that H_n is unitary by reduction. When $n = 2$,

$$H_2 = 2^{-1/2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, H_2^\dagger = 2^{-1/2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Then

$$H_2H_2^\dagger = H_2^\dagger H_2 = 2^{-1}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = 2^{-1}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = I_2.$$

Thus, H_2 is unitary. Suppose that H_n is unitary, then

$$H_{2n} = H_n \otimes H_2 = 2^{-1/2}\begin{pmatrix}H_n & H_n\\ H_n & -H_n\end{pmatrix}, H_{2n}^\dagger = 2^{-1/2}\begin{pmatrix}H_n^\dagger & H_n^\dagger\\ H_n^\dagger & -H_n^\dagger\end{pmatrix}.$$

$$\begin{aligned}
H_{2n}H_{2n}^\dagger = H_{2n}^\dagger H_{2n} &= 2^{-1}\begin{pmatrix}H_n & H_n\\ H_n & -H_n\end{pmatrix}\begin{pmatrix}H_n^\dagger & H_n^\dagger\\ H_n^\dagger & -H_n^\dagger\end{pmatrix}\\
&= 2^{-1}\begin{pmatrix}2H_nH_n^\dagger & 0\\ 0 & -2H_nH_n^\dagger\end{pmatrix}\\
&= 2^{-1}\begin{pmatrix}2I_n & 0\\ 0 & -2I_n\end{pmatrix}\\
&= I_{2n}.
\end{aligned}$$

Thus, H_{2n} is unitary. Therefore, H_n is unitary.

Then we prove that quantum initialisation (1) is given by $H_n\delta_0$.
For $n = 2$,

$$H_2\delta_0 = 2^{-1/2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = 2^{-1/2}\begin{pmatrix}1\\ 1\end{pmatrix}.$$

Suppose that $H_{2^n}\delta_0 = 2^{-n/2}\begin{pmatrix}1\\ 1\\ \vdots\\ 1\end{pmatrix}$ holds, then

$$H_{2^{n+1}}\delta_0 = 2^{-1/2}\begin{pmatrix}H_{2^n} & H_{2^n}\\ H_{2^n} & -H_{2^n}\end{pmatrix}\begin{pmatrix}1\\ 0\\ \vdots\\ 0\end{pmatrix} = 2^{-1/2}\begin{pmatrix}H_{2^n}\\ H_{2^n}\end{pmatrix} = 2^{-(n+1)/2}\begin{pmatrix}1\\ 1\\ \vdots\\ 1\end{pmatrix},$$

which is the initialization of a $(n+1)$-qubit quantum system.
Thus, quantum initialization (1) is given by $H_n\delta_0$.

2. Check that the matrix Φ_ϕ for phase shift is unitary.

Solution.

$$\Phi_\phi = \begin{pmatrix}1 & 0\\ 0 & \mathrm{e}^{\mathrm{i}\phi}\end{pmatrix}, \Phi_\phi^\dagger = \begin{pmatrix}1 & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\phi}\end{pmatrix}.$$

Then we have

$$\begin{aligned}\Phi_\phi\Phi_\phi^\dagger &= \begin{pmatrix}1 & 0\\ 0 & \mathrm{e}^{\mathrm{i}\phi}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\phi}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & \mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}\phi}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &= I_2,\end{aligned}$$

and

$$\begin{aligned}
\Phi_\phi^\dagger \Phi_\phi &= \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= I_2.
\end{aligned}$$

Thus, the matrix Φ_ϕ for phase shift is unitary.

3. Recall the operator T_f. Identify $T_\oplus$, for exclusive or, and show that it is unitary.

Solution.

$$T_\oplus \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, T_\oplus^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
T_\oplus T_\oplus^\dagger = T_\oplus^\dagger T_\oplus &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= I_4,
\end{aligned}$$

So the operator $T_\oplus$ is unitary.

4. What is the result of including in the program for the random bit, evolution by

$$2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}?$$

Solution. It turns a normal state into a quantum state with equal probability of being in the 0 state and the 1 state since

$$2^{-1/2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = 2^{-1/2}\begin{pmatrix}1\\ 1\end{pmatrix},$$

$$2^{-1/2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = 2^{-1/2}\begin{pmatrix}1\\ -1\end{pmatrix}.$$