M3228.000300
SLAM 101

Lecture 03 Uncertainty Representation

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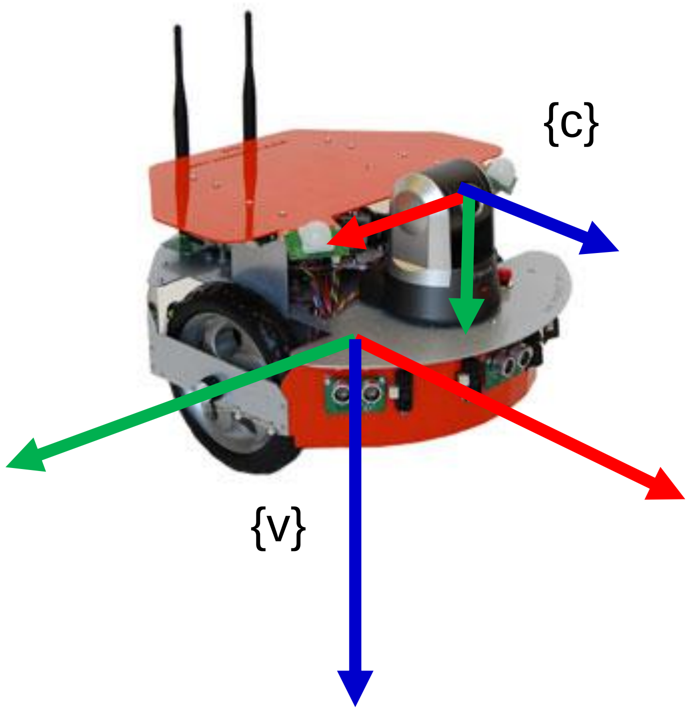
Mobile Robot State Representation

- Mobile robots



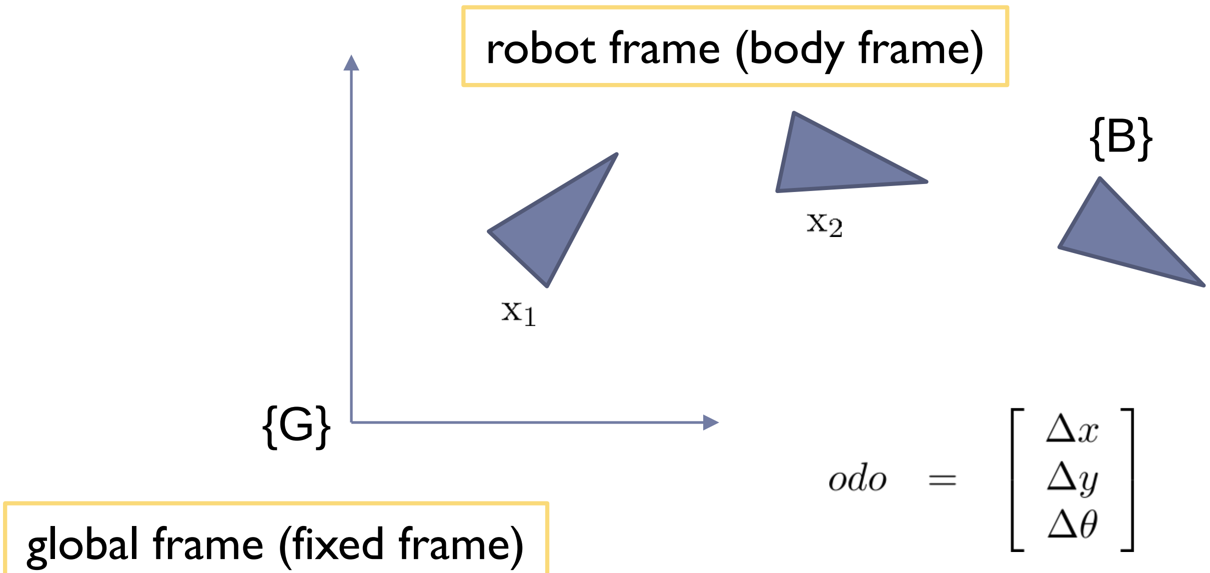
$$X = [x, y, z, r, p, h]$$

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in SE(3)$$

Pose = position and orientation

Poses and Trajectory

- We need a robot trajectory (a set of poses)
  - Pose in SE(3) and/or vector form



$$x_t = [x, y, \theta]$$

$$odo = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta \theta \end{bmatrix}$$

$$T = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) & \Delta x \\ \sin(\Delta\theta) & \cos(\Delta\theta) & \Delta y \\ 0 & 0 & 1 \end{bmatrix}$$

Probabilistic Robotics

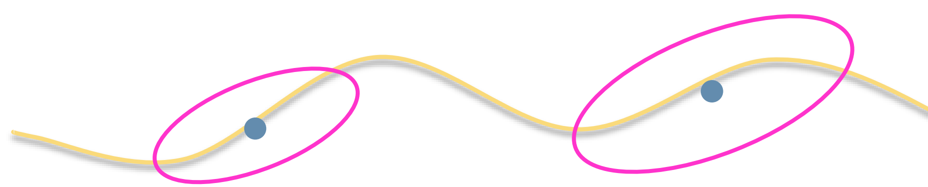
- Robot pose in Gaussian distribution

Discretized robot trajectory

Pose-graph

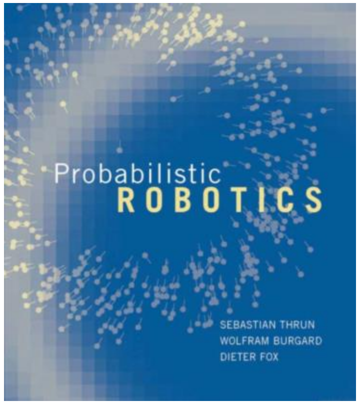
$$x_1 \sim \mathcal{N}(\mu_1, \Sigma_1) \qquad x_2 \sim \mathcal{N}(\mu_2, \Sigma_2)$$

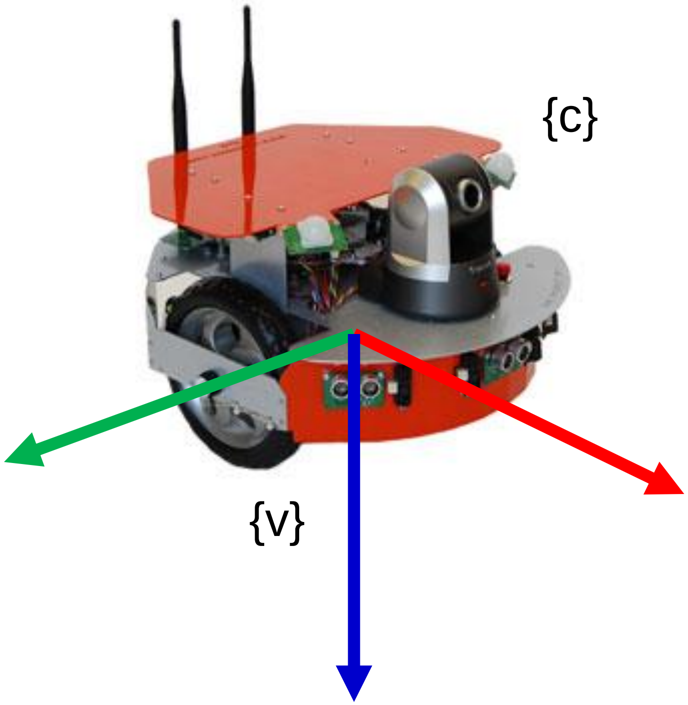
Uncertainty propagation

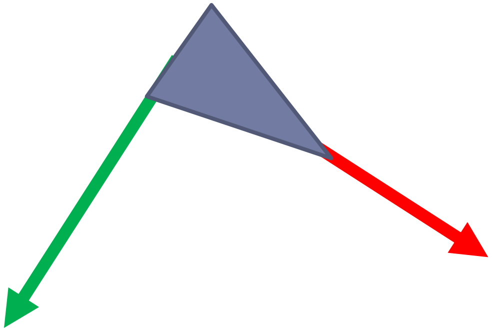


Uncertainty Propagation

- The core of the SLAM
  - Robot pose & map are uncertain
  - Probabilistic robotics

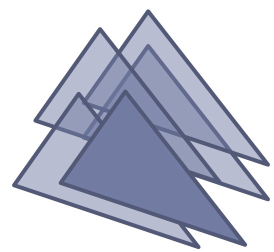




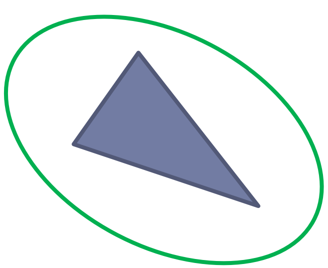


Pose Uncertainty

- Robot pose uncertainty representation



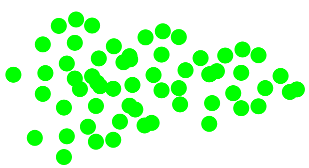
- Using Gaussian
  - Univariate
  - Mean & covariance
- Using sample
  - Arbitrary distribution
  - Need N samples



$$X \sim \mathcal{N}(\mu, \Sigma)$$

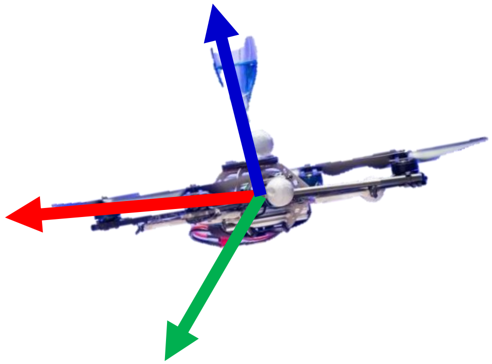
$$\mu = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \Sigma = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

EKF, EIF, SAM



Particle filter

Uncertainty Modeling using Gaussian



Position

$$p = [x, y, z]$$

Orientation

$$[r, p, h]$$
$$R \in SO(3)$$

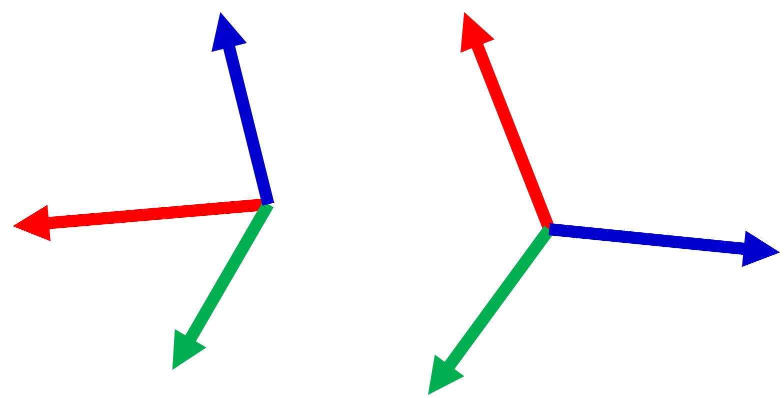
- Vector space
  - Position, Euler, quaternion

$$\tilde{p} = p + \epsilon \quad \epsilon \sim \mathcal{N}(0, \Sigma)$$

$$\begin{aligned} \tilde{r} &= r + \epsilon_r \quad \epsilon_r \sim \mathcal{N}(0, \Sigma_r) \\ \tilde{p} &= p + \epsilon_p \quad \epsilon_p \sim \mathcal{N}(0, \Sigma_p) \\ \tilde{h} &= h + \epsilon_h \quad \epsilon_h \sim \mathcal{N}(0, \Sigma_h) \end{aligned}$$

Additive “noise” ok ← linear

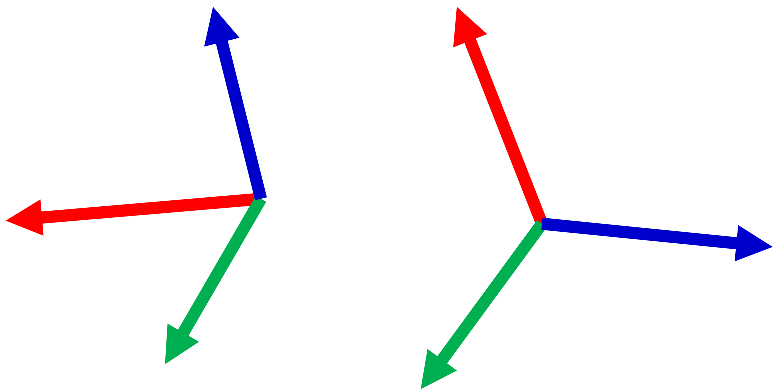
- Manifold
  - Orientation, SE(3), SO(3)



Additive “noise” to a matrix??

Uncertainty Modeling using Gaussian

- Additive noise to SO(3)



$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\tilde{R} = R + \epsilon \quad \epsilon \sim \mathcal{N}(0, \Sigma)$$

Uncertainty Modeling

- Vector space
  - Additive noise to a vector
  - Position
- Manifold
  - Additive noise in a locally linear space
  - Rotation (*not* linear)

$$X = [x, y, z, r, p, h]$$

→ Linearized vector representation

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in SE(3)$$

→ Nonlinear SE(3) representation

Uncertainty Representation

Vector Space

Uncertainty in a Vector Space

$$\tilde{p} = p + \epsilon \quad \epsilon \sim \mathcal{N}(0, \Sigma)$$

$$\begin{aligned} \tilde{r} &= r + \epsilon_r & \epsilon_r &\sim \mathcal{N}(0, \Sigma_r) \\ \tilde{p} &= p + \epsilon_p & \epsilon_p &\sim \mathcal{N}(0, \Sigma_p) \\ \tilde{h} &= h + \epsilon_h & \epsilon_h &\sim \mathcal{N}(0, \Sigma_h) \end{aligned}$$

- R. Smith (1990)
  - Good start paper for SLAM
  - Representation
  - Operation

Estimating Uncertain Spatial Relationships in Robotics*

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In this paper, we describe a representation for spatial information, called the *stochastic map*, and associated procedures for building it, reading information from it, and revising it incrementally as new information is obtained. The map contains the estimates of relationships among objects in the map, and their uncertainties, given all the available information. The procedures provide a general solution to the problem of estimating uncertain relative spatial relationships. The estimates are probabilistic in nature, an advance over the previous, very conservative, worst-case approaches to the problem. Finally, the procedures are developed in the context of state-estimation and filtering theory, which provides a solid basis for numerous extensions.

Uncertainty in a Vector Space

- Mobile robot motion & uncertainty
  - Motion has uncertainty
  - Control action corrupted with Gaussian noise

$$\mathrm{x}_{t+1} = f(\mathrm{x}_t, u_t) + w$$

$$w \sim \mathcal{N}(0, Q)$$

- Mobile robot motion
  - Two operators: oplus and omius (Smith 1998)
  - State = 6 DOF
    - 6 tuple vector
    - Not SE(3)…

Uncertainty in a Vector Space

- Pose (2D case)

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}$$

- Estimate

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{\phi} \end{bmatrix}$$

- Covariance

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{x\phi} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{y\phi} \\ \sigma_{x\phi} & \sigma_{y\phi} & \sigma_\phi^2 \end{bmatrix}$$

Uncertainty in a Vector Space

- State vector = list of poses

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \quad \hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \vdots \\ \hat{\mathbf{x}}_n \end{bmatrix},$$

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} \mathbf{C}(\mathbf{x}_1) & \mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) & \cdots & \mathbf{C}(\mathbf{x}_1, \mathbf{x}_n) \\ \mathbf{C}(\mathbf{x}_2, \mathbf{x}_1) & \mathbf{C}(\mathbf{x}_2) & \cdots & \mathbf{C}(\mathbf{x}_2, \mathbf{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}(\mathbf{x}_n, \mathbf{x}_1) & \mathbf{C}(\mathbf{x}_n, \mathbf{x}_2) & \cdots & \mathbf{C}(\mathbf{x}_n) \end{bmatrix}$$

$$\begin{aligned} \mathbf{C}(\mathbf{x}_i, \mathbf{x}_j) &\triangleq E(\tilde{\mathbf{x}}_i \tilde{\mathbf{x}}_j^T), \\ \mathbf{C}(\mathbf{x}_j, \mathbf{x}_i) &= \mathbf{C}(\mathbf{x}_i, \mathbf{x}_j)^T \end{aligned}$$

Uncertainty Representation

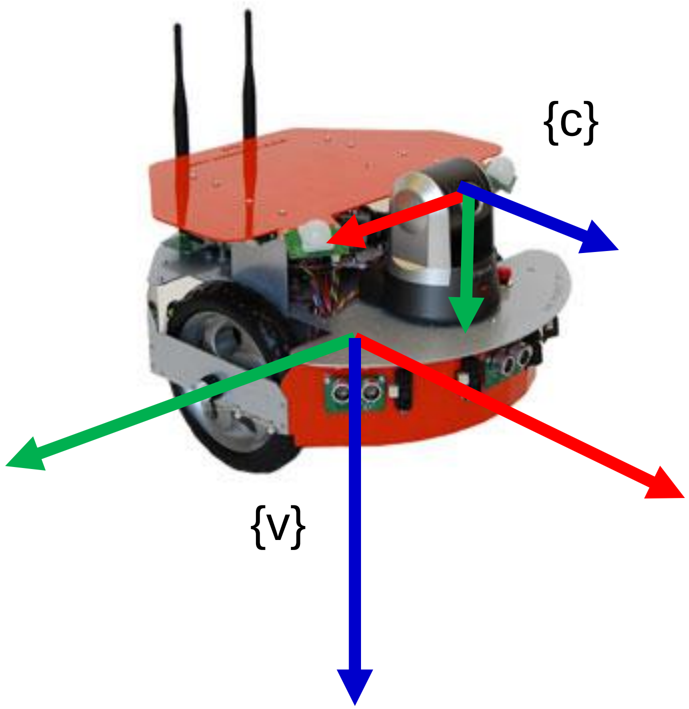
Lie Group

Uncertainty on a Manifold

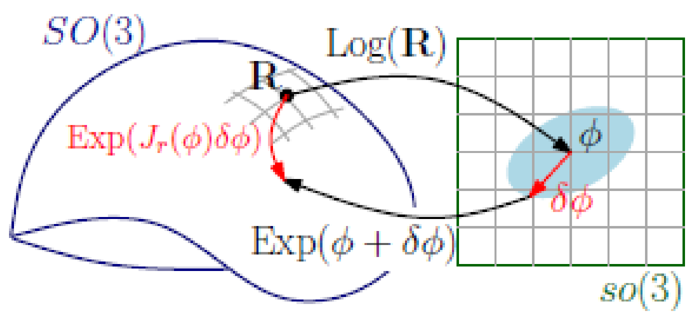
- Pose representation
  - Translation: vector
  - Rotation: SO(3)

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in SE(3)$$

Associated uncertainty for a matrix?

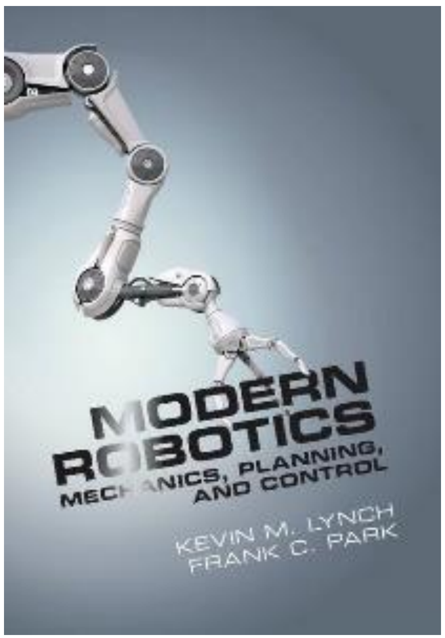


Lie algebra!

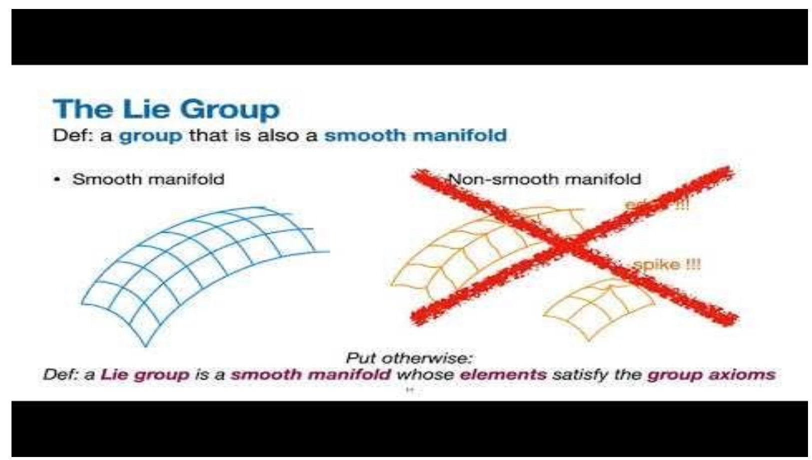


References

- SNUON
  - Introduction to robotics



- Joan Sola's presentation at IROS 2020
  - Lie theory for the Roboticist
  - IROS'20 Workshop on Bringing Geometric Methods to Robot Learning, Optimization and Control.



Review: Angular Velocity

- Angular velocity in space / body coordinate

$$\dot{R} = w_s \times R = [w_s]R$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \Re^3 \qquad [w] = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

Skew-symmetric matrix

$$\omega^\wedge = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}, \quad \forall\, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3$$

$$[w_s] = w_s^\wedge = \dot{R}R^{-1}$$ Space coordinate angular velocity

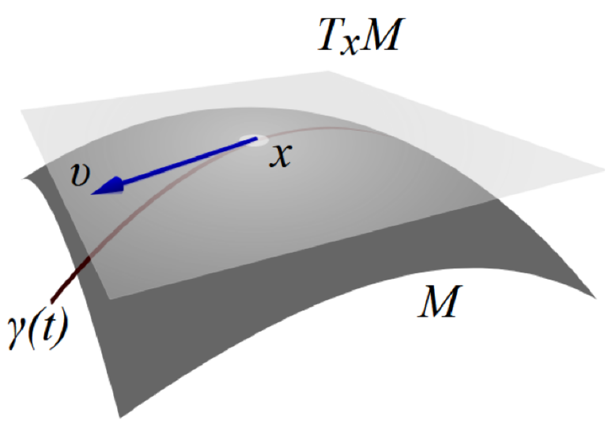
$$[w_b] = w_b^\wedge = R^{-1}\dot{R}$$ Body coordinate angular velocity

Review: Lie Algebra

- Set of skew-symmetric matrices so(3) is Lie Algebra of the Lie Group SO(3)

$$\{A|A^\top + A = 0\} \qquad w^\wedge = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

- Tangent space



If $x \in M$ is a point in the manifold, then the space of *all possible* tangent vectors is called the *tangent space* and is denoted by $T_x M$.

Tangent space at identity I is Lie algebra

Review: Lie Algebra Operator

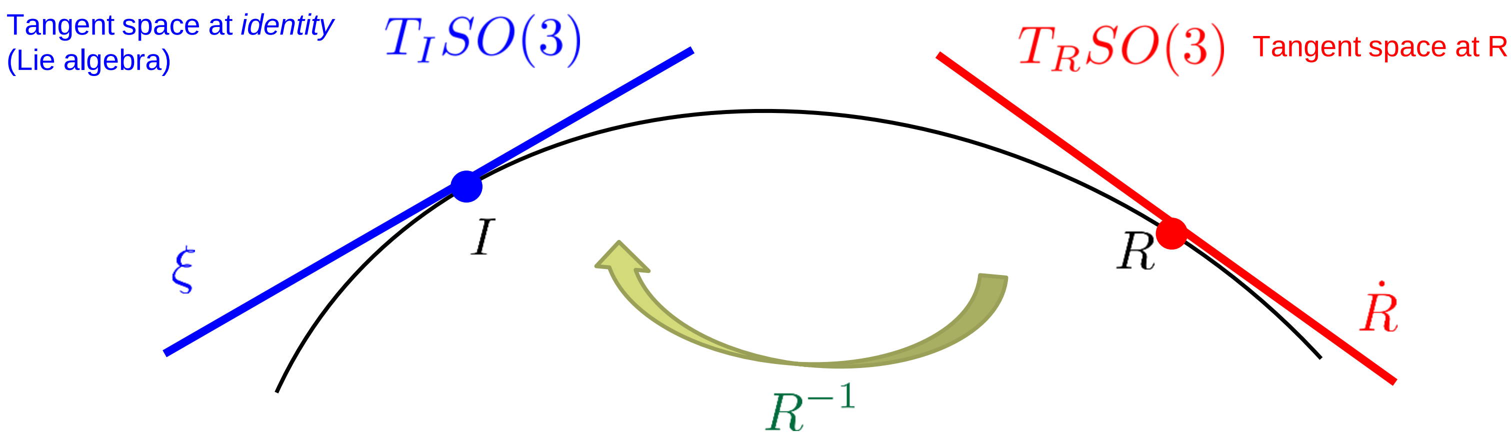
- Lie algebra *hat* operator
  - Vector to skew symmetric matrix

$$\omega^{\wedge} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}^{\wedge} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in \mathfrak{so}(3)$$

- Lie algebra *vee* operator
  - Skew symmetric matrix to vector

Review: Tangent Space & Lie Algebra

- Consider two rotation matrices $I, R \in SO(3)$



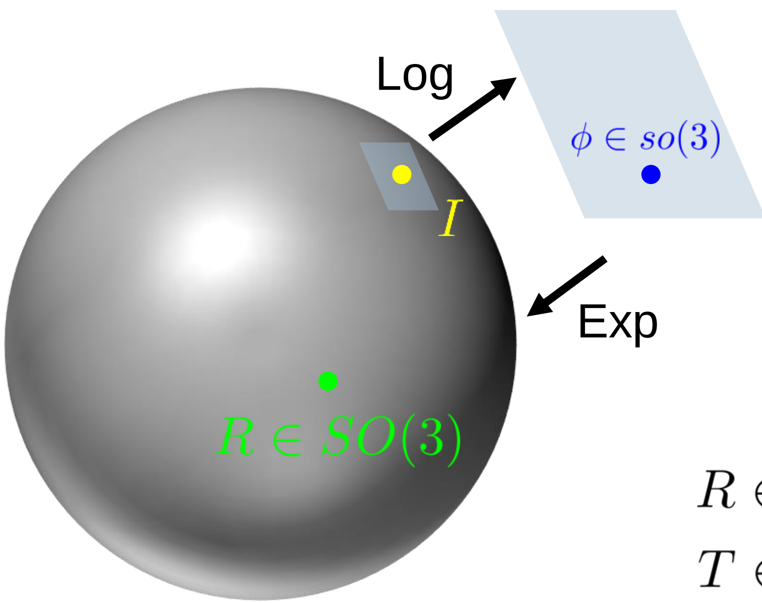
- Using change of basis

$$\dot{R} = R\xi \qquad \xi = R^{-1}\dot{R}$$

- Tangent space is a vector space! → Additive noise!
- Tangent space at identity == Lie algebra
- Lie algebra consists of all possible $\dot{R}$ at $R = I$

Review: Exp/Log

- Exponential and log mapping between Lie Group and Lie Algebra



$$\mathrm{Exp}(\underbrace{\boxed{\phi^\wedge}}_{\text{so(3)}}) = \mathbf{I} + \frac{\sin(\|\phi\|)}{\|\phi\|}\phi^\wedge + \frac{1-\cos(\|\phi\|)}{\|\phi\|^2}\left(\phi^\wedge\right)^2 . \quad =R \rightarrow SO(3)$$

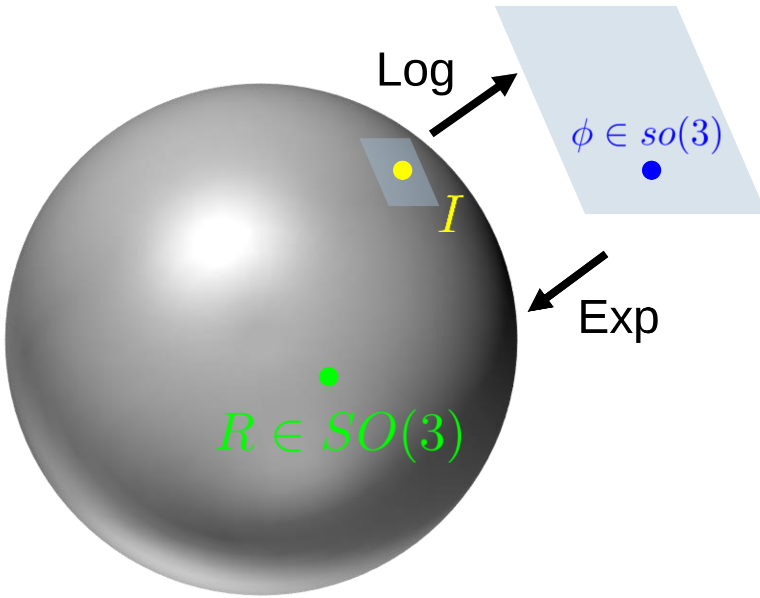
$$\mathrm{Log}(\mathrm{R}) = \frac{\varphi \cdot (\mathrm{R} - \mathrm{R}^\top)}{2\sin(\varphi)} \text{ with } \varphi = \cos^{-1}\left(\frac{\mathrm{tr}\,(\mathrm{R}) - 1}{2}\right)$$

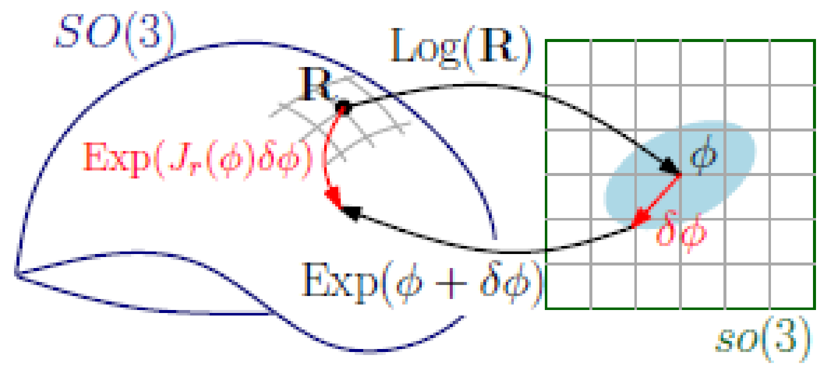
$$R \in SO(3) \xrightarrow{\text{Log}} \phi \in so(3)$$

$$T \in SE(3) \xleftarrow[\text{Exp}]{} \chi \in se(3)$$

Uncertainty on a Manifold

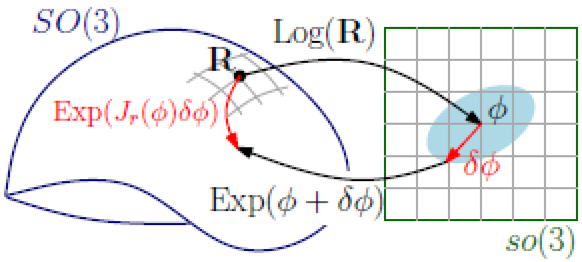
- Log and Exp mapping between SO(3) and so(3)
  - Lie algebra lives in a locally linear space
  - Consider perturbation in Lie algebra





Uncertainty on a Manifold

- First-order approximation for Lie Group SO(3)



Consider noise in Lie algebra

- First-order approximation

$$\mathrm{Exp}(\phi + \delta\phi) \approx \mathrm{Exp}(\phi)\ \mathrm{Exp}(\mathrm{J}_r(\phi)\delta\phi)$$

$$\mathrm{Log}(\ \mathrm{Exp}(\phi)\ \mathrm{Exp}(\delta\phi)\ ) \approx \phi + \mathrm{J}_r^{-1}(\phi)\delta\phi.$$

Right Jacobian

$$\mathrm{J}_r(\phi) = \mathbf{I} - \frac{1-\cos(\|\phi\|)}{\|\phi\|^2}\phi^\wedge + \frac{\|\phi\|-\sin(\|\phi\|)}{\|\phi^3\|}(\phi^\wedge)^2.$$

$$\mathrm{J}_r^{-1}(\phi) = \mathbf{I} + \frac{1}{2}\phi^\wedge + \left(\frac{1}{\|\phi\|^2} + \frac{1+\cos(\|\phi\|)}{2\|\phi\|\sin(\|\phi\|)}\right)(\phi^\wedge)^2$$

- Distribution in the tangent space, then map it to SO(3) via Exp mapping

$$\tilde{\mathrm{R}} = \mathrm{R}\ \mathrm{Exp}(\epsilon), \qquad \boxed{\epsilon \sim \mathcal{N}(0, \Sigma)}$$

Thank you very much !!