Kalman Filter

Grundlagen

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What is the Kalman Filter?

The Kalman filter is a mathematical method for iteratively estimating parameters to describe system states.
In this process, a prediction about a parameter value is repeatedly made, combined with the error-prone measurement, and then used again to make a new prediction.

Kalman Filter

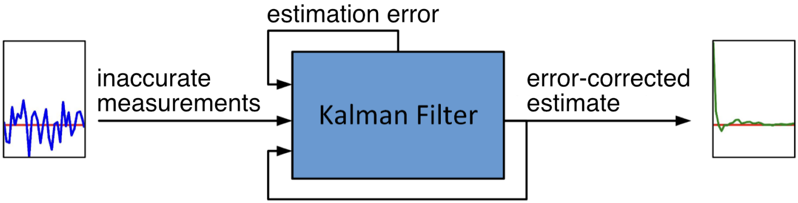

Fig.: Procedure of the Kalman Filter[1]

Sensors susceptible to failure

- Light sensor (photoresistor)
- Ultrasonic sensor
- Infrared sensor
- Temperature sensor (thermoelectric)
- Magnetic field sensor
- Gas and air quality sensor
- Humidity sensor
- Motion sensor (accelerometer)
- Pressure sensor
- Sound sensor

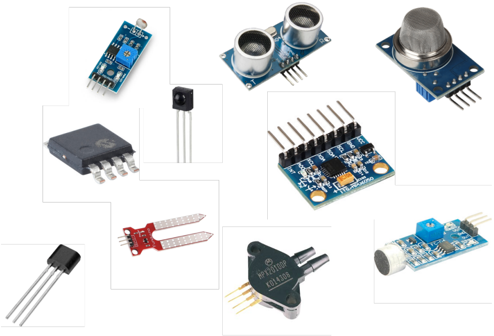

Fig.: sensors

The Mean

$$\mu = \frac{1}{N}\sum_{n=1}^{N} V_n = \frac{1}{5}\left(5+5+10+10+10\right) = 8cent$$

The standarddeviation

	Player1	Player2	Player3	Player4	Player5	Mean
Team A	1.89m	2.10m	1.75m	1.98m	1.85m	1.914m
Team B	1.94m	1.90m	1.97m	1.89m	1.87m	1.194m

$$x_n - \mu = x_n - 1.914m$$

	Player1	Player2	Player3	Player4	Player5
Team A	−0.024m	0.186m	−0.164m	0.066m	−0.064m
Team B	0.026m	−0.014m	0.056m	−0.024m	−0.044m

The variance

	Player1	Player2	Player3	Player4	Player5
Team A	-0.024m	0.186m	-0.164m	0.066m	-0.064m
Team B	0.026m	-0.014m	0.056m	-0.024m	-0.044m

$$(x_n - \mu)^2 = (x_n - 1.914m)^2$$

	Player1	Player2	Player3	Player4	Player5
Team A	0.000576m^2	0.034596m^2	0.026896m^2	0.004356m^2	0.004096m^2
Team B	0.000676m^2	0.000196m^2	0.003136m^2	0.000576m^2	0.001936m^2

The variance

$$\sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(x_{B_n} - \mu)^2$$

$$\sigma_A^2 = \frac{1}{N}\sum_{n=1}^{N}(x_{A_n} - \mu)^2$$
$$= \frac{1}{5}(0.000576 + 0.034596 + 0.026896 + 0.004356 + 0.004096) = 0.014m^2$$

$$\sigma_B^2 = \frac{1}{N}\sum_{n=1}^{N}(x_{B_n} - \mu)^2$$
$$= \frac{1}{5}(0.000676 + 0.000196 + 0.003136 + 0.000576 + 0.001936) = 0.0013m^2$$

Covariance

- **Definition:** Covariance is a measure of how much two random variables change together. If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the lesser values (i.e., the variables tend to show similar behavior), the covariance is positive. In the opposite case, when the greater values of one variable mainly correspond to the lesser values of the other (i.e., the variables tend to show opposite behavior), the covariance is negative.

Mathematical Definition

Given two random variables X and Y, the covariance is defined as:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

where $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ are the expected values (means) of X and Y, respectively.

The covariance

- **Example:** Consider two random variables, X and Y, representing the number of hours studied and the test scores of students. If higher hours of study tend to correspond with higher test scores, the covariance between X and Y would be positive.

Properties

- $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$
- $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$ (Variance of X)
- If X and Y are independent, $\mathrm{Cov}(X, Y) = 0$

The covariance

Prediction

1. State prediction: $\hat{x}_k = A\hat{x}_{k-1} + Bu_{k-1}$
2. Covariance prediction: $P_k = AP_{k-1}A^T + Q$

Correction

3. Kalman Gain Prediction: $K_k = P_kH^T(HP_kH^T + R)^{-1}$
4. State Update: $\hat{x}_k = \hat{x}_k + K_k(z_k - H\hat{x}_k)$
5. Covariance Update: $P_k = (I - K_kH)P_k$

Kalman explained

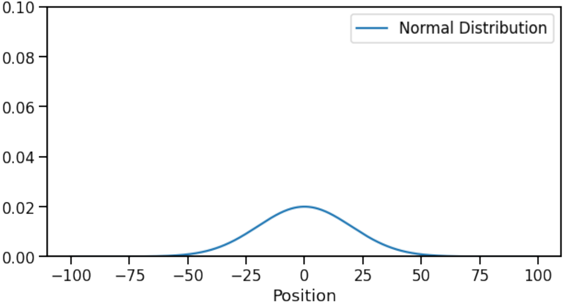

Fig.: Start Postion at t_0

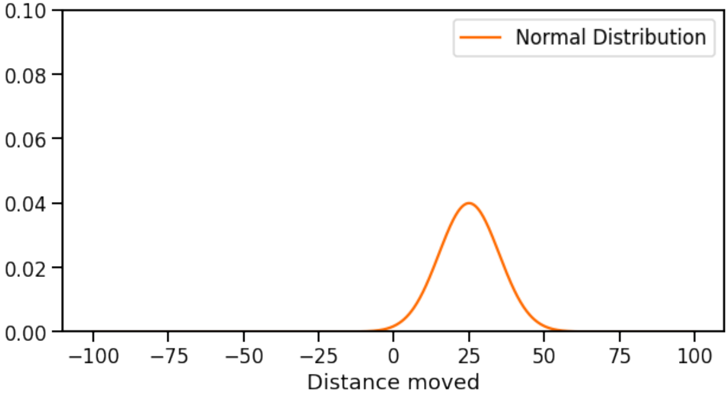

Fig.: Calculated Postion at t_1

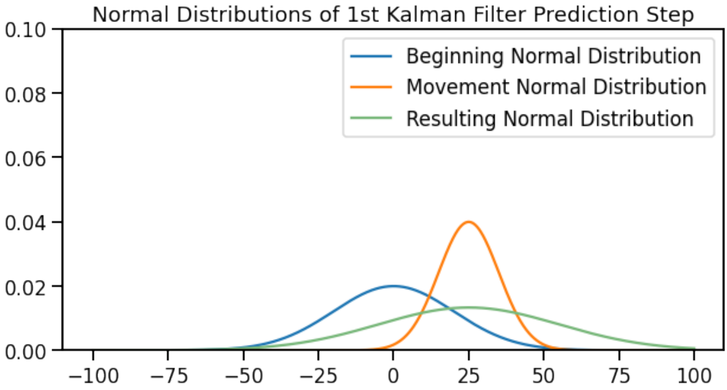

Fig.: Predicted Postion at t_1

Kalman explained

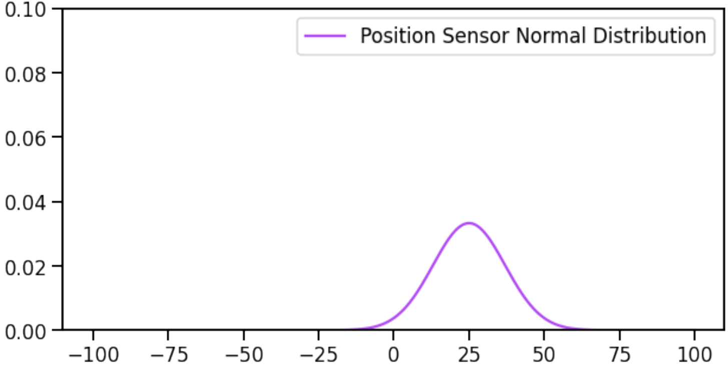

Fig.: Messured Sensor Data of Postion at t_1

Kalman explained

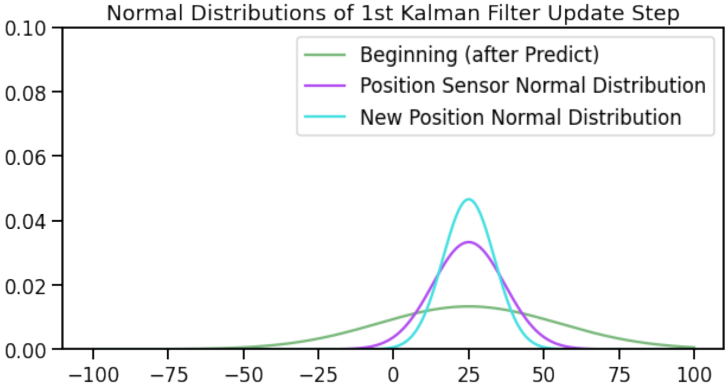

Fig.: Correction of the Kalman Gain after measurement at t_1

Takeaways

- The precision decreases with the prediction
- The precision increases with the correction

State Representation

We define the state vector x to include the position and velocity of the mouse in both x and y directions

$$x = \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix}$$

where:

- x is the position on the x-axis
- y is the postion on the y-axis
- v_x is the velocity in the x-direction
- v_y is the velocity in the y-direction

Formulate the Transtition Model

The state transition model describes how the state evolves from one time step to the next. If we assume a constant velocity model, the state transition can be expressed as:

$$x_{k+1} = Ax_k + w_k$$

where w_k represents the proces noise, assumed to be Gaussian with zero mean and covariance Q.

Create the Transition Matrix

The transition matrix A for a constant velocity Model is:

$$A = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Δt is the time intervall between mesurements.

Observation Model

The observation model realtes the state to the measurements:

$$z_k = Hx_k + v_k$$

where v_k represents the measurement noise, assumed to be Gaussian with zero mean and covariance R.
The observation matrix H for direct measurement of position is:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Prediction Step

The prediction step estimates the next state and its uncertainty:

- Predict the state:

$$x_{k+1} = Ax_k \tag{1}$$

- Predict the error coavriance:

$$P_{k+1} = AP_kA^T + Q \tag{2}$$

Correction Step

The correction step updates the state estimate with the new measurement:

- Compute the Kalman gain:

$$K_k = P_k H^T (H P_k H^T + R)^{-1} \tag{3}$$

- Update the state estimate:

$$x_k = x_k + K_k (z_k - H x_k) \tag{4}$$

- Update the error covariance:

$$P_k = (I - K_k H) P_k \tag{5}$$

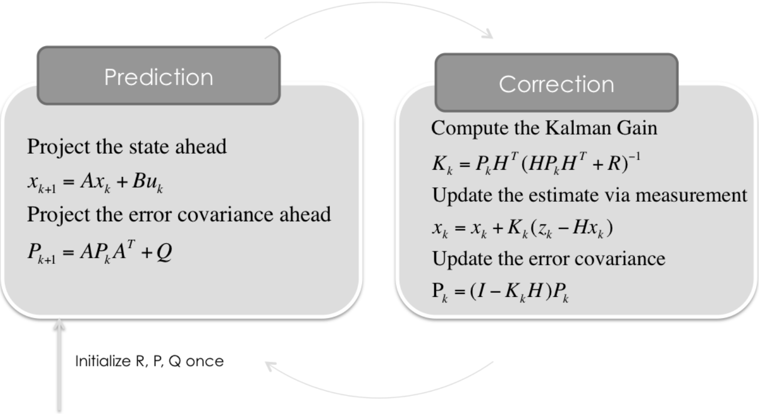

Fig.: Iterative Nature of the Kalman filter

Cheat Sheet - Math Symbols

Symbol	Meaning
k	Interval / iteration of the Kalman filter
z_k	Measurement from the sensor
x_k	Value of the current estimation
x_{k-1}	Value of the previous estimation
P_k	Value of the current error covariance
P_{k-1}	Value of the previous error covariance
R	Variance of the measurements
Q	Process noise covariance
A	Transition Matrix
H	Observation Matrix
Bu_{k-1}	Control signal

Summary

KALMA KALMA !

List of figures

[1] A test footnote in the first column