Comparing Perturbed Optimizer and Barrier Function Method for Differentiating Linear Programs

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28 Oct 2021

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Deep Learning

- Applied in image classification, voice recognition and so much more.
- Core of deep learning: Artificial Neural Networks.
  - Directed graph of parametrised processing nodes [1]

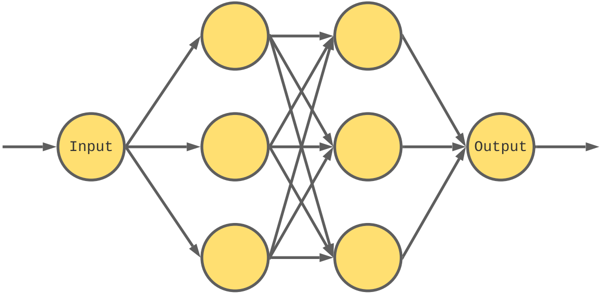

Figure: A conceptual sketch of a neural network

- Learning: Tune node parameters
- Done through back propagation
- Require node gradient

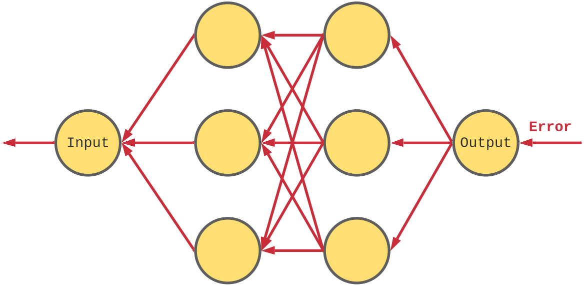

Figure: Conceptual sketch of back propagation of a neural network

Declarative Nodes

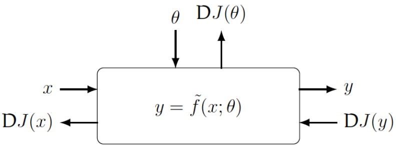

Figure: Imperative Node [1]

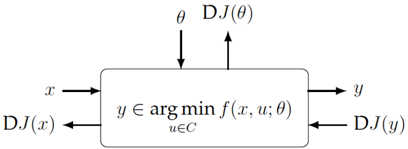

Figure: Declarative Node [1]

- All imperative nodes can be expressed as declarative nodes
- Declarative node has functions beyond the capability of imperative nodes, such as hard-code constraints into nodes.

Linear Program (LP)

- Declarative node fails on Linear Program

Definition

Optimisation problem with linear objective function and constraints. The canonical form of a Linear Program is

$$\text{minimize } c^T x$$
$$\text{subject to } Ax \preceq b$$

where $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$

- The solution to a Linear Program always lies on a vertex of its feasible region

Toy Example

$$\begin{aligned} \text{minimize} \quad & \theta x_1 - 4x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 30 \\ & x_2 \geq 0, x_1 \geq 1 \end{aligned} \tag{1}$$

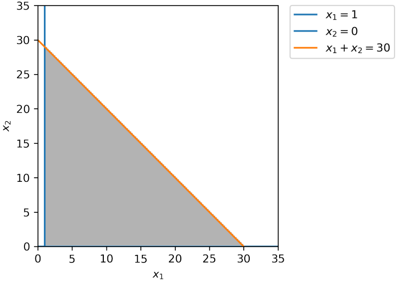

Figure: Feasible region

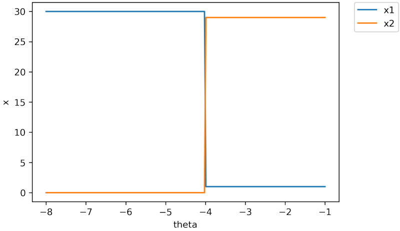

Figure: Solution of LP versus θ

A constant zero gradient defects back propagation, hence model training

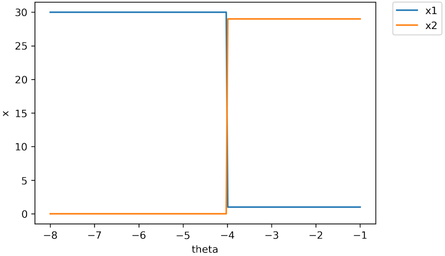

Figure: Solution of LP versus θ

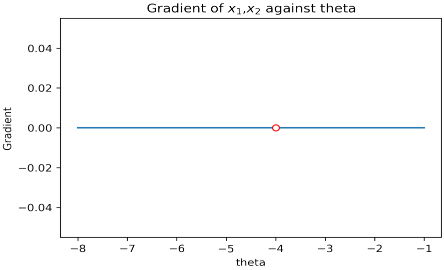

Figure: Gradient of solution versus θ

Research Question

Barrier function method v.s. Perturbed optimizer method

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Barrier Function Method

Combine constraints functions with the objective function through log barrier function.

Definition

Consider the optimisation problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \preceq b \end{aligned}$$

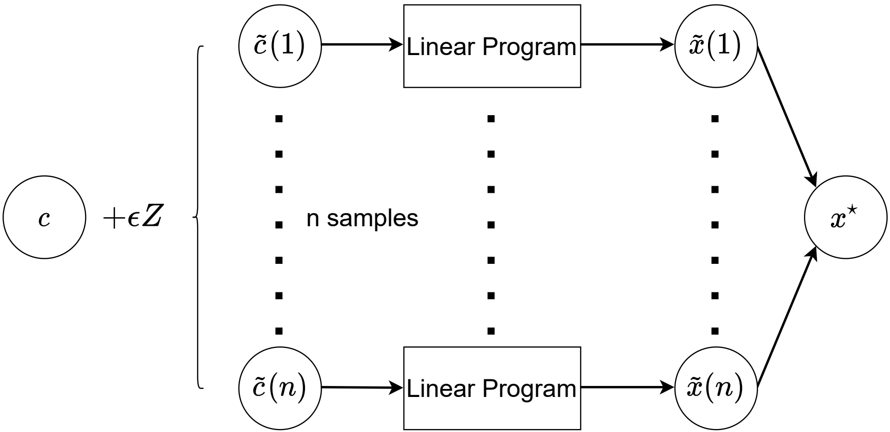
Applying barrier function method transforms this LP to an unconstrained optimisation problem

$$\text{minimize} \quad tc^T x - \sum_{i=1}^{m} \log\left(b_i - a_i^T x\right)$$

where a_i^T is the i_{th} row of A

$\frac{\partial x}{\partial c} = -D_{xx}^{-1} f(x, c) D_{xc} f(x, c)$ [1]

Perturbed Optimiser Method



$\frac{\partial x}{\partial c} = \mathbf{E}\left[x^*(c+\epsilon Z)\nabla_z \nu(Z)^\top/\epsilon\right]$ where Z is the noise, $\nu(z)$ is a function defined from noise distribution and ϵ is the temperature parameter of noise [2].

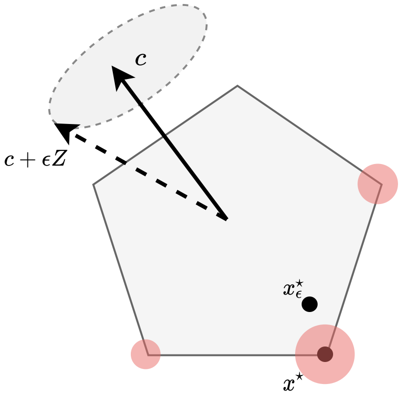


Figure: Geometric interpretation of perturbed optimizer

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Differentiating Toy LP

Recall the toy LP problem shown earlier

Toy Example

$$\begin{aligned} \text{minimize} \quad & \theta u_1 - 4u_2 \\ \text{subject to} \quad & u_1 + u_2 \leq 30 \\ & u_2 \geq 0, u_1 \geq 1 \end{aligned} \tag{2}$$

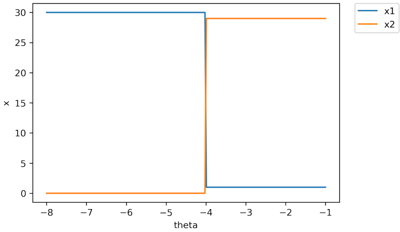


Figure: Solution of LP versus θ

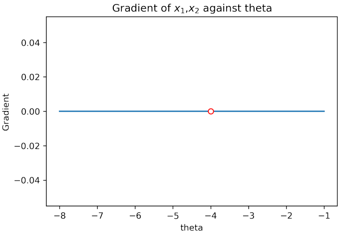


Figure: Gradient of solution versus θ

t = 0.5
Avg. time cost for sol/grad = 0.01s/0.0001s

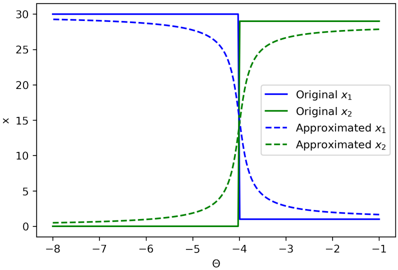


Figure: u against θ

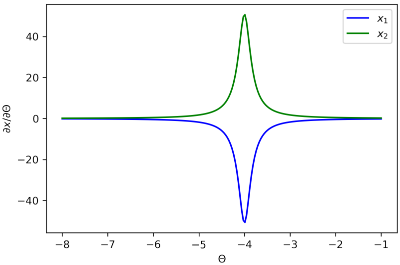


Figure: Gradient of u against θ

Differentiating Toy LP – Perturbed Optimizer Method

$\sigma = 0.7$, Gumbel Noise, Sample Size $= 2{,}000$
Avg. time cost for sol/grad $= 14.82$s$/0.0002$s

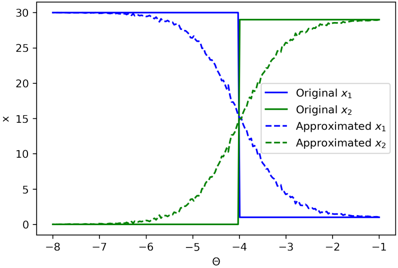


Figure: u against θ

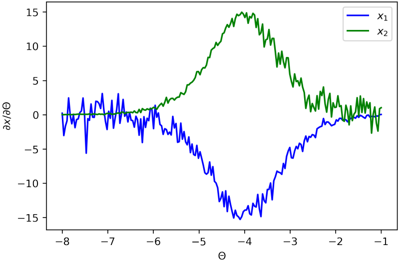


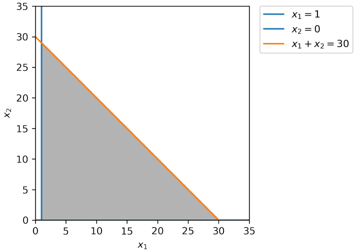
Figure: Gradient of u against θ

Back Propagation through LP Declarative Node

LP Declarative Node

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & u_1 + u_2 \leq 30 \\ & u_2 \geq 0, u_1 \geq 1 \end{aligned} \tag{3}$$

Initialise $c = [-8, -4]$ with $x^\star = [30, 0]$
Aim to update c with approximated gradient to achieve $x^\star = [1, 29]$



Back Propagation through LP Declarative Node
Barrier Function Method

Loss $= \|\bar{x} - [1, 29]\|_2^2$, learning rate $= 0.01$, t=0.1

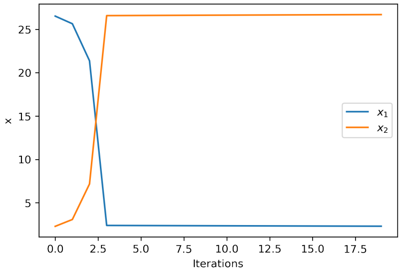


Figure: $\bar{x}$ over training iterations

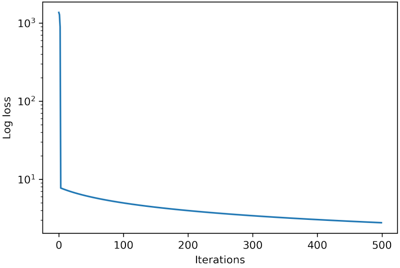


Figure: Log loss of declarative node

Back Propagation through LP Declarative Node
Perturbed Optimizer Method

Loss $= \|\bar{x} - [1, 29]\|_2^2$, learning rate $= 0.01$
$\epsilon = 0.5$, Sample Size $= 200$, Noise type $=$ Gumbel

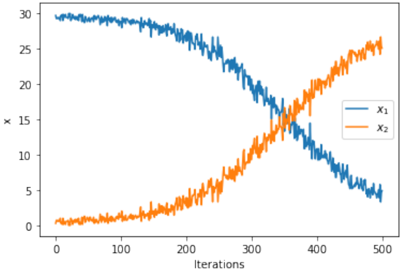


Figure: $\bar{x}$ over training iterations

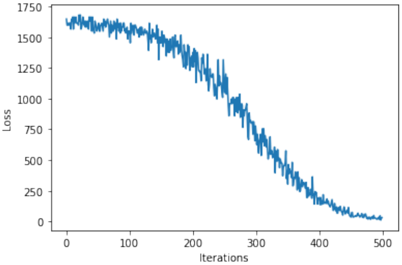


Figure: Loss of declarative node

Trainability of LP declarative node on CIFAR10

- CIFAR10: An image dataset contains 10 classes of object
- Initialise a Convolutional Neural Network and embed an LP declarative node.
- Perform Image classification tasks on CIFAR10
- Not aiming for state of the art results, but to demonstrate the ability of incorporating differentiable LP into an end-to-end learnable model.

Trainability of LP declarative node on CIFAR10

	Original CNN	Plain LP
Mean Average Accuracy	63%	10%

	Perturbed Optimizer	Barrier Function Method
Mean Average Accuracy	30%	N/A

- Proves the trainability of LP declarative node within a complete network
- Barrier function method failed due to limited computational resource
  - Determine the gradient of barrier function method requires inverting a Hessian matrix, in this model the matrix is 400×400 in dimension.

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Conclusion

Barrier Function Method

- Low computational cost for low dimensional LP
- Exponentially raising computational cost for high dimensional LP
- Deterministic approximation
- Smooth gradient approximation
- Require optimisation problem solver capable for log barrier terms

Perturbed Optimiser Method

- Computational cost is sample size dependent, generally it is high
- Relatively Lower computational cost for high dimensional LP
- Stochastic approximation
- Non-smooth gradient approximation, can be improved by increasing sample size
- Capable with any solver

Future Works

- Determine theoretical equivalency between perturbed optimizer and barrier function methods
- Find constructive LP declarative nodes for model training, aim for state of the art results

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