

European Option Pricing with Lévy Processes

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1 Introduction

The objective of this project is primarily educational. Its purpose is to understand in detail the mathematical and statistical mechanisms that operate in return dynamics models based on Lévy processes, as well as their applications to option modelling and pricing. More precisely, the work pursues the following goals:

- Present Lévy processes, their fundamental properties, and the elements required for modelling financial returns;
- Study in detail two classical models from the literature:
 - the Variance–Gamma (VG) model, obtained by gamma subordination of a Brownian motion,
 - the Merton model with Gaussian jumps;
- Implement and compare two parameter estimation methods:
 - maximum likelihood estimation (MLE),
 - the estimator based on the empirical characteristic function (ECF);
- Analyse and compare the goodness of fit of the two models to empirical data;
- Finally, apply the estimated models to the pricing of a European option using the Carr–Madan method based on the Fourier transform.

2 Lévy Processes

Lévy processes provide a natural extension of the Black–Scholes model by introducing trajectories that combine a Brownian component, jumps of small or large magnitude, and a deterministic drift. Unlike the classical BS model, which assumes continuous log-normal returns and constant volatility, Lévy processes capture features observed in financial markets such as sudden price jumps, heavy tails, and skewed return distributions.

In the context of European call pricing, this generalisation offers several advantages: a more realistic representation of return distributions, allowing better option valuation when markets anticipate extreme events; and the possibility to calibrate the model to implied volatilities across maturities and strikes, improving the fit of the volatility surface. The following steps are based on Cont & Tankov.

2.1 Definition

A stochastic process $(X_t)_{t \geq 0}$ is called a *Lévy process* if the following conditions hold:

1. **Independent increments:** for any $0 \leq t_1 < t_2 < \dots < t_n$, the increments

$$X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$$

are mutually independent.

2. **Stationary increments:** for all $0 \leq s < t$,

$$X_t - X_s \stackrel{d}{=} X_{t-s},$$

meaning that the distribution of increments depends only on the length of the interval.

3. **Càdlàg paths:** the trajectory $t \mapsto X_t$ is right-continuous with left limits.

These properties imply several fundamental consequences:

- **Continuity in probability:** for any $\varepsilon > 0$,

$$\lim_{h \rightarrow 0} P(|X_{t+h} - X_t| > \varepsilon) = 0.$$

- **No deterministic jumps:** for any $t \geq 0$,

$$P(X_t = X_{t-}) = 1.$$

- **Nature of trajectories:** a Lévy process generally has jumps. In fact, any Lévy process with almost surely continuous paths must be a Brownian motion with drift.

2.2 Characteristic Function and Lévy Exponent

The characteristic function plays a central role in the study of Lévy processes. For any $u \in R$,

$$E[e^{iuX_t}] = (E[e^{iuX_1}])^t = e^{t \log E[e^{iuX_1}]}.$$

Define the *Lévy exponent* $\psi(u)$ by

$$\psi(u) := \log E[e^{iuX_1}],$$

so that

$$E[e^{iuX_t}] = e^{t\psi(u)}.$$

2.3 Lévy–Itô Decomposition

Any Lévy process can be decomposed as the sum of four independent components: deterministic drift, a Brownian component, small jumps, and large jumps. More precisely:

$$X_1 = \gamma + \sigma W_1 + \int_{|x| < 1} x \tilde{N}(1, dx) + \int_{|x| > 1} x N(1, dx),$$

where:

- W is a standard Brownian motion,
- $N(t, dx)$ is a Poisson random measure with intensity $t \nu(dx)$,
- $\tilde{N}(t, dx) = N(t, dx) - t \nu(dx)$ is the compensated measure for small jumps ($|x| < 1$),
- $\gamma \in R$ and $\sigma \geq 0$,
- ν is the *Lévy measure*, defined on $R \setminus \{0\}$ and satisfying

$$\int_R (1 \wedge x^2) \nu(dx) < \infty.$$

The measure ν describes the frequency and distribution of jumps of size x , including the abundance of small jumps, the possible presence of rare but extreme jumps, asymmetry between upward and downward movements, and tail heaviness.

2.3.1 Large Jumps and Compound Poisson Processes

For $|x| > 1$, the number of jumps in $[0, 1]$ follows a Poisson law with parameter $\nu(|x| > 1)$, and the jump amplitudes are i.i.d. with density proportional to $\nu(dx)$. The characteristic function of this component is:

$$E\left[e^{iu \int_{|x| > 1} x N(1, dx)}\right] = \exp\left(\int_{|x| > 1} (e^{iux} - 1) \nu(dx)\right).$$

2.3.2 Small Jumps and Compensation

If $\nu(|x| < 1) < \infty$, small jumps are finite in number and handled like large jumps. If $\nu(|x| < 1) = \infty$, there are almost surely infinitely many small jumps, and compensation is required:

$$e^{iux} - 1 \longrightarrow e^{iux} - 1 - iux \mathbf{1}_{|x| < 1},$$

ensuring convergence of the integral and càdlàg paths.

2.4 Lévy–Khintchine Formula

Combining all components yields the Lévy–Khintchine formula:

$$\phi_{X_t}(u) = E[e^{iuX_t}] = \exp \left[t \left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \int_R (e^{iux} - 1 - iux\mathbf{1}_{|x|<1})\nu(dx) \right) \right].$$

- Drift: $iu\gamma$.
- Brownian diffusion: $-\frac{1}{2}\sigma^2 u^2$.
- Jump part: $\int_R (e^{iux} - 1 - iux\mathbf{1}_{|x|<1})\nu(dx)$.

Thus, the triplet (γ, σ, ν) fully characterises the process: drift, continuous fluctuations, and jumps.

Derivations of the basic characteristic functions used here, including those of the compound Poisson process and the normal distribution, are collected in Annex A (see A.1 and A.2).

3 Specific Models (Variance Gamma, Merton)

3.1 Variance Gamma

The Variance Gamma (VG) process, originally introduced by Madan and Seneta (1990) and detailed in Fiorani (2004), is a fundamental example of a Lévy process. It is constructed by replacing the time of a Brownian motion with a Gamma subordinator, a procedure known as subordination. The study of the VG process provides a concrete example of a Lévy process with an explicitly computable characteristic function, whose parameters are directly linked to the Lévy triplet.

The VG process is defined by

$$X_t = \theta G_t + \sigma W_{G_t},$$

where:

- $(W_s)_{s \geq 0}$ is a standard Brownian motion,
- $(G_t)_{t \geq 0}$ is a Gamma subordinator, independent of W ,
- $G_t \sim \Gamma(\alpha = t/\nu, \beta = \nu)$,
- $\theta \in R$ is a drift parameter,
- $\sigma > 0$ is the volatility associated with the Brownian component,
- $\nu > 0$ is the jump-activity parameter.

Subordination preserves the fundamental properties of Lévy processes, since G_t is increasing with independent and stationary increments.

3.1.1 Fundamental Properties

Independent increments. Since $(G_{t+s} - G_t)$ is independent of $(G_u)_{u \leq t}$ and W has independent increments,

$$X_{t+s} - X_t = \theta(G_{t+s} - G_t) + \sigma(W_{G_{t+s}} - W_{G_t}),$$

which is independent of the past.

Stationary increments. Because $(G_{t+s} - G_t) \stackrel{d}{=} G_s$, it follows that

$$X_{t+s} - X_t \stackrel{d}{=} X_s.$$

Càdlàg paths and pricing approaches. The VG process is a Lévy process of the *finite-variation infinite-activity* type, with no direct Brownian component. Given estimated parameters (θ, σ, ν) , there are two equivalent ways to price European options:

- **Via the Lévy measure $\nu(dx)$:** Construct the characteristic function $\varphi_{X_T}(u)$ using the Lévy–Khintchine formula

$$\varphi_{X_T}(u) = \exp \left[iu\gamma T + T \int_R (e^{iux} - 1 - iux\mathbf{1}_{|x|<1})\nu(dx) \right],$$

then apply the Carr–Madan Fourier transform to obtain option prices.

- **Directly via the VG characteristic function:**

$$\varphi_{X_T}(u) = \left(1 - i\theta\nu u + \frac{1}{2}\sigma^2\nu u^2 \right)^{-T/\nu},$$

which is then inserted into Carr–Madan. This method is computationally more efficient.

Both methods yield identical option prices, but the first provides deeper insight into the contribution of jumps, whereas the second is more efficient numerically.

3.1.2 Characteristic Function

The characteristic function of X_t is

$$\rho_{X_t}(u) = E[e^{iuX_t}].$$

Conditionally on $G_t = g$:

$$E[e^{iu(\theta g + \sigma W_g)}] = e^{iu\theta g - \frac{1}{2}\sigma^2 u^2 g} = e^{gA}, \quad A := iu\theta - \frac{1}{2}\sigma^2 u^2.$$

Since $W_g \sim \mathcal{N}(0, g)$, this follows from the standard property of Brownian motion.

Expectation with respect to G_t . Because $G_t \sim \Gamma(t/\nu, \nu)$,

$$E[e^{AG_t}] = (1 - \nu A)^{-t/\nu}, \quad \Re(A) < 1/\nu.$$

Thus the characteristic function of the VG process is:

$$\rho_{X_t}(u) = \left(1 - \nu \left(iu\theta - \frac{1}{2}\sigma^2 u^2 \right) \right)^{-t/\nu}.$$

3.1.3 Interpretation of the Parameters

- ν controls jump frequency: small $\nu \Rightarrow$ few jumps (Brownian limit), large $\nu \Rightarrow$ high jump activity.
- σ adjusts jump magnitude: larger σ produces larger jumps.
- θ controls skewness: the sign and direction of asymmetry.

Since subordination implies $E[G_t] = t$, we have $\alpha\beta = t$ with $\alpha = t/\nu$, $\beta = \nu$.

As $\nu \rightarrow 0$,

$$\lim_{\nu \rightarrow 0} (1 - \nu A)^{-t/\nu} = e^{tA},$$

and the VG process converges to a Brownian motion with drift θ and volatility σ .

3.2 The Merton Model

The Merton model (1976), discussed in Cont & Tankov, is a classical Lévy process applied to financial markets. It extends the Black–Scholes model by integrating log-normal jumps, thereby capturing market shocks, skewness, and excess kurtosis observed in empirical returns.

3.2.1 Definition of the Merton Process

The log-price $X_t = \ln S_t$ satisfies:

$$X_t = \left(\mu - \frac{\sigma^2}{2} - \lambda k \right) t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

where:

- W_t is a standard Brownian motion,
- σ is the diffusive volatility,
- $N_t \sim \text{Poisson}(\lambda t)$ is the number of jumps,
- $Y_i \sim \mathcal{N}(\mu_J, \sigma_J^2)$ is the i -th jump size,
- $k = E[e^{Y_i} - 1] = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$,
- γ compensates for drift induced by jumps so that the expected return remains μ .

The price S_t remains strictly positive since jumps are exponentials of normal variables: e^{Y_i} is log-normal.

3.2.2 Interpretation as a Lévy Process

The process decomposes into a continuous part and a jump part:

$$X_t = X_t^{\text{cont}} + X_t^{\text{jumps}}.$$

The increments are independent and stationary, confirming that X_t is a *Lévy process*. Its Lévy triplet is:

$$\left(\mu - \frac{\sigma^2}{2} - \lambda k, \sigma^2, \nu(dy) = \lambda \frac{1}{\sqrt{2\pi\sigma_J^2}} \exp\left[-\frac{(y - \mu_J)^2}{2\sigma_J^2}\right] dy \right),$$

where ν is the Lévy measure associated with Gaussian jumps.

3.2.3 Main Properties

- **Log-normal jumps:** $S_t = S_0 e^{X_t}$ remains positive.
- **Non-zero skewness and kurtosis:** jumps capture asymmetry and heavy tails.
- **Moments:**

$$E[S_t] = S_0 e^{\mu t}, \quad \text{Var}(S_t) = S_0^2 e^{2\mu t} \left(e^{\sigma^2 t + \lambda t(\sigma_J^2 + \mu_J^2)} - 1 \right).$$

3.2.4 Characteristic Function

The characteristic function of X_t is

$$\phi_{X_t}(u) = E[e^{iuX_t}].$$

Brownian part. The continuous component is normal:

$$E[e^{iuX_t^{\text{cont}}}] = \exp\left(iu\left(\mu - \frac{\sigma^2}{2} - \lambda k\right)t - \frac{1}{2}\sigma^2 u^2 t\right).$$

Jump part. Conditionally on $N_t = n$:

$$\sum_{i=1}^n Y_i \sim \mathcal{N}(n\mu_J, n\sigma_J^2),$$

and thus

$$E[e^{iuX_t^{\text{jumps}}}] = \exp\left(\lambda t(e^{iu\mu_J - \frac{1}{2}\sigma_J^2 u^2} - 1)\right).$$

Full characteristic function. Combining the two components:

$$\phi_{X_t}(u) = \exp\left\{iu\left(\mu - \frac{\sigma^2}{2} - \lambda k\right)t - \frac{1}{2}\sigma^2 u^2 t + \lambda t(e^{iu\mu_J - \frac{1}{2}\sigma_J^2 u^2} - 1)\right\}.$$

4 Parameter Estimation and Optimization Algorithm

To estimate the parameters of the models under study, two main methods are used: maximum likelihood estimation (MLE) and a method based on the characteristic function.

4.1 Maximum Likelihood Estimation

Let X be a real-valued random variable, continuous or discrete, whose distribution depends on a parameter θ belonging to a parametric family \mathcal{D}_θ . We define a function f by:

$$f(x; \theta) = \begin{cases} f_\theta(x), & \text{if } X \text{ is continuous,} \\ P_\theta(X = x), & \text{if } X \text{ is discrete.} \end{cases}$$

The *likelihood* of θ for an i.i.d. sample (x_1, \dots, x_n) is given by:

$$L(x_1, \dots, x_n; \theta) = \prod_{i=1}^n f(x_i; \theta).$$

4.1.1 Case of the Variance Gamma Process

For an increment X_t of a Variance Gamma process:

$$X_t = \theta G_t + \sigma W_{G_t}, \quad G_t \sim \Gamma\left(\frac{t}{\nu}, \nu\right),$$

where W_{G_t} is a standard Brownian motion independent of G_t .

The marginal density of X_t is obtained by integrating the conditional density over the law of the Gamma subordinator:

$$f_{VG}(x; \theta, \sigma, \nu) = \int_0^{+\infty} f_{X|G}(x|g) f_G(g) dg,$$

with

$$f_{X|G}(x|g) = \frac{1}{\sqrt{2\pi\sigma^2 g}} \exp\left[-\frac{(x - \theta g)^2}{2\sigma^2 g}\right], \quad f_G(g) = \frac{1}{\Gamma(\alpha)\beta^\alpha} g^{\alpha-1} e^{-g/\beta}, \quad \alpha = \frac{t}{\nu}, \beta = \nu.$$

This integral yields an expression involving the modified Bessel function K_γ :

$$K_\gamma(\zeta) = \frac{1}{2} \int_0^{+\infty} t^{\gamma-1} \exp\left[-\frac{\zeta}{2}\left(t + \frac{1}{t}\right)\right] dt, \quad \gamma = \alpha - \frac{1}{2}.$$

The marginal density is then:

$$f_{VG}(x; \theta, \sigma, \nu) = \frac{2e^{\theta x/\sigma^2}}{\sigma\sqrt{2\pi\nu^\alpha}\Gamma(\alpha)} \left(\frac{|x|}{\sqrt{\theta^2 + 2\sigma^2/\nu}} \right)^{\alpha-1/2} K_{\alpha-1/2} \left(\frac{|x|}{\sigma^2} \sqrt{\theta^2 + 2\sigma^2/\nu} \right).$$

This expression can be found in Fiorani (2004).

The log-likelihood for a sample r_1, \dots, r_n is:

$$\ell(\theta, \sigma, \nu) = \sum_{i=1}^n \ln f_{VG}(r_i; \theta, \sigma, \nu).$$

Optimization The MLE parameters are obtained by solving:

$$\frac{\partial \ell}{\partial \theta} = 0, \quad \frac{\partial \ell}{\partial \sigma} = 0, \quad \frac{\partial \ell}{\partial \nu} = 0,$$

generally using numerical methods.

The Python code used to estimate the VG parameters via MLE is available on GitHub by D. Laptev <https://github.com/dlaptev/VarGamma> (accessed February 10, 2025).

4.1.2 Case of the Merton Model

For the log-price $X_t = \ln S_t$:

$$X_t = \left(\mu - \frac{\sigma^2}{2} - \lambda k \right) t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

with $N_t \sim \text{Poisson}(\lambda t)$ and $Y_i \sim \mathcal{N}(\mu_J, \sigma_J^2)$.

The conditional density for $N_t = n$ is:

$$f_{X_t|N_t=n}(x) = \frac{1}{\sqrt{2\pi(\sigma^2 t + n\sigma_J^2)}} \exp \left[-\frac{\left(x - \left(\mu - \frac{\sigma^2}{2} - \lambda k \right) t - n\mu_J \right)^2}{2(\sigma^2 t + n\sigma_J^2)} \right].$$

The marginal density is obtained by summing over all n :

$$f_M(x) = \sum_{n=0}^{\infty} f_{X_t|N_t=n}(x) e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

The log-likelihood for a sample r_1, \dots, r_n is:

$$\ell(\mu, \sigma, \lambda, \mu_J, \sigma_J) = \sum_{i=1}^n \ln f_M(r_i; \mu, \sigma, \lambda, \mu_J, \sigma_J),$$

and the MLE is obtained by numerically solving:

$$\frac{\partial \ell}{\partial \mu} = 0, \quad \frac{\partial \ell}{\partial \sigma} = 0, \quad \frac{\partial \ell}{\partial \lambda} = 0, \quad \frac{\partial \ell}{\partial \mu_J} = 0, \quad \frac{\partial \ell}{\partial \sigma_J} = 0.$$

4.2 Empirical Characteristic Function Method (ECF)

The empirical characteristic function (ECF) method estimates the parameters of a Lévy process by comparing its theoretical characteristic function with its empirical characteristic function. This approach is particularly useful when the marginal density is complex or difficult to manipulate analytically, while the characteristic function admits a simple closed-form expression.

4.2.1 Theoretical Characteristic Function

For an increment X_t , the following closed-form expressions are available.

- **Variance Gamma (VG).** The characteristic function of the VG process is

$$\phi_{VG}(u; \theta, \sigma, \nu) = E[e^{iuX_t}] = \left(1 - i\theta\nu u + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu}.$$

- **Merton Model.** With $k = E[Y_i - 1]$, the characteristic function is:

$$\phi_M(u; \mu, \sigma, \lambda, \mu_J, \sigma_J) = \exp\left(iu(\mu - \sigma^2/2 - \lambda k)t - \frac{1}{2}\sigma^2 u^2 t + \lambda t(e^{iu\mu_J - \frac{1}{2}\sigma_J^2 u^2} - 1)\right).$$

4.2.2 Empirical Characteristic Function

Given a sample (r_1, \dots, r_n) , the empirical characteristic function is defined by:

$$\hat{\phi}(u) = \frac{1}{n} \sum_{j=1}^n e^{iur_j}.$$

4.2.3 Fitting Criterion

Parameter estimation is based on minimizing the weighted squared distance between the empirical and theoretical characteristic functions:

$$J(\Theta) = \sum_{k=1}^K w_k |\hat{\phi}(u_k) - \phi(u_k; \Theta)|^2,$$

where:

- Θ is the parameter vector (e.g., (θ, σ, ν) for VG or $(\mu, \sigma, \lambda, \mu_J, \sigma_J)$ for Merton),
- $(u_k)_{k=1}^K$ is a grid of frequencies (defined in 6.1),
- (w_k) is a sequence of weights, often constant or decreasing to reduce the influence of high frequencies.

Thus, the ECF estimate is:

$$\hat{\Theta} = \arg \min_{\Theta} J(\Theta).$$

Weighting is necessary because:

- Low frequencies $u \approx 0$ contain the main information about the distribution's moments, while high frequencies $u \gg 1$ are noisier, especially for daily financial data such as the S&P500.
- Without weighting, discrepancies at high frequencies may dominate the loss, degrading parameter estimation.

For this reason, we use a simple and robust decreasing weight, e.g.:

$$w(u) = \frac{1}{1 + u^2},$$

which gradually decreases the weight of high frequencies.

4.2.4 Parameter Initialization

A suitable initialization is crucial for optimization convergence. It is typically based on empirical moments or on a diffusion/jump separation.

Variance Gamma (VG). Instead of using raw moments, initial values can be defined using robust estimates:

1. **Location (c)**

Use the empirical mean of log-returns:

$$c_0 = \bar{r} = \frac{1}{n} \sum_{i=1}^n r_i.$$

2. **Diffusion volatility (σ)**

Use a robust estimate based on the MAD:

$$\sigma_0 = \text{MAD} = \text{median}(|r_i - \bar{r}|),$$

adjusted by the 75% quantile of the standard normal:

$$\sigma_0 = \frac{\text{MAD}}{\Phi^{-1}(0.75)}.$$

3. **Asymmetry / drift of the Gamma clock (θ)**

Initial guess based on empirical skewness, bounded to avoid extreme values:

$$\theta_0 = \text{sign}(\text{skew}) \cdot \min(|\text{skew}| \cdot \sigma_0, \sigma_0).$$

4. **Shape parameter (ν)**

Instead of $\sigma_0^2/(\text{kurtosis} - 3)$, which may blow up when kurtosis is close to 3, use a bounded version:

$$\nu_0 = \text{clip} \left(\frac{\sigma_0^2}{\max(\text{kurtosis} - 3, 0.1)}, 0.01, 10 \right).$$

Merton Model (normal jumps). Parameter initialization is done in several steps to separate the diffusion and jump components and remain robust to extreme values.

1. **Location:** Use the median of log-returns:

$$\hat{\mu} = \text{median}(r_1, \dots, r_n).$$

2. **Robust volatility:** Based on MAD:

$$\text{MAD} = \text{median}(|r_i - \hat{\mu}|), \quad \sigma_{\text{rob}} = \frac{\text{MAD}}{\Phi^{-1}(0.75)}.$$

3. **Jump detection:** Returns exceeding a threshold $\text{thr} \cdot \sigma_{\text{rob}}$ are considered jumps:

$$\mathcal{J} = \{r_i : |r_i - \hat{\mu}| > \text{thr} \cdot \sigma_{\text{rob}}\}, \quad \text{thr} = 3.$$

4. **Jump frequency and parameters:** Empirical jump frequency and jump moments:

$$\hat{\lambda} = \frac{|\mathcal{J}|}{n}, \quad \hat{\lambda} \geq 10^{-6},$$

$$(\hat{\mu}_J, \hat{\sigma}_J) = \begin{cases} \left(\frac{1}{|\mathcal{J}|} \sum_{r_i \in \mathcal{J}} r_i, \sqrt{\frac{1}{|\mathcal{J}| - 1} \sum_{r_i \in \mathcal{J}} (r_i - \hat{\mu}_J)^2} \right), & |\mathcal{J}| \geq 2, \\ (-0.02, 0.05), & \text{otherwise (fallback).} \end{cases}$$

5. **Diffusion volatility:** Estimated on the full sample:

$$\hat{\sigma} = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (r_i - \bar{r})^2}, \quad \text{if } \hat{\lambda} > 0.05 : \quad \hat{\sigma} \leftarrow \max(0.8\hat{\sigma}, 10^{-6}).$$

4.3 Optimization Problem and Nelder–Mead Method

4.3.1 Problem Formulation

Whether using MLE or ECF, parameter estimation amounts to solving

$$f(\Theta) = \begin{cases} -\log L(\Theta) & \text{(MLE),} \\ \sum_{k=1}^m w_k |\hat{\phi}(u_k) - \phi(u_k; \Theta)|^2 & \text{(ECF).} \end{cases}$$

The function f is generally nonlinear, multimodal, and may be nondifferentiable, justifying the use of a gradient-free method such as the Nelder–Mead algorithm.

4.3.2 Principle of the Nelder–Mead Method

Nelder–Mead (Wikipedia (2025)) uses a *simplex* of $n + 1$ points in a space of dimension n .

- **Simplex:** set of $n+1$ points (e.g. 4 points for VG).
- **Operations:** reflection, expansion, contraction (inner or outer), reduction.

The idea is to move the simplex to explore the parameter space:

- *Reflection* tests the direction opposite the worst point;
- *Expansion* extends a promising direction;
- *Contraction* brings the simplex closer to the best point when no improvement is found;
- *Reduction* shrinks the simplex around the best solution.

Simplex Initialization

1. Choose an initial point x_0 (from moments or a robust estimate).
2. Construct n near points to form a simplex.
3. Evaluate $f(x_i)$ and sort the points:

$$f(x_1) \leq f(x_2) \leq \dots \leq f(x_{n+1}).$$

Simplex Operators

- **Centroid:**

$$x_c = \frac{1}{n} \sum_{i=1}^n x_i.$$

- **Reflection:**

$$x_r = x_c + \alpha(x_c - x_h), \quad \alpha > 0.$$

- **Expansion:**

$$x_e = x_c + \gamma(x_r - x_c), \quad \gamma > 1.$$

- **Contraction (outer or inner):**

$$x_{co} = x_c + \beta(x_r - x_c), \quad x_{ci} = x_c - \beta(x_c - x_h), \quad 0 < \beta < 1.$$

- **Reduction:**

$$x_i = x_1 + \delta(x_i - x_1), \quad 0 < \delta < 1.$$

4.3.3 Criteria and Statistical Tests

To evaluate the fit of the Variance Gamma (VG) and Merton models, we use the following criteria:

- **Kolmogorov–Smirnov (KS):** maximum distance between the empirical distribution F_n and the theoretical distribution F :

$$D_n = \sup_x |F_n(x) - F(x; \hat{\Theta})|.$$

- **Wasserstein Distance (order 1):** mean transport distance between two distributions P and Q :

$$W_p(\mu, \nu) = \inf \left\{ \left(E[d(X, Y)^p] \right)^{1/p} \mid P_X = \mu, P_Y = \nu \right\}.$$

5 Pricing of a European Call

5.1 Price of a European Call

We consider a financial asset $(S_t)_{t \geq 0}$ defined on a probability space (Ω, \mathcal{F}, P) , whose underlying dynamics are modeled by a Lévy process. We work in a general framework where the risk-free rate r and the cost of carry (or dividend yield) q are constant.

Definition of the European Call. A European call option with maturity T and strike price K is a derivative product whose payoff at maturity is:

$$\text{Payoff} = (S_T - K)^+ = \max(S_T - K, 0).$$

The buyer of the call therefore holds the right, but not the obligation, to purchase the underlying asset at price K at time T .

Pricing Principle: Absence of Arbitrage. In the absence of arbitrage opportunities, the present value of the call must equal the expected value of its payoff, discounted at the risk-free rate. To express this principle rigorously, we introduce the fundamental notion of a *risk-neutral measure*.

Risk-Neutral Measure and Martingale Property

In an arbitrage-free market, there exists a probability measure Q equivalent to the historical measure P such that the discounted price of the underlying becomes a martingale:

$$e^{-(r-q)t} S_t \text{ is a martingale under } Q.$$

Equivalently,

$$E^Q[S_T \mid \mathcal{F}_t] = e^{(r-q)(T-t)} S_t.$$

This condition determines the drift under the risk-neutral measure. If the underlying follows a Lévy process,

$$S_t = S_0 e^{X_t},$$

the process (X_t) must be adjusted under Q so as to satisfy the martingale condition above. For the particular case of the Variance Gamma model, this leads to an explicit relation between historical and risk-neutral parameters.

General Pricing Formula for the European Call

Under the risk-neutral measure Q , and imposing the martingale condition, the price of the European call is:

$$C(S_0, K, T) = e^{-rT} E^Q[(S_T - K)^+].$$

Expanding the payoff:

$$C(S_0, K, T) = e^{-rT} \int_0^\infty (s - K)^+ f_{S_T}^Q(s) ds,$$

where $f_{S_T}^Q$ denotes the density of the underlying asset price at maturity under Q .

Intuitive Interpretation. This formula states that:

- the call price equals the expected value of its future gains,
- computed in a fictitious world where all assets grow at the risk-free rate,
- and then discounted at rate r .

Under the historical measure P , the underlying carries a risk premium. But under the risk-neutral measure Q , all risk premia disappear, and only the risk-free rate remains. This risk neutralization enables the expectation-based computations used in Fourier methods, in particular in the Carr–Madan method.

5.2 Carr–Madan Method

5.2.1 Motivation for Fourier Methods

In Lévy models such as the Variance Gamma (VG) model, the density $f_{S_T}^Q$ is generally not available in closed form. This prevents direct integration of the payoff, which requires explicit knowledge of $f_{S_T}^Q$. However, the characteristic function of the log-price,

$$X_T = \ln(S_T), \quad \varphi_{X_T}(u) = E^Q[e^{iuX_T}],$$

is always available in closed form for Lévy models. This property motivates the use of Fourier techniques, particularly the Carr–Madan method (developed in Carr & Madan (1999)), which expresses the European call price directly from the characteristic function.

Let $k = \ln(K)$ denote the log-strike. The call price can be written as

$$C(k) = e^{-rT} \int_{\ln(K)}^{+\infty} (e^x - K) f_{X_T}(x) dx.$$

- The payoff grows asymptotically like e^x as $x \rightarrow +\infty$. A Fourier transform requires sufficient integrability (controlled growth).
- To guarantee this integrability, Carr–Madan introduce a damping parameter $\alpha > 0$ and define the transformed quantity

$$C_\alpha(k) = e^{\alpha k} C(k).$$

This damping imposes the condition

$$E^Q[S_T^{\alpha+1}] < \infty, \quad \text{equivalently} \quad E^Q[e^{(\alpha+1)X_T}] < \infty.$$

- The Carr–Madan Fourier transform is then given by

$$\psi_\alpha(\nu) = \int_{-\infty}^{+\infty} e^{i\nu k} C_\alpha(k) dk = \frac{e^{-rT} \varphi_{X_T}(\nu - i(\alpha + 1))}{\alpha^2 + \alpha - \nu^2 + i\nu(2\alpha + 1)}.$$

The proof is detailed in Appendix A.4.

- The damped price is then recovered through inverse Fourier transform:

$$C_\alpha(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\nu k} \psi_\alpha(\nu) d\nu.$$

5.3 Integrability Conditions and the Risk–Neutral Measure for the Variance–Gamma Model

5.3.1 Existence of Exponential Moments

Under the Variance–Gamma model, the log–return process X_t has characteristic function:

$$\varphi_{X_t}(u) = \left(1 - i\theta\nu u + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu}.$$

The moment generating function (MGF) is obtained by substituting $u \mapsto -iu$:

$$E[e^{uX_t}] = \left(1 - \theta\nu u + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu}.$$

It is well defined only if

$$\boxed{1 - \theta\nu u + \frac{1}{2}\sigma^2\nu u^2 > 0.}$$

This condition for the existence of exponential moments plays a central role in the Carr–Madan method as well as in the construction of the risk–neutral measure.

5.3.2 Condition on the Damping Parameter α (Carr–Madan)

The Carr–Madan method requires the existence of the exponential moment

$$E\left[e^{(\alpha+1)X_T}\right] < \infty,$$

which imposes that the MGF be valid at $u = \alpha + 1$, i.e.

$$\boxed{1 - \theta\nu(\alpha + 1) + \frac{1}{2}\sigma^2\nu(\alpha + 1)^2 > 0.}$$

This condition ensures convergence of the Fourier transform of the regularized payoff and numerical stability of the FFT.

5.3.3 Risk–Neutral Measure and Computation of the Correction Term

To price options, one must work under a risk–neutral measure Q such that the discounted underlying price is a martingale:

$$e^{-(r-q)t}S_t \text{ is a martingale under } Q.$$

Writing $S_t = S_0 e^{X_t}$, this condition is equivalent to

$$E^Q[e^{X_t}] = e^{(r-q)t}.$$

A standard approach is to introduce a deterministic correction term w_t and write

$$\ln S_t = \ln S_0 + (r - q)t + X_t - w_t.$$

This yields

$$e^{-(r-q)t}S_t = S_0 e^{X_t - w_t}, \quad \text{and the martingale condition becomes } E^Q[e^{X_t - w_t}] = 1.$$

For a stationary Lévy process, one has the representation

$$E[e^{uX_t}] = e^{tk(u)},$$

where $k(u)$ is the cumulant exponent. For the VG model,

$$k(u) = -\frac{1}{\nu} \ln\left(1 - \theta\nu u + \frac{1}{2}\sigma^2\nu u^2\right).$$

The condition $E[e^{X_t - w_t}] = 1$ gives

$$e^{t(k(1) - w)} = 1 \implies w = k(1) = -\frac{1}{\nu} \ln\left(1 - \theta\nu + \frac{1}{2}\sigma^2\nu\right).$$

Thus, the log–underlying dynamics under the risk–neutral measure are:

$$X_t^Q = X_t - wt.$$

Condition for the Existence of the Unit Moment

The correction term w is well defined only if

$$1 - \theta\nu + \frac{1}{2}\sigma^2\nu > 0.$$

This is exactly the condition for the existence of the exponential moment $E[e^{X_t}]$, consistent with the condition imposed on the damping parameter α in the Carr–Madan method. This consistency ensures the global validity of the mathematical framework (existence of the MGF, convergence of the FFT, absence of arbitrage).

Remark on the Choice of Method

The deterministic correction w_t is the simplest way to obtain a risk-neutral measure in the VG model, as it does not alter the Lévy measure: jump parameters remain unchanged from their historical values. By comparison, a common alternative is the Esscher transform, which changes the VG parameters by tilting the probability measure. This approach is consistent but more involved, as it modifies the Lévy measure and produces a different model. In this report, we adopt the deterministic correction w .

6 Numerical Application: Pricing an SPX European Call

6.1 Construction of the Frequency Grid for the ECF and the Carr–Madan Method

Estimating the parameters of the Variance–Gamma and Merton models via the ECF, as well as pricing European options using the Carr–Madan method, requires defining a frequency grid

$$u = \{u_0, u_1, \dots, u_{N-1}\},$$

that is sufficiently rich to capture tail behaviour while remaining compatible with the constraints of the Fast Fourier Transform (FFT).

Estimation of the Tail Exponent ξ via the Hill Estimator

Let $(X_t)_{t=1}^n$ denote a series of log-returns. We consider the absolute values $|X_t|$ in order to capture both tails symmetrically. After sorting in decreasing order,

$$X_{(1)} \geq X_{(2)} \geq \dots \geq X_{(n)},$$

we select k extreme observations. The Hill estimator of the tail parameter ξ is given by:

$$\hat{\xi} = \frac{1}{k} \sum_{i=1}^k \ln \left(\frac{X_{(i)}}{X_{(k+1)}} \right).$$

A small $\hat{\xi}$ signals heavier tails, implying slower decay of the characteristic function and the need to extend the frequency grid.

Definition of the Maximum Frequency Bound

The estimated tail exponent is used to set an upper bound for the frequency grid by the heuristic:

$$u_{\max} = \text{clip} \left(\frac{10}{\hat{\xi}}, 1, 20 \right),$$

to maintain a balance between:

- sufficiently covering the tails (high frequencies required when $\hat{\xi}$ is small),
- excluding very high-frequency regions where the empirical CF becomes strongly noisy.

Frequency Grid for the ECF

The final grid is defined as

$$u_k = k \Delta u, \quad k = 0, 1, \dots, N-1, \quad u_k \leq u_{\max},$$

where Δu is imposed by the FFT via the classical relation $\Delta u \Delta x = \frac{2\pi}{N}$. Only the frequencies satisfying $u_k \leq u_{\max}$ are used in the ECF estimation.

6.2 Parameters and Calibration

6.2.1 Estimation Results (Variance Gamma vs Merton)

Parameters	MLE	ECF
c	0.000916	0.000807
σ	0.011431	0.012259
θ	-0.000570	-0.000471
ν	1.296766	4.189263

Table 1: Estimated VG model parameters

Statistic	Data	MLE	ECF
KS statistic	-	0.029628	0.129648
Wasserstein	-	0.000644	0.001903
Mean	0.000337	0.000454	0.000200
Std. deviation	0.012297	0.011541	0.012298
Skewness	-0.476663	-0.076712	-0.728582
Kurtosis	12.871057	3.397900	13.485884

Table 2: Statistical comparison for the VG model

Parameters	MLE	ECF
c	0.001215	0.000587
σ	0.005919	0.008389
λ	0.368159	0.067692
μ_J	-0.002387	-0.003699
σ_J	0.016712	0.034358

Table 3: Estimated Merton model parameters

Statistic	Data	MLE	ECF
KS statistic	-	0.030026	0.061444
Wasserstein	-	0.000581	0.000936
Mean	0.000337	0.000388	0.000342
Std. deviation	0.012297	0.011579	0.011827
Skewness	-0.476663	-0.371951	-0.664946
Kurtosis	12.871057	3.945544	10.316793

Table 4: Statistical comparison for the Merton model

Variance–Gamma (VG) Model MLE and ECF produce notably different parameter estimates, especially for the shape parameter ν . The mean and standard deviation are relatively similar across methods, but the ECF slightly overestimates volatility. KS and Wasserstein distances show that MLE fits the empirical distribution of log–returns much more accurately.

MLE thus reproduces the general shape of the distribution better, especially the moderate tails, providing a more coherent fit to the overall data. The ECF tends to misrepresent the central and intermediate distribution regions despite achieving a kurtosis closer to the empirical value.

Graphical comparisons are given in Appendix B.1.

Merton Model Differences between MLE and ECF are also present for the Merton model. MLE estimates a higher jump frequency λ with more moderate jump parameters, better capturing jump behaviour in the data. KS and Wasserstein distances confirm this observation. Simulated moments from MLE reproduce empirical skewness and kurtosis more faithfully, whereas ECF underestimates jump frequency and overestimates jump dispersion.

Graphical comparisons are given in Appendix B.2.

General Comparison and Conclusion Comparing the two models and two estimation methods, we retain the Variance–Gamma parameters obtained via Maximum Likelihood Estimation (MLE) as the reference for pricing our European option.

6.2.2 Price via FFT Carr–Madan

For the numerical example, to compute a 6-month European call option, we consider:

- Spot: $S_0 = 6870.399$,
- Maturity: $T = 108/365 \approx 0.296$ years,
- Risk-free rate: $r = 3.83\%$, *U.S. Department of the Treasury (2025)*
- Dividend yield: $q = 0$,
- VG parameters: $\theta = -0.000564$, $\sigma = 0.011433$, $\nu = 1.296790$.

The drift correction w is imposed to ensure the martingale property under the risk–neutral measure Q :

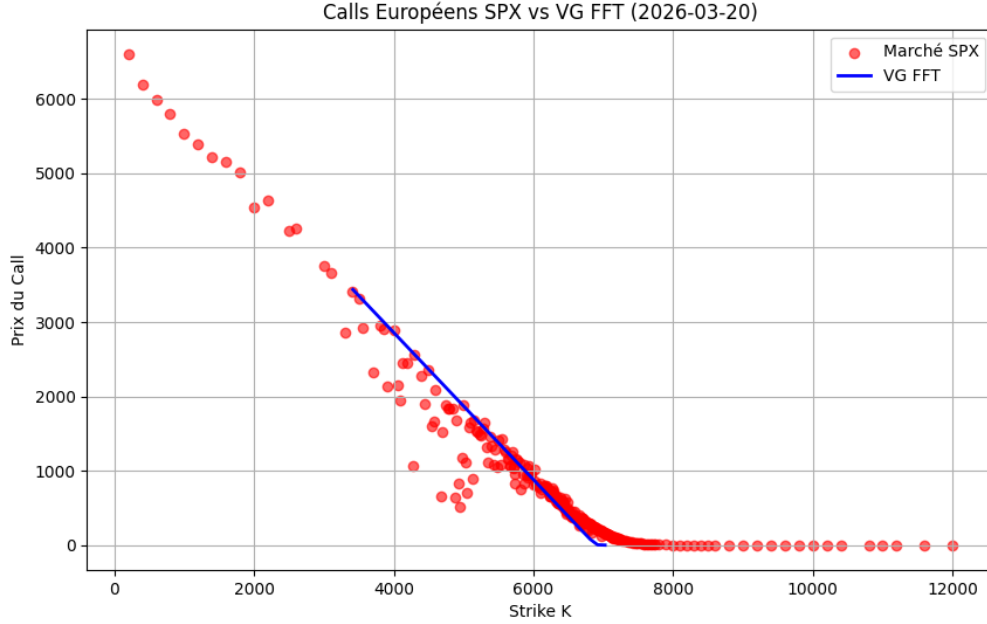
$$w = \ln \left[(1 - \theta\nu + \tfrac{1}{2}\sigma^2\nu)^{-1/\nu} \right].$$

To ensure convergence of the Fourier transform, we introduce a damping factor $\alpha = 1.2$. The strike grid is centred around S_0 , and the FFT parameters are:

$$N = 2^{14}, \quad \eta = 0.025.$$

The choice of η within a reasonable range (typically 0.01–0.05) ensures a sufficiently fine grid and numerical stability. Hence $\eta = 0.025$ is a methodological choice needed to properly capture the Fourier transform of the payoff.

6.2.3 Results



Interpretation The figure shows the prices of 6-month European call options computed using the Carr–Madan FFT method with the Variance–Gamma parameters calibrated via MLE. The strike interval relative to the spot price is: $0.5S_0$ to $1.5S_0$.

- Option prices decrease as the strike K increases, which is consistent with the expected behavior of European calls.
- The estimated option prices are consistent across the entire strike interval considered.

Conclusion These results confirm that the MLE-calibrated Variance–Gamma model is suitable for pricing European options, effectively capturing tail behavior and return asymmetry observed in the market.

6.2.4 Summary: analysis of the European call and graphical interpretation

Reminder: A European call is a derivative instrument that grants its holder the right, but not the obligation, to buy an underlying asset at a predetermined strike price K at a fixed maturity T .

Intuitively, for a given strike K :

- The investor considers the probability that the future underlying price S_T is above K .
- If $S_T > K$, the call will be **in the money** at maturity, generating a positive payoff.
- In such a case, purchasing the call today may be beneficial.

The Variance–Gamma (VG) model represents the probability distribution of S_T , incorporating heavy tails and asymmetry in market returns. The Carr–Madan FFT method provides an efficient tool to compute the current option price C_0 for all strikes, using the information embedded in the distribution of the underlying.

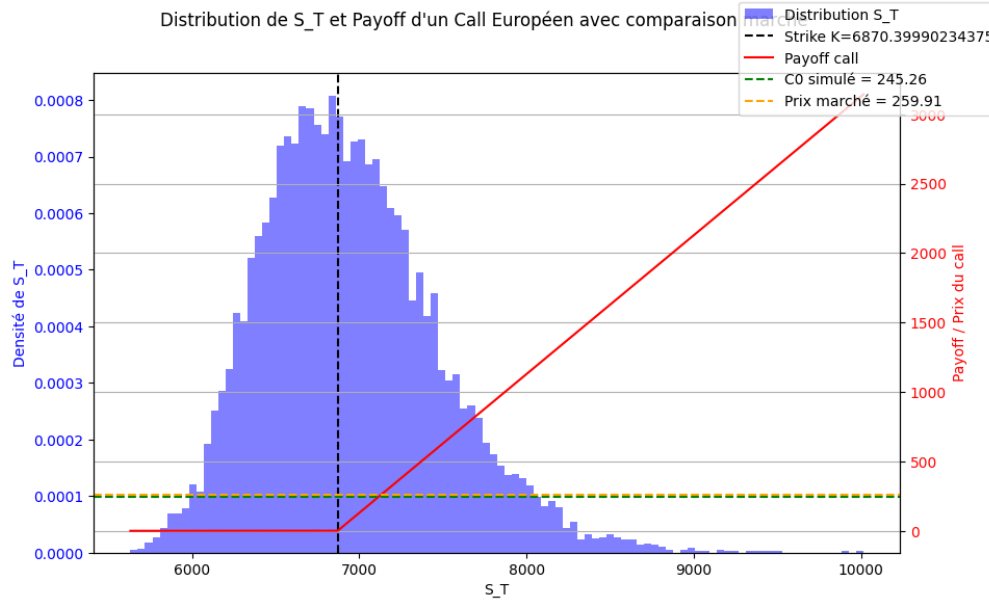
Comparing the model-implied price C_0 with the observed market price allows one to determine whether the option is:

- **Undervalued**, if the model price is lower than the market price;
- **Overvalued**, if the model price is higher than the market price;

- **Fairly valued**, if both prices are close.

Hence, the call represents a future right to buy the asset at K , and the present value C_0 corresponds to the fair amount to pay today for this right. If projections indicate that the option will likely end up in the money and C_0 remains reasonable, purchasing the call may be an attractive investment.

Graphical interpretation The accompanying figure combines the simulated distribution of S_T with the call payoff:



Observations and interpretation

- Most of the histogram's mass lies to the right of the strike, indicating that under the VG-simulated distribution, the call has a high probability of finishing in the money.
- The simulated price C_0 is lower than the market price, suggesting that the option may be undervalued by the market compared to the model's projection.
- The more the probability mass concentrates above K , the higher the expected payoff contribution, increasing the option's potential attractiveness.

Conclusion Given these observations, purchasing the 6-month European call at this strike appears attractive. The combination of a high probability of finishing in the money and a simulated price below the market price indicates a favorable gain potential for the investor.

7 Appendix A - Fundamental Derivations

7.1 A.1 Characteristic function of a compound Poisson process

- Consider a compound Poisson process:
 - The number of jumps $N \sim \text{Poisson}(\gamma)$.
 - The jump sizes X_j are i.i.d. with law $\mu(dx)$.

In a Lévy process, there is not a single type of event but a **Lévy intensity measure** ν distributed over different jump sizes x , corresponding to an infinite collection of localized (γ, μ) pairs.

- The characteristic function is:

$$\phi(u) = E\left[e^{iu \sum_{j=1}^N X_j}\right] = \exp\left[\gamma \int_R (e^{iux} - 1) \mu(dx)\right].$$

- Detail the calculation using conditional expectation on N :

$$E\left[e^{iu \sum_{j=1}^N X_j}\right] = \sum_{n=0}^{\infty} E\left[e^{iu \sum_{j=1}^N X_j} \mid N = n\right] P(N = n).$$

- Conditional on $N = n$:

$$E\left[e^{iu \sum_{j=1}^n X_j}\right] = E\left[\prod_{j=1}^n e^{iu X_j}\right] = \prod_{j=1}^n E[e^{iu X_j}] = (E[e^{iu X_1}])^n.$$

- Recall that $P(N = n) = e^{-\gamma} \frac{\gamma^n}{n!}$. Hence:

$$E\left[e^{iu \sum_{j=1}^N X_j}\right] = \sum_{n=0}^{\infty} (E[e^{iu X_1}])^n e^{-\gamma} \frac{\gamma^n}{n!} = e^{-\gamma} \sum_{n=0}^{\infty} \frac{(\gamma E[e^{iu X_1}])^n}{n!}.$$

- Note: the exponential series satisfies

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} = e^{\alpha}, \quad \text{with } \alpha = \gamma E[e^{iu X_1}].$$

- Finally we obtain:

$$E\left[e^{iu \sum_{j=1}^N X_j}\right] = e^{-\gamma} e^{\gamma E[e^{iu X_1}]} = \exp\left[\gamma (E[e^{iu X_1}] - 1)\right] = \exp\left[\gamma \int_R (e^{iux} - 1) \mu(dx)\right].$$

Remark: This formula shows that the characteristic function of a compound Poisson process is naturally expressed in terms of the Lévy measure $\nu(dx) = \gamma \mu(dx)$. It forms the basis for decomposing Lévy processes into jump components of different sizes.

7.2 A.2 Characteristic function of a normal distribution

- Let

$$X \sim \mathcal{N}(\mu, \sigma^2), \quad f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right), \quad x \in R.$$

- We want to compute the characteristic function:

$$\rho_X(u) = E[e^{iux}] = \int_R e^{iux} f_X(x) dx.$$

Thus,

$$\rho_X(u) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{iux} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) dx.$$

- Change of variable:

$$y = x - \mu, \quad x = \mu + y, \quad dx = dy.$$

Then:

$$\rho_X(u) = e^{iu\mu} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp\left(iuy - \frac{y^2}{2\sigma^2}\right) dy.$$

- Complete the square:

$$iuy - \frac{y^2}{2\sigma^2} = -\frac{1}{2\sigma^2}(y^2 - 2iu\sigma^2 y) = -\frac{1}{2\sigma^2}[(y - iu\sigma^2)^2 - (iu\sigma^2)^2].$$

Therefore

$$e^{iuy - \frac{y^2}{2\sigma^2}} = \exp\left(-\frac{(y - iu\sigma^2)^2}{2\sigma^2}\right) \exp\left(\frac{(iu\sigma^2)^2}{2\sigma^2}\right) = \exp\left(-\frac{(y - iu\sigma^2)^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2}u^2\sigma^2\right).$$

- The integral becomes:

$$\int_{-\infty}^{+\infty} e^{iuy} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy = e^{-\frac{1}{2}u^2\sigma^2} \int_{-\infty}^{+\infty} \exp\left(-\frac{(y - iu\sigma^2)^2}{2\sigma^2}\right) dy.$$

The integral above is the integral of a Gaussian density centered at $iu\sigma^2$, whose value is simply $\sqrt{2\pi}\sigma$.

- Finally,

$$\rho_X(u) = \exp\left(iu\mu - \frac{1}{2}u^2\sigma^2\right).$$

Remark on the normalization factor. The factor $\sqrt{2\pi}\sigma$ appears because, even if the Gaussian center is translated in the complex plane ($y - iu\sigma^2$), the width and shape remain that of a normal density with variance σ^2 . The integral of a normal density over R is always $\sqrt{2\pi}\sigma$, regardless of its (real or complex) center. This property allows the direct derivation of the final result.

7.3 A.3 Characteristic function of the Gamma distribution

- The density of a Gamma variable with parameters (α, β) is:

$$f_G(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0, \alpha > 0, \beta > 0.$$

- In our case, we use the parametrization:

$$\alpha = \frac{1}{v}, \quad \beta = v.$$

- The characteristic function of G is:

$$\phi_G(u) = E[e^{iuG}] = \int_0^{+\infty} e^{iux} f_G(x) dx.$$

Thus:

$$\phi_G(u) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^{+\infty} e^{iux} x^{\alpha-1} e^{-x/\beta} dx.$$

- Combine exponentials:

$$e^{iux} e^{-x/\beta} = e^{-(\frac{1}{\beta} - iu)x}.$$

Let

$$A = \frac{1}{\beta} - iu, \quad \text{with } \Re(A) = \frac{1}{\beta} > 0.$$

The integral becomes:

$$\phi_G(u) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^{+\infty} x^{\alpha-1} e^{-Ax} dx.$$

- Using the classical Euler integral property:

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx, \quad \alpha > 0.$$

- Show that:

$$\int_0^{+\infty} x^{\alpha-1} e^{-Ax} dx = A^{-\alpha} \Gamma(\alpha).$$

- Substitution: $u = Ax$, $x = u/A$, $dx = du/A$:

$$\int_0^{+\infty} x^{\alpha-1} e^{-Ax} dx = \int_0^{+\infty} \left(\frac{u}{A}\right)^{\alpha-1} e^{-u} \frac{du}{A} = A^{-\alpha} \int_0^{+\infty} u^{\alpha-1} e^{-u} du = A^{-\alpha} \Gamma(\alpha).$$

- Deduce the characteristic function:

$$\phi_G(u) = \frac{1}{\Gamma(\alpha)\beta^\alpha} A^{-\alpha} \Gamma(\alpha) = (1 - iu\beta)^{-\alpha}, \quad \text{with } \Re(1 - iu\beta) > 0.$$

Remark: interpretation of the Gamma law in a Lévy process. In a Lévy process, the Gamma law often appears as a **waiting time distribution** in a Poisson process:

- If Poisson events occur at rate λ , the waiting time between two events is exponential with parameter λ .
- The time needed to observe α *independent events* then follows a $\text{Gamma}(\alpha, 1/\lambda)$ law.

In other words:

- α (shape) measures concentration: larger α yields a more concentrated distribution.
- β (scale) controls dispersion: it stretches or contracts the distribution.

7.4 A.4 Detailed derivation of the Carr–Madan Fourier transform

We want to compute:

$$\psi_\alpha(v) = \int_{-\infty}^{\infty} e^{ivk} c_\alpha(k) dk.$$

Substitute

$$c_\alpha(k) = e^{\alpha k} C(e^k), \quad C(e^k) = e^{-rT} E[(e^X - e^k)^+],$$

and using the density $f_X(x)$:

$$C(e^k) = e^{-rT} \left(\int_k^{\infty} e^x f_X(x) dx - e^k \int_k^{\infty} f_X(x) dx \right),$$

the transform becomes

$$\psi_\alpha(v) = e^{-rT} \int_{-\infty}^{\infty} e^{(iv+\alpha)k} \left[\int_k^{\infty} e^x f_X(x) dx - e^k \int_k^{\infty} f_X(x) dx \right] dk.$$

By interchanging the order of integration (Fubini's theorem, valid due to damping), we obtain two terms.

3.1 First term

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} e^{(iv+\alpha)k} \left(\int_k^{\infty} e^x f_X(x) dx \right) dk \\
&= \int_{-\infty}^{\infty} f_X(x) e^x \left(\int_{-\infty}^x e^{(iv+\alpha)k} dk \right) dx \\
&= \int_{-\infty}^{\infty} f_X(x) e^x \frac{e^{(iv+\alpha)x}}{iv+\alpha} dx \\
&= \frac{1}{iv+\alpha} \int_{-\infty}^{\infty} e^{(iv+\alpha+1)x} f_X(x) dx \\
&= \frac{\phi_T(v - i(\alpha+1))}{iv+\alpha},
\end{aligned}$$

where $\phi_T(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} f_X(x) dx$ is the characteristic function of X , and $u = v - i(\alpha+1)$.

3.2 Second term

$$\begin{aligned}
I_2 &= \int_{-\infty}^{\infty} e^{(iv+\alpha)k} \left(-e^k \int_k^{\infty} f_X(x) dx \right) dk \\
&= - \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^x e^{(iv+\alpha+1)k} dk \right) dx \\
&= - \int_{-\infty}^{\infty} f_X(x) \frac{e^{(iv+\alpha+1)x}}{iv+\alpha+1} dx \\
&= - \frac{\phi_T(v - i(\alpha+1))}{iv+\alpha+1}.
\end{aligned}$$

3.3 Combination of the two terms

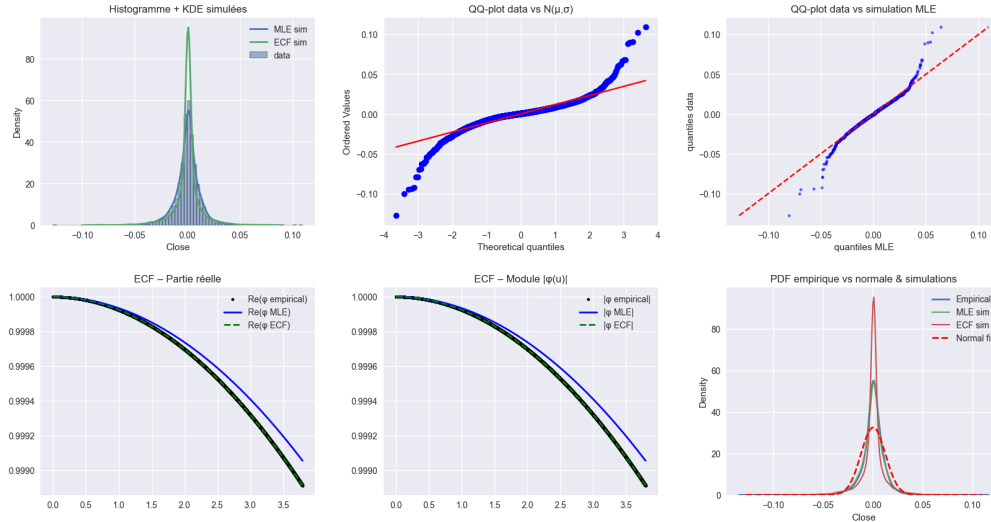
$$\begin{aligned}
\psi_{\alpha}(v) &= e^{-rT} \phi_T(v - i(\alpha+1)) \left(\frac{1}{iv+\alpha} - \frac{1}{iv+\alpha+1} \right) \\
&= e^{-rT} \phi_T(v - i(\alpha+1)) \frac{1}{(iv+\alpha)(iv+\alpha+1)}.
\end{aligned}$$

This corresponds to the standard Carr–Madan formula:

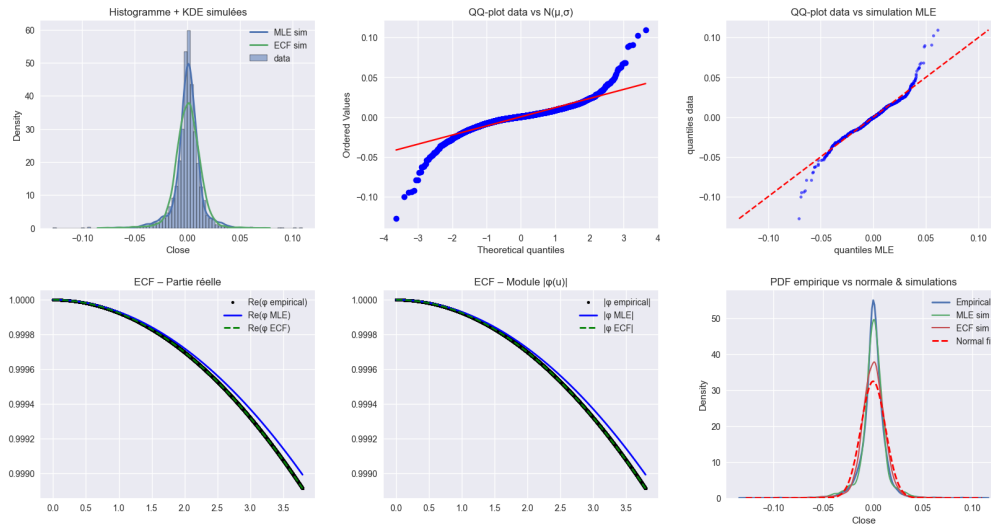
$$\boxed{\psi_{\alpha}(v) = \frac{e^{-rT} \phi_T(v - i(\alpha+1))}{(iv+\alpha)(iv+\alpha+1)}}.$$

8 Annexe B - Graphes

8.1 B.1 Comparaison of results for Variance Gamma



8.2 B.2 Comparaison of results for Merton



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