M3228.000300

SLAM 101

Lecture 00 Probability

Ayoung Kim

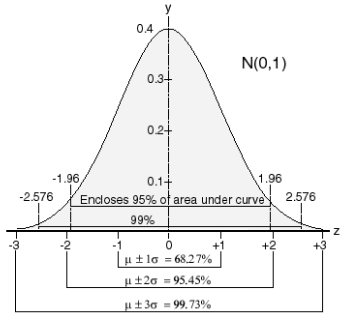
Probability and SLAM

- Bayes rule

$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}$$

- Gaussian

$$\mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$



Bayes Rule

Conditional Probability

- Conditional probability

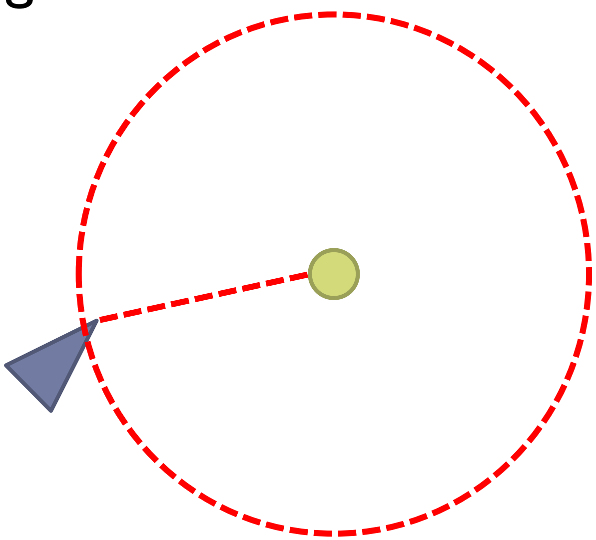
  Probability that X will take vale x_i given Y will take value y_i
  Consider only instance for which $Y = y_i$ then the fraction of such instance for which $X = x_i$

  $$p(X = x_i|Y = y_i) = \frac{p(X=x_i, Y=y_i)}{p(Y=y_i)} = \frac{p(X \bigcap Y)}{p(Y)}$$

  $$\boxed{p(X, Y) = p(X|Y)p(Y): \text{ Product rule}}$$

- . Given a dice gave number 3, prob of getting H?
  - p(X=H | Y =3)

- SLAM
  - p(robot position | sensor measurement)



Bayes Theorem

- Bayes theorem

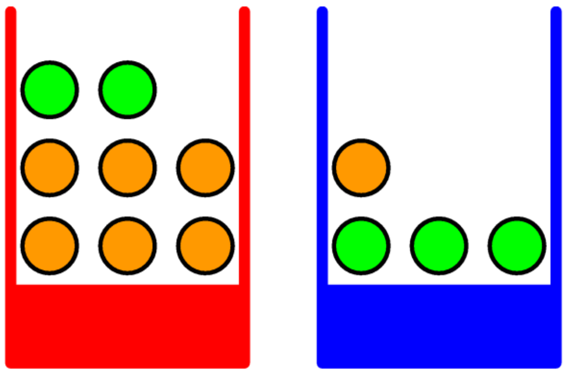
$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}$$

  - Central rule in pattern recognition and machine learning
  - Robotics estimation and computer vision
  - Using sum rule, it becomes

$$p(Y|X) = \frac{p(X|Y)p(Y)}{\sum_Y p(X|Y)p(Y)}$$

Probability – Toy Example

- Blue box and red box



$$\text{Prob(Picking red box)} = \text{P(B = red)} = 0.4$$

$$\text{Prob(Picking blue box)} = \text{P(B = blue)} = 0.6$$

$$\text{P(B=red)} + \text{P(B=blue)} = 1$$

Select either red or blue

$$p(F = green|B = red) \quad =$$

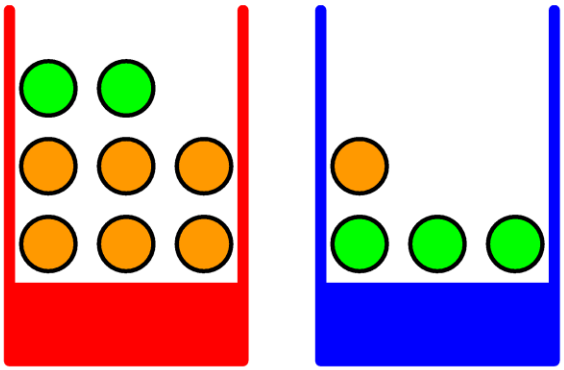
$$p(F = orange|B = red) \quad =$$

$$p(F = green|B = blue) \quad =$$

$$p(F = orange|B = blue) \quad =$$

$$p(B = r|F = o)$$

Probability – Toy Example



Q. What is p(F=orange)?

Q. Also what is P(B=red | F = orange)?

$$
\begin{aligned}
p(F = orange) &= p(F = o|B = r)p(B = r) \\
&\quad + p(F = o|B = b)p(B = b) \\
&= \frac{3}{4} \times \frac{4}{10} + \frac{1}{4} \times \frac{6}{10} \\
&= \frac{9}{20}
\end{aligned}
$$

Overall (no matter what the box is) the probability of choosing green ball

$$
\begin{aligned}
p(B = r|F = o) &= \frac{p(F = o|B = r)p(B = r)}{p(F = o)} \\
&= \frac{\frac{3}{4} \times \frac{4}{10}}{\frac{9}{20}} \\
&= \frac{2}{3}
\end{aligned}
$$

If the ball you've chosen in orange, the probability that the box is red

Prior and Posterior Probability

- In the previous example

likelihood

Prior probability

$$\boxed{p(B = r|F = orange)} = \frac{\boxed{p(F = o|B = r)}\boxed{p(B = r)}}{p(F = o)}$$

Posterior probability

- Prior probability
  - Available before we observe the event
- Posterior probability
  - Probability after we observe an event that the ball is orange

$$\boxed{\text{posterior} \propto \text{likelihood} \times \text{prior}}$$

Probability w.r.t. Continuous Variable

- Previous example probability over discrete set
  - Probability mass function (pmf) $p(X = x_i)$
  - Probability dense function (pdf) $p(X = x_i) = 0$ or $f(X = x_i) = 0$

$$p(x \in (a,b)) = \int_a^b p(x)dx$$

  - Cumulative distribution function

$$P(z) = \int_{-\infty}^{z} p(x)dx$$

$$p(x) \geq 0$$

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

$$p(x) = \int p(x,y)dy$$

$$p(x,y) = p(y|x)p(x)$$

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)} = \frac{p(x|y)p(y)}{\int p(x,y)dy}$$

Expectation and Variance

- Expectation of a function

$$E[f] = \sum p(x) f(x) \quad \text{or} \quad E[f] = \int p(x) f(x) dx$$

  - Conditional expectation

$$E[f|y] = \sum p(x|y) f(x)$$

- Variance of a function $\mathrm{var}[f] = E[(f(x)^2] - E[f(x)]^2$
  - Variance of a random variable

$$\mathrm{var}[x] = E[x^2] - E[x]^2$$

  - Covariance of two random variables

$$\mathrm{cov}[x, y] = E[xy] - E[x]E[y]$$

(Note: $\mathrm{cov}[x, y] = 0$ if x and y are indep.)

Review Other Rules

- Law of total probability
  - Random variable X,Y. and their conditional probability is known

$$P(X) = \sum_{i=1}^{n} P(X|Y = y_i) = P(X|Y = y_1) + P(X|Y = y_2) + \cdots + P(X|Y = y_n)$$

Sum over all possible Y

- Marginalization
  - Random variable X,Y. and their joint probability is known

$$P(X = x_i) = \sum_{j=1}^{n} P(X = x_i, Y = y_j)$$

- Marginal probability written as expected value

$$\begin{aligned} P(X = x) &= \int_y P(X = x, Y = y)dy \\ &= \int_y P(X = x|Y = y)P(Y = y)dy \\ &= E[P(X = x|Y = y)] \end{aligned}$$

Y \ X	x_1	x_2	x_3	x_4	$p_Y(y)$ ↓
y_1	4/32	2/32	1/32	1/32	**8/32**
y_2	3/32	6/32	3/32	3/32	**15/32**
y_3	9/32	0	0	0	**9/32**
$p_X(x)$ →	**16/32**	**8/32**	**4/32**	**4/32**	**32/32**

Wikipedia

Gaussian

Gaussian Distribution

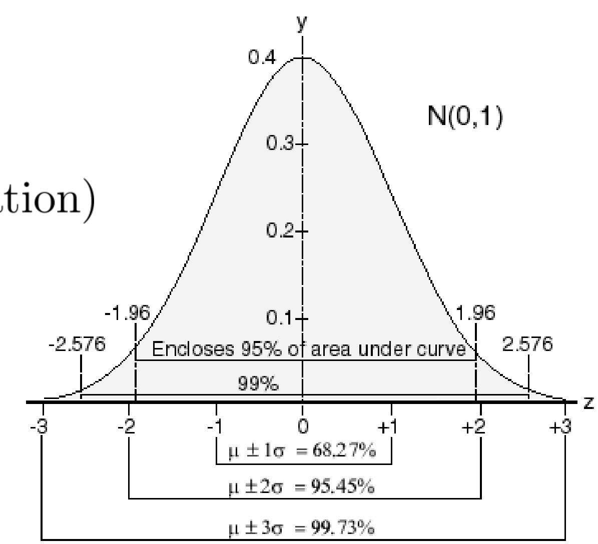
- Continuous random variable
  - Gaussian (Normal) distribution
  - if it is governed by two parameter mean and variance and is in the following form

$$\mathcal{N}(x;\mu,\sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

Random variable

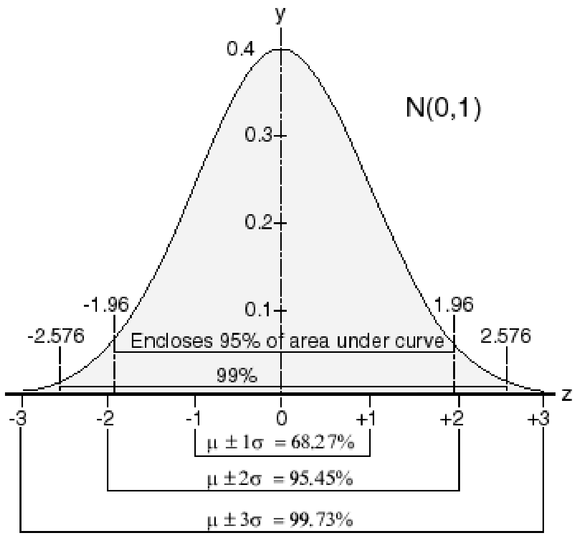
Parameters

μ is mean, σ^2 is variance (σ is standard deviation)



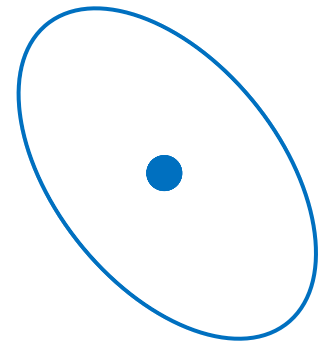
Gaussian Distribution

- 1D

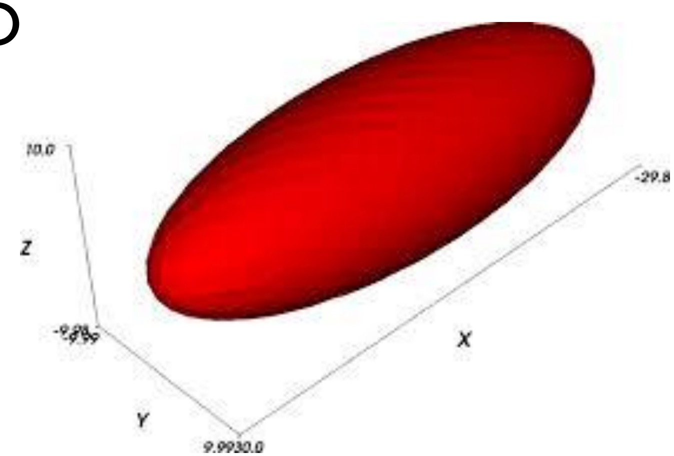


68–95–99.7 rule (Wikipedia)

- 2D



- 3D



Multivariate Gaussian Distribution

- For a vector x

$$\mathcal{N}(\mathrm{x}; \mu, \Sigma) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}\underbrace{(\mathrm{x}-\mu)^\top \Sigma^{-1}(\mathrm{x}-\mu)}_{\text{Mahalanobis distance}}\right\}$$

μ is D dimensional mean vector
Σ is $D \times D$ variance matrix

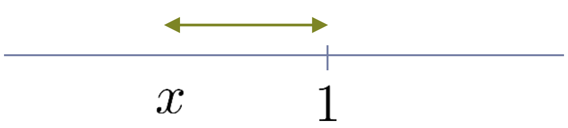
- Mahalanobis distance

$$\boxed{D^2 = (\mathrm{x}-\mu)^\top \Sigma^{-1}(\mathrm{x}-\mu)}$$

If $x \in \mathcal{R}$ (scalar) and $\Sigma = I = 1$

$D^2 = (x-\mu)^2 \rightarrow$ Euclidean distance

If $\mu = 1$ then $D^2 = (x-1)^2$: Euclidean distance to 1

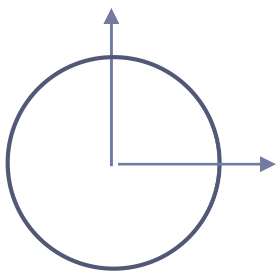


Mahalanobis distance

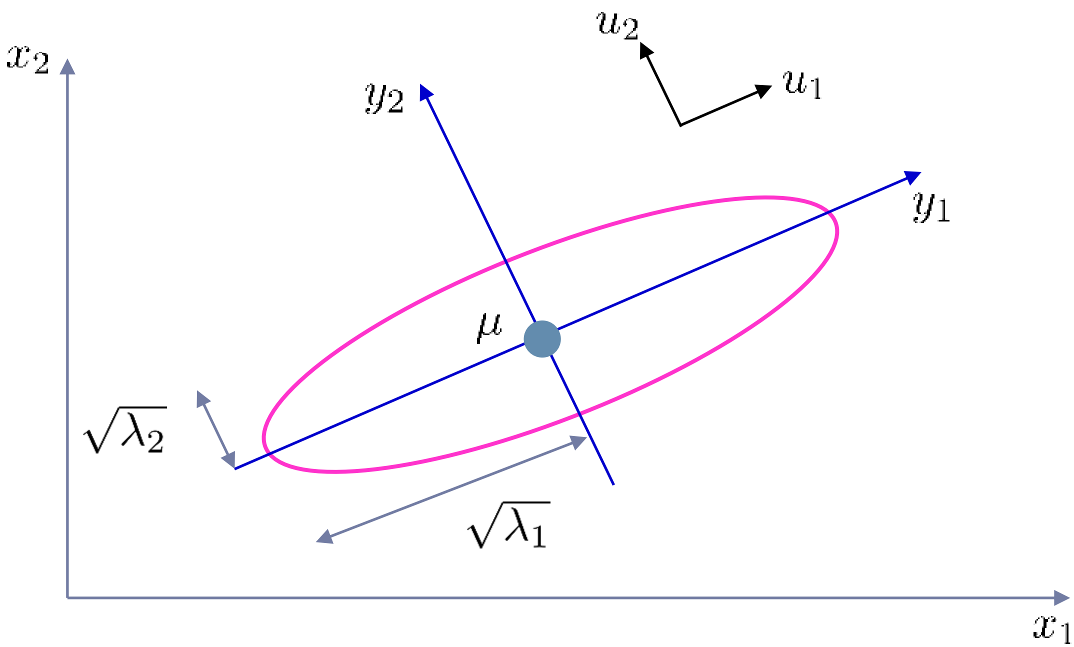
- For general 2D matrix Σ, apply eigenvalue decomposition

$$\Sigma = U\Lambda U^{-1} \text{ and } \Sigma u_i = \lambda_i \mathrm{u}_i$$

$$\Sigma = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_1^\top \\ u_2^\top \end{bmatrix}$$



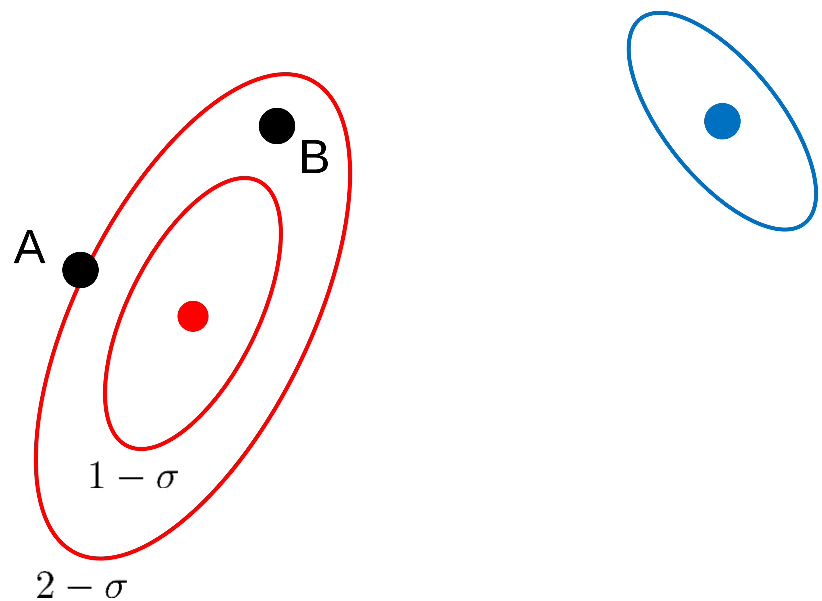
$$\boxed{D^2 = (\mathrm{x} - \mu)^\top \Sigma^{-1} (\mathrm{x} - \mu)}$$



Mahalanobis distance

- What is the unit? How do we measure?
  - Euclidean distance 2 meters, 5 feet

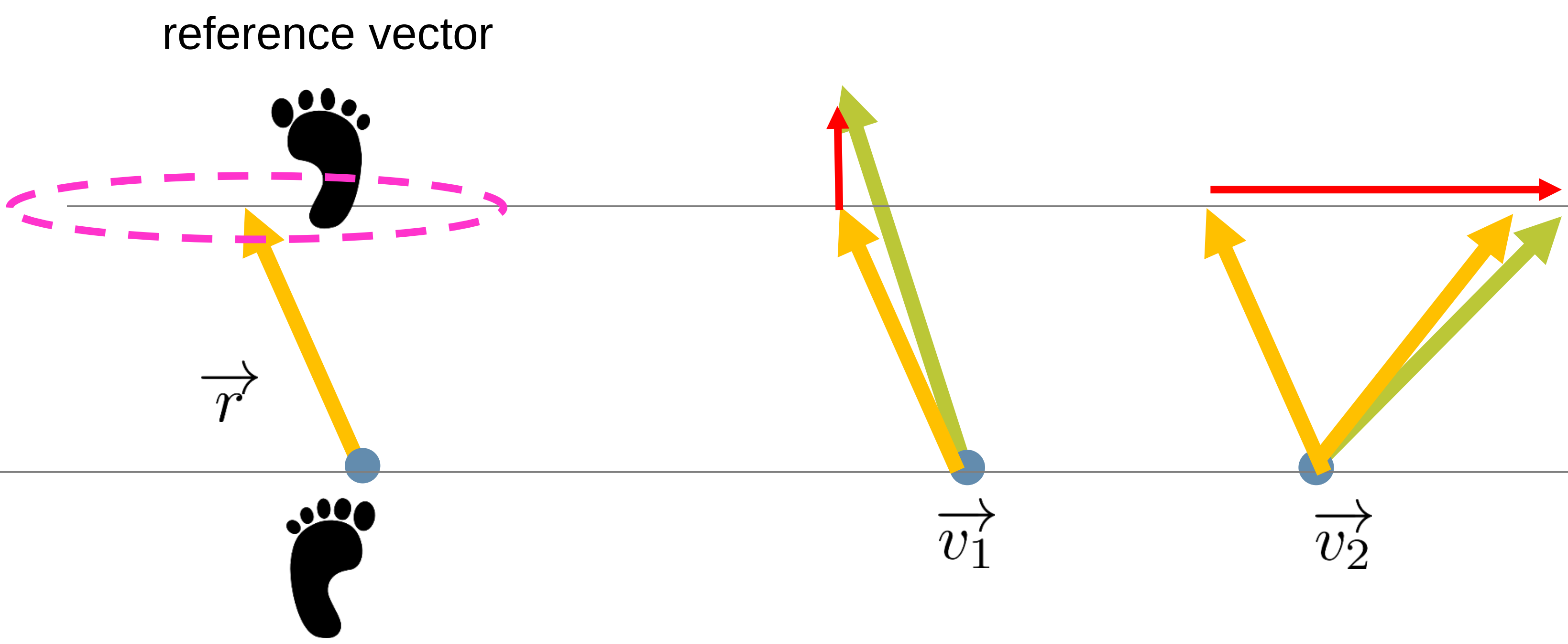
$$D = \sqrt{(\mathrm{x} - \mu_1)^\top \Sigma_1^{-1} (\mathrm{x} - \mu_1)}$$



$x_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$

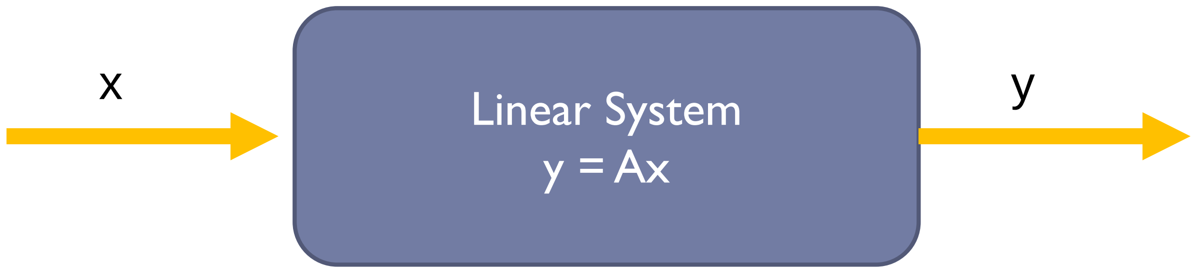
Mahalanobis distance

- Example: measurement validation



Gaussian and Linearity

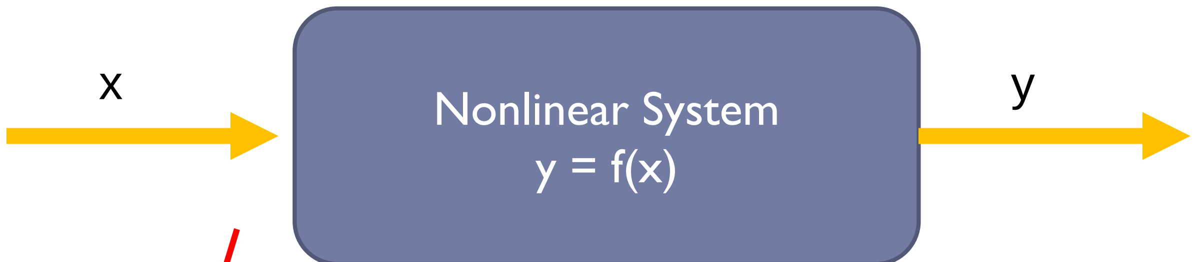
- Random variable x is Gaussian $x \sim \mathcal{N}(\mu_x, \sigma_x^2)$
  - How about y ?

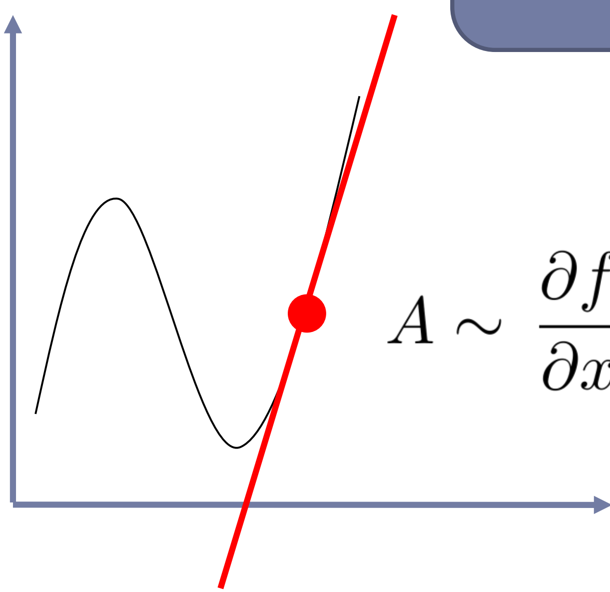


$$\begin{aligned} \mu_y &= A\mu_x \\ \sigma_y^2 &= A\sigma_x^2 A^\top \end{aligned}$$

Gaussian and Linearity

- Random variable x is Gaussian $x \sim \mathcal{N}(\mu_x, \sigma_x^2)$
  - How about y for nonlinear system?





$$A \sim \left.\frac{\partial f}{\partial x}\right|_{x_0}$$

$$\begin{aligned}\mu_y &= f(\mu_x) \\ \sigma_y^2 &= \left.\frac{\partial f}{\partial x}\right|_{x_0} \sigma_x^2 \left.\frac{\partial f}{\partial x}\right|_{x_0}^{\top}\end{aligned}$$

Probability and SLAM

Simultaneous Localisation and Mapping (SLAM):
Part I The Essential Algorithms

Hugh Durrant-Whyte, *Fellow, IEEE*, and Tim Bailey

- Solve for localization and mapping $P(\mathbf{x}_k, \mathbf{m} \mid \mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0)$

- Motion propagation
$$P(\mathbf{x}_k, \mathbf{m} \mid \mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k}, \mathbf{x}_0) = \int P(\mathbf{x}_k \mid \mathbf{x}_{k-1}, \mathbf{u}_k) \times P(\mathbf{x}_{k-1}, \mathbf{m} \mid \mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k-1}, \mathbf{x}_0) \mathrm{d}\mathbf{x}_{k-1}$$

- Measurement update
$$P(\mathbf{x}_k, \mathbf{m} \mid \mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0) = \frac{P(\mathbf{z}_k \mid \mathbf{x}_k, \mathbf{m}) P(\mathbf{x}_k, \mathbf{m} \mid \mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k}, \mathbf{x}_0)}{P(\mathbf{z}_k \mid \mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k})}$$

Thank you very much !!