Kalman Filter

Grundlagen

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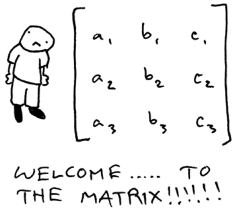

Rudolf Emil Kalman

- was born in 1930 in Hungary
- Bachelor of Science and Master of Science at the MIT
- PHD 1957 at the Columbia
- developed the filter arond the years 1960 to 1961
- died in 2016 in Florida

Fig.: Rudolf Emil Kalman

What is the Kalman Filter?

The Kalman filter is a mathematical method for iteratively estimating parameters to describe system states.
In this process, a prediction about a parameter value is repeatedly made, combined with the error-prone measurement, and then used again to make a new prediction.

Kalman Filter

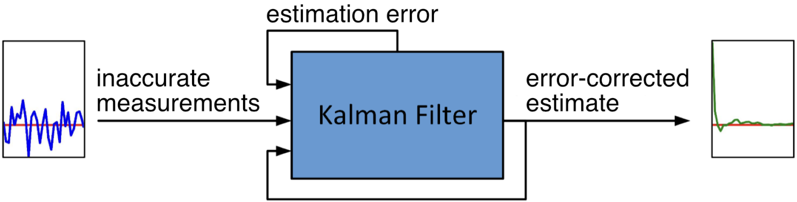

Fig.: Procedure of the Kalman Filter[1]

Sensors susceptible to failure

- Light sensor (photoresistor)
- Ultrasonic sensor
- Infrared sensor
- Temperature sensor (thermoelectric)
- Magnetic field sensor
- Gas and air quality sensor
- Humidity sensor
- Motion sensor (accelerometer)
- Pressure sensor
- Sound sensor

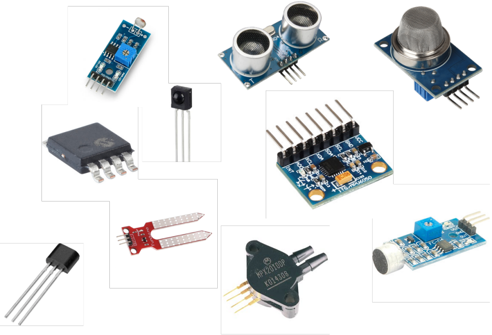

Fig.: sensors

The Mean

$$\mu = \frac{1}{N}\sum_{n=1}^{N} V_n = \frac{1}{5}\,(5+5+10+10+10) = 8 \text{ Cent}$$

The standarddeviation

	Player1	Player2	Player3	Player4	Player5	Mean
Team A	1.89m	2.10m	1.75m	1.98m	1.85m	1.914m
Team B	1.94m	1.90m	1.97m	1.89m	1.87m	1.194m

$$x_n - \mu = x_n - 1.914m$$

	Player1	Player2	Player3	Player4	Player5
Team A	−0.024m	0.186m	−0.164m	0.066m	−0.064m
Team B	0.026m	−0.014m	0.056m	−0.024m	−0.044m

The variance

	Player1	Player2	Player3	Player4	Player5
Team A	−0.024m	0.186m	−0.164m	0.066m	−0.064m
Team B	0.026m	−0.014m	0.056m	−0.024m	−0.044m

$$(x_n - \mu)^2 = (x_n - 1.914m)^2$$

	Player1	Player2	Player3	Player4	Player5
Team A	0.000576m^2	0.034596m^2	0.026896m^2	0.004356m^2	0.004096m^2
Team B	0.000676m^2	0.000196m^2	0.003136m^2	0.000576m^2	0.001936m^2

The variance

$$\sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(x_{B_n} - \mu)^2$$

$$\sigma_A^2 = \frac{1}{N}\sum_{n=1}^{N}(x_{A_n} - \mu)^2 = \frac{1}{5}(0.000576 + 0.034596 + 0.026896 + 0.004356 + 0.004096) = 0.014m^2$$

$$\sigma_B^2 = \frac{1}{N}\sum_{n=1}^{N}(x_{B_n} - \mu)^2 = \frac{1}{5}(0.000676 + 0.000196 + 0.003136 + 0.000576 + 0.001936) = 0.0013m^2$$

Covariance

- **Definition:** Covariance is a measure of how much two random variables change together. If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the lesser values (i.e., the variables tend to show similar behavior), the covariance is positive. In the opposite case, when the greater values of one variable mainly correspond to the lesser values of the other (i.e., the variables tend to show opposite behavior), the covariance is negative.

Mathematical Definition

Given two random variables X and Y, the covariance is defined as:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

where $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ are the expected values (means) of X and Y, respectively.

- **Example:** Consider two random variables, X and Y, representing the number of hours studied and the test scores of students. If higher hours of study tend to correspond with higher test scores, the covariance between X and Y would be positive.

Properties

- $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
- $\text{Cov}(X, X) = \text{Var}(X)$ (Variance of X)
- If X and Y are independent, $\text{Cov}(X, Y) = 0$

Process of Kalman Filters

Prediction

1. State prediction: $\hat{x}_k = A\hat{x}_{k-1} + Bu_{k-1}$
2. Covariance prediction: $P_k = AP_{k-1}A^T + Q$

Correction

3. Kalman Gain Prediction: $K_k = P_k H^T (HP_k H^T + R)^{-1}$
4. State Update: $\hat{x}_k = \hat{x}_k + K_k(z_k - H\hat{x}_k)$
5. Covariance Update: $P_k = (I - K_k H)P_k$

Kalman explained

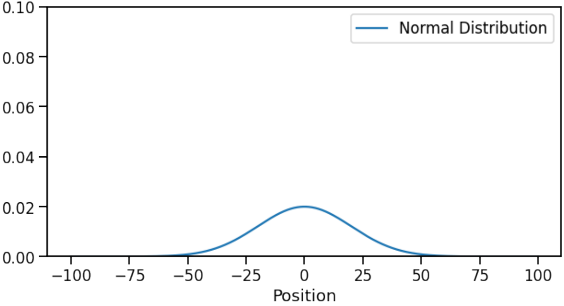

Fig.: Start Postion at t_0

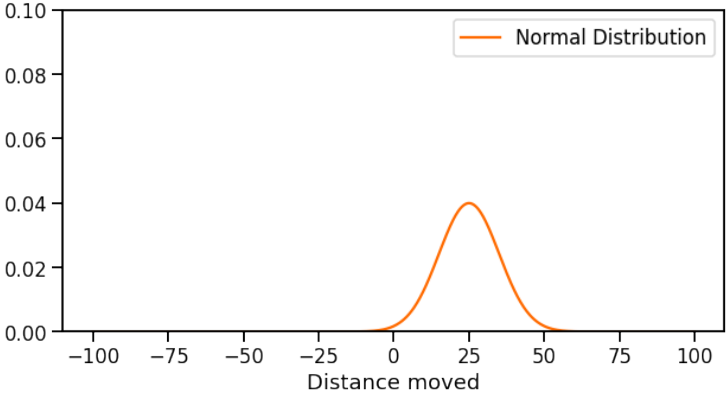

Fig.: Calculated Postion at t_1

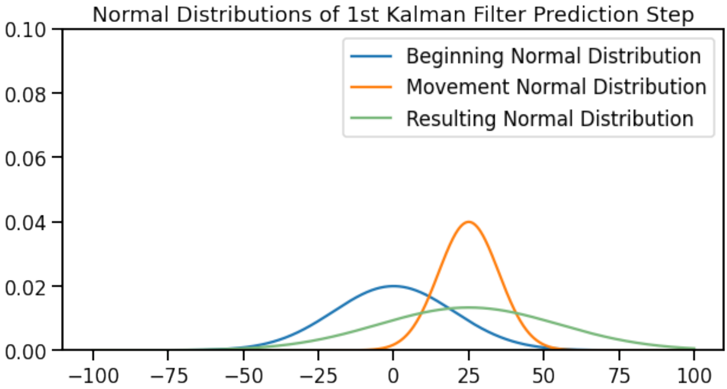

Fig.: Predicted Postion at t_1

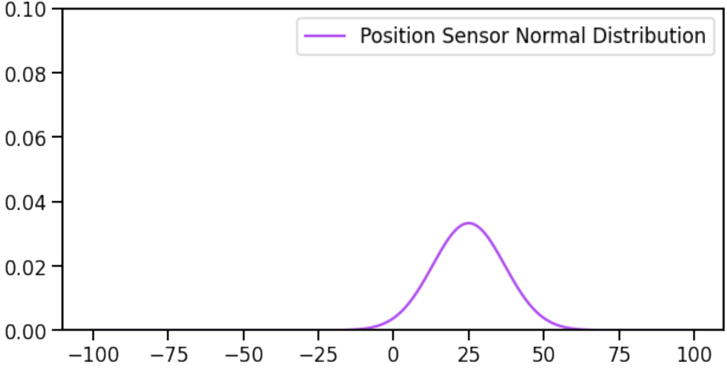

Fig.: Messured Sensor Data of Postion at t_1

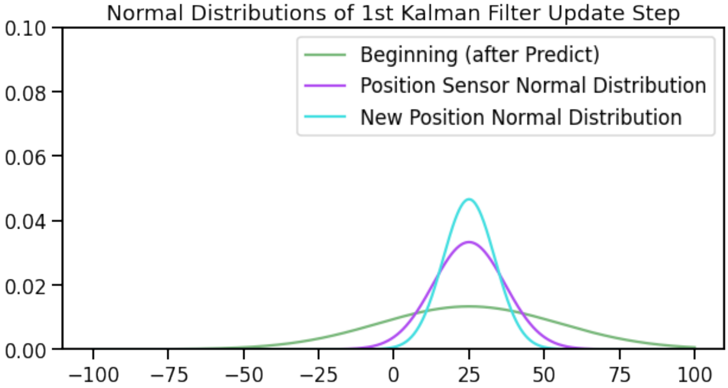

Fig.: Correction of the Kalman Gain after measurement at t_1

Takeaways

- The precision decreases with the prediction
- The precision increases with the correction

Einbindung in einen konkreten Anwendungsfall

State Representation

We define the state vector x to include the position and velocity of the mouse in both x and y directions

$$x = \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix}$$

where:

- x is the position on the x-axis
- y is the postion on the y-axis
- v_x is the velocity in the x-direction
- v_y is the velocity in the y-direction

Formulate the Transtition Model

The state transition model describes how the state evolves from one time step to the next. If we assume a constant velocity model, the state transition can be expressed as:

$$x_{k+1} = Ax_k + w_k$$

where w_k represents the proces noise, assumed to be Gaussian with zero mean and covariance Q.

Create the Transition Matrix

The transition matrix A for a constant velocity Model is:

$$A = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Δt is the time intervall between mesurements.

Observation Model

The observation model realtes the state to the measurements:

$$z_k = Hx_k + v_k$$

where v_k represents the measurement noise, assumed to be Gaussian with zero mean and covariance R.
The observation matrix H for direct measurement of position is:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The prediction step estimates the next state and its uncertainty:

- Predict the state:

$$x_{k+1} = Ax_k \tag{1}$$

- Predict the error coavriance:

$$P_{k+1} = AP_kA^T + Q \tag{2}$$

Correction Step

The correction step updates the state estimate with the new measurement:

- Compute the Kalman gain:

$$K_k = P_k H^T (H P_k H^T + R)^{-1} \tag{3}$$

- Update the state estimate:

$$x_k = x_k + K_k (z_k - H x_k) \tag{4}$$

- Update the error covariance:

$$P_k = (I - K_k H) P_k \tag{5}$$

Kalman Filter

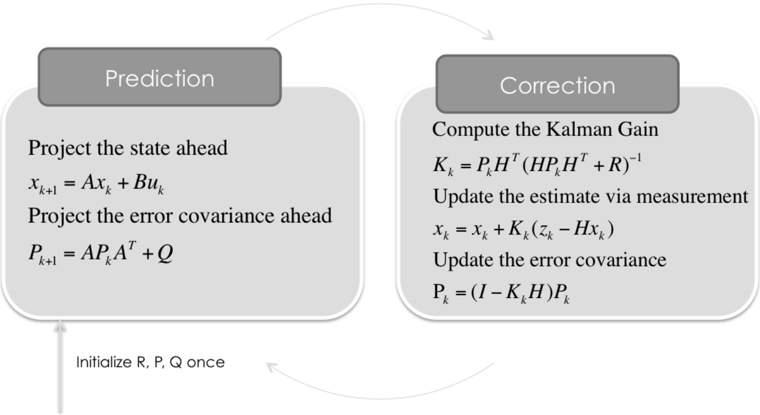

Fig.: Iterative Nature of the Kalman filter

Cheat Sheet - Math Symbols

Symbol	Meaning
k	Interval or iteration of the Kalman filter
z_k	Measurement vector. This contains the real-world measurement we received in this time step.
x_k	Newest estimate of the current "true" state.
P_k	Newest estimate of the average error for each part of the state.
R	Estimated measurement error covariance. Finding precise values for Q and R are beyond the scope of this guide.
Q	Estimated process error covariance. Finding precise values for Q and R are beyond the scope of this guide.
A	State transition matrix. Basically, multiply state by this and add control factors, and you get a prediction of the state for the next time step.
H	Observation matrix. Multiply a state vector by H to translate it to a measurement vector.
B	Control matrix. This is used to define linear equations for any control factors.
u_{k-1}	Control vector. This indicates the magnitude of any control system's or user's control on the situation.

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List of figures

[1]A test footnote in the first column