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Quantitative Asset Management

Robust Asset Allocation

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1 Introduction

Asset management; and the broader discipline of investing, brought up many rich and talented men and women in the past, and made even more of the very same unlucky and poor. Their strategies and approaches were as diverse as the academic backgrounds they stemmed from and most had glorious but also less lucrative times. Some of the most renowned are probably Warren Buffet, following the doctrines of his and Mr Graham's school; George Soros, taking a macro-driven approach; Peter Lynch, that focuses more on fundamentals; Robert Levine and his strong-horse method; David Swensen, that invests in alternative assets; or the brains behind the Renaissance Medallion Fund, that do algorithmic trading.

However, regardless of which strategy investors ultimately follow, probably all of them can agree on the benefits of diversification and the importance of asset allocation. Since the second half of the twentieth century, the economic theory underlying portfolio choice is well understood. Thanks to pioneering work like the one of Markowitz 1952 and Tobin 1969, mean-variance optimized portfolios became the norm. This was further improved to settings with uncertainty by Merton 1969 and dynamic settings by Samuelson 1969. While the discipline evolved further into multiple directions such as index funds in the 1970s; factor investing; and smart beta strategies in recent years, this basic dogma remained.

While in theory an easy endeavour, optimal asset allocation is sensitive to the true parameters in terms of covariance and conventional static approaches unsuitable to capture the complex and dynamic real-world. Despite these well-known pitfalls and shortcomings, little advances have made it into the investment practice of asset asset managers. Therefore, this paper aims to illustrate the approach for robust asset allocation suggested by Anderson and Cheng 2016 and replicate their strategy with real stock data from an extended period, reaching over fifteen years and thereby, covering two major equity sell-offs.

2 Literature Review

As mentioned above, the sensitivity of portfolio optimization to its input parameters is one of the key challenges of mean-variance theory in practice. In practice, out-of-sample validity is often overseen to the advantage of in-sample fit and optimal choices are by construction only the best choices in a given sample. Until today, practitioners often estimate constant means and covariances from historical data and assume them to be the correct values, which is obviously not true out-of-sample. Allowing for parameter uncertainty is, therefore, important, to allow for wrong input parameters and avoid over-fitting.

Puzzled by practitioners affection for outdated methods, academics proposed various sophisticated approaches to tackle this problem and tried to shift out-of-sample results into the focus (see Albrecher et al. 2009 for a detailed treatment). However, only Richard Michaud and Robert

Michaud's simplistic resampling approach made it to somewhat an industry standard and will therefore be briefly discussed in the following, followed by a treatment of this work's main reference, Anderson and Cheng's 'Robust Bayesian Portfolio Choices' and some alternative methods like the ones proposed by Ceria and Stubbs 2006 and Tütüncü and Koenig 2004. However, to start with, we will briefly discuss the origin of mean-variance optimization on the basis of Markowitz's seminal work.

2.1 Modern Portfolio Theory

Mean-Variance optimization, better known as modern portfolio theory, describes the school of Markowitz, whereby the trade-off between risk and reward is solved, such that the expected return is maximized given a certain level of risk (cf Markowitz 1952). This mathematical approach to asset allocation is the formalization of diversification, which allows the portfolio to reduce all asset specific risk to the level that only market risk remains. This implies that a portfolio has a lower risk to the same expected return compared to a non optimally diversified strategy and therefore, arguably the most efficient way to asset allocation. However, this only holds true given the true expected returns and covariances are known, since those are crucial for an optimal allocation.
Historical correlations might indicate future ones but are not perfectly matching. Thus, distortions arise and portfolios are non-optimally diversified in future periods. This parameter uncertainty is a key problem to solve and was addressed in multiple papers, including the ones mentioned later in this chapter. Other scholars that criticised modern portfolio theory for its high sensitivity to input changes are Chopra and Ziemba 1993, that prove the issue mathematically; Best and Grauer 1991, that assessed the impact of such on expected returns; Jobson and Korkie 1981, that promote using sharpe ratios for optimizing mean-variance portfolios; and Broadie 1981, who discusses the overestimation of expected returns through varying estimation errors.

2.2 Resampling

Among the practitioners that tackled the aforementioned problem was Richard Michaud, who patented his Resampled Efficiency techniques (cf. Richard Michaud 1998) in 1998. Just under a decade later, Richard Michaud and Robert Michaud 2007 extended this Resampled Efficiency solution formally, by showing that it is a Bayesian-based generalization of Markowitz's approach. Their solution is based on Monte Carlo simulations to account for parameter uncertainty in mean-variance portfolio optimization. In their later paper (Richard Michaud and Robert Michaud 2007), the scholars argue that, while the theoretical idea behind Markovitz's portfolio optimization is correct, the model relies too heavily on inadequate data and is therefor fairly susceptible. In their opinion, models tend to rely too extremely on data that does not have sufficient predictive power and do not account for parameter and estimation uncertainty. To address these issues, their proposed Resampled Efficiency portfolios uses bootstrapping and are constructed by a conventional Monte Carlo algorithm that follows the following basic steps:

1. Randomly sample means and covariance matrix of returns, with the center of these estimations being the regular values used in conventional mean-variance optimizations
2. Calculate the mean-variance efficient frontier based on these sampled risk and return estimates
3. Do this repeatedly to construct a big data set
4. Average portfolio weights from all these generated data points

The scholars show, that the resulting portfolio is more conservative and underperforms in-sample; but outperforms out-of-sample. Optionally, additional constrains can be added.

2.3 'Robust Bayesian Portfolio Choices'

One of the more sophisticated approaches is the one of Anderson and Cheng 2016. They propose a Bayesian-averaging portfolio choice strategy to overcome the problem of parameter uncertainty. Anderson and Cheng's method accounts for parameter uncertainty by using a Bayesian averaging approach over various models. In each period, they recalculate the covariance matrix and thereby optimal weights and reevaluate all historical models using Bayesian updating. The authors include robust mean-variance optimization by creating the optimization problem in a way that includes uncertainty about the correctness of a model. The advantage over rolling window approaches is that a model never fully dies out and no information is lost. Additionally, they allow investors to be in doubt that predictions are correct and thus, to focus on the worst scenario that is somewhat reasonable, given a range of predictions. They include a model parameter that lets them set the agents level of aversion against mis-specification; with the limit taking the normal form of the standard mean-variance problem.
Their proposed technique follows the following steps to arrive at their optimal weights in each period respectively:

1. At date t-1, there are t-1 existing models which make mean and covariance predictions of excess returns on time t. Also, each model at date t-1 is assigned a prior probability.
2. At date t, a new model is born. Therefore, the prior probabilities of all t models are adjusted for the inclusion of this new model.
3. The excess returns on date t are observed. The parameters of each model are updated using the new information. Prior probabilities are also updated using Bayesian Rules. Optimal portfolio choices at time t are computed.

4. At date t+1, the excess returns on t+1 are observed. Therefore, the excess portfolio returns are obtained by the product between the optimal portfolio choices at time t and the excess returns on t+1.

Important to note, that each model assumes constant mean and covariance for future periods. At each point in time, all the historical models get modified by using the new gained knowledge from this point in time. Each single one of them produces a new prediction at that point and the probability that the model is correct is calculated, based on the information known up to that day. By following this sophisticated approach, Anderson and Cheng produce robust predictions on future means and variances by accounting for parameter and model uncertainty, thus produce better estimations. See chapter 4 of this paper for a detailed treatment of their, and our, methodology. During the empirical part of their paper, the authors show that their strategy yields superior results in real stock data and artificial return series, what we will be replicating with own data at a later stage.

2.4 'Incorporating estimation errors into portfolio selection: Robust portfolio construction'

As seen previously, two potential improvements to Markowitz's approach are an increased risk aversion parameter, which would minimise the over-estimation of the efficient frontier; and statistical re-sampling, that considers errors by averaging individual optimal portfolio's with respect to randomly generated expected return and risk estimators. However, both approaches have some flaws. An increased risk-parameter still assumes that the covariance matrix of the estimation error is a constant multiple of the matrix of returns; and the re-sampling approach is based on ad-hoc methodology, that is fairly time-consuming and doesn't satisfy all constraints.
Therefore, Ceria and Stubbs 2006 propose robust optimisation to consider uncertainty in parameters. The authors show, that with conventional methods, tiny estimation errors yield dramatic changes in the weights of the assets in the portfolio. They call this the error-maximisation effect. To solve this, they suggest the following maximization problem (here shown with conditions for a long-only robust portfolio that satisfies a budget and a variance constraint):

$$\begin{aligned} \max_{w} \quad & \bar{\alpha}^T w - \kappa \left\| \Sigma^{1/2} w \right\| \\ \text{s.t.} \quad & e^T w = 1 \\ & w^T Q w \leq v \\ & w \geq 0, \end{aligned} \tag{1}$$

whereby $\bar{\alpha}$ stands for an estimate of expected returns; w for the vector of weights; κ for a confidence parameter, Σ for the variance-covariance matrix; e for a budget; v for a variance target and Q for the covariance matrix of returns, not obtained by the estimates but in practice obtained from a risk model provider.

This formula is the same as the one for a classical mean-variance optimization problem, aside from the inclusion of $\kappa \left\|\Sigma^{1/2} w\right\|$. This term reduces the effect of the estimation error on the optimal portfolio. In relation to the conventional optimization problem, the term's presence would adjust the expected return of assets with positive weights downwards and vice versa.

The scholars also propose alternative forms of their robust portfolio optimization. This is necessary, because the basic version only adjust estimates of expected returns downwards if long-only constraints are present. Otherwise, an alpha downwards-adjustment for an asset with an already negative weight, would become overly negative. The solution is a zero net alpha-adjustment frontier (see equation 15 of Ceria and Stubbs 2006 for a detailed description), whereby, in the case of being fully invested, a portfolio weight which is above the optimum gets alpha-adjusted downwards and vice versa. Another alternative is the robust active return/active risk frontier (equation 18 of the Ceria and Stubbs paper), that uses benchmark weights as a determinant for holdings in an asset and whether to adjust up or downwards. Lastly, the authors suggest to use what they call a general robust optimization framework (equation 19 of Ceria and Stubbs 2006), that compiles all previously mentioned models and permits for the construction of other alternatives.

In their paper, the authors demonstrate mathematically that the true and estimated frontiers lie closer together with robust optimization. Also, they show empirically how both curves move closer to the true efficient frontier with data from 30 US equities with various strategies (long-short dollar neutral strategy, long-only maximum return strategy and long-only active strategy) and how they outperform classical approaches.

2.5 'Robust Asset Allocation'

To solve the problem of unreliable structure of returns, Tütüncü and Koenig 2004 choose a way to robust asset allocation by choosing the most pessimistic view of robustness to find a solution that has the best performance under its worst case. In the paper, instead of the point estimates used in classical mean-variance problem, they use an uncertainty set which covers most possible values that the input parameter can reach to describe the uncertainty of the optimization problem.

The intuition of the method used in this paper is to address the following problem: What choice of the variables will optimize the worst case objective value? That is, under different choices of variables, this approach considers the worst case inputs in the uncertainty set and pick the set of variable which gives the best worst-case performances.

The steps of the worst-case strategy developed in this paper are as follows:

1. Solve

$$\begin{aligned} \min_{Q\in U_Q} \max \quad & x^T Q x \\ \text{s.t.} \quad & x \in \chi \\ & w^T Q w \leq v \end{aligned} \tag{2}$$

$$\min_{\mu\in U_\mu} \mu^T x \geq R, \tag{3}$$

without the expected return constraint, using a saddle point algorithm. x_{min} denotes the optimal solution and use $R_{min} = (\mu^L)^T x_{min}$ going forward

2. Solve

$$\max \min_{Q\in U_Q, \mu\in U_\mu} \mu^T x - \lambda x^T Q x \tag{4}$$

with $\lambda = 0$. x_{max} denotes the optimal solution and use $R_{max} = (\mu^L)^T x_{max}$ going forward

3. Choose K as the desired points on the efficient frontier and divide the interval $[R_{min}, R_{max}]$ into K pieces

4. Use the endpoints of intervals as the expected return constraint to solve the optimization problem using a saddle point algorithm

The authors stretch, that this technique possesses three properties. It takes a very conservative view by assuming a significant worst-case scenario; it provides stability over time since the portfolio weights remain relatively unchanged over a period of time; and it concentrates weights on a small set of asset classes. The first two properties are desirable and expected, but the third one comes with a surprise.
The worst-case robust asset management strategy overcomes some of the weakness of classical mean-variance optimization, especially suitable for conservative investors and investors who prefer to buy and hold the position for a long horizon.

3 Research Question

Based on the literature mentioned in the previous chapter, we aim to contribute to the field by assessing, whether Anderson and Cheng's proposed approach yields superior results in times of extreme events, such as the Great Financial Crisis of 2009 or the equity sell-off following the Covid-19 related shutdowns in 2020. We do so by extending Anderson and Cheng's scope to all permanent constitutes of the SP 500 from 01/01/2005 to 10/16/2020.
Therefore, we aim to answer the following research question:

Can 'Robust Bayesian Portfolio Choices' yield superior returns
in an environment of extreme financial markets?

4 Theoretical Framework

In the following the mathematical framework for our replication study of Anderson and Cheng 2016 is outlined. For further information and more detail we refer to their paper.
In each time period t both parameters and model probabilities are updated. After this process the assets are allocated accordingly where the optimization function accounts for uncertainty.

4.1 Initialization

In period 1 we initialize mean, covariance and model- and scaling parameters.

$$\bar{\mu}_{t-1} = \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{t-1}\sum_{s=1}^{t-1} R_{i,s})$$
$$\bar{\lambda}_{t-1} = \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{t-2}\sum_{s=1}^{t-1}(R_{i,s} - \bar{\mu}_{i,t-1})^2)$$
$$\kappa_{t,t-1} = 1, \delta_{t,t-1} = 1, \tau = 4 \text{ and } \alpha = 1$$

4.2 Updating parameters in each t

At each point in time the mean and covariance matrix for all existing previous models is updated, using the newly observed excess returns R_t.

$$\mu_{m,t} = \frac{\kappa_{m,t-1}\mu_{m,t-1} + R_t}{\kappa_{m,t}}$$
$$\Sigma_{m,t} = \frac{\delta_{m,t-1}\kappa_{m,t}\Sigma_{m,t-1} + \kappa_{m,t-1}(R_t - \mu_{m,t-1})(R_t - \mu_{m,t-1})'}{\delta_{m,t}\kappa_{m,t}}$$

With $\kappa_{m,t} = \kappa_{m,t-1} + 1$ and $\delta_{m,t} = \delta_{m,t-1} + 1$, which can be viewed as scaling parameters that define the weight put on more recent inputs.

4.3 Updating probabilities for each model m in each t

At each point in time the probability that a model is correct gets updated for each existing model, using the newly observed excess returns R_t. This can be done with Bayes rule.

$$P_t(m|F_t) = \frac{L(R_t|m, F_{t-1})P_t(m|F_{t-1})}{\sum_{m\in M_t} L(R_t|m, F_{t-1})P_t(m|F_{t-1})}$$

where the likelihood can be expressed as follows.

$$L(R_t|m, F_{t-1}) = \frac{\kappa_{m,t-1}^{n/2}\det(\Lambda_{m,t-1}^{\nu_{m,t-1}/2})\Gamma_n(\nu_{m,t-1}/2)}{\pi^{n/2}\kappa_{m,t}^{n/2}\det(\Lambda_{m,t}^{\nu_{m,t}/2})\Gamma_n(\nu_{m,t-1}/2)}$$

With $\Lambda_{m,t} = \delta_{m,t}\Sigma_{m,t}$ and $\nu_{m,t} = \delta_{m,t} + n + 1$ and $\Gamma_n(x)$ stands for the multivariate Gamma function.

As for the prior we chose the "Sharing prior" which allows to set the fraction α that each model shares with its own past and with the other models. We chose $\alpha = 1$ which represents the perfect sharing prior where older and newer models have equal probabilities.

$$P_t(m|F_{t-1}) = \begin{cases} (1-\alpha)P_{t-1}(m|F_{t-1} + \alpha[\sum_{q=1}^{m}(\frac{1}{t-q+1})P_{t-1}(q|F_{t-1})]), & \text{if } m < t \\ \alpha[\sum_{q=1}^{t-1}(\frac{1}{t-q+1})P_{t-1}(q|F_{t-1})], & \text{if } m = t \end{cases}$$

4.4 Allocate assets in each t

The model accounts for potential miss specification and future uncertainty by taking into account estimation error and parameter uncertainty. To optimally allocate assets in each time period we solve the following robust mean-variance optimization problem.

$$\max_{\phi_t}(\phi_t'\hat{\mu} + R_{ft+1} - \frac{1}{\tau}\int \exp[-\tau(\phi_t' z_{t+1})^2] f(z_{t+1}|F_t)dz_{t+1})$$

where ϕ_t represent the optimal portfolio weights. This integral can be approximated numerically with the following equations.

$$\int \exp[-\tau(\phi_t' z_{t+1})^2] f(z_{t+1}|F_t)dz_{t+1} = \sum_{m\in M_t} U_{m,t}P_t(m|F_t)$$

where

$$U_{m,t} = \begin{cases} \frac{1}{\sqrt{q_{m,t}}}\exp[\frac{\tau^2\phi_t'\bar{\Sigma}_{m,t}\phi_t - 2\tau\xi_{m,t} + \theta\tau\xi_{m,t}^2}{2q_{m,t}}], & \text{when } q > t \\ +\infty, & \text{when } q_{m,t} \leq 0 \end{cases}$$

with $q_{m,t} = 1 - \theta\tau\phi_t'\bar{\Sigma}_{m,t}\phi_t$ and $\xi = \phi_t'(\mu_{m,t} - \hat{\mu_t})$.

The optimal portfolio weights ϕ_t can be calculated as follows.

$$\phi_t = \frac{1}{\theta}\hat{\Sigma}_t^{-1}\hat{\mu}_t \qquad \text{where}$$

$$\hat{\Sigma}_t = V(R_{t+1}|F_t) = \sum_{m\in M_t}(\bar{\Sigma}_{m,t} + \mu_{m,t}\mu_{m,t}')'P_t(m|F_t) - \hat{\mu}_t\hat{\mu}_t{}' \qquad \text{with}$$

$$\bar{\Sigma}_{m,t} = V(R_{t+1}|m, F_t) = \Big(\frac{1+\kappa_{m,t}}{\kappa_{m,t}}\Big)\Sigma_{m,t} \qquad \text{and}$$

$$\hat{\mu}_t = E(R_{t+1}|F_t) = \sum_{m\in M_t}\mu_{m,t}P_t(m|F_t)$$

Next periods excess returns are observed based on the portfolio created in this instance.

5 Empirical Analysis

5.1 Data

For this analysis we used stocks from the S&P 500 Index. We did not include the full index as we removed stocks that were not included for the whole time period. This simplifies the process, however it can lead to a survivorship bias as the stocks are not truly randomly picked but rather are already a subset of well performing firms. As we used the same data for all different models, namely the robust Bayesian and the Historical expectations based allocation, we deem it appropriate. For the risk-free rate, we use the yield of a 10-year treasure bill and convert it to a daily rate.

We analyzed different time ranges and observed the models performances on the more stable post crisis time period as well as a full range time frame that starts in 2005. A good model should perform well under different market conditions and not succumb to higher volatility periods.

For the parameters in this paper, we use the same ones as Anderson and Cheng 2016. For risk aversion ($\theta = 1$). The model uncertainty aversion is four for the Bayesian robust model ($\tau = 4$). In the step of updating model probability when a new model is born, we use perfect sharing prior ($\alpha = 1$).

5.2 Benchmark models

As our benchmark we chose to compare the model to an asset allocation strategy that uses historical expectations as a direct measurement for future asset allocations as well as a simple equal weighted strategy. The historical expectations model assumes that future means and variance are approximated by their full past. This model is non-Bayesian and does not include prior information. All past information is treated as equal. The model can be described as follows.

$$E(R_{t+1}|F_t) = \hat{\mu}_t \text{ and } V(r_{t+1}|F_t) = \hat{\Sigma}_t, \text{ where}$$
$$\hat{\mu}_t = \frac{1}{t}\sum_{s=1}^{t} R_s,$$
$$\hat{\Sigma}_t = \frac{1}{t-1}\sum_{s=1}^{t}(R_s - \hat{\mu}_t)(R_s - \hat{\mu}_t)'$$

Using this method to calculate the mean and covariance, the actual portfolio weights are computed based on the regular Markovitz mean-variance portfolio optimization approach where he aims to maximize

$$E(\phi' R_{t+1} + R_f t + 1|F_t) - \frac{\theta}{2} V(\phi' R_{t+1} + R_f t + 1|F_t)$$

where $R_f t + 1$ denotes the risk free rate of return. We evaluated the models both with constant risk free rate and time-dependent risk free rate and got very similar results. The numbers outlined below are calculated with a time-dependent risk-free rate of return. θ is a risk aversion parameter. This yields the following optimal portfolio weights:

$$\phi_t = \frac{1}{\theta}\hat{\Sigma}_t^{-1}\hat{\mu}_t \qquad \text{Historical expectations model}$$

$$\phi_t = \left(\frac{1}{n}\right)\mathbf{1} \qquad \text{1/N model}$$

5.3 Evaluation measure

This paper uses sharpe ratio to evaluate the out-of-sample performance of the portfolio. The sharpe ratio is obtained by dividing the mean excess return of portfolio by its standard deviation.

$$\frac{\bar{E}(R_{pt+1})}{\bar{\sigma}(R_{pt+1})}$$

R_{pt+1} is portfolio excess return at time t+1. R_{t+1} is individual asset excess return at time t+1. R_{pt+1} can be obtained by the multiplication of weight allocated at date t and the excess return at t+1. $R_{pt+1} = \phi_t' R_{t+1}$

The time-dependent risk free rate of return is calculated using the daily movements of a 10 year T-bill.

5.4 Evaluation results

To evaluate the performance of the robust Bayesian portfolio model we compared its returns and sharpe ratio to the standard non-robust mean variance Markovitz portfolio selection model earlier described as the historical expectations model $\phi_t = \frac{1}{\theta}\hat{\Sigma}_t^{-1}\hat{\mu}_t$ and the equally weighted model that simply weights all assets with $\phi_t = \left(\frac{1}{n}\right)\mathbf{1}$.

Metric	Time Frame	Equally weighted	std. Markovitz	Robust Bayesian
ann. sharpe ratio	05-12	0.00802	-0.0014	0.00915
cumulative returns	05-12	0.19109	-0.03256	0.22016
Lowest return	05-12	-0.08994	-0.08087	-0.093974675
25% percentile return	05-12	-0.00648	-0.00713	-0.00725
50% percentile return	05-12	0.00056	0.000429	0.000185
75% percentile return	05-12	0.00687	0.00754	0.00774
Highest return	05-12	0.11353	0.10401	0.10480
ann. sharpe ratio	12-20	0.06283	0.05790	0.04325
cumulative returns	12-20	0.80254	0.81403	0.59229
Lowest return	12-20	-0.04312	-0.04561	-0.04320
25% percentile return	12-20	-0.00373	-0.00400	-0.00419
50% percentile return	12-20	0.00054	0.00054	0.00029
75% percentile return	12-20	0.00533	0.00613	0.00583
Highest return	12-20	0.03098	0.03588	0.03688

Metric	Timeframe	Equally weighted	std. Markovitz	Robust Bayesian
ann. sharpe ratio	05-12	0.00802	-0.00241	0.01194
cumulative returns	05-12	0.19109	-0.06285	0.32455
Lowest return	05-12	-0.08994	-0.11561	-0.10405
25% percentile return	05-12	-0.00648	-0.00840	-0.00848
50% percentile return	05-12	0.00056	-0.00008	-0.00007
75% percentile return	05-12	0.00687	0.00857	0.00922
Highest return	05-12	0.08033	0.08033	0.14091
ann. sharpe ratio	12-20	0.06283	0.02710	0.02813
cumulative returns	12-20	0.80254	0.44596	0.44965
Lowest return	12-20	-0.04312	-0.05478	-0.04718
25% percentile return	12-20	-0.00373	-0.00572	-0.00560
50% percentile return	12-20	0.00054	0.00003	0.00029
75% percentile return	12-20	0.00533	0.00708	0.00699
Highest return	12-20	0.03098	0.04339	0.03919

During year 2005 to 2010, both sharpe ratio and cumulative return show that Bayesian model cannot outperform the other two. The equally weighted strategy has the highest sharpe ratio and cumulative return, and it has the lowest volatility among three strategies. Both standard Markovitz and Bayesian model have a negative cumulative return, indicating that they underperform risk-free rate in this time frame.

However, an interesting phenomenon happens from October 2008 to March 2009 (See Figure 1 and Figure 2). It is well known that on September 15 2008, the bankruptcy of Lehman Brothers triggered global panic. Economies nearly collapsed during this period, and governments, central banks, and Federal Reserve cooperated together to save the economy. During this period, Bayesian model has a better performance than the other two. While equally weighted and non-robust mean variance strategies are struggling with negative sharpe ratios, Bayesian model is in the positive domain. A possible explanation could be that, during crisis period, prior information is very important since a stock has negative return yesterday may trigger more investors to sell it today. Bayesian model has a storage of prior information which could be a possible reason why it beats the other two models. This could serve as an evidence that Bayesian model perform better during financial crisis. However, it is arbitrary to reach a conclusion and further evaluation need to be performed.

During more the stable years 2010 to 2015 robust Bayesian was able to outperform std. Markovitz in terms of sharpe ratio and cumulative returns while having a smaller drawdown, however itself got outperformed by the simple equally weighted strategy in all of these metrics besides drawdown which the equally weighted shows the biggest. These mixed results show that now clear winner emerges and that depending on time period and evaluation metric one might outperform the other. In any case all of the models are similar to each other, both in terms of magnitude and sign.

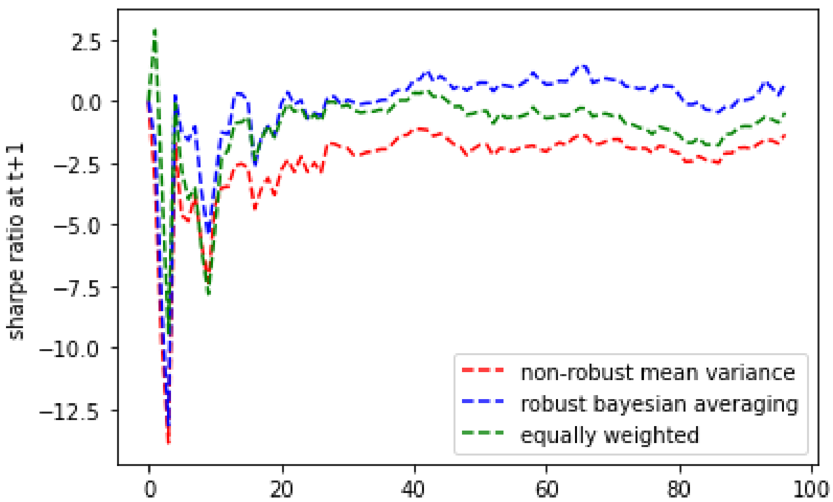

Figure 1: sharpe-ratio at t+1 from October 2008 to March 2009

During the most recent five years, the performance of the three strategies mirrored the behaviour of the earliest time frame. Robust Bayesian was outperformed by both other strategies and yielded a disastrous return, considering that the period saw two major equity rushes. Fair to say, that it also saw a major equity sell-off. The robust Bayesian strategy seems to be the most conservative in this period, since the lowest return is the highest among the three strategies and it has a small

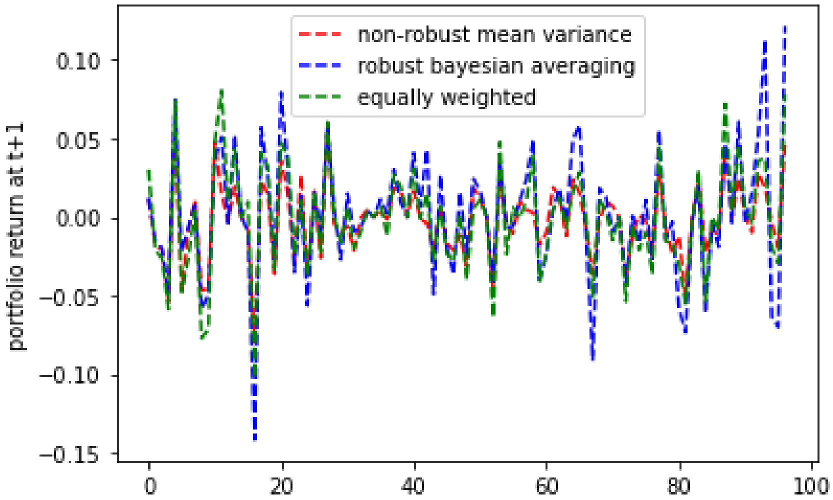

Figure 2: portfolio return at t+1 from October 2008 to March 2009

variance.

To test the robustness of these results we also tested it on a separate data set. This data represents the typical swiss pension funds asset classes and is entirely comprised of different indices and bonds. It is monthly data set starting in 1960 and ending in 2020. The results of this additional analysis are summarized in the following table. With this data the robust Bayesian model clearly outperforms the others, which is somewhat surprising, given that the inputed data is generally considered to be less risky and volatile than the previously tested stock data. Also of note is that the results for the equally weighted approach and the std. Markovitz approach are identical, indicating that with such few stocks the model ends up allocating as the equally weighted method.

Metric	**Timeframe**	**Equally weighted**	**std. Markovitz**	**Robust Bayesian**
ann. sharpe ratio	1960-2020	1.72765	1.72765	2.38331
cumulative returns	1960-2020	1.93389	1.93389	4.35451
Lowest return	1960-2020	-0.14934	-0.14934	-0.22996
25% percentile return	1960-2020	-0.007892	-0.007892	-0.008979
50% percentile return	1960-2020	0.003534	0.003534	0.004137
75% percentile return	1960-2020	0.01672	0.01672	0.02433
Highest return	1960-2020	0.1007	0.1007	0.0.2858

We find that the Bayesian model only uses the last couple of time steps as informative inputs. After a short amount of time the probability of earlier models converges to zero. Exemplary some the probability of some priors are presented in the following figures. One can see that most often the most recent model is assigned the most weight and that after about 5 time steps no weight is assigned to earlier models at all. This indicates that the stocks do not have extensive ”memories” which is part of the reason why modeling future behaviour and trends is generally very difficult.

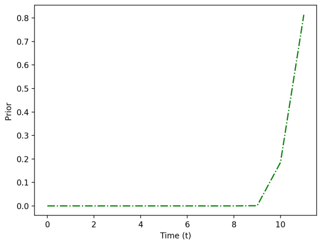

Figure 3: Bayesian Prior 1

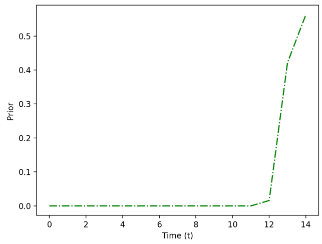

Figure 4: Bayesian Prior 2

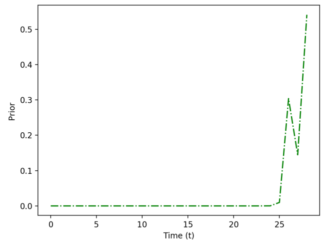

Figure 5: Bayesian Prior 3

6 Conclusion

In this paper we look at different methods to allocate assets in a portfolio that go beyond simple Markovitz portfolio optimization. These robust approaches aim to be more stable in out-of-sample performance by accounting for real world uncertainty and frictions. Out of all the proposed methods we chose to replicate Anderson and Cheng 2016 approach that introduces a Bayesian probability framework. In each time period a new model is born and all probabilities of all previous models are updated to account for this new information. This information then gets aggregated to allocate the assets accordingly.

We compare this model to a simple Markovitz portfolio strategy and an equal-weight strategy. We find that depending on the time frame and performance metric, different models perform best. During higher volatility periods the Bayesian approach achieves the best results, while under more stable conditions the equally weighted model is superior. Therefore, in regard to our research question, whether ‘Robust Bayesian Portfolio Choices’ yield superior returns in an environment of extreme financial markets, we can conclude that this holds true based on our focus on sharpe ratios, max drawdowns and returns, even though the strategy was outperformed otherwise. We look at sharpe ratio, max drawdown and return in a timeframe from 2005 until 2020 to evaluate these models.

While no clear winner emerged it is interesting to note that the Bayesian model might be a good method in more volatile times such as recessions.

Potential extensions would be the inclusion of GARCH specifications to model time varying volatility in the variance-covariance estimations, a combination of Anderson and Cheng’s and Richard Michaud and Robert Michaud’s approach, wherby the ‘Robust Bayesian Portfolio Choices’ technique is enforced by repeatedly bootstrapping past observations.

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