SLAM 101

Lecture 00 Linear Algebra

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System of Linear Equations

- Linear equations
  - # of unknowns / # of equations

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

    - 3 unknowns and 3 equations
  - Matrix form

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \qquad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$Ax = b$$

Linear Span

- Set S of vectors

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \middle| k \in \mathbb{N}, v_i \in S, \lambda_i \in \mathbf{K} \right\}.$$

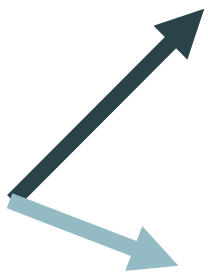
$$\begin{bmatrix} \boxed{\begin{matrix} 2 \\ 4 \\ -2 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -6 \\ 7 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 0 \\ 2 \end{matrix}} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

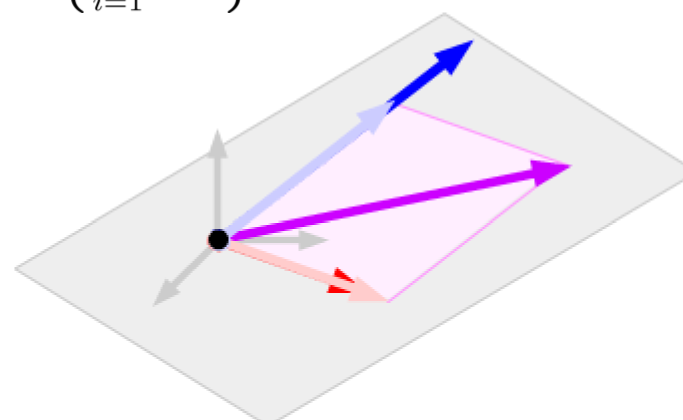
$$v_1 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Linear Span

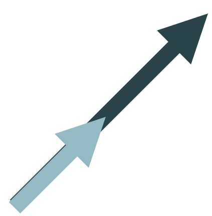
- Span of two vectors
  - Linearly independent vectors

$$\mathrm{span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \right\}$$





  - Linearly dependent vectors



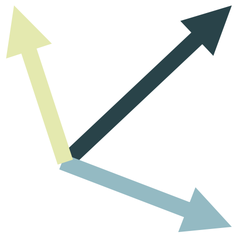
$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Linear Equation and Span

- Examine the linear equation

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \quad v_1 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$u \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$



Finding a scale !

Linear Independence

- If three vectors are linearly indep. to each other
  - Spans 3D space = represent any 3 dim. Vector
- Checking lin. indep.
- Rank
  - Rank of a matrix = # of linearly indep. columns of the matrix

Rank of a Matrix

- Examples
  - (1) $$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$
  - (2) $$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \end{bmatrix}$$
  - (3) $$A^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 2 & -2 \end{bmatrix}$$

$$\boxed{\operatorname{rank}(A) = \operatorname{rank}(A^\top)}$$

Matrix Inversion

- Solution exist = matrix inversion exist
  - Matrix is invertible
  - Matrix has full rank

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$u \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

System of Linear Equations

- Solving $\mathrm{y} = A\mathrm{x}$
- A matrix (m by n)
  - m = number of equation, n = number of unknown

$$A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \qquad A = \begin{bmatrix} & & & \\ & & & \end{bmatrix} \qquad A = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

square $m = n$	fat $m < n$	thin $m > n$
Exact solution	Under constrained	Over constrained

  - Solution $\mathrm{x} = A^{-1}\mathrm{y}$ $\qquad A^+ = A^*(AA^*)^{-1}\,. \qquad A^+ = (A^*A)^{-1}A^*\,.$
  - Pseudo inverse

Let's look at some decompositions

Matrix Properties

Matrix Properties

- Symmetric
  - A square matrix that is equal to its transpose

$$A = A^\top \qquad (AB)^\top = B^\top A^\top$$

- Orthogonal (or orthonormal)
  - A square matrix with real entries
    whose columns and rows are orthogonal unit vectors

$$A = A^\top \qquad AA^\top = A^\top A = I$$

- Unitary
  - Orthogonal matrix
  - Complex conjugate transpose (complex)

Matrix Decomposition

Eigen Value Decomposition

- ## Eigenvalues and eigenvectors

For a $N \times N$ square matrix A,
an eigenvector of A is a N dimensional vector v that

$A\mathrm{v} = \lambda \mathrm{v}$
λ is the corresponding eigenvalue

$$\begin{aligned} A\mathrm{v} &= \lambda \mathrm{v} \\ 0 &= \det(A - \lambda I) = p(\lambda) \end{aligned}$$

$$p(\lambda) = \lambda^N + a_1 \lambda^{N-1} + \cdots = 0$$

Solutions λ_i are the eigenvalues

- ## Matrix representation

$$A = Q\Lambda Q^{-1} \qquad Q = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_N \\ | & & | \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}$$

$$A^{-1} = Q\Lambda^{-1}Q^{-1}$$

Singular Value Decomposition

- Singular value decomposition (SVD)

A $M \times N$ matrix B,
can be factorized into multiplication of three matrices

$$B = U\Sigma V^*$$

U is an $M \times M$ unitary matrix
Σ is an $M \times N$ diagonal matrix
V is an $N \times N$ unitary matrix

SVD – Example

- Consider 4x5 matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

Check unitary

$$\mathbf{U}\mathbf{U}^{\mathbf{T}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_4$$

$$\mathbf{V}\mathbf{V}^{\mathbf{T}} = \begin{bmatrix} 0 & 0 & \sqrt{0.2} & 0 & -\sqrt{0.8} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{0.8} & 0 & \sqrt{0.2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_5$$

SVD is not unique

$$\mathbf{V}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ \sqrt{0.4} & 0 & 0 & \sqrt{0.5} & -\sqrt{0.1} \\ -\sqrt{0.4} & 0 & 0 & \sqrt{0.5} & \sqrt{0.1} \end{bmatrix}$$

SVD – Matrix Inverse

- Application of SVD
  - Matrix inverse $B^+ = V\Sigma^+U^*$
  - Pseudo inverse and optimization

$$A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

square $m = n$

Exact solution

$$A = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}$$

fat $m < n$

Under constrained

$$A = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

thin $m > n$

Over constrained

SVD – Optimization

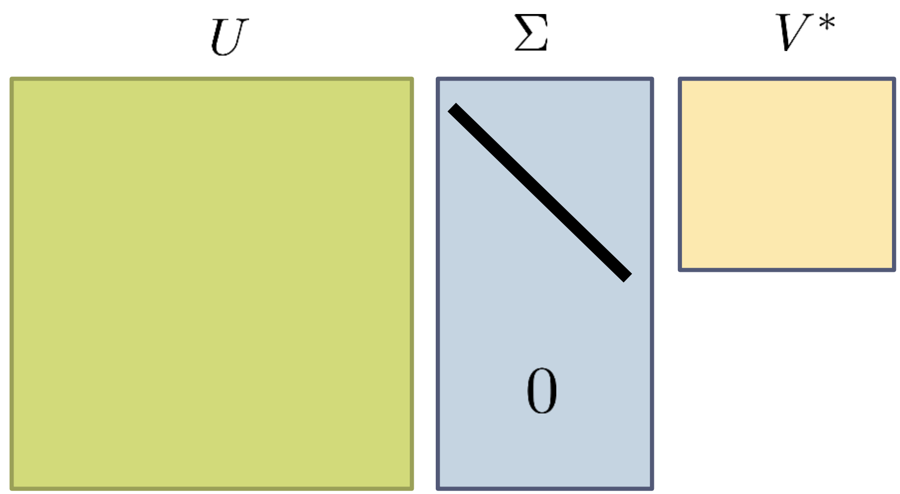
- Optimization
  - Least square problem $\min ||Ax - b||^2$
  - More equation than unknown (data fitting)

$$A = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

thin $m > n$

Over constrained

$$A = \begin{bmatrix} \quad & \quad \end{bmatrix}$$



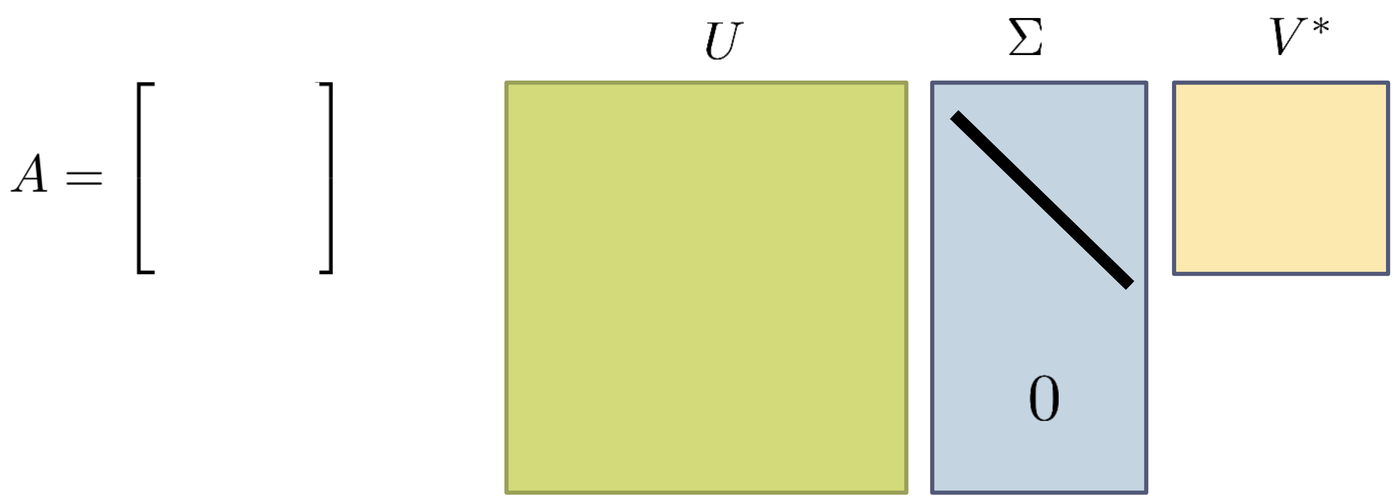
$$\min ||Ax - b||^2 \rightarrow \min ||\Sigma V^* x - U^* b||^2$$

SVD – Optimization

- Homogeneous case when b=0

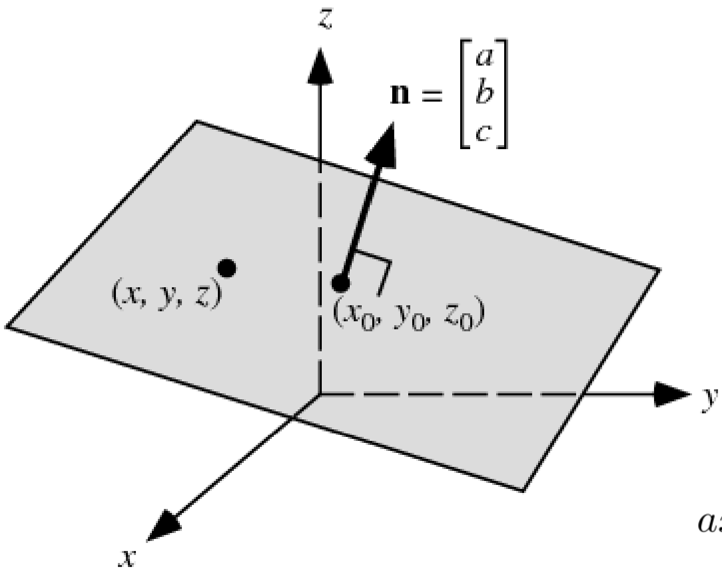
$$\min ||Ax - b||^2 \rightarrow \min ||\Sigma V^* x||^2$$

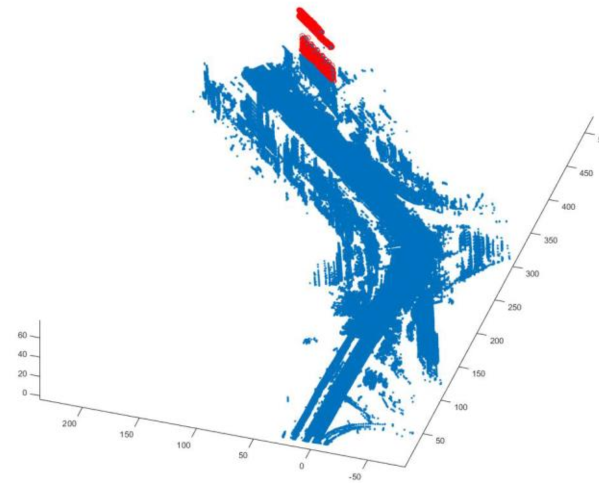
- Optimal solution x is the last column of the V matrix
- Column of V that corresponds to zero(or minimum) singular value



SVD for Plane Fitting

- Plane fitting from point cloud





$$ax + by + cz + d = 0$$

$$\begin{aligned} ax_1 + by_1 + cz_1 + d &= 0 \\ ax_2 + by_2 + cz_2 + d &= 0 \\ ax_3 + by_3 + cz_3 + d &= 0 \\ &\vdots \\ ax_n + by_n + cz_n + d &= 0 \end{aligned}$$

$$\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & z_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

Summary

- Linear algebra
  - System of Linear Equations
  - Matrix decomposition
  - Matrix properties

- Next lecture
  - Visual features

Thank you very much !!