Discrete Choice Model

DCM

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Outline

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Introduction

- This is a demand-side model, which we use to estimate the demand
    - We cannot get "demand" in the usual sense, because demand is a random variable. The "best" we can do is the conditional expectation of individual demand
- When and how to use a Discrete Choice Model?
    - case 1. Ignoring the supply side
    - case 2. Modelling consumer demand while simplifying the supply side using an IO model (such as perfectly competitive or a monopolistic competitive market setting)
- Some examples. Farronato and Fradkin (2022); Nevo (2001); Berry et al. (1995); Train (1998)

Outline

Logit

- Setup
  - A decision maker n faces J alternatives,

$$U_{nj} = V_{nj} + \epsilon_{nj}$$

  - ϵ_{nj} follows type-I extreme value distribution ($\epsilon_{nj} \sim Gumbel(0, 1)$), i.e., $F(\epsilon_{nj}) = \exp(-\exp(-\epsilon_{nj}))$, of which the variance is $\pi^2/6$
  - Assuming variance equaling $\pi^2/6$ implicitly normalizes the scale of utility (homothetic property)
  - ϵ_{nj} is i.i.d. for any n and j

Type-I Extreme Value Distribution

- CDF

$$\text{Gumbel}(\mu, \beta) : F(x) = e^{-e^{-(x-\mu)/\beta}}$$

  where $x \in R$

- location parameter, μ, and scale parameter $\beta > 0$
- Graph exposition (see python codes)

Choice Probabilities

- Consumer n's demand for an alternative $i \in J$

$$P_{ni} = \frac{e^{V_{ni}}}{\sum_{j=1}^{J} e^{V_{nj}}} \tag{1}$$

Choice Probabilities

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$$P_{ni} = \frac{e^{V_{ni}}}{\sum_{j=1}^{J} e^{V_{nj}}} \tag{1}$$

- Proof hint

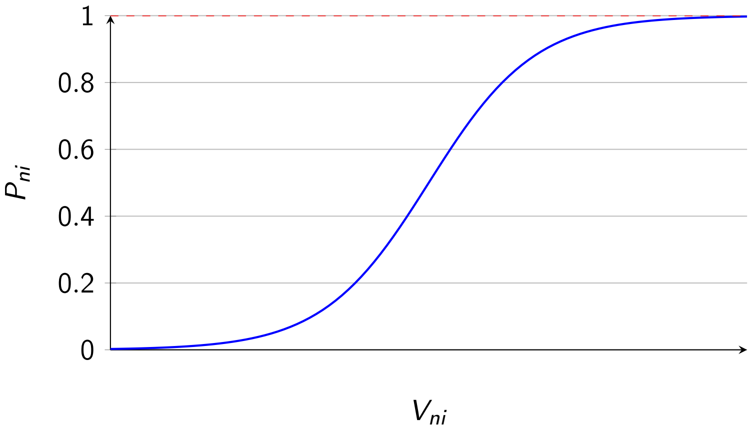
$$\begin{aligned}
\Pr(\text{n chooses i}) =& \Pr(U_{ni} > U_{nj}, \forall j \neq i) \\
=& \Pr(V_{ni} + \epsilon_{ni} > V_{nj} + \epsilon_{nj}, \forall j \neq i) \\
=& \Pi_{j \neq i} \Pr(V_{ni} + \epsilon_{ni} > V_{nj} + \epsilon_{nj}) \\
=& \int_{-\infty}^{\infty} \Pi_{j \neq i} \Pr(\epsilon_{nj} < V_{ni} - V_{nj} + \epsilon) dF(\epsilon)
\end{aligned}$$

where $F(\epsilon)$ is the CDF of Gumbel(0,1). In the equations above, we use following theorems

- i.i.d. property; total probability theorem ($\Pr(A) = \int_{\Omega} \Pr(A|X = x) dF(x)$). In our case, event $A = \Pi_{j \neq i} \Pr(V_{ni} + \epsilon_{ni} > V_{nj} + \epsilon_{nj})$

Choice Probabilities (Cont'd)

- Properties
  - P_{ni} is between 0 and 1



Choice Probabilities (Cont'd)

- Properties
  - the choice probabilities for all alternatives sum to one for one decision maker
  - Independence from Irrelevant Alternatives (IIA)

$$\frac{P_{ni}}{P_{nk}} = \frac{\exp(V_{ni}/\sum_j \exp(V_{nj}))}{\exp(V_{nk}/\sum_j \exp(V_{nj}))} = \exp(V_{ni} - V_{nk})$$

    where the ratio does not depend on any alternatives other than i and k
  - Proportional substitution

$$\frac{\partial P_{ni}/P_{ni}}{\partial z_{nj}/z_{nj}} = E_{iz_{nj}} = -\beta_z z_{nj} P_{nj}$$

    where z_{nj} is the attribute of alternative j as faced by person n and β_z is its coefficient. This cross-elasticity is the same, $\forall i$
  - The log-likelihood function is globally concave in parameters β, when the utility function is linear in parameters: $V_{nj} = \beta' x_{nj}$ MCFADDEN (1974)

Panel Data Application

- Utility function. $U_{njt} = V_{njt} + \epsilon_{njt}, \ \forall j, t$
- Choice probabilities. $P_{nit} = \frac{e^{V_{nit}}}{\sum_j e^{V_{njt}}}$
- Adding dynamic patterns
  - If representative utility for each period is specified to depend on variables for that period; for example, $V_{njt} = \beta' x_{njt}$, then there is essentially no difference between the logic model with panel data and with purely cross-sectional data
  - Dynamic aspects of behavior can be captured by specifying representative utility in each period to depend on observed variables from other period. For example, a lagged price response is represented by entering the price in period $t-1$ as an explanatory variable in the utility for period t

Consumer Surplus Calculation

- Exercise target. Estimate the change in consumer surplus that is associated with a particular policy
- Consumer surplus. $CS_n = (1/\alpha_n) \times \max_j(U_{nj})$, where α_n is the marginal utility of income, $\frac{dU_n}{dY_n} = \alpha_n$, with Y_n the income of person n.
- The expected consumer surplus

$$
\begin{aligned}
E(CS_n) =& \frac{1}{\alpha_n} E[\max_j(V_{nj} + \epsilon_{nj})] \\
=& \frac{1}{\alpha_n} \ln(\sum_j e^{V_{nj}}) + C.
\end{aligned}
$$

Prove the Emax term is a log-sum form.

Consumer Surplus Calculation (Cont'd)

- For example, the change in consumer surplus that results from a change in the alternatives and/or the choice set

$$\Delta E(CS_n) = \frac{1}{\alpha_n}[\ln(\sum_{j}^{J^1} \exp(V_{nj}^1) - \ln(\sum_{j}^{J^0} \exp(V_{nj}^0)]$$

where the superscripts 0 and 1 refer to before and after the change.

- Total consumer surplus
  - $E(CS_n)$ is the average consumer surplus in the subpopulation of people who have the same representative utilities as person n. The total consumer surplus in the population is calculated as the weighted sum of $E(CS_n)$ over a sample of decision makers, with the weights reflecting the numbers of people in the population who face the same representative utilities as the sampled person

Estimation: Maximum Likelihood Estimates (MLE)

- Intuition. Observed sample has the largest likelihood to happen given population distribution
- Likelihood function of independent sample (joint probability)

$$L(\beta) = \Pi_{n=1}^{N} \Pi_i (P_{ni})^{y_{ni}}$$

where β is a vector containing the parameters of the model.

- Log-likelihood function

$$LL(\beta) = \sum_{n=1}^{N} \sum_{i} y_{ni} \ln P_{ni}$$

and the estimator is the value of β that maximizes this function; its derivative with respect to each of the parameters is zero. The MLE are therefore the values of β that satisfy this first-order condition

Optimization

- One-dimensional minimization

$$\min_{x \in R} f(x) \tag{2}$$

where $f : R \rightarrow R$

- Multi-dimensional minimization

$$\min_{x \in R} f(x) \tag{3}$$

where $f : R^n \rightarrow R$

- Gradient

$$\nabla f(x) = (\partial f(x)/\partial x_1, \partial f(x)/\partial x_2, ..., \partial f(x)/\partial x_n)$$

- Hessian matrix

$$H(x) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)_{i,j=1}^{n}$$

Optimization: Steepest Descent

- Steepest descent
  - the steepest ascent direction is gradient
  - in the steepest method the search direction is which f falls most rapidly per unit length
  - the steepest method chooses the search direction:

$$s^{k+1} = -(\nabla f(x^k))'$$

  - Graph interpretation:
    one-dimensional $y = f(x)$; two-dimensional $z = f(x, y)$
  - Gradient only points out the direction but ignores the step length

Examples: Compare Steepest Descent and Newton's Method

- **Example 1:** $y = x^2$
  - Gradient $\nabla f(x) = 2x$ points out the direction.
  - Let $x_0 = 3$ or $x_0 = 5$.
  - Gradient descent method, the step is -6 and -10, respectively.
  - So $x_1 = -3$ or -5, not closer to the solution. Gradient descent method fails.
  -
  - Newton's method. $H(\cdot) = 2$; the step size is -3 and -5.
  - One iteration obtains the solution.
  - **Theorem:** Newton's method reaches the optimum in one iteration (since a 2-degree Taylor expansion of a quadratic form is an accurate approximation).
- **Example 2:** $y = x^3$
  - What about cubic form?
  - Similarly, $\nabla f(x) = 3x^2$, $H(x) = 6x$.
  - Let $x_0 = 5. \nabla f(5) = 75$, $H(5) = 30$.
  - Newton's method: Step size $= -75/30 = -2.5$.
  - By iteration: $x_1 = 2.5$, $x_2 = 1.25$, $x_3 = 0.625$, $x_4 = 0.3125$, ...

Optimization: Newton's Method (One-Dimension)

- Taylor's expansion
  - for C^2 functions $f(x)$, Newton's method is often used. We define:

$$p(x) \equiv f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

  - The idea behind Newton's method is to start at a point a and find quadratic polynomials $p(x)$, that approximate $f(x)$ at a to the 2nd degree. We next approximately minimize $f(\cdot)$ by finding a point x_m that minimizes $p(x)$.
- Newton's method iteration process

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

- Convergence: quadratic convergence when approaching x^*: $\lim_{n\to\infty} \frac{|x_{n+1}-x^*|}{|x_n-x^*|^2} = \frac{1}{2}\left|\frac{f'''(x^*)}{f''(x^*)}\right| < \infty$
  - proof hint. Expand $f'(x)$ at x_0 and set $x_0 = x_n$

Optimization: Newton's Method (Multi-Dimension)

- Taylor's expansion

$$f(x) \doteq f(x^k) + \nabla f(x^k)(x - x^k) + \frac{1}{2}(x - x^k)^T H(x^k)(x - x^k)$$

- Newton's method iteration process

$$x^{k+1} = x^k - H^{-1}(x^k)(\nabla f(x^k))^T$$

- Add line search in Newton's method
  - find iterating step $s = -H^{-1}(x^k)(\nabla f(x^k))^T$
  - solve $\lambda_k = \arg\min_\lambda f(x^k + \lambda s^k)$
  - iterate $x^{k+1} = x^k + \lambda_k s^k$
  - intuition: λ_k modifies the step length in each iteration, ensuring the largest decrease in the target function.
  - Newton's method (without line search) implicitly sets step length, $\lambda_k = 1$

Optimization: Quasi Newton's Method

- Drawback of Newton's method
  - Newton's method needs to calculate Hessian in every iteration
- BFGS method
  - initially set weight matrix to be a diagonal I matrix
  - update Hessian while iterating instead of calculating the accurate Hessian
- Stochastic gradient descent (SGD) method is utilized in deep learning
- for your references: chapter 4 in Judd (1998), and Le et al. (2011)

BFGS Algorithm

Algorithm 1: BFGS Algorithm

Input : Choose initial guess x^0, initial Hessian guess H^0, stopping parameters δ and $\epsilon > 0$

Output: Optimal point x^*

for $k = 0, 1, 2, \ldots$ **do**

  Solve $H_k s^k = -(\nabla f(x^k))^T$ for the search direction s^k;

  Solve $\lambda_k = \arg\min_\lambda f(x^k + \lambda s^k)$;

  $x^{k+1} = x^k + \lambda_k s^k$;

  Update H_k:

$$z_k = x^{k+1} - x^k,$$

$$y_k = (\nabla f(x^{k+1}))^T - (\nabla f(x^k))^T,$$

$$H_{k+1} = H_k - \frac{H_k z_k z_k^T H_k}{z_k^T H_k z_k} + \frac{y_k y_k^T}{y_k^T z_k}$$

  if $\|x^k - x^{k+1}\| < \epsilon(1 + \|x^k\|)$ **then**

    if $\|\nabla f(x^{k+1})\| < \delta(1 + |f(x^{k+1})|)$ **then**

      stop and report success;

    else

      stop and report convergence to non-optimal point;

Limitations

- Plain Logit can represent systematic taste variation but not random taste variation (cannot be linked to observed characteristics)
- IIA property of the choice probability
- The cross-elasticity is the same for all i
  - a change in an attribute of alternative j changes the probabilities for all other alternatives by the same percent

Code Example

- See Python script

Outline

Introduction

- The standard logit model exhibits independence from irrelevant alternatives (IIA), which implies proportional substitution across alternatives.
- Generalized Extreme Value (GEV) models constitute a broad class of models that accommodate a variety of substitution patterns.
- The unifying attribute of GEV models is that the unobserved portions of utility for all alternatives are jointly distributed as a generalized extreme value.
- This distribution allows for correlations among alternatives and is a generalization of the univariate extreme value distribution used in standard logit models.
- GEV models offer the advantage of providing choice probabilities that usually take a closed form, making them both flexible and practical for various applications.

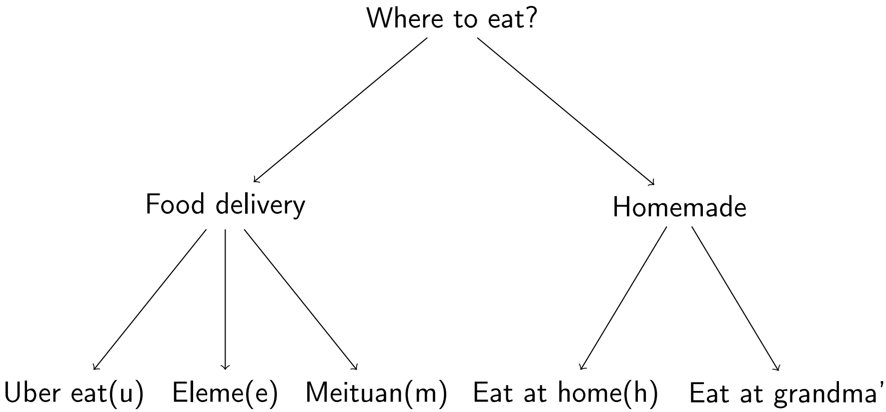
Nested Logit: Overview

- The nested Logit model is the most widely used member of the GEV family
- This model has been applied by many researchers in a variety of fields
- A nested Logit model is appropriate when the set of alternatives faced by a decision maker can be partitioned into subsets, called **nests**.
    1. independence of Irrelevant Alternatives (IIA) holds within each nest
    2. for any two alternatives in different nests, IIA does not hold in general
    3. the tree diagram representation

Food Delivery Choice Example

- **Choice set** = {Eleme (e), Meituan (m), Uber eat (u), Eating at home (h), Eating at grandma's (g)}
- If we use the plain Logit, when the price of eleme, p_e, increases, the probability of choosing Uber eat, P_{nu}, choosing Meituan, P_{nm}, choosing eating at home, P_{nh}, and choosing eating at grandma's, P_{ng}, would increase proportionately.
- **Is this realistic?**
- A more realistic scenario: P_{nu} and P_{nm} rise more relative to P_{nh} because the consumer may prefer placing a delivery order

Food Delivery Choice Example (Cont'd)



Choice Probabilities

- The utility that person n derives from alternative j in nest k is expressed as $U_{nj} = V_{nj} + \epsilon_{nj}$, where V_{nj} is observed by the researcher and ϵ_{nj} is an unobserved random variable. The nested Logit model assumes that the vector of unobserved utilities $\epsilon_n = \langle \epsilon_{n1}, \epsilon_{n2}, \epsilon_{n3}, \ldots, \epsilon_{nJ} \rangle$ follows a cumulative distribution given by:

$$\exp\left(- \sum_{k=1}^{K} \left(\sum_{j \in B_k} e^{-\epsilon_{nj}/\lambda_k} \right)^{\lambda_k} \right) \tag{4}$$

- This distribution belongs to the Generalized Extreme Value (GEV) family and generalizes the distribution used in the plain Logit model

Choice Probabilities (Cont'd)

- The choice probability for alternative $i \in B_k$ is:

$$P_{ni} = \frac{e^{V_{ni}/\lambda_k} \left(\sum_{j \in B_k} e^{V_{nj}/\lambda_k} \right)^{\lambda_k - 1}}{\sum_{l=1}^{K} \left(\sum_{j \in B_l} e^{V_{nj}/\lambda_l} \right)^{\lambda_l}}. \tag{5}$$

- For alternatives $i \in B_k$ and $m \in B_l$, the ratio of their choice probabilities is:

$$\frac{P_{ni}}{P_{nm}} = \frac{e^{V_{ni}/\lambda_k} \left(\sum_{j \in B_k} e^{V_{nj}/\lambda_k} \right)^{\lambda_k - 1}}{e^{V_{nm}/\lambda_l} \left(\sum_{j \in B_l} e^{V_{nj}/\lambda_l} \right)^{\lambda_l - 1}}. \tag{6}$$

- If $k = l$ (i.e., i and m are in the same nest), the terms in parentheses cancel out, yielding:

$$\frac{P_{ni}}{P_{nm}} = \frac{e^{V_{ni}/\lambda_k}}{e^{V_{nm}/\lambda_k}}. \tag{7}$$

Choice Probabilities (Cont'd)

- Proof hint for the choice probability
  - decompose utility function

$$U_{nj} = V_{nj} + W_{nk} + \xi_{nj} + \epsilon_{nj}$$

    where $\epsilon \sim i.i.d.Gumbel(0,1)$
  - Conditional probability
    - $\Pr_n(i) = \Pr_n(i|nest\ k)\Pr_n(nest\ k)$ where i is a choice in nest k
    - log-sum term

Remarks

- Independence from Irrelevant nests (IIN). The probability ratio does not depend on the attributes of alternatives in nests those NOT containing the two alternatives
- Loosely stated, the probability of choosing nest k depends on the expected utility that the person receives from that nest
- The log-likelihood function is NOT globally concave and is NOT close to a quadratic even in concave areas.
- Instead, nested Logit models can be estimated consistently, though not efficiently, in a sequential fashion

$$\text{For } i \in B_k, \quad \text{Prob}(i) = \sum_{l=1}^{L} \text{Prob}(i \mid \text{nest } l) \times \text{Prob}(\text{nest } l)$$
$$= \text{Prob}(i \mid \text{nest } k) \times \text{Prob}(\text{nest } k)$$
$$\text{since } \text{Prob}(i \mid \text{nest } l) = 0, \quad l \neq k \text{ for } i \in B_k$$

Code Example

- See Python script

Outline

Introduction

- The Logit model is limited in three important ways:
  - It cannot represent random taste variation.
  - It exhibits restrictive substitution patterns due to the IIA property, even though GEV models relax this restriction.
  - It cannot be used with panel data when unobserved factors are correlated over time for each decision maker.
- In a RUM model, the explanatory variables did not vary over decision makers, but the coefficient is random over consumers
- The observed dependent variable was often market shares rather than individual customers' choices
- Two families:
  - Probit (multi-normal distributed unobserved factor)
  - Mixed logit (random coefficients)

Mixed Logit (with individual-level data)

- The decision maker faces a choice among J alternatives.
- Utility of person n from alternative j,

$$U_{nj} = \beta_n' x_{nj} + \epsilon_{nj}$$

where x_{nj} are observed variables that relate to the alternative and decision maker, β_n is a vector of coefficients of these variables for person n representing that person's tastes, and ϵ_{nj} is a random term that is iid extreme value. The coefficients vary over decision makers in the population with density $f(\beta;\theta)$. This density is a function of parameters θ that represent, for example, the mean and covariance of the β's in the population

Choice Probabilities

- This specification is the same as for plain Logit except that β varies over decision makers rather than being fixed.
- The decision maker knows the value of his own β_n and ϵ_{nj}'s for all j and chooses alternative i iff $U_{ni} > U_{nj}, \forall j \neq i$.
- The researcher observes the x_{nj}'s but not β_n or the ϵ_{nj}'s. That is, the probability conditional on β_n is,

$$L_{ni}(\beta_n) = \frac{e^{\beta_n' x_{ni}}}{\sum_j e^{\beta_n' x_{nj}}}$$

- If the researcher observed β_n, then the choice probability would be plain Logit

Choice Probabilities (Cont'd)

- However, the researcher does not know β_n and therefore cannot condition on β
- How to solve this?

Choice Probabilities (Cont'd)

- However, the researcher does not know β_n and therefore cannot condition on β
- How to solve this?
- Total Probability Theorem. The unconditional choice probability is the integral of $L_{nj}(\beta)$ over all possible variables of β_n.

$$P_{ni} = \sum_{\beta \in \text{Support}\{\beta\}} \text{Prob}(i|\beta)\text{Prob}(\beta_n = \beta); \text{ discrete PMF}$$

$$\Longleftrightarrow \int_{\beta \in \mathcal{B}} \frac{e^{\beta' x_{ni}}}{\sum_j e^{\beta' x_{nj}}} f(\beta) d\beta; \text{ continuous PDF}$$

- Usually, $f(\beta)$ has been specified to be Normal or Lognormal: $\beta \sim N(b, W)$ or $\ln \beta \sim N(b, W)$, with parameters b and W that are estimated.

Estimation

- Representative Utility

$$U_{nj} = \beta_n' x_{nj} + \epsilon_{nj},$$

- Choice probabilities

$$P_{ni} = \int L_{ni}(\beta) f(\beta|\theta) d\beta,$$

where

$$L_{ni}(\beta) = \frac{e^{\beta' x_{ni}}}{\sum_{j=1}^{J} e^{\beta' x_{nj}}}.$$

- The coefficients β_n are distributed with density $f(\beta|\theta)$, where θ refers collectively to the parameters of this distribution (such as the mean and covariance of β). The researcher specifies the functional form $f(\cdot)$ and wants to estimate the parameters θ.
- The probabilities are approximated through simulation for any given value of θ:

Maximum Simulated Likelihood

Algorithm 2: Mixed Logit Probability Simulation

Input : Choose a parameter vector θ and set draw times R

Output: Simulated probability $\hat{P}_{ni}$

for *each draw r from 1 to R* **do**

  Draw a value of β^r from $f(\beta|\theta)$. Calculate the Logit formula $L_{ni}(\beta^r)$ with this draw.

Average the results to obtain the simulated probability:

$$\hat{P}_{ni} = \frac{1}{R}\sum_{r=1}^{R} L_{ni}(\beta^r).$$

Maximum Simulated Likelihood (Cont'd)

- Two key properties of $\hat{P}_{ni}$.
  - $\hat{P}_{ni}$ is an unbiased estimator of P_{ni} by construction. Its variance decreases as R increases.
  - it is strictly positive, so that $\ln \hat{P}_{ni}$ is defined, approximating the log-likelihood function
  - $\hat{P}_{ni}$ is smooth (twice differentiable) in the parameters θ and the variables x, which facilitates the numerical search for the maximum likelihood function and the calculation of elasticities.
- The simulated probabilities are inserted into the log-likelihood function to give a simulated log likelihood:

$$\text{SLL} = \sum_{n=1}^{N} \sum_{j=1}^{J} d_{nj} \ln \hat{P}_{nj},$$

where $d_{nj} = 1$ if person n chose j and zero otherwise. The maximum simulated likelihood (MSL) estimator is the value of θ that maximizes SLL.

Code Example

- See Python script

Outline

Introduction

- Question 1: What if we do not have individual-level data?
- Question 2: What if there are other observed characteristics correlated with price? (Endogeneity)
- Solution: Use market-level data and BLP method.

BLP: Representative Utility Function

- The indirect utility of consumer i from consuming product j in market t, $U(x_{jt}, \xi_{jt}, p_{jt}, \tau_i; \theta)$, is a function of observed and unobserved (by the researcher) product characteristics, x_{jt} and ξ_{jt}, respectively
- Consider a particular specification

$$u_{ijt} = \alpha_i(y_i - p_{jt}) + x_{jt}\beta_i + \xi_{jt} + \epsilon_{ijt},$$

$$i = 1, 2, \ldots, I_t, \quad j = 1, \ldots, J, \quad t = 1, \ldots, T$$

where y_i is the income of consumer i, p_{jt} is the price of product j in market t, x_{jt} is a K-dimensional (row) vector of observable characteristics of product j, ξ_{jt} is the unobserved product characteristic, and ϵ_{ijt} is a mean-zero stochastic term

Representative Utility Function (Cont'd)

- α_i is consumer i's marginal utility from income, and β_i is a K-dimensional (column) vector of individual-specific taste coefficients
- Observed characteristics vary with the product. BLP (1995) examines the demand for cars and include as observed characteristics like horsepower, size, and air conditioning
- Depending on the structure of the data, some components of the unobserved characteristics can be captured by dummy variables. For example, we can model $\xi_{jt} = \xi_j + \xi_t + \Delta\xi_{jt}$ and capture ξ_j and ξ_t by brand- and market-specific dummy variables
- Source of endogeneity. ξ_{jt} is unobserved and potentially correlated with p_{jt}

Representative Utility Function (Cont'd)

- Implicit assumption in the particular specification
  - The form of the indirect utility can be derived from a quasilinear utility function, which is free of wealth effects. Including wealth effects alters the way the term $y_i - p_{jt}$ enters the equation above.
    - For instance, BLP(1995) builds on a Cobb-Douglas utility function to derive an indirect utility that is a function of $\log(y_i - p_{jt})$.
  - ξ_{jt} is the same for all consumers.
  - All consumers face the same product characteristics.
  - In particular, all consumers are offered the same price.

Decompose Utility Function: Consumer Preferences and Characteristics

- Consumer preferences $\langle \alpha_i, \beta_i \rangle$ vary as a function of individual characteristics
  - The individual characteristics consist of two components: observed demographics, D_i, and additional unobserved characteristics, v_i
  - Even though we do not observe individual data, we know something about the distribution of the demographics and additional characteristics
  - Demographics: i.e. income, age, family size, race, and education Information we might have includes large samples we can use to estimate some features of the distribution (i.e., Census data, Current Population Survey)

Consumer Preferences and Characteristics (Cont'd)

- Formally, this will be modeled as:

$$\begin{bmatrix}\alpha_i \\ \beta_i\end{bmatrix} = \begin{bmatrix}\alpha \\ \beta\end{bmatrix} + \Pi D_i + \sigma v_i,$$

$$v_i \sim P_v^*(v), \quad D_i \sim \hat{P}_D^*(D),$$

where D_i is a $d \times 1$ vector, Π is a $(K+1) \times d$ matrix of parameters, and Σ is a $(K+1) \times (K+1)$ matrix of parameters

- If we assume that $P_v^*(\cdot)$ is a standard multivariate normal distribution, then the matrix Σ allows each component of v_i to have a different variance and allows for correlation between these characteristics
- For simplicity, we assume that v_i and D_i are independent

Decompose Utility Function (Cont'd)

- Let $\theta = (\theta_1, \theta_2)$ be a vector containing all the parameters of the model
- The vector $\theta_1 = (\alpha, \beta)$ contains the *linear* parameters, and the vector $\theta_2 = (\Pi, \Sigma)$ contains the *nonlinear* parameters. We have:

$$u_{ijt} = \alpha_i y_i + \delta_{jt}(x_{jt}, p_{jt}, \xi_{jt}; \theta_1) + \mu_{ijt}(x_{jt}, p_{jt}, v_i, D_i; \theta_2) + \epsilon_{ijt}$$

$$\delta_{jt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt}$$

$$\mu_{ijt} = [-p_{jt}, x_{jt}] \cdot (\Pi D_i + \Sigma v_i)$$

where $[-p_{jt}, x_{jt}]$ is a $1 \times (K+1)$ vector

Decompose Utility Function (Cont'd)

- Explanation
  - The first term, $\alpha_i y_i$, is given only for consistency with the indirect utility function, which will be cancelled out.
  - The second term, δ_{jt}, which is referred to as the mean utility, is common to all consumers.
  - The last term, $\mu_{ijt} + \epsilon_{ijt}$, represents a mean-zero heteroskedastic deviation from the mean utility that captures the effects of the random coefficients.
- A comment: Is it realistic that consumers choose no more than one good, which is the assumption in BLP model
- Response:
  - even though many of us buy more than one brand at a time, less actually consume more than one at a time, so discreteness of choice can be sometimes defended by defining the choice period appropriately;
  - in some cases, the researcher has to model the choice of multiple products, or continuous quantities, explicitly

Normalization: Outside Goods

- The specification of the demand system is completed with the introduction of an outside good: consumers may decide not to purchase any of the brands
- The indirect utility from this outside option is given by:

$$u_{i0t} = \alpha_i y_i + \xi_{0t} + \pi_0 D_i + \sigma_0 v_{i0} + \epsilon_{i0t}$$

- For simplicity, we can regard the outside option as "buy nothing." Hence, the consumer would not get the characteristics bundle and does not need to pay
- The standard practice is to set ξ_{0t}, π_0, and σ_0 all to zero. Since the term $\alpha_i y_i$ will eventually cancel out (common to all products), this is equivalent to normalizing the utility from the outside good to zero

Market Shares

- Iindividual is defined as a vector of demographics and product-specific shocks $(D_i, v_i, \epsilon_{i0t}, \ldots, \epsilon_{ijt})$, this implicitly defines the set of individual attributes that lead to the choice of good j.
- Formally, the set is

$$A_{jt}(x_t, p_t, \delta_t; \theta_2) = \\ \{(D_i, v_i, \epsilon_{i0t}, \ldots, \epsilon_{ijt}) \mid u_{ijt} \geq u_{ilt}, \forall l = 0, 1, \ldots, J\}$$

where $x_t = (x_{1t}, \ldots, x_{Jt})'$, $p_t = (p_{1t}, \ldots, p_{Jt})'$, and $\delta_t = (\delta_{1t}, \ldots, \delta_{Jt})'$ are observed characteristics, prices, and mean utilities of all brands, respectively. The set A_{jt} defines the individuals who choose brand j in market t.

Market Shares (Cont'd)

- Assuming ties occur with zero probability, the market share of the j-th product is just an integral over the mass of consumers in the region A_{jt}

$$
\begin{aligned}
s_{jt}(x_t, p_t, \delta_t; \theta_2) &= \int_{A_{jt}} dP^*(D, v, \epsilon) \\
&= \int_{A_{jt}} dP^*(\epsilon | D, v) dP^*(v | D) dP^*_D(D) \qquad (8) \\
&= \int_{A_{jt}} dP^*(\epsilon | D, v) dP^*(v | D) dP^*_D(D)
\end{aligned}
$$

- where $P^*(\cdot)$ denotes population distribution functions. Given assumptions on the distribution of the (unobserved) individual attributes, we can compute the integral, either analytically or numerically

Market Shares (Cont'd)

- Therefore, for a given set of parameters, the equation above predicts the market share of each product in each market, as a function of product characteristics, prices, and unknown parameters
- One possible estimation strategy is to choose parameters that minimize the distance between the market shares predicted by the equation and the observed shares
- This estimation strategy does not account for the correlation between prices and the unobserved product characteristics
- The BLP method accounts for this correlation

Straightforward Approach

- As previously pointed out, a straightforward approach to the estimation is to solve

$$\min_{\theta} \| s(x, p, \delta(x, p, \xi; \theta_1); \theta_2) - S \|,$$

where $s(\cdot)$ are the market shares given by Equation 8, and S are the observed market shares

- Unobserved variables include individual-level characteristics, (D_i, v_i, ϵ_i) and unobserved product characteristics, ξ_j
    - (D_i, v_i, ϵ_i) were integrated over. The econometric error term will be the unobserved product characteristics, ξ_{jt}
    - prices are potentially correlated with this term, the econometric estimation will have to take account of this

Straightforward Approach (Cont'd)

- Straightforward approach is usually not taken due to costly minimization
  - all the parameters enter the minimization problem nonlinearly. In some applications the inclusion of brand and time dummy variables resultsin a large number of parameters and a costly nonlinear minimization problem
- BLP method avoids this problem by transforming the minimization problem so that some (or all) of the parameters enter the objective function linearly

BLP Method

- Let $Z = [z_1, \ldots, z_M]$ be a set of instruments such that

$$E[Z_m \omega(\theta^*)] = 0,$$

  where ω, a function of the model parameters, is an error term, and θ^* denotes the "true" parameters
- GMM estimate

$$\hat{\theta} = \arg\min_{\theta} \omega(\theta)' Z \Phi^{-1} Z' \omega(\theta), \tag{9}$$

  where Φ is a consistent estimate of $E[Z'\omega\omega' Z]$
- The error term is defined as the structural error, ξ_{jt} as mentioned
- In order to use Equation 9, we need to express the error term as an explicit function of the parameters of the model and the data
  - The key insight is that the error term ξ_{jt} enters only the mean utility level, $\delta(\cdot)$

BLP Method Procedure

- The mean utility level is a linear function of ξ_{jt} We solve for each market the implicit system of equations:

$$s(\delta_t; \theta_2) = S_t, \quad t = 1, \ldots, T$$

where $s(\cdot)$ are the market shares given by 8, and S are the observed market shares.

- First, predict market share

$$s_{jt}(p_t, x_t, \delta_t, P_{ns}; \theta_2) = \frac{1}{ns}\sum_{i=1}^{ns} s_{jti} = \frac{1}{ns}\sum_{i=1}^{ns}$$

$$\frac{\exp\left(\delta_{jt} + \sum_{k=1}^{K} x_{jt}^k(\sigma_k v_i^k) + \pi_{k1} D_{i1} + \cdots + \pi_{kd} D_{kd}\right)}{1 + \sum_{m=1}^{J} \exp\left(\delta_{mt} + \sum_{k=1}^{K} x_{mt}^k(\sigma_k v_i^k) + \pi_{k1} D_{i1} + \cdots + \pi_{kd} D_{kd}\right)}$$

BLP Method Procedure (Cont'd)

- $(v_i^1, \ldots, v_i^K)$ and $(D_{i1}, \ldots, D_{id})$, $i = 1, 2, \ldots, ns$, are draws from $P_v^*(v)$ and $P_D^*(D)$, respectively, while x_{jt}^k, $k = 1, \ldots, K$, are the variables that have random slope coefficients.
- Second, using the computation of the market share, we invert the system of equations by the contraction mapping

$$\delta_t^{h+1} = \delta_t^h + \ln S_t - \ln s(\delta_t^h; p_t, x_t, \delta_t, P_{ns}; \theta_2), \quad h = 0, \ldots, H$$

where $s(\cdot)$ are the predicted market shares, H is the smallest integer such that $\|\delta_t^H - \delta_t^{H-1}\|$ is smaller than the tolerance level, and δ_t^H is the approximation to δ_t.

BLP Method Procedure (Cont'd)

- Third, define the error term:

$$\omega_{jt} = \delta_{jt} - \delta_{jt}(\theta_2) - (x_{jt}\beta - \alpha p_{jt}) \equiv \xi_{jt}$$

- Note
  - the observed market shares, S, enter this equation.
  - The reason for distinguishing between θ_1 and θ_2: θ_1 enters this term, and the GMM objective, in a linear fashion, while θ_2 enters nonlinearly.

Detailed Procedure

- See Algorithm.pdf

Welfare Change

- Recall utility function

$$u_{ijt} = \alpha_i y_i + \delta_{jt}(x_{jt}, p_{jt}, \xi_{jt}; \theta_1) + \mu_{ijt}(x_{jt}, p_{jt}, v_i, D_i; \theta_2) + \epsilon_{ijt}$$

$$\delta_{jt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt}$$

$$\mu_{ijt} = [-p_{jt}, x_{jt}] \cdot (\Pi D_i + \Sigma v_i)$$

Welfare Change (Cont'd)

1. **Estimate Parameters Before Policy Shock**
   - Estimate δ_j^{pre}, β^{pre}, α^{pre}
   - Calculate $\mathbb{E}[\max_j u_{ij}^{\text{pre}}]$:

$$\mathbb{E}[\max_j u_{ij}^{\text{pre}}] = \ln\left(\sum_j \exp(\delta_j^{\text{pre}} + \mu_{ij}^{\text{pre}})\right)$$

2. **Estimate Parameters After Policy Shock**
   - Estimate δ_j^{post}, β^{post}, α^{post}s
   - Calculate $\mathbb{E}[\max_j u_{ij}^{\text{post}}]$:

$$\mathbb{E}[\max_j u_{ij}^{\text{post}}] = \ln\left(\sum_j \exp(\delta_j^{\text{post}} + \mu_{ij}^{\text{post}})\right)$$

3. **Calculate Welfare Change**
   - Compute the difference in expected maximum utility:

$$\Delta W_n = \mathbb{E}[\max_j u_{nj}^{\text{post}}] - \mathbb{E}[\max_j u_{nj}^{\text{pre}}]$$

Python Example

- See Python script

Thanks!

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