C1: 1st-order Differential Equation

1st-order Linear Differential Equation

$y' + py = q \Leftrightarrow (ye^{\int p dx})' = qe^{\int p dx} \Leftrightarrow y = e^{-\int p)dx} \int qe^{\int p dx}$ (∵ Integrating factor e^{pdx})

Directly-separable Differential Equation

$y' = \frac{f(x)}{g(y)} \Leftrightarrow \int g(y)dy = \int f(x)dx \wedge g(y) \neq 0$

$y' = f(\frac{y}{x}) \Leftrightarrow \frac{1}{f(u)-u}du = \int \frac{1}{x}dx \vee f(u) = u = u'$ (∵ Plug $u = \frac{y}{x} \Rightarrow y' = u + xu'$)

Homogeneous equation of degree 0

Definition: $(\forall x, y \in D)(\forall t \in \mathbb{R})(f(tx, ty) = t^0 f(x, y) = f(x, y))$

$\forall y' = f(x, y) \Leftrightarrow \int \frac{1}{f(u)-u}du = \int \frac{1}{x}dx \vee f(1, u) = u$ (∵ Plug $u = \frac{y}{x} \Rightarrow f(x, y) = f(x, ux) = f(1, u)$)

Remark: $(\forall t \neq 0)(f(x, y) = x^n f(1, u))$ which results in inseparable terms.

$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$. By case-partitioning:

Case 1: If $a_1 b_2 = a_2 b_1$ (can be 0) $\wedge b_2 \neq 0$, plug $u = a_2 x + b_2 y \Rightarrow \frac{u'-a_2}{b_2} = \frac{b_1}{b_2} + \frac{c_1 - \frac{b_1 c_2}{b_2}}{u+c_2} \Rightarrow u' = \frac{b_2 c_1 - b_1 c_2}{u+c_2} + a_2 + b_1$

∴ Edge case: $u' = 0 \Leftrightarrow u = \frac{a_2+b_1}{b_1 c_2 - b_2 c_1} - c_2$

∴ General case: $\int \frac{1}{\frac{b_2 c_1 - b_1 c_2}{u+c_2} + a_2+b_1} du = \int dx \Leftrightarrow \frac{u}{a_2+b_1} + \frac{b_1 c_2 - b_2 c_1}{(a_2+b_1)^2}\ln|b_2 c_1 - b_1 c_2 + (a_{2+b_1})(u+c_2)| - x = c$

case 2: If $a_1 b_2 = a_2 b_1 \wedge b_2 = 0$, then $a_2 = 0 \vee b_1 = 0$.

If $B_1 = 0$, then $y' = \frac{a_1 x + c_1}{a_2 x + c_2}$ is directly integrable. Else if $a_2 = 0$, then $y' = \frac{b_1 y + c_1}{a_2 x + c_2}$ is directly separable.

Case 3: If $a_1 b_2 \neq a_2 b_1$, then plug $(x, y) = (u + h, v + g)$ s.t. $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} h \\ g \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$

∴ $y' = \frac{a_1 u + b_1 v}{a_2 u + b_2 v}$ is homogeneous of degree 0.

If $a_1 \neq 0$, then $(h, g) = (-\frac{c_1}{a_1} + \frac{b_1}{a_1} \cdot \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}, -\frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1})$. Else, solve (h, g) by back substitution.

Exact 1st-order Differential Equation

Definition: $(\exists u(x, y))(M(x, y)dx + N(x, y)dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du) \Rightarrow u = c$. Beware edge case(s) $y = c$.

Necessary and sufficient condition: If M, N and 1st partial derivatives continuous, then $M_y = N_x$ (∵ Clairaut's Theorem).

Else if $\exists$ integrating factor $\mu(x, y)$ s.t. $(\mu M)_y = (\mu N)_x$ by definition, then $N\mu_x - M\mu_y = \mu(M_y - N_x)$.

Plug suitable $v(x, y)$ into $\mu = \mu(v)$ s.t. $\mu_x = v_x \frac{d\mu}{dv}, \mu_y = v_y \frac{d\mu}{dv}$, then $\frac{1}{\mu}\frac{d\mu}{dv} = \frac{M_y - N_x}{Nv_x - Mv_y}$, suitability by if R.H.S. is pure $\phi(v)$. Eg: $v = x^\alpha y^\beta$

Likely-solvable Nonlinear Differential Equation of degree 1

Bernoulli (Solvable): $(\forall n \in \mathbb{R} \backslash \{0, 1\})(y' + py = qy^n \Rightarrow (y^{1-n}e^{\int p dx})' = qe^{\int p dx} \Rightarrow y = (e^{-\int p dx}\int qe^{\int p dx}dx)^{\frac{1}{1-x}})$

∵ Integrating factor $= y^{-n}e^{\int p dx}$. Edge case $y = 0$ holds. Both $n = 0 \vee n = 1$ make DE directly solvable.

Riccati (May be unsolvable): $y' = p + qy + ry^2$.

Case 1: If $\exists y_p$ (can be 0), then $y_G = y_p + \frac{1}{e^{-\int(q+2ry_p)dx}(c - \int e^{\int(q+2ry_p)dx}rdx)}$. ∴ single y_p suffices for solvability.

Case 2: If $\exists y_{p_1}, y_{p_2}$ being distinct, then $\ln|\frac{y_G - y_{p_1}}{y_G - y_{p_2}}| = \int(y_{p_1} - y_{p_2})rdx + c$.

Case 3: If $\exists y_{p_1}, y_{p_2}, y_{p_3}$ being distinct, then cross-ratio $\frac{(y_{p_1}-y_{p_2})(y_{p_3}-y_G)}{(y_{p_1}-y_G)(y_{p_3}-y_{p_2})} = c$.

Case 4: If $\exists y_{p_1}, y_{p_2}, y_{p_3}, y_{p_4}$ being distinct, then cross-ratio $\frac{(y_{p_1}-y_{p_2})(y_{p_3}-y_{p_4})}{(y_{p_1}-y_{p_4})(y_{p_3}-y_{p_2})} = c$, y_G derivable from Cases 1, 2, 3.

Abel of the 1st kind (May be unsolvable): $y' = p + qy + ry^2 + sy^3$. Try reduce to Riccati if possible. Eg: $u = \frac{y}{x}$.

1st-order Implicit Differential Equation (May be unsolvable vs 1st-order explicit DE)

If $y = f(x, y')$, then plug $p = y' \Leftrightarrow pdx = dy = df = f_x dx + f_p dp \Leftrightarrow (f_x - p)dx + f_p dp = 0$.

Clairaut's Equation (May be unsolvable): If $y = xy' + f(y')$, then plug $p = y' \Leftrightarrow (x + f'(p))p' = 0$.

If $f(x, y', y'') = 0$, then plug $p = y' \Leftrightarrow f(x, p, p')$ is implicit 1st-order DE.

If $f(y, y', y'') = 0$, then plug $p = y' \Leftrightarrow f(y, p, p\frac{dp}{dy}) = 0$ is implicit 1st-order DE.

C2: General nth-order Linear Differential Equation

Generalised Picard-Lindelöf Theorem to nth-order linear ODE (Existence and Uniqueness Theorem)

Derivative operator: $L(D)y = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f(x)$ | Initial conditions: $y^{(n-1)}(x_0) = y_{n-1}, \cdots, y(x_0) = y_0$

If $(\forall i \in [1, n])(a_i(x)$ Lipschitz-continuous on $[p, q]) \wedge f$ continuous on $[p, q] \wedge f \neq 0)$, then $(n-1)$-Initial-Value Problem (IVP) has unique solution y_p on $[p, q]$ (∵ degree of freedom= 1).

1st Consequence: If $f \equiv 0 \wedge (\exists x_0 \in [p, q])(y(x_0) = \cdots = y^{(n-1)}(x_0) = 0)$, then $y_H \equiv 0$ on $[p, q]$ uniquely.

Remark: Do not mix IVP with k-point Boundary-Value Problem (BVP) where $x_1, \cdots, x_k \in \partial D_x$ boundary set:

$\begin{cases} b_{10}y^{(n)}(x_1) + b_{11}y^{(n-1)}(x_1) \cdots + b_{1n}y(x_1) = d_1 \\ \cdots \\ b_{k0}y^{(n)}(x_k) + b_{k1}y^{(n-1)}(x_k) \cdots + b_{kn}y(x_k) = d_k \end{cases}$

3 notable IVP vs BVP distinctions:

1. BVP's $x_i \in \partial D_x$; IVP's $x_0 \in D_x$.

2. BVP condition may comprise ≥ 1 distinct y's derivative powers; IVP condition each has single y's derivative powers↑.

3. Homogeneous BVP may prematurely lack non-trivial y_H iff inconsistent but has $y_H \equiv 0$; $(n-1)$-point homogeneous IVP has unique $y_H \equiv 0$.

Fundamental Theorem of Algebra to ODE

$L(D)y = 0$ has exactly n linearly-independent solutions $\phi_1, \cdots, \phi_n$ giving a fundamental solution set $y_H = c_1\phi_1 + \cdots + c_n\phi_n$

If $\deg(a_1) = \cdots = \deg(a_n) = 0 \Leftrightarrow a_1, \cdots, a_n$ constant coefficients, then consider auxiliary (characteristic) equation:

$\lambda^n + \cdots + a_{n-1}\lambda + a_n = 0$ (∵ $y = e^{\lambda x}$) $\Rightarrow (\exists k$ eigenvalues $\lambda_1 \cdots \lambda_k$ of algebraic multiplicities $m_1 \cdots m_k$ s.t. $\sum m_i = n$)

∴ Fundamental solution set y_H's construction:

1. $(\forall \lambda_i \in \mathbb{R})(\forall j \in [0, m_i - 1])(y = c_{ij}x^j e^{\lambda_i x})$ (∴ $n = 2 \wedge \exists 1$ unique $\lambda \in \mathbb{R} \Leftrightarrow y_H = c_{i1}e^{\lambda_i x} + c_{i2}xe^{\lambda_i x}$)

2. $(\forall \lambda_i \notin \mathbb{R})(\forall j \in [0, m_i - 1])(y = c_{ij1}x^j e^{\mathbb{Re}(\lambda_i)x}\cos(\mathbb{Im}(\lambda_i)x) + c_{ij2}x^j e^{\mathbb{Re}(\lambda_i)x}\sin(\mathbb{Im}(\lambda_i)x))$ (∵ Conjugate Root Theorem)

Euler-Cauchy Equation (2nd-order linear non-constant homogeneous $x^2 y'' + a_1 xy' + a_2 y = 0, x > 0$)

∵ Auxiliary equation: $\lambda(\lambda - 1) + a_1\lambda + a_2 = 0$ (∵ plug $y = x^\lambda$)

∴ 1. If $\exists \lambda_1 \neq \lambda_2 \in \mathbb{R}$, fundamental solution set $y_H = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$

2. If $\exists \lambda_1 = \lambda_2 \in \mathbb{R}$, fundamental solution set $y_H = c_1 x^{\lambda_1} + c_2 x^{\lambda_1}\ln(x)$

3. If $\exists \lambda_1 = \lambda_2 \notin \mathbb{R}$, fundamental solution set $y_H = c_1 x^{\mathbb{Re}(\lambda_1)}\cos(\mathbb{Im}(\lambda_1)\ln x) + c_2 x^{\mathbb{Re}(\lambda_1)}\sin(\mathbb{Im}(\lambda_1)\ln x)$

"Wronskian" Theorem

Definition: Wronskian $W(\phi_1, \cdots, \phi_n)(x) = \det\left(\begin{bmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{bmatrix}\right)$. Property: anti-symmetric.

Statement: $(\forall n$ full solution to $L(D)y = 0)(W = 0 \Leftrightarrow W \equiv 0 \Leftrightarrow$ dependence$\wedge W \neq 0 \Leftrightarrow W$ not vanish$\Leftrightarrow$independence)

This may fail specifically in linear-dependence clause for m candidates$<$ order n. Counter-eg: $\phi_1 = x^2, \phi_2 = x|x|$.

1st-Consequence [**Abel's Theorem**]: $n = 2 : y'' + a_1(x)y' + a_2(x)y = 0 \Rightarrow (\forall x_0 \in D_x)(W(x) = W(x_0)e^{-\int_{x_0}^x a_1(t)dt})$

Given independent ϕ_1, ϕ_2, verifiable $a_1(x) = -\frac{\phi_1\phi_2'' - \phi_1''\phi_2}{W(\phi_1, \phi_2)}, a_2(x) = \frac{\phi_1'\phi_2'' - \phi_1''\phi_2'}{W(\phi_1, \phi_2)}$

Variation of Parameters for 2nd-order linear non-homogeneous $y'' + a_1(x)y' + a_2(x)y = f(x)$

1. If $\exists y_{H_1}$, then $y_{H_2} = y_{H_1}\int y_{H_1}^{-2}e^{-\int a_1(x)dx}dx$. $W(y_{H_1}, y_{H_2}) = e^{-\int a_1(x)dx} \neq 0$.

2. If $\exists y_{H_1}, y_{H_2}$, then $y_P = -y_{H_1}\int \frac{y_{H_2}}{W(y_{H_1}, y_{H_2})}fdx + y_{H_2}\int \frac{y_{H_1}}{W(y_{H_1}, y_{H_2})}fdx$ (∵ WLOG solve u_1, u_2 in $u_1'y_1 + u_2'y_2 = 0$)

∴ 2nd-order linear non-homogeneous equation is solvable if $\exists y_{H_1}(x)$.

Undetermined Coefficients for 2nd-order linear constant non-homogeneous $y'' + a_1(x)y' + a_2(x)y = f(x)$

1. ∵ If $f(x) = P_s(x)e^{tx}, P_s =$degree-s ≥ 0 polynomial, plug $y = Q(x)e^{tx} \Rightarrow Q'' + (2t + a_1)Q' + (t^2 + a_1 t + a_2)Q = P_s$

∴ 1(a). If coefficient of $Q = t^2 + a_1 t + a_2 \neq 0$, $y_P = Q_s(x)e^{tx}$. Compare coefficients

∴ 1(b). If coefficient of $Q = t^2 + a_1 t + a_2 = 0 \wedge$coefficient of $Q' = 2t + a_1 \neq 0$, $y_P = xQ_s(x)e^{tx}$. Compare coefficients

∴ 1(c). If coefficient of $Q = t^2 + a_1 t + a_2 = 0 \wedge$coefficient of $Q' = 2t + a_1 = 0$, $y_P = x^2 Q_s(x)e^{tx}$. Compare coefficients

2. ∵ If $f(x) = P_s e^{\alpha x}\sin(\beta x) \vee f(x) = P_s e^{\alpha x}\cos(\beta x)$,

∴ 2(a). If $\alpha + i\beta$ doesn't solve auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$, then $y_P = Q_s(x)e^{\alpha x}\cos(\beta x) + R_s(x)e^{\alpha x}\sin(\beta x)$

∴ 2(b). If $\alpha + i\beta$ solves auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$, then $y_P = xQ_s(x)e^{\alpha x}\cos(\beta x) + xR_s(x)e^{\alpha x}\sin(\beta x)$

6 Linear Derivative Operator Properties

1. $D^{-1}f = \int f dx$ (∵ Fundamental Theorem of Calculus)

2. $(D - a)^{-1}f = ce^{ax} + e^{ax}\int e^{-ax}fdx$ (∵ $y' - ay = f \Leftrightarrow (ye^{\int -adx})' = fe^{adx}$)

3. $L(D)^{-1}(e^{ax}f) = e^{ax}L(D + a)^{-1}f$

4. $L(D)(e^{ax}f) = e^{ax}L(D + a)f$

5. $(D - a)^{-1}f = -(\frac{1}{a} + \frac{1}{a^2}D + \frac{1}{a^3}D^2 + \cdots + \frac{1}{a^{n+1}}D^n)f$ if f is degree-n polynomial (∵ Power-series expansion)

6. If $L(D)y = f = e^{\alpha x}\cos(\beta x)$, solve for $\mathbb{Re}(y_P)$ where $L(D)y_P = e^{(\alpha+i\beta)x}$ (∵ de-Moivre's Theorem).

If $L(D)y = f = e^{\alpha x}\sin(\beta x)$, solve for $\mathbb{Im}(y_P)$ where $L(D)y_P = e^{(\alpha+i\beta)x}$.

C3: 2nd-order Linear Differential Equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x)$

Terminologies

∵ Equivalent formulation: $(a_0 y' - a_0'y)' + (a_1 y)' + (a_0'' - a_1' + a_2)y = f(x)$

∴ Exact 2nd-order linear DE: $a_0'' - a_1' + a_2 \equiv 0 \Leftrightarrow a_0 y' + (-a_0' + a_1)y = \int f(x)dx$ is solvable 1st-order explicit linear DE.

(Self-transpose) Adjoint DE: $(a_0 y)'' - (a_1 y)' + a_2 y = 0 \Leftrightarrow a_0 y'' + (2a_0' - a_1)y' + (a_0'' - a_1' + a_2)y = 0$

Note: ∴ $y_{H-adjoint} =$integrating factor to "exactify" initial DE.

Self-adjoint DE: DE $\equiv$ Adjoint DE $\Leftrightarrow a_0'(x) = a_1(x) \Leftrightarrow (a_0 y')' + a_2 y = f(x)$.

Note: ∴ Integrating factor to "self-adjointify" initial DE$= \frac{1}{a_0}e^{\int \frac{a_1}{a_0}dx}$

Lagrange's Identity & Green's 2nd Identity

Denote $L[y] = a_0 y'' + a_1 y' + a_2 y$, then formal-adjoint differential operator $L^+[y] = (a_0 y)'' - (a_1 y)' + a_2 y$

Lagrange's Identity: $(\forall$ twice-differentiable $u(x), v(x))(vL[u] - uL^+[v] = \frac{d}{dx}M(u, v) = \frac{d}{dx}(ua_1 v - u(a_0 v)' + u'a_0 v))$

Green's 2nd Identity: $(\forall$ bounds $x_0, x_1 \in D_L \cap D_{L^+})((v, L[u])_{x_0}^{x_1} - (u, L^+[v])_{x_0}^{x_1} = M(u, v))$ (∵ Integral analogue ↑)

Note: $(\forall$ continuous real-valued g, h on $[x_0, x_1])((g, h) = \int_{x_0}^{x_1} ghdx)$

If $L[y] = 0$ self-adjoint, then $L \equiv L^+$ by definition $\Rightarrow vL[u] - uL[v] = -\frac{d}{dx}(a_0 W(u, v)) \Leftrightarrow (v, L[u]) - (u, L[v]) = -a_0 W$.

1st-Consequence: If u, v are homogeneous (not necessarily independent) solution to initial DE $(a_0 y')' + a_2 y = 0$, then $L[u] = L[y] = 0 \Rightarrow a_0 W$ constant on $[x_0, x_1]$ (∵ Lagrange's Identity)

Sturm-Liouville Equation (Special 2nd-order linear self-adjoint homogeneous $(a_0(x)y')' + a_2(x)y = -\lambda f(x)y$)

Note: In most literature for BVP, $L[y] = (a_0 y')' + a_2 y$, though pedantically, $\lambda f(x)$ is part of y's coefficient for linearity.

Regular 2-point Sturm-Liouville BVP:

$\begin{cases} (\forall x \in \text{open}(a, b))(L[y] + \lambda f(x)y = (a_0 y')' + a_2 y = 0) \\ c_{11}y(a) + c_{12}y'(a) = 0 \\ c_{21}y(b) + c_{22}y'(b) = 0 \end{cases}$

∴ Eigenvalue λ varies associated (non-trivial by definition) eigenfunction y to DE; $y \equiv 0$ holds.

∴ Eigenfunction y must fulfill both DE and all boundary conditions (∴ if system consistent, restrict family's scope)

Left Column

3 **Properties of eigenvalues and eigenfunctions to 2-point Sturm-Liouville BVP:**
1. All λ are real, simple (algebraic multiplicity= $1 \Leftrightarrow \lambda \mapsto y$ injective), have real-valued y | 2. λ countably-incr to ∞.
3. Associated eigenfunctions y_2 of distinct eigenvalues $\lambda_1 \neq \lambda_2$ are distinct orthogonal w.r.t given weight function
$f(x) > 0$ in Sturm-Liouvuille BVP's DE, i.e. $\int_a^b y_1 y_2 dx = 0$
1st-Consequence: Normalising all eigenfunctions $(\forall n)(\int_a^b y_n^2 dx = 1)$, then ($\forall g$ fulfilling boundary conditions (not necessarily DE))$(g = \sum_{i=1}^n (y_n \int_a^b gy_n f dx)$ recursively where infinite series converges uniformly on $[a, b]$.

Non-homogeneous Self-Adjoint Regular BVP (Not necessarily Sturm-Liouville)

$$\begin{cases} (\forall x \in \text{open}(a, b))(L[y] = (a_0 y')' + a_2 y = f(x)) \\ c_{11} y(a) + c_{12} y'(a) = 0 \\ c_{21} y(b) + c_{22} y'(b) = 0 \end{cases}$$

y_P **construction**: Non-homogeneous system has unique $y_P = y_1 \int_x^b \frac{y_2 f}{W a_0} dt + y_2 \int_a^x \frac{y_1 f}{W a_0} dt$ (aka **Green's Solution** via Variation of Parameters) iff associated homogeneous system has $y_H \equiv 0$ only, where non-unique y_1 and y_2 solve:

$$\begin{cases} L[y_1] = (a_0 y_1')' + a_2 y_1 = 0 \\ c_{11} y_1(a) + c_{12} y_1'(a) = 0 \end{cases} \quad \begin{cases} L[y_2] = (a_0 y_2')' + a_2 y_2 = 0 \\ c_{21} y_2(b) + c_{22} y_2'(b) = 0 \end{cases}$$

Note: 1. y_1, y_2 each belong to 1 of 2 distinct families ($\because$ Fundamental Theorem of Algebra) so non-unique. Nonetheless, constant coefficients are cancelled in Green's solution, imaginable as a "sliding window" on $[a, b]$.
    2. Some lit:*Fredholm Theorem*, successor:*Fredholm-Alt Theorem*| 3. ($\forall$self-adj L)$(W a_0 = c)$ ($\because$ Lagrange)
Fredholm Theorem: Non-homogeneous system has non-unique $y_P = y_1 \int_x^b \frac{y_2 f}{W a_0} dt + y_2 \int_a^x \frac{y_1 f}{W a_0} dt$ (aka **Green's Solution** via Variation of Parameters) iff associated homogeneous system fulfills ($\forall$non-trivial y_H)$((y_H, f) = \int_a^b y_H f dx = 0)$, where y_1 =any non-trivial y_H fulfilling both boundary conditions$\wedge L[y_2] = 0 \wedge W(y_1, y_2) \neq 0$ everywhere.
Note: Single family of homogeneous solution exists so maintaining y_1, y_2' linear-independence ($\because y_2$ cannot fulfill both conditions together while $W(y_1, y_2) \neq 0$) for applicability of Variation of Parameters suffices.

Sturm Comparison Theorem

Standard Form: $y'' + \frac{a_1}{a_0} y' + \frac{a_2}{a_0} y = 0$|Normal Form: $u'' + [\frac{a_2}{a_0} - \frac{1}{4}(\frac{a_1}{a_0})^2 - \frac{1}{2}(\frac{a_1}{a_0})']u = 0$ ($\because$ Plug $u = ye^{\frac{1}{2}\int \frac{a_1}{a_0} dx}$)
$(\exists q_1(x), q_2(x))(\forall x \in [a, b] \subseteq \mathbb{R}(q_1(x) \leq q_2(x)) \Rightarrow y'' + q_1 y = 0$ is **Sturm-minorant**$\wedge y'' + q_2 y = 0$ is **Sturm-majorant**.
Sturm Comparison Theorem: Let $y_1 \neq 0$ be Sturm-minorant's solution$\wedge y_2 \neq 0$ be Sturm-majorant's solution, then ($\forall$ consecutive zeroes $x_L < x_H$ of y_1)$(\exists$zero $x_c \in [x_L, x_H]$ of y_2)
$\therefore q_1 \equiv q_2$ in I $\Rightarrow$ **Sturm Separation Theorem**: zeroes of y_1, y_2 coincide iff dependent, else alternate iff independent.
Sturm-Separation Theorem's Corollaries
I bdd$\Rightarrow$ y oscillatory in I iff $y \equiv 0$ ($\because$ Bolzano-Weierstrass theorem)|I unbdd$\wedge y_2 \neq 0$ oscillatory $\Rightarrow y_1 \neq 0$ oscillatory
$\therefore q_1 \leq 0 \Rightarrow y_1$ has ≤ 1 zero ($\because$ Take Sturm-majorant $y_2'' = 0$)
$\therefore \frac{a_1}{a_0}, \frac{a_2}{a_0}$ continuous on closed $I = [a, b] \wedge \frac{a_2}{a_0} < 0)(\forall y \neq 0)(y$ has ≤ 1 zero in I) ($\because$ Rolle's Theorem in Standard Form)
$\therefore (\forall q$ monotonically-increasing on $[0, \infty))(\forall u \neq 0)$(zero gap monotonically-decreasing) ($\because$ Sturm-Comparison Theorem on $\omega(x + x_{y+1} - x_t) = u(x)$ in Normal Form)
$\therefore$ Prove y oscillatory on unbdd $I \Leftarrow$ Make DE Sturm-majorant+Find Sturm-majorant's oscillatory solution
$\therefore$ Prove y has $\leq k$ zeroes on any $I \Leftarrow$ Make DE Sturm-minorant+Find Sturm-majorant with $\leq (k-1)$zeroes

C4: 1st-order Linear Differential System
Heuristics to solving associated homogeneous system $\vec{x}' = A(t)\vec{x}$
1. Compute eigenvalues of $n \times n$ matrix A as $\lambda_1 \cdots \lambda_k$ with algebraic multiplicities $m_1 \cdots m_k$ ($\therefore \sum m = n$) from characteristic equation $\det(A - \lambda I_n) = 0$. $\therefore m_i = 1 \Rightarrow \lambda_i$ simple.
2. $\forall i \in [1, k]$ Compute geometric multiplicity $= \dim(\text{Null}(A - \lambda_i I_n)) = $ (linearly-independent solutions' count to $(A - \lambda_i I_n)\vec{x} = \vec{0})$ = (free variables' count in $\text{RREF}(A - \lambda_i I_n))$
$\therefore m_i > 1 \wedge$geometric multiplicity=algebraic multiplicity$\Rightarrow \lambda_i$ quasi-simple
3. $\forall$ simple/quasi-simple λ_i, solve for m_i eigenvectors in $\vec{v}$ to $(A - \lambda_i I_n)\vec{v} = \vec{0}$, then $\vec{x} = e^{\lambda_i t}\vec{v}$.
Else, Cayley-Hamilton Theorem: $p_A(\lambda) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I_n = \vec{0} \Rightarrow \forall i \in [1, k](A - \lambda_i I_n)^{m_i}\vec{v} = \vec{0}$ has exactly m_i linearly-independent solutions in $\vec{v}$, hence $\vec{x}$, attributable to it.
Crucially, **Construction**: $\forall j \in [1, m_i](\vec{x_{ij}}(t) = e^{\lambda_i t}\sum_{k=0}^{m_i-1}\frac{t^k}{k!}(A - \lambda_i I_n)^k \vec{v_{ij}}(t))$
Non-failsafe shortcut
WLOG pick arbitrary $\vec{v_{ij}} \neq \vec{0}$. Let $S = \{m \leq m_i | (A - \lambda_i I_n)^m \vec{v_{ij}} = \vec{0} \wedge (A - \lambda_i I_n)^{m-1}\vec{v_{ij}} \neq \vec{0} \wedge m \in \mathbb{N}^+\}$. If $S = \emptyset$, shortcut fails. Else let $r = \max\{S\}$, then $\vec{v_{ij}}$ alone offers r (vs 1) linearly-independent solutions in $\vec{x}$.
Crucially, **Construction**: $\forall k \in [1, r](\vec{x_{ijk}}(t) = e^{\lambda_i t}\sum_{\ell=k-1}^{r-1}\frac{t^{\ell-k+1}}{(\ell-k+1)!}(A - \lambda_i I_n)^\ell \vec{v_{ij}}(t))$
Particular solution $\vec{x_P}$ given $\vec{x}' = A(t)\vec{x} + \vec{g}(t)$
$\because$ Fundamental matrix $\Phi(t) = \begin{pmatrix} | & \cdots & | \\ \vec{x_1}(t) & \cdots & \vec{x_n}(t) \\ | & \cdots & | \end{pmatrix}$ invertible $\therefore$ Variation of parameters: $\vec{x_P}(t) = \Phi(t)\int_{t_0}^t \Phi^{-1}(s)\vec{g}(s)ds$

Phase Plane Diagram of 2×2 homogeneous system $\vec{x}'(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{x}(t), \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

Terminologies: (x_c, y_c) critical$\Leftrightarrow ax_c + by_c = cx_c + dy_c = 0$. $\therefore (0, 0)$ critical. Node=approached & entered critical point
Stability$\Leftrightarrow \forall R > 0 \exists \epsilon \in (0, R]\forall$integral curve $C_1(\exists t_0)(C_1 \cap N_\epsilon(t_0) \neq \emptyset \Rightarrow C_1 \subset N_R(t_0))$.
Asymptotic stability$\Leftrightarrow$stability$\wedge \exists R_{cr} > 0 \forall C_1(\exists t_0(C_1 \cap N_{R_{cr}}(t_0) \neq \emptyset \Rightarrow C_1$ tends to critical point as $t \to \infty)$
Proper node$\Leftrightarrow (\forall$direction through node)$(\exists C_1)$ ($\because$ necessary cond: critical point is node)| $\therefore$ Saddle, spiral centre aren't.
Cayley-Hamilton Method
$A^{-1} = \frac{1}{\det(A)}[\text{tr}(A)I_n - A]$ where trace scalar $\text{tr}(A) = \sum a_{ii}$. $\therefore n = 2 \Rightarrow A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Right Column

Case 1a: λ distinct real positive: Improper unstable node, Half-lines: $y = \frac{A_1}{B_1}x, y = \frac{A_2}{B_2}x$
Case 1b: λ distinct real negative: Improper asymptotically-stable node, same Half-Lines↑
Case 2: λ distinct real opposite signs: Unstable saddle point, same Half-Lines↑
Case 3a: λ real equal positive: Unstable node, Half-Lines: $y = \frac{A}{B}x$ ($\because$ no-t term), $y = \frac{A_1}{B_1}x$ only if λ not quasi-simple
Case 3b: λ real equal negative: Asymptotically-stable node, same Half-Lines↑. In both Cases, proper iff λ quasi-simple
Case 4: $\lambda = \pm i\omega$ purely-imaginary: Stable conic centre, $\vec{v}(t) = c_1\cos(\omega t)\vec{v_1} + c_2\sin(\omega t)\vec{v_2}$
Case 5a: λ complex, $\mathbb{Re}(\lambda) > 0$: Unstable div spiral| **Case 5b**: λ complex, $\mathbb{Re}(\lambda) < 0$: Asymptotically-stable conv spiral

C5: Power Series
Analysis of 2nd-order Linear DEs
$y'' + p(x)y' + q(x)y = 0 \Leftrightarrow \frac{d^2 y}{dt^2} + (\frac{2}{t} - \frac{1}{t^2}p(\frac{1}{t}))\frac{dy}{dt} + q(\frac{1}{t})y = 0$ ($\because$ Plug $xt = 1$)
Differentiability: Let open $U \subseteq \mathbb{C}. \forall f : U \to \mathbb{C}$(fcomplex-differentiable in $z_0 \in U \Leftrightarrow \lim_{z \to z_0}\frac{f(z)-f(z_0)}{z-z_0} \in \mathbb{C}$
$\therefore$ fcomplex-differentiable (aka holomorphic) in $U \Leftrightarrow \forall z_0 \in U$(fcomplex-differentiable in z_0)|f holomorphic in open $V \subseteq U \Leftrightarrow \forall z_0 \in V$(fcomplex-differentiable in z_0)
Analyticity: f analytic in $z_0 \in U \Leftrightarrow (\exists \epsilon > 0)(\exists S(z) = \sum_0^\infty a_n(z - z_0)^n)(\forall z \in N_\epsilon(z_0))(f \equiv S), \epsilon =$radius of convergence
$\therefore$ fanalytic in open $V \subseteq U \Leftrightarrow \forall z \in V$(fanalytic in z)|Analyticity$\Rightarrow$Differentiability ($\because S(z)$ differentiable), converse fails
Eg: $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \forall x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ has Taylor series $T \equiv 0$, not $f(x)$, despite f infinitely-differentiable.
x_0 ordinary$\Leftrightarrow p(x_0), q(x_0)$analytic|$x_0$ regular-singular $\Leftrightarrow x_0$ non-ordinary$\wedge(x - x_0)p(x)$analytic$\wedge(x - x_0)^2 q(x)$analytic
If $x_0 \to \pm\infty$, equivalently check $t = 0$ in above transformed DE.
Uniqueness construction to 2nd-order linear IVP $y'' + p(x)y' + q(x)y = 0, y'(x_0) = a_1, y(x_0) = a_0$
Necessary condition: x_0 ordinary point$\Rightarrow p(x) = \sum_0^\infty p_n(x - x_0)^n \wedge q_x = \sum_0^\infty q_n(x - x_0)^n \forall |x - x_0| \leq R$
$\therefore$ Seek $y = \sum_0^\infty a_n(x - x_0)^n$, **Recurrence Relation** on $a_0, a_1: \forall n \in \mathbb{N}_0(n+1)(n+2)a_{n+2} = -\sum_{k=0}^n[(k+1)p_{n-k}a_{k+1} + q_{n-k}a_k]$
Method of Frobenius to 2nd-order linear DE $x^2 y'' + xp(x)y' + q(x)y = 0$
Necessary condition: $x_0 = 0$ regular-singular point$\Rightarrow y = x^r \sum_0^\infty a_n x^n, a_0 \neq 0$.
$\therefore a_0$ - omitted x^r coefficient $= r(r-1) + rp_0 + q_0 = 0$ ($\because$ Indicial Equation)
Recurrence Relation: $\forall n \in \mathbb{N}_0 a_n[(r+n)(r+n-1) + (r+n)p_0 + q_0] = -\sum_{k=0}^{n-1}a_k[(r+k)p_{n-k} + q_{n-k}]$
Analysis: $r_1 = r_2 \vee (r_1 - r_2 \in \mathbb{Z} \wedge \sum_{k=0}^{n-1}a_k[(r+k)p_{n-k} + q_{n-k}] \neq 0) \Rightarrow \exists$exactly 1 (larger $\mathbb{Re}(r)$'s) Frobenius solution, solve 2nd by variation of parameters. Else guaranteed 2. If $r \notin \mathbb{R}$, take $x^r = x^\alpha e^{i\ln\beta}$ and omit imaginary coeffs only in last step

C6: Fundamental Theory of ODEs
Lipschitz Property
$f(t, x) : U \to \mathbb{R}$ meets Lipschitz cond with Lipschitz const $L \Rightarrow \exists L > 0(\forall(t, x_1), (t, x_2) \in U)(|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|)$
Picard-Lindelof Theorem
Rectangular $D_f = \{(t, x)|t_0 - \alpha \leq t \leq t_0 + \alpha, x_0 - \beta \leq x \leq x_0 + \beta\} \wedge |f(t, x)| \leq M$ bounded $\wedge$f meets Lipschitz cond with Lipschitz c $L \Rightarrow$ IVP $\frac{dx}{dt} = f(x, t), x(t_0) = x_0 \Leftrightarrow x(t) = x_0 + \int_{t_0}^t f(t, x)dt$ has unique sol in $[t_0 - \omega, t_0 + \omega], \omega = \min\{\alpha, \frac{\beta}{M}\}$
Crucially, **Successive Approximation Construction**: $\phi_0(t) = x_0$ const, $\forall k \in \mathbb{N}_0(\phi_{k+1}(t) = \phi_0(t) + \int_{t_0}^t f(s, \phi_k(s))ds)$
$\therefore \forall k \in \mathbb{N}_0|\phi_k(t) - \phi_0(t)| \leq M|t - t_0| \Rightarrow M =$Lipschitz c of $\phi_k =$asymptote $x - x_0 = M(t - t_0)$'s slope in D_f for $\{\phi_k\}_{k \in \mathbb{N}_0}$
$\therefore$ 1. Generalising D_f to open-strip $S = \{(t, x)|t_0 - \alpha \leq t \leq t_0 + \alpha\}$, $f(t, x)$ is not necessarily bounded. Discarding bdd condition, IVP still has unique sol on S similarly |2. Further generalising D_f to $\mathbb{R}^2$, IVP has unique sol on $\mathbb{R}$ w.r.t. t.
Gronwall Inequality Derivation
Necessary conditions: f, g, h continuous non-negative $\forall t \geq t_0$
$\therefore$ 1st Inequality: $f(t) \leq h(t) + \int_{t_0}^t g(s)f(s)ds \Rightarrow f(t) \leq h(t) + \int_{t_0}^t g(s)h(s)e^{\int_s^t g(u)du}ds$
$\therefore$ Gronwall Inequality: If $h(t) = k, f(t) \leq k + \int_{t_0}^t g(s)f(s)ds \Rightarrow f(t) \leq ke^{\int_{t_0}^t g(s)ds}$| $\therefore \forall t \leq t_0(f(t) \leq ke^{\int_{t_0}^t g(s)ds}) \Rightarrow f(t) \equiv 0$
$\therefore \forall$ successive approximation constructions $\phi(t), \varphi(t), \phi \equiv \varphi$. $\therefore$ Generalise D_f to S, $|\phi(t) - \varphi(t)| \leq \epsilon\omega e^{L\alpha} \to 0^+$ as $\epsilon \to 0^+$
Existence and Uniqueness Theorem to Linear System
$\forall i \in [1, n](x_i' = f_i(t, x_1 \cdots x_n) \Leftrightarrow \vec{x}' = f(t, \vec{x})$. Then, $f(t, \vec{x}) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ fulfills Lipschitz condition w.r.t. x in $\mathbb{R}^{n+1} \Leftrightarrow \exists L > 0|f(t, \vec{x}) - f(t, \vec{y})| \leq L|\vec{x} - \vec{y}|$ | Metric independent of Lipschitz. MA3220 opts Manhattan dist $|\vec{x}| = \sum_{i=1}^n |x_i|$

C7: MF26 Formulas
Integration by Reduction Formula (assuming $n \in \mathbb{N}^+, ab \neq 0$)
$I_n = \int \sin^n(ax)dx \Rightarrow anI_n = a(n-1)I_{n-2} - \sin^{n-1}(ax)\cos(ax)$|$I_n = \int \csc^n(ax)dx \Rightarrow (n-1)I_n = (n-2)I_{n-2} - \frac{\cos(ax)}{a\sin^{n-1}(ax)}$
$I_n = \int \cos^n(ax)dx \Rightarrow anI_n = a(n-1)I_{n-2} + \cos^{n-1}(ax)\sin(ax)$|$I_n = \int \sec^n(ax)dx \Rightarrow (n-1)I_n = (n-2)I_{n-2} + \frac{\sin(ax)}{a\cos^{n-1}(ax)}$
$I_n = \int e^{ax}\sin^n(bx)dx \Rightarrow I_n = \frac{n(n-1)b^2}{a^2+b^2n^2}I_{n-2} + \frac{e^{ax}\sin^{n-1}(bx)}{a^2+b^2n^2}(a\sin(bx) - bn\cos(bx))$ $\therefore$ Formula for $\int \sin^n(bx)dx$ follows.
$I_n = \int e^{ax}\cos^n(bx)dx \Rightarrow I_n = \frac{n(n-1)b^2}{a^2+b^2n^2}I_{n-2} + \frac{e^{ax}\cos^{n-1}(bx)}{a^2+b^2n^2}(a\cos(bx) + bn\sin(bx))$ $\therefore$ Formula for $\int \cos^n(bx)dx$ follows.
$I_n = \int x^n e^{ax}dx \Rightarrow I_n = -\frac{n}{a}I_{n-1} + \frac{x^n e^{ax}}{a}$|$I_n = \int x^{-n}e^{ax}dx \Rightarrow I_n = \frac{a}{n-1}I_{n-1} + \frac{-e^{ax}}{(n-1)x^{n-1}}\forall n \in \mathbb{N}^+\backslash\{1\}$
Some MF26 Formula [Integration by Parts *Descending* Preference: LiATE]
$\int \frac{1}{\sqrt{(x+b)^2 \pm a^2}}dx = \ln|(x+b) + \sqrt{(x+b)^2 \pm a^2}| + c$ ($\because$ Substitute $x = a\tan\theta + b, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$)
$\int \frac{1}{\sqrt{a^2-(x+b)^2}}dx = \sin^{-1}(\frac{x+b}{a}) + c$ | $\int \frac{1}{a^2+(x+b)^2}dx = \frac{1}{a}\tan^{-1}(\frac{x+b}{a}) + c$ | $\cosh x = \frac{e^x+e^{-x}}{2} \leq e^{\frac{x^2}{2}}$ by Taylor expansion
$\int \sin^{-1}(x)dx = x\sin^{-1}(x) + \sqrt{1-x^2} + c$ ($\because \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}(x)\forall x \in [-1, 1]$) | $\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + c$
$\int \sec^{-1}(x)dx = x\sec^{-1}(x) - \ln|x + x\sqrt{1 - \frac{1}{x^2}}| + c$ | $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$ $(\forall x \in (-a, a))(\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}(\frac{x}{a}) + c)$
$(\forall x > a)(\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln|\frac{x-a}{x+a}| + c)$ $(\forall x \in (-a, a))(\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln|\frac{a+x}{a-x}| + c)$