Exercise 2 & 3

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Exercise 2

1. If Schrödinger's cat happens to be in state $\frac{1}{\sqrt{2}}(-1,-1)$ on the unit circle, what is the observed result?

Solution. Let $\chi = \frac{1}{\sqrt{2}}(-1,-1)$, then $|\langle\chi,\delta_0\rangle| = \frac{1}{2}$ and $|\langle\chi,\delta_1\rangle| = \frac{1}{2}$, where $\delta_0 = (1,0)$ and $\delta_1 = (0,1)$. So the observed result is 50% in state alive and 50% in state dead.

2. What about in state $(0,-1)$?

Solution. Assume $\delta_0 = (1,0)$ means "alive" and $\delta_1 = (0,1)$ means "dead". Let $\chi = (0,-1)$, then $|\langle\chi,\delta_0\rangle| = 0$ and $|\langle\chi,\delta_1\rangle| = 1$. So the observed result is 100% in state dead. The state $(0,-1)$ is same as the state $(0,1)$, since the square of their projections on the axes are the same.

3. Suppose that instead of the unit circle in the plane, the quantum state of Schrödinger's cat, a qubit, is represented as the set of complex numbers with modulus 1,

$$\{z : \mathbb{C} \mid z = \mathrm{e}^{\mathrm{i}\theta}, 0 \le \theta \le 2\pi\}$$

so θ is in radians. How are the states in the first two questions represented? What results if the cat is observed in state $\mathrm{e}^{\mathrm{i}3.0}$?

Solution. $\frac{1}{\sqrt{2}}(-1,-1)$ can be represented as $\frac{1}{\sqrt{2}}(-1-i)$. It is equivalent to $\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$, according to Euler's formula. So the state in question 1 is $\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$.

In the same way, $(0,-1)$ can be represented as $\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}$.

$e^{i3.0} = \cos 3.0 + i \sin 3.0$ represents the state $(\cos 3.0, \sin 3.0)$. So the observed result is dead of probability $\sin^2 3.0 (\approx 0.02)$ and alive of probability $\cos^2 3.0 (\approx 0.98)$. (Assume $(1,0)$ means alive and $(0,1)$ means dead.)

Exercise 3

1. Show how to construct the standard basis of $\mathbb{R}^8$ from the standard basis of $\mathbb{R}^2$ using tensor product.

SOLUTION. The standard basis of $\mathbb{R}^2$ is

$$\delta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then the standard basis of $\mathbb{R}^8$ is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T$$

$$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\end{pmatrix}^T$$

$$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}^T$$

$$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}^T$$

2. Let I_n be the identity matrix of dimension n. What is $I_2 \otimes I_2$? And $I_2 \otimes I_2 \otimes I_2$?

Solution.

$$I_2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$I_2 \otimes I_2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix} = I_4$$

$$I_2 \otimes I_2 \otimes I_2 = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix} = I_8$$

$$\underbrace{I_2 \otimes I_2 \otimes \cdots \otimes I_2}_{n \text{ times}} = I_{2^n}$$

3. Let $H_2 := \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. What is $H_2 \otimes H_2$? $H_2 \otimes H_2 \otimes H_2$?

Solution.

$$H_2 \otimes H_2 = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$$

$$\begin{aligned}H_2 \otimes H_2 \otimes H_2 &= \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &= \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1\\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1\end{pmatrix}\end{aligned}$$

4. Show that quantum states are orthogonal if they are antipodal, i.e., diametrically opposite, on the Bloch sphere (like δ_0 and δ_1).

Solution. Let $\psi_1 = (\theta, \phi)$ be a state on the Bloch sphere. Then the antipodal state is $\psi_2 = (\pi - \theta, \phi + \pi)$. $\psi_1 = (\theta, \phi)$ can be represented as $\cos\frac{\theta}{2}\delta_0 + \sin\frac{\theta}{2}e^{\phi i}\delta_1$(I learned this from the internet). Then $\psi_2 = \cos\frac{\pi-\theta}{2}\delta_0 + e^{(\phi+\pi)i}\sin\frac{\pi-\theta}{2}\delta_1 = \sin\frac{\theta}{2}\delta_0 - e^{\phi i}\cos\frac{\theta}{2}$. Their inner product is

$$\begin{aligned}\langle \psi_1, \psi_2\rangle &= \sin\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{(\phi-\phi)i}\\ &= \sin\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ &= 0\end{aligned}$$

So they are orthogonal.

(Actually, I do not really understand this problem. I used some knowledge I learned from the internet to solve it.)

5. Show how to initialise the system consisting of a pair of Schrödinger cats. What is the result after evolution by $H_2 \otimes H_2$?

Solution. A Schrödinger's cat can be represented as $\frac{1}{\sqrt{2}}(1,1)$. Then a pair of Schrödinger's cats can be represented as

$$\frac{1}{\sqrt{2}}(1,1) \otimes \frac{1}{\sqrt{2}}(1,1) = \frac{1}{2}(1,1,1,1)$$

After evolution by $H_2 \otimes H_2$, the state becomes

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$