Some Ideas about Category Theory

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Overview of Contents

- What is Category Theory
- Isomorphism
- Monics and Epics
- Initial and Terminal Objects
- Products and Coproducts
- My Feelings

What is Category Theory

- "Mathematics of Mathematics"

Data: Objects and Arrows(Morphisms)

Structure: Identities and Composition

$$a \to b \to c \qquad a \to c$$

Properties: Unit Laws,Associativity

Categories:	**Set**	**Mat**	**Poset(Z^+,|)**
Objects:	All Sets	$\boldsymbol{N}$	$\boldsymbol{Z}^+$
Morphisms:	Function	Matrix	Multiplication
Compositions:	Composition of Functions	Matrix Multiplication	Comp of Multiplication

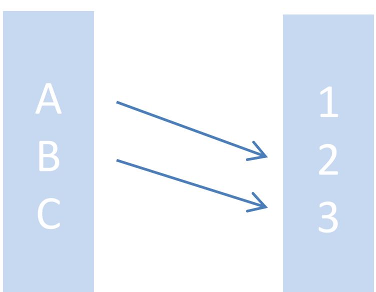
Isomorphism

Isomorphism is an invertible morphism in a category.

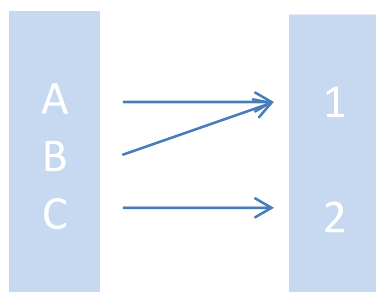
g and f are inverses if $g \circ f = I_a \;\; f \circ g = I_b$ given that

$$f: a \to b \;\; g: b \to a$$

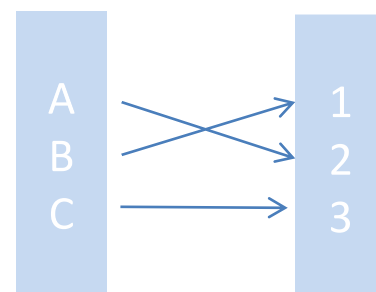
In **Set**, Isomorphisms is equivalent to bijection



Injection, 1 has no inverse



Surjection, 1 has no inverse



Bijection, every morphism is invertible

Monics and Epics

- Monic is "Categorical Injectivity"

$f: a \to b$ is monic if $\forall\ s, t: c \to a\ , f{\circ}s = f{\circ}t \Longrightarrow s = t$

- Epic is "Categorical Surjectivity"

$f: a \to b$ is epic if $\forall\ s, t: b \to c\ , s{\circ}f = t{\circ}f \Longrightarrow s = t$

For **Poset($\boldsymbol{Z^+}$,|)**

Is it monic?

Yes! We can at most have one morphism between any two objects in a Poset.

Is it epic?

Yes! By same idea…

Then we know it is both monic and epic,is it isomorphism?

Both Monic and Epic does not mean isomorphism in **Poset**

For **Poset(Z^+,|)**

Suppose we have 5

By morphism $\times 2$

We got 10

No Integer to reverse this!

Initial and Terminal Objects

For an object a in $\mathcal{C}$, we have an unique $a \rightarrow b \quad \forall b$ in $\mathcal{C}$

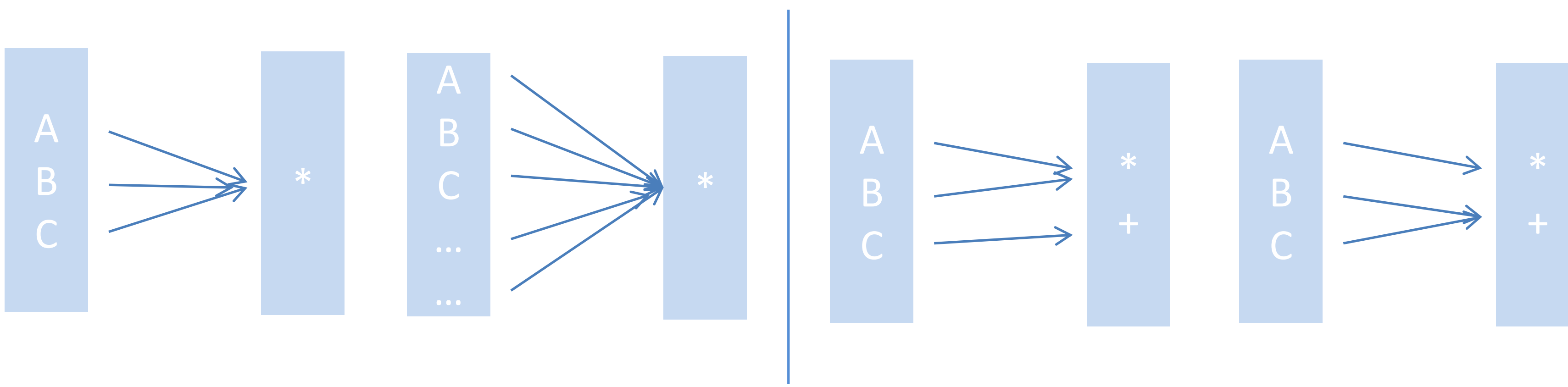
Then a is an initial object.

For an object a in $\mathcal{C}$, we have an unique b $\rightarrow a \quad \forall b$ in $\mathcal{C}$

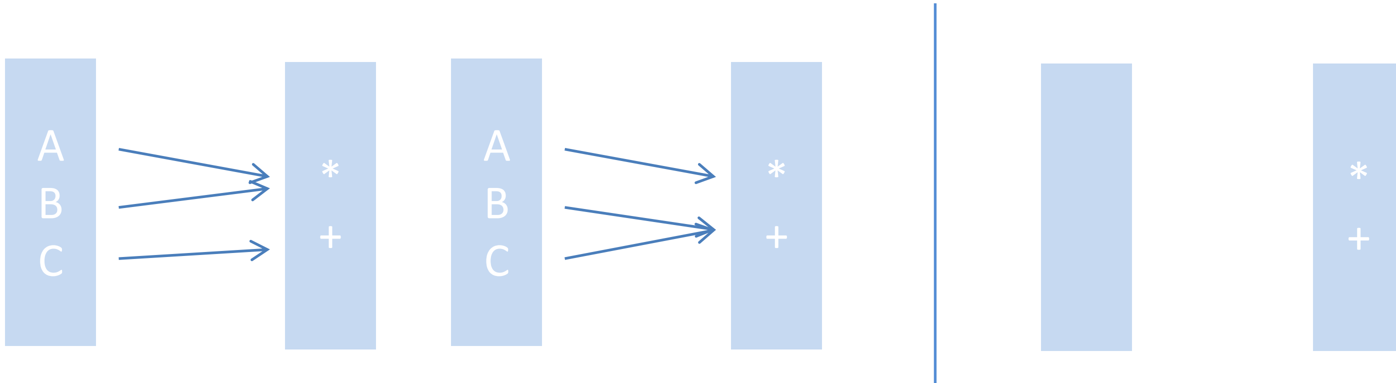
Then a is a terminal object.

In **Set**,

Any set with one element is terminal.



Empty Set is initial.



Only “Empty Function”

Also for **Poset(Z^+,|)**

1 is initial

What about a terminal object in this category?

There is no!

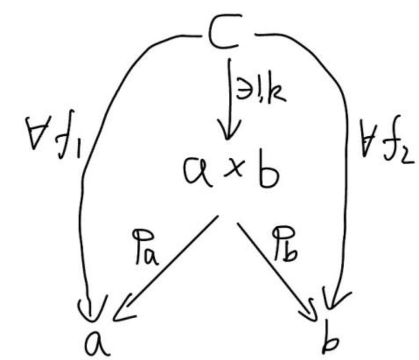
Constrain the objects of this **Poset** to factors of positive integer Z, and up to Z

Then Z is a terminal object.

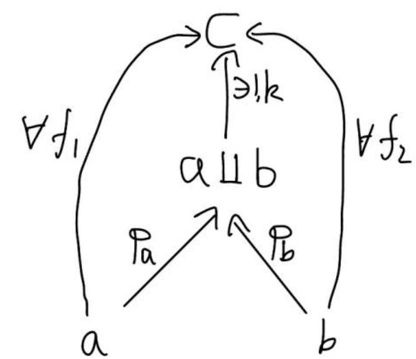
(Z is a least common multiple of all the elements here)

Products and Coproducts

For $a, b \in \mathcal{C}$, the product of a, b is an object $a \times b \in \mathcal{C}$ with morphisms $P_a: a \times b \to a, P_b: a \times b \to b$ such that $\forall c \in \mathcal{C}$, morphisms $f_1: c \to a, f_2: c \to b$, there exist an unique morphism $k: c \to a \times b$, which satisfies $P_a \circ k = f_1, P_b \circ k = f_2$



For $a, b \in \mathcal{C}$, the coproduct of a, b is an object $a \coprod b \in \mathcal{C}$ with morphisms $P_a: a \to a \coprod b, P_b: b \to a \coprod b$ such that $\forall c \in \mathcal{C}$, morphisms $f_1: a \to c, f_2: b \to c$, there exist an unique morphism $k: a \coprod b \to c$, which satisfies $k \circ P_a = f_1, k \circ P_b = f_2$



For **Set**,

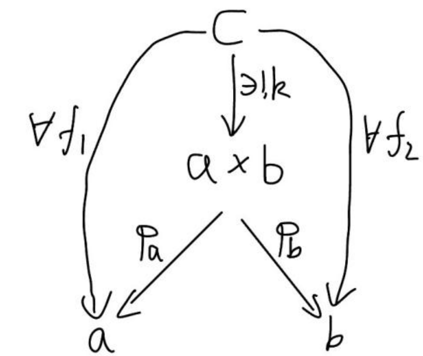
If have two sets A and B,

The product is (a, b)

Construct:

C is the any other set

$$P_a(a,b) = a;\ P_b\,(a,b) = b$$
$$k(c) = (a,b)$$
$$f_1(c) = a;\ f_2(c) = b$$



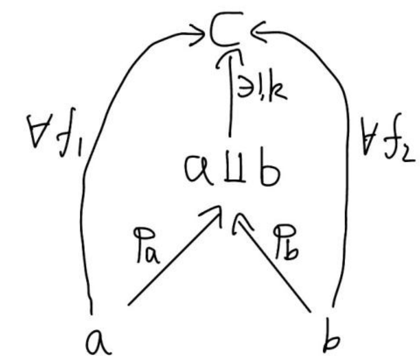
The coproduct is a **disjoint union** of A and B.

Example of disjoint union:

$$If\ \ a,b \in X; b,c \in Y, X\coprod Y = \{a_X, b_X, b_Y, c_Y\}$$

Construct:

$$P_a(a) = a;\ P_b\,(b) = b \text{ "result in a}\coprod\text{b"}$$
$$k = (f_1(x)\ if x \in A)\text{or}\ (f_2(x)\ if x \in B)$$
$$f_1(a) = c;\ f_2(b) = c$$



For **Poset(Z^+,|)**,

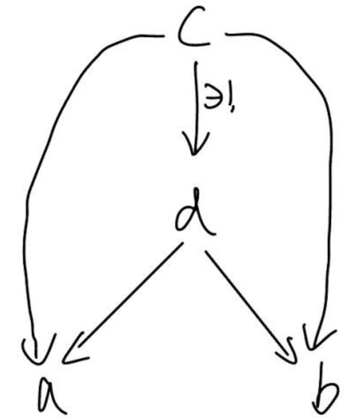
In product,

d is a factor of a and b,

c are any factors of a and b,

If any factors of a and b are factors of d…

d=gcd(a,b)



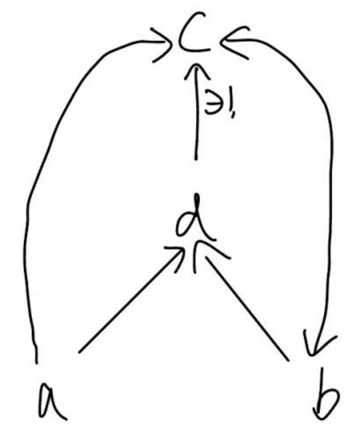
In coproduct,

d is a multiplier of a and b,

c are any multipliers of a and b,

If any other multiplier of a and b are multipliers of d…

d=lcm(a,b)



The book I read is The Joy Of Abstraction – *An Exploration of Math, Category Theory, and Life* by Eugenia Cheng

Really Good Book:

Introduced some background knowledge

Used easy understanding examples

Also I am really thankful to my mentor Daniel.

Thanks!