C0: Real Numbers and Functions

Function definition

Relation (rule) $f : X \to Y$ is a function (mapping) $\Leftrightarrow (\forall x \in \text{domain } X)(\exists \text{unique } y \in \text{codomain } Y)(f(x) = y)$.
Range (image) = Output value set $\subseteq Y$. Vertical Line Test: Rule is a function iff no vertical line cuts graph at > 1 point.

Injectivity, Surjectivity, Bijectivity

Injectivity (1-to-1) $X \preceq Y : (\forall a \neq b \in X)(f(a) \neq f(b))$
Surjectivity (Onto) $X \succeq Y : (\forall y \in Y)(\exists x \in X)(y = f(x)) \Leftrightarrow \text{range} = Y$
Bijectivity (1-to-1 correspondence) $X \approx Y$: Injectivity $\wedge$ Surjectivity
Horizontal Line Test: Function is injective iff no horizontal line intersects its graph at 1 point.
Invertibility: $(\exists g : Y \to X)(g \circ f = id_X \wedge f \circ g = id_Y) \Leftrightarrow f$ is bijective.

Function types

Real-valued f : range $\subseteq \mathbb{R}$. Real f : Real-valued function where $X \subseteq \mathbb{R}$
Maximal (natural) domain: Largest input set where f is real-valued. In MA1521, implicit domain=maximal domain.
Rational $f : (\exists \text{ polynomials } P(x), Q(x))(f(x) = \frac{P(x)}{Q(x)})$.
Piecewise (hybrid) f : Has > 1 subfunctions with disjoint intervals. Step f :Piecewise f of constant subfunctions.

Function composition

$f \circ g$ exists $\Leftrightarrow R_g \subseteq D_f . D_{f \circ g} = D_g, R_{f \circ g} \subseteq R_g$

Fundamental Theorem of Algebra

Polynomial of degree $n \in \mathbb{N}_0$ has exactly n complex roots, multiplicity included.

C1: Limits and Continuity

Precise $(\epsilon - \delta)$ definition of deleted function limit [Excluded from MA1521]

Well-defined limit: $\lim_{x \to p, x \in S \subseteq \mathbb{R}} f(x) = L \in \mathbb{R} \Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in S)(0 < |x - p| < \delta \Rightarrow |f(x) - L| < \epsilon)$
Note: Deleted limit doesn't need $f(p) \in \mathbb{R}$ vs non-deleted limit.
Well-defined limit at infinity: $\lim_{x \to \infty} f(x) = L \Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(x > \delta \Rightarrow |f(x) - L| < \epsilon)$
Well-defined limit at minus infinity: $\lim_{x \to -\infty} f(x) = L \Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(x < -\delta \Rightarrow |f(x) - L| < \epsilon)$
Infinite limit: $\lim_{x \to p, x \in S \subseteq \mathbb{R}} f(x) = \infty \Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in S)(0 < |x - p| < \delta \Rightarrow f(x) > \epsilon)$
Minus infinite limit: $\lim_{x \to p, x \in S \subseteq \mathbb{R}} f(x) = -\infty \Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in S)(0 < |x - p| < \delta \Rightarrow f(x) < -\epsilon)$

Imprecise metric definition of deleted function limit [Included in MA1521]

$$\lim_{x \to p} f(x) = L \in \mathbb{R} \Leftrightarrow \begin{cases} \lim_{x \to p^+} f(x) = L & \text{if } p \text{ is a left end-point} \\ \lim_{x \to p^-} f(x) = L & \text{if } p \text{ is a right end-point} \\ \lim_{x \to p^+} f(x) = \lim_{x \to p^-} f(x) = L & \text{if } p \text{ is an internal point} \end{cases}$$

Continuity definition

f continuous at a point $p \Leftrightarrow \lim_{x \to p} f(x) = f(p)$. $\therefore$ This is essentially the definition of a non-deleted limit.
f continuous on interval $I \Leftrightarrow (\forall p \in I)(\lim_{x \to p} f(x) = f(p))$

Law of Limits

Sufficient condition: $\lim_{x \to p} f(x), \lim_{x \to p} g(x) \in \mathbb{R}$
Sum Law: $\lim_{x \to p}(f(x) + g(x)) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$
Product Law: $\lim_{x \to p}(f(x) \cdot g(x)) = \lim_{x \to p} f(x) \cdot \lim_{x \to p} g(x)$. Quotient Law: $\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{\lim_{x \to p} f(x)}{\lim_{x \to p} g(x)}$
Composition Law: g continuous at $\lim_{x \to p} f(x) \Leftrightarrow \lim_{x \to p} g(f(x)) = g(\lim_{x \to p} f(x))$

Squeeze (Sandwich, 2 Officer) Theorem

$p \in I \subseteq X \Rightarrow (\forall x \in I - p)(g(x) \leq f(x) \leq h(x) \wedge \lim_{x \to p} g(x) = \lim_{x \to p} h(x) = L \in \mathbb{R} \Rightarrow \lim_{x \to p} f(x) = L)$
Corollary [**Replacement Rule**]: $p \in I \subseteq X \Rightarrow (\forall x \in I - p)(g(x) = h(x) \Rightarrow \lim_{x \to p} g(x) = \lim_{x \to p} h(x))$

Intermediate Value Theorem (IVT)

Sufficient condition: f continuous on $[a, b] \in X$
$(\forall y \in (\min(f(a), f(b)), \max(f(a), f(b))))(\exists c \in (a, b))(y = f(c))$ i.e. $[\min(f(a), f(b)), \max(f(a), f(b))] \subseteq f([a, b])$
Corollary [**Bolzano's Theorem**]: Additionally $\min(f(a), f(b)), \max(f(a), f(b))$ has opposite signs$\Rightarrow f$ has a root in $[a, b]$.

Big O Simplification

$(\forall \text{rational } f(x) = \frac{P(x)}{Q(x)})(\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{O(P(x))}{O(Q(x))})$

C2: Derivatives

Limit definition of total function derivative

$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$
$\therefore f$ differentiable at $x_0 \Rightarrow \lim_{x \to x_0} f(x) = \lim_{x \to x_0} (\frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0)) = f(x_0) \Rightarrow f$ continuous at x_0. Converse fails
(Pathological eg: Weierstrass function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ where $a \in (0, 1), b \in \mathbb{N}, b \equiv 1 (\bmod 2), ab > 1 + \frac{3\pi}{2}$).
f differentiable on interval $I \Leftrightarrow (\forall p \in I)(f'(p) \in \mathbb{R})$

Inverse Function Theorem

Sufficient condition: f differentiable on open $I \subseteq X$
$(\forall x \in I)(f^{-1}(f(x)) = x \Leftrightarrow (f^{-1})'(f(x)) = \frac{1}{f'(f^{-1}(f(x)))})$ i.e. $(\forall y \in f[I])((f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))})$

Cartesian & Parametric Conic Equations (Polar form: $r = \frac{\ell}{1 + e \cos \delta}, \ell =$ semi-latus rectum, $e =$ eccentricity)

Circle: $(x - x_0)^2 + (y - y_0)^2 = r^2 \Leftrightarrow (\forall t \in [0, 2\pi))((x, y) = (r \cos(t) + x_0, r \sin(t) + y_0))$. $e = 0$
Ellipsis: $\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1 \Leftrightarrow (\forall t \in [0, 2\pi))((x, y) = (a \cos(t) + x_0, b \sin(t) + y_0))$. $e = \sqrt{1 - \frac{b^2}{a^2}}$
Parabola: $(y - y_0)^2 = 4a(x - x_0) \Leftrightarrow (\forall t \in \mathbb{R})((x, y) = (at^2 + x_0, 2at + y_0))$. $e = 1$
Horizontal hyperbola: $\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1 \Leftrightarrow (\forall t \in [-\pi, \pi] \backslash \{-\frac{\pi}{2}, \frac{\pi}{2}\})((x, y) = (a \sec(t) + x_0, b \tan(t) + y_0))$. $e = \sqrt{1 + \frac{b^2}{a^2}}$
Vertical hyperbola: $\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = -1 \Leftrightarrow (\forall t \in [-\pi, \pi] \backslash \{-\frac{\pi}{2}, \frac{\pi}{2}\})((x, y) = (a \tan(t) + x_0, b \sec(t) + y_0))$. $e = \sqrt{1 + \frac{a^2}{b^2}}$

C3: Applications of Differentiation

Function Trends for f on $I = (a, b)$

Increasing: $(\forall x \in I)(f'(x) \geq 0)$. Decreasing: $(\forall x \in I)(f'(x) \leq 0)$. Strict inequalities guarantee injective f
Concavity (*concave downward*, convex upward): $(\forall t \in [0, 1])(f(ta + (1 - t)b) \geq tf(a) + (1 - t)f(b))$. Eg: logarithmic
Convexity (*convex downward*, concave upward): $(\forall t \in [0, 1])(f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b))$. Eg: exponential
Concavity definition omits differentiability. $f' \in \mathbb{R} \Rightarrow (f \text{ concave} \Leftrightarrow f' \text{ decreasing})$. $f'' \in \mathbb{R} \Rightarrow (f \text{ concave} \Leftrightarrow f'' \leq 0)$.
Beware f' decreasing$\not\Rightarrow f'' \leq 0$ (eg: if $f'' \notin \mathbb{R}$) and f strictly convex$\not\Rightarrow f'' < 0$ (eg: $x = 0$ for $f(x) = x^4$).
Jensen's Inequality: $(\forall i \in [1, n])(a_i > 0) \wedge f \text{ convex} \Leftrightarrow f(\frac{\sum a_i x_i}{\sum a_i}) \leq \frac{\sum a_i f(x_i)}{\sum a_i}$ $\therefore n = 2$ gives convexity definition.

Extrema

Local/relative max: $(\forall x \in I)(f(x) \leq f(p))$. Local/relative min: $(\forall x \in I)(f(x) \geq f(p))$. Global/absolute definitions follow.
Extrema definition omits continuity. $f' \in \mathbb{R} \Rightarrow f'(p) = 0$. Converse fails (eg: stationary inflection point).
Extreme Value Theorem: f continuous on closed $I \Rightarrow (\exists\ c, d \in I)(\forall x \in I)(\text{local min} = f(c) \leq f(x) \leq f(d) = \text{local max})$
Theorem fails for open I or f having ≥ 1 point of discontinuity.

1st Derivative Test for local extrema [4th scenario excluded from MA1521]

$$(p, f(p)) = \begin{cases} \text{min} & \Leftrightarrow (\exists r > 0)((\forall x \in (p - r, p)(f(x) \leq 0)) \wedge (\forall x \in (p, p + r))(f(x) \geq 0)) \\ \text{max} & \Leftrightarrow (\exists r > 0)((\forall x \in (p - r, p)(f(x) \geq 0)) \wedge (\forall x \in (p, p + r))(f(x) \leq 0)) \\ \text{non} - \text{extrema} & \Leftrightarrow (\exists r > 0)(\forall x \in (p - r, p + r) \backslash \{p\})(f(x) \neq 0) \\ \text{inconclusive} & \text{otherwise (eg: } x = 0 \text{ for } f(x) = x^2 \sin(\frac{1}{x}) \text{ oscillating infinitely in that neighbourhood)} \end{cases}$$

2nd Derivative Test for local extrema

$$(p, f(p)) = \begin{cases} \text{min} & f'(p) = 0 \wedge f''(p) > 0 \\ \text{max} & f'(p) = 0 \wedge f''(p) < 0 \\ \text{non} - \text{extrema} & f'(p) \in \mathbb{R} \backslash \{0\} \\ \text{inconclusive} & \text{otherwise, particularly if } f'(p) = f''(p) = 0 \end{cases}$$

Critical and inflection points

Critical point p (critical value$= f(p)$): p is internal point $\wedge (f'(p) = 0 \vee f'(p) \notin \mathbb{R})$ $\therefore f'(p) \in \mathbb{R} \Rightarrow p$ is stationary point.
Inflection point: Concavity changes sign. Sufficient condition [Excluded from MA1521]: $(\exists r > 0)(\frac{f''(p+r)}{f''(p-r)} < 0)$
Necessary condition [Included in MA1521]: $f''(p) = 0$.

Mean Value Theorem (MVT)

Sufficient condition: f differentiable on $(a, b) \neq \emptyset$ | Statement: $(\exists c \in (a, b))(f'(c) = \frac{f(b) - f(a)}{b - a})$
Corollary [**Rolle's Theorem**]: Additionally $f(a) = f(b) \Rightarrow (\exists c \in (a, b))(f'(c) = 0)$

l'Hopital's Rule [Provable by MVT]

3 sufficient conditions:$\lim_{x \to p} f(x) = \lim_{x \to p} g(x) \notin \mathbb{R} \backslash \{0\}) \wedge (\forall x \in I \backslash \{p\})(f'(x) \in \mathbb{R}, g'(x) \neq 0) \wedge (\lim_{x \to p} \frac{f'(x)}{g'(x)} \in \mathbb{R})$
$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)}$. Excluded from MA1521: Rule fails if $g'(x) = 0$ (Eg: $f(x) = x + \frac{\sin(2x)}{2}, g(x) = f(x)e^{\sin(x)}$)

C4: Integrals

Fundamental Theorem of Calculus [Order of parts reversed in MA1521]

For arbitrary real-valued function f continuous on $[a, b]$:
Part I : $\int_a^b f dx = F(b) - F(a)$ i.e. integration is inverse of differentiation.
Part II : for f differentiable on (a, b), $\frac{d}{dx} \int_a^x f dt = f(x)$ i.e. differentiation is inverse of integration.
Part II holds even if f is discontinuous and Riemann integrable, thus non-trivially implying Part I. Part II+ Chain Rule
$\frac{d}{dx} \int_a^{g(x)} f dt = f(g(x))g'(x)$

Riemann Integrals

$$\int_a^b f dx = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{b - a}{n} f(a + k \cdot \frac{b - a}{n}) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{b - a}{2n}(f(a + (k - 1) \cdot \frac{b - a}{n}) + f(a + k \cdot \frac{b - a}{n})) \text{ (Trapezoidal Rule)}$$

In general, $\int_a^b f dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^{\infty} \Delta x_k f(x_k^*)$. Simpson's approximations are derivable but excluded from MA1521.
Double (surface) integral:$\iint_D f dA = \lim_{\max \Delta A_k \to 0} \sum_{k=1}^{\infty} \Delta A_k f(x_k^*, y_k^*)$. If $f \geq 0$, $\iint f dA =$volume V.
Triple (volume) integral:$\iint_D f dV = \lim_{\max \Delta V_k \to 0} \sum_{k=1}^{\infty} \Delta V_k f(x_k^*, y_k^*, z_k^*)$
Simpson's $\frac{1}{3}, \frac{3}{8}$ Rules: $\int_a^b f dx \approx \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] \approx \frac{b-a}{8}[f(a) + 3f(\frac{2a+b}{3}) + 3f(\frac{a+2b}{3}) + f(b)]$

Improper Integrals

Type I : f continuous on $[a, \infty) \Rightarrow \int_a^{\infty} f dx = \lim_{b \to \infty} \int_a^b f dx$. Type II : f discontinuous at $c \Rightarrow \int_a^b f dx = \int_a^c f dx + \int_c^b f dx$

Integration by Reduction Formula (assuming $n \in \mathbb{N}^+, ab \neq 0$)

$I_n = \int \sin^n(ax)dx \Rightarrow anI_n = a(n - 1)I_{n-2} - \sin^{n-1}(ax) \cos(ax)$ | $I_n = \int \csc^n(ax)dx \Rightarrow (n - 1)I_n = (n - 2)I_{n-2} - \frac{\cos(ax)}{a \sin^{n-1}(ax)}$
$I_n = \int \cos^n(ax)dx \Rightarrow anI_n = a(n - 1)I_{n-2} + \cos^{n-1}(ax) \sin(ax)$ | $I_n = \int \sec^n(ax)dx \Rightarrow (n - 1)I_n = (n - 2)I_{n-2} + \frac{\sin(ax)}{a \cos^{n-1}(ax)}$
$I_n = \int e^{ax} \sin^n(bx)dx \Rightarrow I_n = \frac{n(n-1)b^2}{a^2+b^2n^2} I_{n-2} + \frac{e^{ax} \sin^{n-1}(bx)}{a^2+b^2n^2}(a \sin(bx) - bn \cos(bx))$ $\therefore$ Formula for $\int \sin^n(bx)dx$ follows.
$I_n = \int e^{ax} \cos^n(bx)dx \Rightarrow I_n = \frac{n(n-1)b^2}{a^2+b^2n^2} I_{n-2} + \frac{e^{ax} \cos^{n-1}(bx)}{a^2+b^2n^2}(a \cos(bx) + bn \sin(bx))$ $\therefore$ Formula for $\int \cos^n(bx)dx$ follows.
$I_n = \int x^n e^{ax} dx \Rightarrow I_n = -\frac{n}{a} I_{n-1} + \frac{x^n e^{ax}}{a}$ | $I_n = \int x^{-n} e^{ax} dx \Rightarrow I_n = \frac{a}{n-1} I_{n-1} + \frac{-e^{ax}}{(n-1)x^{n-1}} \forall n \in \mathbb{N}^+ \backslash \{1\}$

Some MF26 Formula [Integration by Parts *Descending* Preference: L*i*ATE]

$\int \frac{1}{\sqrt{(x+b)^2 \pm a^2)}} dx = \ln |(x + b) + \sqrt{(x + b)^2 \pm a^2)}| + c$ ($\because$ Substitute $x = a \tan \theta + b, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$)
$\int \frac{1}{\sqrt{a^2 - (x+b)^2}} dx = \sin^{-1}(\frac{x+b}{a}) + c$ | $\int \frac{1}{a^2 + (x+b)^2} dx = \frac{1}{a} \tan^{-1}(\frac{x+b}{a}) + c$ | $\cosh x = \frac{e^x + e^{-x}}{2} \leq e^{\frac{x^2}{2}}$ by Taylor expansion

C5: Applications of Integration

Volume V of revolution on rotation about x-axis & Arc Length

Disk Method: $V = \pi \int_{x_0}^{x_1} f(x)^2 dx$ ($\because \pi f^2$ = disk radius, dx = disk thickness.

Cylindrical Shell Method: $V = 2\pi \int_{y_0}^{y_1} y|f(y)|dy$ ($\because 2\pi y$ = shell circumference, $|f|$ = height, dy = shell thickness.

Arc length $= \int_a^b \sqrt{1+(f')^2}dx$. By Pythagora's Theorem, each infinitesimal arc segment$= \sqrt{(dx)^2+(dy)^2}$

C6: Sequences and Series

Sequence and Series Terminologies

Sequence $f: n \mapsto a_n, n \in \mathbb{N}^+$ (Conventionally $\{a_n\}_{n\in\mathbb{N}^+}$): Bijection from 1-indexed set of consecutive positive ints to multiset. Infinite seq: domain$=\mathbb{N}^+$ Partial sum $S_n = \sum_{k=1}^{n} a_k$. Series (infinite series)$= \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k$ is a value

Cauchy (fundamental) sequence: $(\forall\epsilon > 0)(\exists N \in \mathbb{N}^+)(\forall m, n \geq N)(|x_m - x_n| < \epsilon)$ i.e. arbitrarily close consecutive terms

Convergence & Divergence Tests

Fundamental Theorem Test: $(\exists m \in \mathbb{N}^+)(\forall n \geq m)(a_n \geq 0) \Rightarrow$ (S_n bounded$\Leftrightarrow S_n$ converges) i.e. S_n is finite or infinite.

1. **nth-term Divergence Test**: $\lim_{n\to\infty} a_n \neq 0 \Rightarrow S_n$ diverges. Converse fails (eg: harmonic series)
2. **Integral Test**: $(\exists n \in \mathbb{N}^+)$($f$ continuous, positive and decreasing on $[n,\infty)) \Rightarrow (\int_1^\infty f dx$ converges$\Leftrightarrow S_n$ converges)
   1. Corollary [**p-series Test**]: $p > 1 \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. $(p,k) = (1,1)$: Divergent Harmonic series.
   2. $p > 1, k = 1$: Convergent Riemann zeta function $\zeta(p), \zeta(2)$ = Basel's series $\frac{\pi^2}{6}$
3. **Direct Comparison Test**: $(\exists m \in \mathbb{N}^+)(\forall n \geq m)(0 \geq a_n \geq b_n) \Rightarrow (\sum_{n=1}^{\infty} a_n$ diverges$\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges).
4. **Ratio (d'Alembert's) Test**: $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$. | **Root (Cauchy's) Test**: $L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$.

   Series: $\begin{cases} \text{Absolutely convergent} & \text{if } L \in [0,1) \\ \text{Divergent} & \text{if } L > 1 \\ \text{Inconclusive} & \text{if } L = 1 \end{cases}$ Root Test is stronger than Ratio Test (eg: divergent Cauchy seq).
   1. Determine radius of convergence R of power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ by Ratio/Root Test, and convergence at bounds $-R+a, R+a$ by 1st principle. $\therefore (-R+a, R+a) \subseteq$ Interval of convergence $\subseteq [-R+a, R+a]$.
5. **Alternating Series (Leibniz's) Test**: $(\exists m \in \mathbb{N}^+)(\forall n \geq m)(a_n > 0 \wedge \lim_{n\to\infty} a_n = 0 \wedge a_{n+1} \leq a_n) \Rightarrow \sum_{n=1}^{\infty}(-1)^n a_n, \sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge. Converse fails (eg: $\{-\frac{1}{n^2}\}$)
6. **Absolute Convergence Test**: $\sum_{n=1}^{\infty} |a_n|$ converges $\Leftrightarrow \sum_{n=1}^{\infty} a_n$ converges i.e. Absolute convergence defined

MF26 Taylor Series

Analytic function on open X: $(\forall x_0 \in X)(\forall k \in \mathbb{N}_0)(\exists a_k \in \mathbb{R})(f(x) = \text{convergent } \sum_{k=0}^{\infty} a_k(x-x_0)^k)$.

Taylor series at x_0 for differentiable $f : f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ $\therefore$ Taylor series is analytic on interval of convergence.

Maclaurin series: Taylor series centred at 0 i.e. $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$. $\because$ f-value is independent of choice of pivot x_0.

$\forall x \in (-1,1) : \ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}$ | $\ln(1-x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$ | $\frac{1}{(1-x)^m} = \sum_{k=m-1}^{\infty} \binom{k}{m-1} x^{k-m+1}$

$\forall x \in \mathbb{R} : e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ | $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ | $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ | $\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ | $\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$

$\forall x \in [-1,1] : \sin^{-1} x = \sum_{k=0}^{\infty} \frac{(2k)! x^{2k+1}}{4^k (k!)^2 (2k+1)}$ | $\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$ | $\forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}) : \tan x = \sum_{k=0}^{\infty} \frac{\mathbb{B}_{2k}(-4)^k(1-4^k)x^{2k-1}}{(2k)!}$

Recursive definition of divergent Bernoulli series $\mathbb{B}_n^+ = 1 - \sum_{k=0}^{n-1} \binom{n}{k} \frac{\mathbb{B}_k^+}{n-k+1}, \mathbb{B}_0 = 1 \therefore (\mathbb{B}_2, \mathbb{B}_4, \mathbb{B}_6, \mathbb{B}_8) = (\frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30})$.

C7: Vectors and Geometry of Space

Projections

Scalar projection of $\vec{v}$ onto $\vec{u}$, $\text{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}$ | Vector projection of $\vec{v}$ onto $\vec{u}$, $\text{proj}_{\vec{u}}\vec{v} = (\text{comp}_{\vec{u}}\vec{v})(\frac{\vec{u}}{\|\vec{u}\|}) = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}$

$\therefore$ By first principle, perpendicular distance from point $P(x_0, y_0, z_0)$ to plane $Q : \vec{r} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = d$ is $\frac{|d-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}}$

C8: Functions of 2-3 Variables

Contours

Level set $L_c(f)$: Input set s.t output=constant c. Level curve: Bivariate f's level set. Level surface: Trivariate f's level set

Contour plot: Graph of level sets for constants at regular intervals.

Level Set Theorem: Gradient ∇ is orthogonal to level set of differentiable f at any point.

Cylindrical surface Equations [Non-exhaustive]

Cylindrical surface: Set of points on all lines parallel to a given line and contained in a given plane. $\therefore$ 1 of 3 input variables absent in parametric form $\Rightarrow$ surface is cylindrical. Converse fails iff cylinder is oblique.

Cylinder: 3D-solid bound by 1 cylindrical surface and 2 parallel planes. $\therefore$ Cylinder is right or oblique.

Elliptic: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ | Parabolic: $x^2 + 2ay = 0$ | Hyperbolic: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Quadric surface Equations [Non-exhaustive]

Quadric surface: Graph of deg-2 equation in 3 input variables i.e. $Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+Gx+Hy+Iz+J = 0$

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ | Elliptic (double) cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ | Elliptic paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$

Hyperboloid of 2 sheets: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ | Hyperboloid of 1 sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ | Hyperbolic paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} \pm z = 0$

Limit definition of partial function derivative [MA1521 uses only real-valued (scalar) function $f : \mathbb{R}^n \to \mathbb{R}$]

$(\forall i \in [1,n])(f_{x_i}|_{\vec{x}=\vec{a}} = \frac{\partial f(\vec{a})}{\partial x_i} = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{e_i})-f(\vec{a})}{h} = \lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x})-f(\vec{a})}{\|\vec{x}-\vec{a}\|})$ | Gradient $\nabla f(a_1, \cdots, a_n) = \langle f_{a_1}, \cdots, f_{a_n} \rangle$

f differentiable at $\vec{a} \Leftrightarrow (\forall i \in [1,n])((f_{a_i} \in \mathbb{R}) \wedge (\forall \Delta x_i > 0)(\exists \epsilon_i > 0)((\Delta x_i \to 0 \Rightarrow \epsilon_i \to 0) \wedge (\Delta f = \sum_{i=1}^{n} \Delta x_i f(x_i + \epsilon_i)))$

f differentiable on interval $I \subseteq X \Leftrightarrow (\forall \vec{a} \in I)$(f differentiable at point a corresponding to $\vec{a}$) i.e. I is a smooth surface.

$J((\nabla f)^{\mathbb{T}}) = H(f)$ | Directional derivative (rate of change) $D_{\hat{\mathbf{u}}}f(a_1 \cdots a_n)$ at point a in unit $\hat{\mathbf{u}}$'s direction $= \nabla f(a_1 \cdots a_n) \cdot \hat{\mathbf{u}}$. $\mathbf{J}, \mathbf{H}$ given as follows.

$m \times n$ Jacobian matrix $\mathbf{J}$ of 1st-order partial derivatives of vector-valued $f : \mathbb{R}^n \to \mathbb{R}^m = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$

$n \times n$ Hessian matrix $\mathbf{H}$ of 2nd-order partial derivatives of real-valued $f : \mathbb{R}^n \to \mathbb{R} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ \frac{\partial^2 f}{\partial x_n x_1} & \frac{\partial^2 f}{\partial x_n x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$

Clairaut's Theorem on equality of mixed partials

Sufficient condition: $f_{x_{ij}}, f_{x_{ji}}$ continuous on neighbourhood $\ni$ a | Statement: $f_{x_{ij}} = f_{x_{ji}}$

Corollary: $(\forall i, j \in [1,n])(f_{x_{ij}}, f_{x_{ji}}$ continuous$) \Leftrightarrow H(f)$ is symmetric.

Derivation of equation of tangent plane to surface $S : z = f(x,y)$ at point $P(a,b,c)$

$\because$ tangent // x-axis: $(\forall \lambda \in \mathbb{R})(\vec{r} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ f_x \end{bmatrix})$, tangent // y-axis: $(\forall \mu \in \mathbb{R})(\vec{r} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ f_y \end{bmatrix})$

$\therefore$ normal $\hat{\mathbf{n}}$ to tangent plane // $\begin{bmatrix} 0 \\ 1 \\ f_y \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ f_x \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ -1 \end{bmatrix} \Rightarrow$ Tangent plane equation: $\vec{r} \cdot \begin{bmatrix} f_x \\ f_y \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} f_x \\ f_y \\ -1 \end{bmatrix}$

$\therefore$ Generalised tangent hyperplane equation: $\vec{r} \cdot \begin{bmatrix} \nabla f \\ -1 \end{bmatrix} = d$

Chain Rule for multivariable function

$J_{f\circ g)}(\vec{a}) = J_f(g(\vec{a}))J_g(\vec{a}) \Leftrightarrow \frac{\partial(y_1 \cdots y_k)}{\partial(x_1 \cdots x_n)} = \frac{\partial(y_1 \cdots y_k)}{\partial(u_1 \cdots u_m)} \frac{\partial(u_1 \cdots u_m)}{\partial(x_1 \cdots x_n)}$. $k = 1 : (\forall i \in [1,n])(\frac{\partial y}{\partial x_i} = \sum_{k=1}^{m} \frac{y}{u_k} \frac{uk}{x_i})$ is a depth-2 tree.

Implicit Differentiation Theorem: $f(x_1 \cdots x_n) = 0, x_n$ defined by $x_1 \cdots x_{n-1} \Rightarrow (\forall i \leq n-1)(\frac{\partial x_n}{\partial x_i} = -\frac{f_{x_i}}{f_{x_n}})$ if $f_{x_n} \neq 0$

Extrema, Critical and saddle points

Extrema definition follow from 1-variable analog. Critical point: $(\forall i \in [1,n])(f_{x_i}(\vec{a}) \notin \mathbb{R}$ or $0)$.

Saddle point: $\vec{a}$ is critical point $\wedge(\exists \vec{b}, \vec{c}$ in neighbourhood of $\vec{a})(f(\vec{c}) < f(\vec{a}) < f(\vec{b}))$

2nd Partial Derivative Test [Fails for ≥ 3 independent variables, use 1st principle instead]

$\because (\forall \hat{\mathbf{u}}$ corresponding to $(h,k))(D_{\hat{\mathbf{u}}}^2(f) = k^2(f_{xx}\frac{h^2}{k^2} + 2f_{xy}\frac{h}{k} + f_{yy})$ by Clairaut's Theorem.

$\therefore$ Reduced discriminant $d = f_{xx}f_{yy} - f_{xy}^2 = |H(f)|.(\vec{a}, f(\vec{a})) = \begin{cases} \text{saddle point} & \text{if } d < 0 \\ \text{local min} & \text{if } d > 0 \wedge f_{xx}(\vec{a}) > 0 \\ \text{local max} & \text{if } d > 0 \wedge f_{xx}(\vec{a}) < 0 \\ \text{inconclusive} & \text{if } d = 0 \; (\because f_{xx}(\vec{a}) \neq 0 \text{ if } d > 0) \end{cases}$

C9: Double (Iterated) Integrals

Surface Area: $\iint_D f dS = \iint_D \sqrt{f_x^2 + f_y^2 + 1} dA$

Fubini-Tonelli Theorem

Sufficient condition: f bounded on domain $X \times Y$ | Statement: $\iint_{X\times Y} f d(x,y) = \int_X(\int_Y f dy)dx = \int_Y(\int_X f dx)dy$

Special case (Separable-form): $f(x,y) = g(x)h(y) \Rightarrow \iint_{X\times Y} f d(x,y) = (\int_{x_0}^{x_1} g dx)(\int_{y_0}^{y_1} h dy)$

Domain types for Iterated Integrals [Trace upward for Type I and trace rightward for Type II]

Type I: $D = \{(x,y)|x \in [a,b], y \in [g_1(x), g_2(x)]\} \Rightarrow \iint_D f dA = \int_a^b(\int_{g_1(x)}^{g_2(x)} f dy)dx$ i.e. 2 continuous graphs in x bound f

Type II: $D = \{(x,y)|x \in [h_1(y), h_2(y)], y \in [c,d]\} \Rightarrow \iint_D f dA = \int_c^d(\int_{h_1(y)}^{h_2(y)} f dx)dy$ i.e. 2 continuous graphs in y bound f

Cartesian-polar Conversion

$\because dA = dxdy = |J|drd\theta, |J| = |\frac{\partial(x,y)}{\partial(r,\theta)}| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} (\because x = r\cos\theta, y = r\sin\theta) = r \Rightarrow dA = rdrd\theta$

$\therefore \iint_D f dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$, polar rectangle domain $D = \{(r,\theta)|r \in [a,b], \theta \in [\alpha,\beta]\}$

C10: Separable, reducible, 1st-order or Bernoulli Ordinary Differential Equations

ODE Terminologies

DE: Equation of ≥ 1 function and derivatives | partial DE: DE of > 1 input variables | ordinary DE: DE of 1 input variable

Linear ordinary DE: $\sum_{k=0}^{n} A_k(x)y^{(k)} = D(x)$ forcing function, $y^{(k)}$ = output's kth derivative

Separable & reducible ODEs [Beware denominator = 0 by checking cases $g(y) = 0, f(u) = u$ or $bf(u) + a = 0$]

$y' = f(x)g(y) \Rightarrow \int \frac{1}{g(y)}dy = \int f(x)dx$. Substitute $u = \frac{y}{x}$ or $u = ax + by$ in respective reducible forms below.

Reducible I: $y' = f(\frac{y}{x}) \Rightarrow \int \frac{1}{f(u)-u}du = \ln|x| + c$ | Reducible II$(b \neq 0)$: $y' = f(ax+by) \Rightarrow \int \frac{1}{bf(u)+a}du = x + c$

Standard-form linear 1st-order ODEs

$y' + P(x)y = Q(x)$. Multiply integrating factor $e^{\int P(x)dx}$. $\therefore ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx$. No boundary cases.

Bernoulli ODEs

$y' + P(x)y = Q(x)y^n, n \neq 0, 1$. Substitute $u = y^{1-n}$. $\therefore \frac{u'}{1-n} + P(x)u = Q(x) \Rightarrow \int \frac{1}{Q(x)-P(x)u}du = (1-n)x + c$

Physics Applications

Newton's Law of Cooling: $Q' = hA(T - T_{env}), h$ = heat transfer coefficient, A = surface area.

Linear drag (viscous resistance): $\vec{F_d} = -b\vec{v}, b$ = drag constant> 0. Small spheres' Stokes drag: $b = 6\pi\eta r, \eta$ = viscosity

Drag equation for high Reynolds number (large $\vec{v}$): $\vec{F_d} = -\frac{1}{2}\rho\|\vec{v}\|^2 C_d A\hat{\mathbf{v}} \Rightarrow \left\|\vec{F_d}\right\| \propto -\|\vec{v}\|^2$

C11: More on ODEs [Excluded from MA1521 AY2023-2024 Semester 1]

Verhulst Population (N) Model: $(\exists s > 0)$(death rate $D = sN$). $N' = BN - sN^2 \Rightarrow N = \frac{N_\infty}{1+(\frac{N_\infty}{N_0}-1)e^{-Bt}}$,capacity $N_\infty = \frac{B}{s}$

$\frac{N_\infty}{N_0} > 2 \Rightarrow (\exists$ unique inflection point $(-\frac{1}{B}\ln(\frac{N_\infty}{N_0} - 1), \frac{N_\infty}{2})$. Naive static Malthus Model: $N = N_0 e^{(B-D)t}$, D constant.