How to Avoid Singularity For Euler Angle Set?

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In this paper, an algorithm is proposed to avoid singularity associated with famous minimum element attitude parametrization, Euler angle set. The proposed algorithm makes use of method of sequential rotation to avoid singularity associated with Euler angle set. Further, a switching algorithm is also proposed to switch between different Euler angle sets to avoid the singularity while integrating the kinematic equations corresponding to Euler angles for spacecraft motion. The algorithm is numerically validated by simulation tests.

1   Introduction

Spacecraft attitude determination is the process of estimating the orientation of a spacecraft by making on-board observations of other celestial bodies or reference points, with respect to some reference frame. Attitude parameters are the set of parameters that completely describes the orientation of the spacecraft with respect to a given reference frame, as for instance the inertial reference frame. Leonard Euler in 1775 has shown [1, 2] that the configuration of a rigid body can be fully defined by locating a cartesian set of coordinate fixed in rigid body (called body frame) relative to some inertial coordinate axes. Three parameters are needed to define the origin of body frame and another three parameters are needed to specify the orientation of body frame with respect to external inertial frame. That's why, generally two co-ordinate systems are defined for attitude determination process: one on the vehicle body called the body frame and second is the inertial reference frame. For most problems, the reference frame is a non-moving an inertial frame fixed to the center of Earth. The projection of the body frame axes onto the image frame axes

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is given by an orthogonal matrix ($\mathbf{C}$) called the Direction Cosine Matrix (DCM) or orientation matrix or attitude matrix. A large number of parameterizations[3] are listed in the literature for attitude matrix. But Euler Angles (EA), Euler's principal axis and angle, Euler-Rodrigues Symmetric Parameters (ERSP) or quaternions, Rodrigues Parameters (RP), and Modified Rodrigues Parameters (MRP), are the most popular ones. Every parametrization has its own advantages and disadvantages. The main properties of some classical parameterizations of attitude rotational matrix are listed in Table 1:

Table 1: Classical Parameterizations of Attitude Rotation Matrix

Paramet-rization	Dimension	Attitude Matrix	Kinematic Equations	Singularities	Constraints
DCM, (C_{ij})	9	$\mathbf{C} = [C_{ij}]$	$\dot{\mathbf{C}} = -[\tilde{\boldsymbol{\omega}}]\mathbf{C}$	None	$\mathbf{C}^T\mathbf{C} = \mathbf{I}$
EA (θ_i)	3	$\mathbf{C} = \begin{bmatrix} \text{transcendental} \\ \text{functions of} \\ \theta_i's \end{bmatrix}$	$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \text{transcendental} \\ \text{functions of} \\ \theta_i's \end{bmatrix}\boldsymbol{\omega}$	$\theta_2 = \pm\frac{\pi}{2}$	None
ERSP (q_i)	4	$\mathbf{C} = \begin{bmatrix} \text{algebraic} \\ \text{functions of} \\ q_i's \end{bmatrix}$	$\dot{\mathbf{q}} = \begin{bmatrix} \text{linear} \\ \text{functions of} \\ q_i's \end{bmatrix}\left\{\begin{matrix} 0 \\ \boldsymbol{\omega} \end{matrix}\right\}$	None	$\mathbf{q}^T\mathbf{q} = 1$
RP (r_i)	3	$\mathbf{C} = \begin{bmatrix} \text{quadratic} \\ \text{functions of} \\ r_i's \end{bmatrix}$	$\dot{\mathbf{r}} = \begin{bmatrix} \text{non-linear} \\ \text{functions of} \\ r_i's \end{bmatrix}\boldsymbol{\omega}$	$\phi = \pm\pi$	None
MRP (σ_i)	3	$\mathbf{C} = \begin{bmatrix} \text{quartic} \\ \text{functions of} \\ \sigma_i's \end{bmatrix}$	$\dot{\boldsymbol{\sigma}} = \begin{bmatrix} \text{non-linear} \\ \text{functions of} \\ \sigma_i's \end{bmatrix}\boldsymbol{\omega}$	$\phi = \pm2\pi$	None

From Table 1, we can infer that all the minimal sets of parameterizations of attitude rotation matrix have some non-linearity associated with their differential kinematic equations and some kind of geometric singularity except two redundant set of parameterizations, one is attitude matrix and the other is the ERSP. Both of them have linear and universally applicable differential kinematic equations. The attitude matrix has 6 redundant parameters whereas the quaternion has only one redundant parameter, which makes it more preferable with respect to all the other representation of attitude.

However, the two main reasons that blessed the quaternion as the king of attitude parametrization, namely the fact that subsequent rotations do not involve use of transcendental functions *and* the fact that all the minimum parametrization of attitude present singularity, both of them appear to fall. The reason relies on the fact that the problem of avoiding the use of transcendental functions belongs to the time when

these functions were computed by series expansions, which is, indeed, computationally expensive while now they are much more efficiently evaluated by look-up tables.[§] The second reason comes out from the capability of the Shuster's Method of Sequential Rotations[4] (MSR) to avoid the *un-avoidable* singularity affecting all the minimum attitude parameter. Not only the original QUEST[4] algorithm has taken advantage from the MSR technique, but also some recent attitude determination approaches, like ESOQ2[5] and OLAE[6].

The attention to the MSR has increased not only in the applications, but also the complete theory of it has been developed[7]. For these two reason the MSR deserves more attention than that given.

The basic idea of the method of sequential rotation lies in performing an artificial rotation $\mathbf{R}$ of the n observed directions $\mathbf{b}_i$ or of the associated reference directions $\mathbf{r}_i$. This, in general, would vary the value of computed attitude. If the original observed directions $\mathbf{b}_i$ correspond to the attitude matrix $\mathbf{C}$ and $\mathbf{R}$ represents the sequential rotation matrix, then the computed attitude parameter set will correspond to the attitude matrix $\mathbf{C}_R = \mathbf{C}\mathbf{R}^\mathrm{T}$. Thus, if the original data set is such that the attitude estimation algorithm is singular, then the rotated data set can be defined in such a way that it is not singular. It looks very simple but main problem is how to find out that when particular attitude parameter set is going to encounter a singularity at a particular instant and *"how to get the original attitude parameter set?"*. In reference [7], this problem has been discussed in detail for Rodrigues and modified Rodrigues parameters. However, the process of sequential rotation do not work for Euler angle set as the process of finding back the original Euler angle parameter is very tedious and generally do not provide any simple, compact expression. But there is always an immense interest in Robotics and spacecraft control problems to parameterize the attitude matrix in terms of Euler angle sets as they provide most probably the easiest geometric recipe to describe the motion of any object with respect to a reference frame.

In this paper, we present an efficient algorithm to estimate the spacecraft attitude in terms of non-singular Euler angle set. First, we present a procedure to detect the singular Euler angle set from vector observations only and then present an approach to estimate the non-singular Euler angle set from these vector observations by solving a *Generalized Wahba Problem* [8]. The procedure presented in this paper provides a rigorous linearization of attitude estimation problem, if three or more vector observations are available, and can be used as a starting algorithm for any nonlinear estimation algorithm.

[§]A look-up table technique consists of evaluating a function just by interpolation between two close pre-computed values. For instance, in MATLAB, only one floating point operation is required to evaluate $\sin x$.

2 Euler Angle Sets

In this section, we list a procedure to find which Euler angle set is singular based upon vector observations. But before discussing the algorithm, we give a brief introduction to Euler angle sets.

Euler angles are one of the most commonly used sets of attitude parameters. They describe the attitude of reference frame relative to inertial frame by three successive rotation angles $(\theta_1, \theta_2, \theta_3)$ about the body fixed axes. The angles $(\theta_1, \theta_2, \theta_3)$ are called the Euler angles. The order of the axes about which the reference frame is rotated is important here. These successive rotation provide an instantaneous geometrical recipe for inertial frame therefore for a reference frame undergoing a general motion, the Euler angles are time varying.

Since all rotations are performed about the principal axes of the reference frame, we define $\mathbf{M}_i = \exp\left(-[\tilde{e}_i]\theta_i\right)$ as an elementary rotation matrix about the $\mathbf{e}_i$-body axis. Here, $[\tilde{e}_i]$ represents the skew-symmetric cross product matrix given by the following expression:

$$[\tilde{\mathbf{a}}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{1}$$

From the expression of $\mathbf{M}_i$, we can construct the following three elementary rotation matrices:

$$\mathbf{M}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

$$\mathbf{M}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{3}$$

$$\mathbf{M}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

if i,j, and k, indicate the coordinate axes about which each subsequent rotation is performed, that is, they can be any integer from 1-3, provided that $i \neq j$ and $j \neq k$, are satisfied then the resultant direction cosine matrix can be written as

$$\mathbf{C}_{ijk}(\theta_1, \theta_2, \theta_3) = \mathbf{M}_k(\theta_3)\,\mathbf{M}_j(\theta_2)\,\mathbf{M}_i(\theta_1) \tag{5}$$

In this way it is possible to define twelve sequences of Euler angles, associated with 12 attitude matrices, denoted by $\mathbf{C}_{ijk}$. Accordingly with the chosen Euler angle sequence, we may distinguish the "symmetric set" $\mathbf{C}_{iji}$ (when two of the three rotations occur about the same axis, and the "asymmetric sets" $\mathbf{C}_{ijk}$ with $i \neq j \neq k$, when all the three rotations occur about three distinct axes.

The Euler angle sets represents the minimum parameters attitude parametrization therefore they all have the disadvantage of being singular but they all are easy to visualize which makes them popular for many attitude determination problems. For the asymmetric Euler angle sets $\theta_2 = \pm\frac{\pi}{2}$ results in geometric singularity. For the symmetric Euler angle sets the geometric singularity occurs when $\theta_2 = 0$ or $\pm\pi$ radians. We mention that all Euler angle sets encounter singularity for $\theta_2 = \pm\pi$ or $\theta = \pm\frac{\pi}{2}$ only. This geometric singularity further induces the singularity in the corresponding Euler angle kinematic differential equations.

It is a fundamental topological fact that singularities can never be eliminated in any 3-dimensional representation of orientation. But we can avoid this singularity by describing the attitude at a particular instant by the Euler angle set *which is farthest away from singularity*. In this paper, we present an algorithm to switch between different sets of Euler angles to avoid this singularity.

3   How to Detect Singularity?

In the previous section, we have mentioned that the singularity in all "symmetric sets" (*i-j-i*), of Euler angle corresponds to $\theta_2 = 0$ or $\theta_2 = \pm\pi$ which further correspond to elementary rotation matrix $\mathbf{M}_j = \mathbf{I}_{3\times3}$. Geometrically, this singularity corresponds to rotation about single axis $\mathbf{e}_i$ by an angle $\theta_1 + \theta_3$. Further, if we look at the elements of the attitude matrix then this singularity can be described by the diagonal element, C_{ii} being ±1. For example the direction cosine matrix for Euler angle set 1-2-1 is given by [9]:

$$\mathbf{C}_{121} = \begin{bmatrix} c\theta_2 & s\theta_2 s\theta_1 & -s\theta_2 c\theta_1 \\ s\theta_3 s\theta_2 & -s\theta_3 c\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 c\theta_1 + c\theta_3 s\theta_1 \\ c\theta_3 s\theta_1 & -c\theta_3 c\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 c\theta_1 - s\theta_3 s\theta_1 \end{bmatrix} \tag{6}$$

When $\theta_2 = 0$ or $\pm\pi$, C_{11} element of the attitude matrix, $\mathbf{C}_{121}$ becomes ±1. Therefore, for a general symmetric Euler angle set *i-j-i* the singularity $\theta_2 = 0, \pm\pi$ corresponds to $C_{ii} = \pm1$. That is why none of the symmetric Euler angle set can be used to describe the identity rotation i.e. $\mathbf{C} = \mathbf{I}$.

Similarly, in case of "asymmetric" Euler angle set, "*i-j-k*" ($i \neq j \neq k$), the singularity corresponding to $\theta_2 = \pm\frac{\pi}{2}$ that is described by the following expression for elementary rotation matrix $\mathbf{M}_j$:

$$\mathbf{M}_j = exp(\mp[\tilde{e}_j]\frac{\pi}{2}) \tag{7}$$

Using Rodrigues' formula[3, **?**], we can rewrite the following expression for $\mathbf{M}_j$:

$$\mathbf{M}_j = \mathbf{I} \pm [\tilde{e}_j] \tag{8}$$

It should be noticed that j^{th} row of matrix, $[\tilde{e}_j]$, is equal to zero due to the fact that $\mathbf{e}_j$ is a 3×1 unit vector with only j^{th} element being non-zero. Further, keeping in mind the structure of elementary rotation matrices $\mathbf{M}_i$ and $\mathbf{M}_k$, we can easily show that the kth row of attitude matrix $\mathbf{C}_{ijk} = \mathbf{M}_k \mathbf{M}_j \mathbf{M}_i$ is equal to $\pm e_i^{\mathrm{T}}$, and i^{th} column is equal to $\pm \mathbf{e}_k$, where, $k \neq i$. For example, the attitude matrix for Euler angle set 3-2-1 is given by the following expression:

$$\mathbf{C}_{321} = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix} \tag{9}$$

In above written expression for $\mathbf{C}_{321}$, it is easier to check that when $\theta_2 = \pm\frac{\pi}{2}$, the 1st row equals to $-e_1^{\mathrm{T}}$ and 3rd column is equal to $\mathbf{e}_1$.

Therefore, we can state following theorem to capture the singularity condition for a general Euler angle set:

Singularity Condition. *If "i-j-k" denotes the given Euler angle sequence then it is singular iff the kth row of attitude matrix $\mathbf{C}$ is equal to $\pm e_i^{T\dagger}$ and the ith column of attitude matrix is equal to $\pm e_k$. Further, if $\mathbf{b}$ and $\mathbf{r}$ represents the body frame and inertial frame reference vector observations then the "i-j-k" Euler angle sequence is singular iff the r_i component of inertial reference vector $\mathbf{r}$ is equal to the b_k component of body frame reference vector $\mathbf{b}$.*

Proof. The proof of the first part of this theorem follows from the discussion in last section while the proof of second part follows from the fact that $\mathbf{b} = \mathbf{Cr}$. $\qquad\square$

As a consequence of this theorem and the fact that attitude matrix, $\mathbf{C}$, is an orthonormal matrix, the singularity condition for "i-j-k" Euler angle set can be described by C_{ki} element of attitude matrix, $\mathbf{C}$, being ± 1. Table 2 lists all the singular Euler angle sets when $C_{ij} = \pm 1$.

4 How to Avoid Singularity

It is a fundamental topological fact that singularities can never be eliminated in any 3-dimensional representation of orientation. However, in this section, we discuss two approaches to avoid the singularity in any Euler angle sequence "i-j-k".

The first approach is based upon the method of Sequential rotations [7] as discussed in section 1. According to MSR, the singularity in any minimal parametrization of attitude can be avoided by artificially composing the true attitude matrix, $\mathbf{C}$ with a known rotation matrix, $\mathbf{R}$. If the true attitude matrix $\mathbf{C}$, the artificial sequential

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$^{\dagger}e_i \in \mathcal{R}^3$ is a unit vector with only i^{th} element being non-zero

Table 2: C_{ij} and Corresponding Singular Euler angle Set

$C_{ij} = \pm 1$	Singular Euler Angle Set	Relationship between θ_2 and C_{ij}
C_{11}	1-2-1, 1-3-1	$\theta_2 = \cos^{-1} C_{11} = \pm\pi, 0$
C_{12}	2-3-1	$\theta_2 = \sin^{-1} C_{12} = \pm\frac{\pi}{2}$
C_{13}	3-2-1	$\theta_2 = \sin^{-1} C_{13} = \pm\frac{\pi}{2}$
C_{21}	1-3-2	$\theta_2 = \sin^{-1} C_{21} = \pm\frac{\pi}{2}$
C_{22}	2-1-2, 2-3-2	$\theta_2 = \cos^{-1} C_{22} = \pm\pi, 0$
C_{23}	3-1-2	$\theta_2 = \sin^{-1} C_{23} = \pm\pi, 0$
C_{31}	1-2-3	$\theta_2 = \sin^{-1} C_{31} = \pm\frac{\pi}{2}$
C_{32}	2-1-3	$\theta_2 = \sin^{-1} C_{32} = \pm\frac{\pi}{2}$
C_{33}	3-1-3, 3-2-3	$\theta_2 = \cos^{-1} C_{33} = \pm\pi, 0$

rotation matrix $\mathbf{R}$, and the computed attitude matrix $\mathbf{C}_R$ correspond to the Euler angle vectors Θ, Φ, and Θ_R, respectively, then the problem of finding the original Euler angle vector corresponds to find Θ with Φ and Θ_R assigned. We mention that this process is very tedious and generally do not provide any simple, compact expression [9]. However, for the case when the Euler angle set is symmetric, then we can describe Θ in terms of Φ and Θ_R using spherical geometry relationships [9, 10]:

$$
\begin{aligned}
\Theta_1 &= -\Phi_3 + \tan^{-1}\left(\frac{\sin\Phi_2 \sin\Theta_{R_2} \sin(\Theta_{R_1} - \Phi_1)}{\cos\Phi_2 \cos\Theta_2 - \cos\Theta_{R_2}}\right) \\
\Theta_2 &= \cos^{-1}\left(\cos\Phi_2 \cos\Theta_{R_2} + \sin\Phi_2 \sin\Theta_{R_2} \cos(\Theta_{R_1} - \Phi_1)\right) \qquad (10)\\
\Theta_3 &= \Theta_{R_3} - \tan^{-1}\left(\frac{\sin\Phi_2 \sin\Theta_{R_2} \sin(\Theta_{R_1} - \Phi_1)}{\cos\Phi_2 - \cos\Theta_{R_2} \cos\Theta_2}\right)
\end{aligned}
$$

But, unfortunately, in case of "asymmetric" Euler angle sets, no such expression exist to our knowledge. Apparently, the method of Sequential rotation does not add a useful mechanism to avoid the singularity while integrating the Euler angle kinematic equations.

In the second approach, the fact that elements of direction cosine matrix $\mathbf{C}$ are transcendental functions of Euler angles and therefore C_{ij} lies between -1 and 1 i.e. $\|C_{ij}\| \leq 1$, is used. For instance, Fig. 1 shows the plot of non-diagonal element C_{ij} vs θ_2 for "asymmetric Euler angle" set "j-k-i" and figure 2 shows the plot of diagonal element C_{ii} vs θ_2 for "symmetric" Euler angle set "i-k-i". The red lines in the figures corresponds to the singular points, $C_{ij} = \pm 1$, while the green lines corresponds to the points farthest away from singularity. From figure 1, it is clear that an "asymmetric" Euler angle set "j-k-i" will be farthest away from singularity if $\theta_2 = 0, \pm\pi$ i.e. $C_{ij} = 0$.

Similarly from figure 2, we can conclude that a "symmetric" Euler angle set "*i*-*k*-*i*" will be farthest away from singularity if $C_{ii} = 0$ i.e. $\theta_2 = \pm\frac{\pi}{2}$.
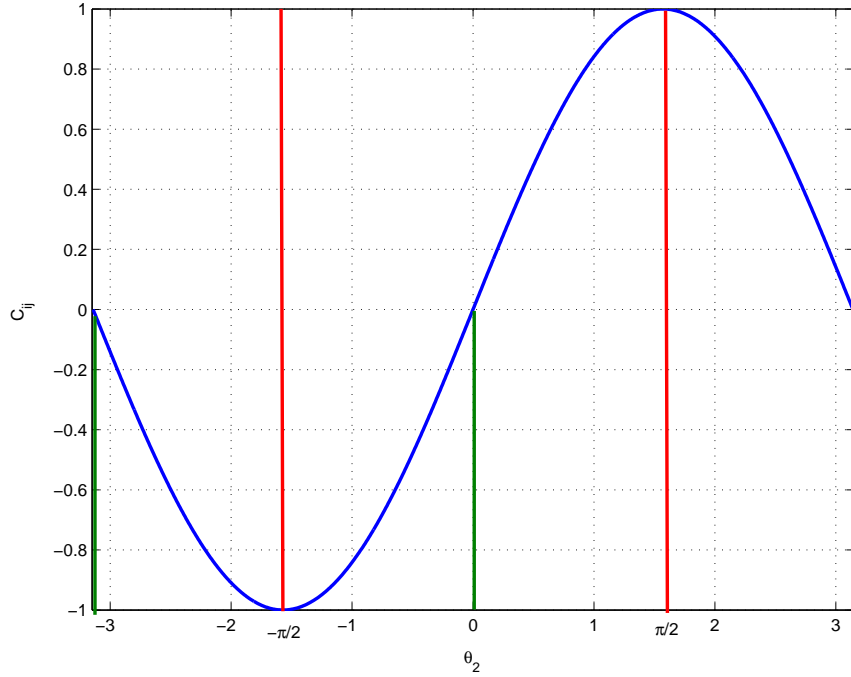

Figure 1: C_{ij} vs θ_2 for asymmetric Euler angle set $j - k - i$

Now, that means we should choose Euler angle set "3-2-1" if element C_{13} of attitude matrix, $\mathbf{C}$, is closest to zero at a particular instant of time. From equation (9), it is clear that it will happen only if $\theta_2 \approx 0, \pm\pi$. Further, from equation (9), we can see that this situation also corresponds to elements C_{11}, C_{12}, C_{23} and C_{33} being equal to ± 1. From table 2, we can conclude that this corresponds to Euler angle sets 1-2-1, 1-3-1, 2-3-1, 3-1-2, 3-1-3, and 3-2-3, being singular. Table 3 lists all the Euler angle sets which best describes the attitude when $C_{ij} = \pm 1$.

From table 3, it is clear that more than one Euler angle set are farthest away from singularity whenever a particular Euler angle set approaches singularity. In case of "asymmetric" Euler angle sets the singularity can be avoided by either switching to another "asymmetric" or "symmetric" Euler angle set however, in case of "symmetric" Euler angle set the singularity can not be avoided by just switching to another "symmetric" Euler angle set. In other words, the "symmetric" Euler angle sets can not be used to represent all possible attitude matrices. For example the identity rotation can not be parameterized by any "symmetric" Euler angle set. Therefore, if our purpose is just to avoid singularity, then we need to switch between an "asymmetric" and a "symmetric" or two "asymmetric" Euler angle sets but, if we want to make sure that at every instant we use the Euler angle set that is farthest away from singularity,
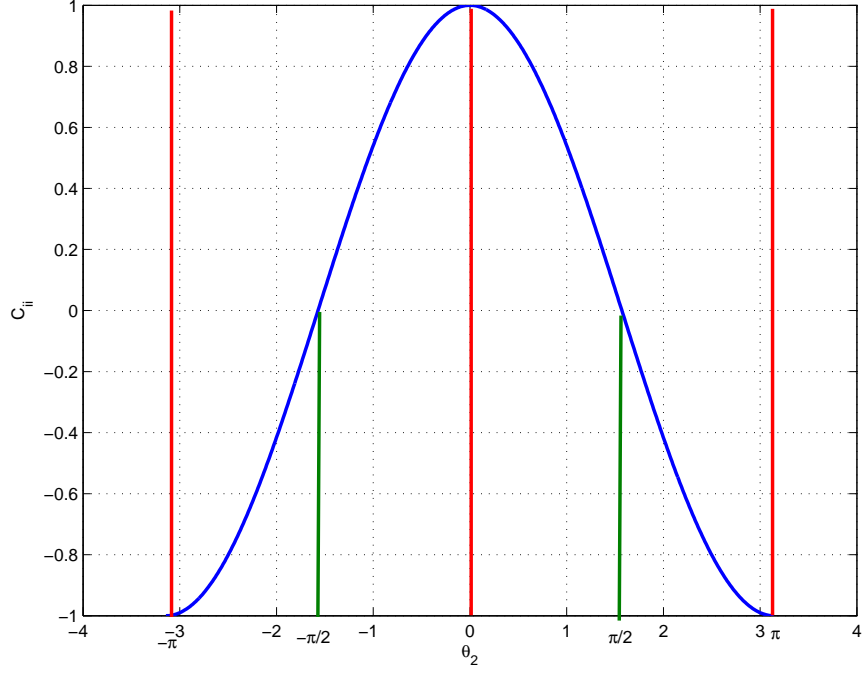
Figure 2: C_{ii} vs θ_2 for symmetric Euler angle set $i - j - i$

Table 3: C_{ij} and Corresponding Singular and Non-singular Euler angle Set

$C_{ij} = \pm 1$	Singular Euler Angle Set	Euler angle sets farthest away from singularity
C_{11}	1-2-1,1-3-1	2-3-1, 3-2-1, 1-3-2, 1-2-3
C_{12}	2-3-1	2-1-2, 2-3-2, 2-1-3, 1-3-2, 3-1-2, 3-2-1
C_{13}	3-2-1	1-2-1, 1-3-1, 2-3-1, 3-1-2, 3-1-3 3-2-3
C_{21}	1-3-2	2-1-2, 2-3-2, 3-1-2, 1-2-1, 1-3-1 3-2-1
C_{22}	2-1-2,2-3-2	3-1-2, 3-2-1, 1-3-2, 2-3-1
C_{23}	3-1-2	3-2-1, 3-1-3, 3-2-3, 2-1-2, 2-3-2, 1-3-2
C_{31}	1-2-3	3-1-3, 3-2-3, 2-1-3, 1-3-1, 1-2-1, 1-3-2
C_{32}	2-1-3	3-1-3, 3-2-3, 2-1-2, 2-3-2, 1-2-3, 2-3-1
C_{33}	3-1-3,3-2-3	2-1-3, 1-2-3, 3-1-2, 3-2-1

then we do need to switch between all the twelve Euler angle sets. For example, the singularity for 3-2-1 Euler angle set can be avoided by either switching to 1-2-3 or 3-1-3 Euler angle set and similarly, the singularity in the case of 1-2-3 or 3-1-3 Euler angle sets can be avoided by switching to 3-2-1 Euler angle set. Finally, we mention that switching between different Euler angle sets is also useful in integrating the corresponding Kinematic equations where the method of sequential rotation is not very useful.

5   Euler Angle Estimation

Due to various reasons discussed in section 1, there is an immense interest in parameterize the attitude matrix in terms of a non-singular Euler angles set. In this section, we present an efficient algorithm to estimate the non-singular Euler angle set from given set of vector measurements. Most of the single frame algorithms described in literature [11], are fully complying with the Wahba's [8] attitude optimality definition[¶] i.e. to find the orthogonal matrix $\mathbf{C}$ with determinant $+1$ that minimizes the following loss function:

$$J(\mathbf{C}) = \frac{1}{2} \sum_{i=1}^{N} a_i |\mathbf{b}_i - \mathbf{C}\mathbf{r}_i| \tag{11}$$

Where, a_i are non-negative weights, $\mathbf{b}_i$ and $\mathbf{r}_i$ are reference unit vectors measured in spacecraft and inertial frame respectively. Generally, non-negative weights, a_i, are normalized to unity but if the weights are chosen to be the inverse of measured error variance i.e. $a_i = \sigma_i^{-2}$ then the solution to Wahba problem is same as minimum variance solution.

Attitude estimation problem, as posed by Wahba, is nonlinear in nature irrespective of which attitude parameters are used, and generally, require the use of some iterative procedure to determine the attitude parameters of interest. The total number of iterations required to solve the Wahba problem within desired accuracy depend upon the initial guess of the attitude parameter of interest. Many closed form solutions [13, 4, 14, 15, 5] to Wahba's problem have been presented in literature but most of them are either in terms of quaternion or Rodrigues parameters. To our knowledge, no closed-form solution to Wahba's problem exist in terms of any Euler angle set. So it becomes really important to have a good initial guess for Euler angle set to solve the Wahba problem using some iterative methods. In this section, we present an efficient way to come up with the initial guess for particular Euler angle set of interest using some basic concepts of linear algebra.

[¶]With the word *optimal attitude* is usually intended an attitude satisfying the Wahba's definition of attitude optimality. To our knowledge, only the Euler-q algorithm[12] is based on a different definition of optimality.

Equation (11) represents a linear optimization problem in terms of the elements of the attitude matrix, $\mathbf{C}$ subject to nonlinear orthogonality constraint which can be restated as follows:

$$J(\mathbf{x}) = \frac{1}{2}|\mathbf{b} - \mathbf{A}\mathbf{x}|^2 \tag{12}$$

$$\mathbf{C}^{\mathrm{T}}\mathbf{C} = \mathbf{I} \tag{13}$$

Where, $\mathbf{b} = \left\{ \ \mathbf{b}_1^{\mathrm{T}}, \ \ldots, \ \mathbf{b}_N^{\mathrm{T}} \ \right\}^{\mathrm{T}} \in \mathcal{R}^{3N}$, $\mathbf{x} = \left\{ \ C_{11} \ \ C_{12} \ \ C_{13} \ \ \ldots \ \ C_{31} \ \ C_{32} \ \ C_{33} \ \right\}^{\mathrm{T}}$ and $\mathbf{A}$ is a $3N \times 9$ matrix given by following expression:

$$\mathbf{A} = \begin{bmatrix} \mathbf{r}_1^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{r}_1^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{r}_1^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{r}_N^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{r}_N^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{r}_N^{\mathrm{T}} \end{bmatrix} \tag{14}$$

The number of solutions to a linear system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be determined by looking at the row echelon form of augmented matrix formed by putting together $\mathbf{A}$ and $\mathbf{b}$, $\left[\ \mathbf{A} \ \vdots \ \mathbf{b} \ \right]$.

1. If $\mathbf{b}$ has a nonzero element where $\mathbf{A}$ has a row consisting entirely of zeros, then the system of equations is inconsistent.

2. If the system is consistent and all the variables are lead variables then there is exactly one solution.

3. If the system is consistent and some of the variables are free variables, then there are an infinite number of solutions.

Now for the attitude determination problem, it can be easily proved that if the vector observations are perfectly known, then for $N > 2$ the row echelon form of augmented matrix $\left[\ \mathbf{A} \ \vdots \ \mathbf{b} \ \right]$ is given by $\begin{bmatrix} \mathbf{I}_{9\times9} & \mathbf{x} \\ \mathbf{O}_{(3N-9)\times9} & 0 \end{bmatrix}$. Therefore, according to rule 2, the attitude determination problem has a unique solution and, if the solution to Wahba problem exists, then this unique solution is also the solution of Wahba problem.

We state the following theorem to capture the whole idea of above discussion:

Solution to Wahba's Problem. *If the solution to Wahba problem exists and matrix $\mathbf{A}$ has column rank 9 and $\mathbf{b}$ lies in range space of $\mathbf{A}$, then the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ has exactly one solution which satisfy the orthogonal constraint.*

So, once we know the attitude matrix $\mathbf{C}$, we can estimate the non-singular Euler angle set using the formulas listed in Ref. [16]. But in real life, the attitude estimation problem is not so simple, and the vector observations, generally body vector measurements, are subject to sensor errors. Due to errors in vector measurements, the vector listing the body reference vectors, $\mathbf{b}$, may not lie in the range space of matrix $\mathbf{A}$, and so we can not even guarantee the existence of the solution to the equation $\mathbf{Ax} = \mathbf{b}$. In this case, we can determine the approximate attitude using deterministic method like TRIAD[17] and, later on, one can use the TRIAD solution as an initial guess to find the optimal solution. But the greatest drawbacks of the TRIAD method is that it is applicable only for two vector measurements and a tedious approach is required to combine the solution for the various vector measurements pairs when more than two vector measurements are available. So to overcome this drawback of TRIAD solution, we suggest to find the suboptimal least square solution to equation (12), by discarding the orthogonality constraint given in equation (13):

$$\mathbf{x} = (\mathbf{A}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{b} \tag{15}$$

Where, $\mathbf{R}$ is the measurement error covariance matrix. It should be noticed that if $\mathbf{b}$, lies in the range space of matrix $\mathbf{A}$ then the least square solution, $\mathbf{C}_{ls}$ will be equivalent to the unique solution listed above. Further, we can pose the Procrustes problem i.e. to find the orthogonal matrix $\hat{\mathbf{C}}$ that is closest to least square solution in the sense of Frobenius norm.

$$\text{Minimize: } ||\hat{\mathbf{C}} - \mathbf{C}_{ls}|| \tag{16}$$

The solution to Procrustes's problem can be found in ref. [18] and is given by:

$$\hat{\mathbf{C}} = \mathbf{U}\mathbf{V}^{\mathrm{T}} \tag{17}$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices obtained by singular value decomposition of least square solution, $\mathbf{C}_{ls}$.

It should be noticed that even the solution to Procrustes's problem is not the optimal solution to Wahba's problem. Therefore, to find the optimal solution in terms of non-singular Euler angle set the concept of small rotation is used. The true solution to the Wahba problem in terms of any non-singular Euler angle sequence can be written as a small rotation to the estimated attitude matrix either by Procrustes's problem or least square method, $\hat{\mathbf{C}}^{\|}$ i.e.

$$\mathbf{C} = \left(\mathbf{I} - \left[\tilde{\boldsymbol{\delta\theta}}\right]\right)\hat{\mathbf{C}} \tag{18}$$

Where, $\left[\tilde{\boldsymbol{\delta\theta}}\right]$ is the cross product matrix given by following expression:

$$\left[\tilde{\boldsymbol{\delta\theta}}\right] = \begin{bmatrix} 0 & -\delta\theta_3 & \delta\theta_2 \\ \delta\theta_3 & 0 & -\delta\theta_1 \\ -\delta\theta_2 & \delta\theta_1 & 0 \end{bmatrix} \tag{19}$$

$^{\|}$To avoid any confusion, $\hat{\mathbf{C}}$ will be used to denote the initial guess for attitude matrix

Now, modelling the sensor errors as Gaussian white noise and making use of the fact that $\mathbf{b}_i = \mathbf{C}\mathbf{r}_i$, we can write:

$$\tilde{\mathbf{b}}_i = \mathbf{C}\mathbf{r}_i + \boldsymbol{\nu}_i \tag{20}$$

where, $\tilde{\mathbf{b}}_i$ represents the measured vector observation in the body reference frame. Further, substitution of equation (18) in equation (20) yields the following expression:

$$\tilde{\mathbf{b}}_i = \left(\mathbf{I} - \left[\hat{\delta\boldsymbol{\theta}}\right]\right)\hat{\mathbf{C}}\mathbf{r}_i + \boldsymbol{\nu}_i \tag{21}$$

After some algebraic manipulations, we can rewrite equation (21) as:

$$\tilde{\mathbf{b}}_i - \hat{\mathbf{b}}_i = \left[\hat{\tilde{\mathbf{b}}}_i\right]\delta\boldsymbol{\theta} + \boldsymbol{\nu}_i \tag{22}$$

where, $\hat{\mathbf{b}} = \hat{\mathbf{C}}\mathbf{r}_i$. Now the set of small rotation angles $\delta\boldsymbol{\theta}$ can be estimated from equation (22) using Gaussian Least squares:

So the whole Euler angle estimation algorithm can be summarized as follows:

1. Given vector measurements $\mathbf{b}_i$ and $\mathbf{r}_i$ find which euler angle set is non-singular.

2. Find the least square solution for attitude matrix, $\mathbf{C}$ and call it $\mathbf{C}_{ls}$.

3. Find the closest orthogonal matrix, $\hat{\mathbf{C}}$ to least square solution $\mathbf{C}_{ls}$ by solving Procrustes problem.

4. Find the estimate for non-singular Euler angle set and call it $\hat{\boldsymbol{\theta}}_1$.

5. set $i = 1$.

6. find $\hat{\boldsymbol{\theta}}_{i+1} = \hat{\boldsymbol{\theta}}_i + \delta\boldsymbol{\theta}_i$ using equation (22).

7. Repeat step 6 till desired accuracy is obtained.

The above mentioned procedure permits the computation of optimal Euler angle set to arbitrarily high accuracy. The procedure to find the initial guess for Euler angle set is of practical interest and very useful in reducing the total number of iterations required to find the optimal solution to the Wahba problem in terms of particular Euler angle set. Further, we mention that this procedure can be used to start any nonlinear estimation algorithm like "Extended Kalman Filter" to estimate the Euler angle set in real time.

6 Error Analysis

In this section, We determine a relationship between the attitude error and the measurement noise using the computed attitude. This is accomplished by using the

Wahba's loss function:

$$J = \frac{1}{2} \sum_{i=1}^{n} (\tilde{\boldsymbol{b}}_i - \hat{\boldsymbol{b}}_i)^\mathrm{T} \mathbf{R}_i (\tilde{\boldsymbol{b}}_i - \hat{\boldsymbol{b}}_i) \tag{23}$$

where, n is the total number of vector measurement available. According to the Gauss Markov equation [19] the error attitude can be computed as

$$\delta\boldsymbol{\theta} = (\mathbf{H}^\mathrm{T} \mathbf{R}^{-1} \mathbf{H}) \mathbf{H}^\mathrm{T} \mathbf{R}^{-1} \boldsymbol{\nu} \tag{24}$$

where,

$$\mathbf{H} = \begin{bmatrix} -\left[\hat{\tilde{\mathbf{b}}}_1\right] \\ -\left[\hat{\tilde{\mathbf{b}}}_2\right] \\ \cdot \\ \cdot \\ \cdot \\ -\left[\hat{\tilde{\mathbf{b}}}_n\right] \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{O} & \cdot & \cdot & \cdot & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_2 & \cdot & \cdot & \cdot & \mathbf{O} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \mathbf{O} & \mathbf{O} & \cdot & \cdot & \cdot & \mathbf{R}_n \end{bmatrix} \tag{25}$$

We recognize that $\mathbf{H}^\mathrm{T} \mathbf{R}^{-1} \mathbf{H}$ is the attitude error covariance matrix $\mathbf{P}$ derived from the least square formulation. Therefore, equation (24) can be re-written as

$$\delta\boldsymbol{\theta} = \mathbf{P} \mathbf{H}^\mathrm{T} \mathbf{R}^{-1} \boldsymbol{\nu} \tag{26}$$

Now, let us consider the residual covariance matrix $\boldsymbol{\Gamma}_k$

$$\boldsymbol{\Gamma}_k = \mathbf{E}[(\tilde{\boldsymbol{b}}_k - \hat{\boldsymbol{b}}_k)(\tilde{\boldsymbol{b}}_k - \hat{\boldsymbol{b}}_k)^\mathrm{T}] \tag{27}$$

Further, substitution of equation (22) in equation (27) leads to following expression:

$$\begin{aligned} \boldsymbol{\Gamma}_k &= \mathbf{E}(\boldsymbol{\nu}_k \boldsymbol{\nu}_k^\mathrm{T} + \left[\hat{\tilde{\mathbf{b}}}_k\right] \delta\boldsymbol{\theta} \boldsymbol{\nu}_k^\mathrm{T} + \boldsymbol{\nu}_k \delta\boldsymbol{\theta}^\mathrm{T} \left[\hat{\tilde{\mathbf{b}}}_k\right]^\mathrm{T} \\ &\quad + \left[\hat{\tilde{\mathbf{b}}}_k\right] \delta\boldsymbol{\theta} \delta\boldsymbol{\theta}^\mathrm{T} \left[\hat{\tilde{\mathbf{b}}}_k\right]^\mathrm{T}) \end{aligned} \tag{28}$$

Next, since the vector measurement errors are uncorrelated, we have

$$\mathbf{E}(\delta\boldsymbol{\theta} \boldsymbol{\nu}_k^\mathrm{T}) = \mathbf{P} \left[\hat{\tilde{\mathbf{b}}}_k\right]^\mathrm{T} \tag{29}$$

$$\mathbf{E}(\boldsymbol{\nu}_k \delta\boldsymbol{\theta}^\mathrm{T}) = \left[\hat{\tilde{\mathbf{b}}}_k\right]^\mathrm{T} \mathbf{P} \tag{30}$$

Finally, substitution of equations (29) and (30) in equation (28) and using the fact that $\left[\hat{\tilde{\mathbf{b}}}_k\right]$ is a skew symmetric matrix leads to the following expression for $\boldsymbol{\Gamma}_k$

$$\boldsymbol{\Gamma}_k = \mathbf{R}_k - \left[\hat{\tilde{\mathbf{b}}}_k\right] \mathbf{P} \left[\hat{\tilde{\mathbf{b}}}_k\right]^\mathrm{T} \tag{31}$$

The matrix $\mathbf{\Gamma}_k$ can be computed independently by following recursive expression:

$$\mathbf{\Gamma}_i(k+1) = \mathbf{\Gamma}_i(k) + \frac{1}{k+1}\{\frac{k}{k+1}[\mathbf{e}_i(k+1) - \bar{\mathbf{e}}_i(k)][\mathbf{e}_i(k+1) - \bar{\mathbf{e}}_i(k)]^{\mathrm{T}} - \mathbf{\Gamma}_i(k)\} \quad (32)$$

where

$$\bar{e}_i(k+1) = \bar{e}_i(k) + \frac{1}{k+1}[\mathbf{e}_i(k+1) - \bar{e}_i(k)] \quad (33)$$

where $\mathbf{e}_i = \tilde{\boldsymbol{b}} - \hat{\boldsymbol{b}}$ is the residual error and $\bar{e}_i$ is the mean of residual error. Equation (31) can be used to check the validity of the attitude determination algorithm mentioned in last section. It should be made sure that, the error covariance is inline with the projected random error magnitude and the left hand side of equation (31) matches well with the right hand side.

7   Numerical Simulation

In this section, we test the algorithm, developed in this paper, by simulating star camera images. The spacecraft body frame star vectors and the reference frame star vectors are simulated by using the J-2000 star catalog with stars of visual magnitude brighter than $M_v \sim 6.4$. The FOV of the star camera is assumed to be $8^0 \times 8^0$ and the true spacecraft body frame vectors are corrupted by centrioding error of 17μ radian. The angular motion of the spacecraft, for simulation purposes is established by prescribing an exact nominal Euler 3-2-1 angle motion.

$$\theta_1(t) = \sin(t/4), \theta_2(t) = \frac{\pi}{2}\sin(t/5), \theta_3(t) = \cos(t/5) \quad (34)$$

Figure 3 shows the plot of the true 3-2-1, Euler angles with time. It should be noticed that for the choice of angular motions, the 3-2-1 Euler angle set can not used to estimate the spacecraft attitude at each time. Therefore, whenever 3-2-1 Euler angle set reaches singularity we switch to 3-1-3 Euler angle set to compute the spacecraft attitude. Initially, least square solution given by equation (15) is used to compute the non-singular Euler angle set, 3-2-1, in this case. The method of sequential least squares [19] and small rotation, as described in previous section, is used to find the spacecraft attitude at subsequent times. Figure 4 shows the plot of the attitude error**angle with time. From this figure, it is clear that the procedure listed in previous section works well for the attitude estimation in terms of non-singular Euler angle set.

**Attitude Error is defined as the principal rotation angle corresponding to error attitude matrix i.e. $\cos^{-1}(\frac{1}{2}Tr(\mathbf{C}_{true}\mathbf{C}_{estimated}^{\mathrm{T}} - \mathbf{I}))$
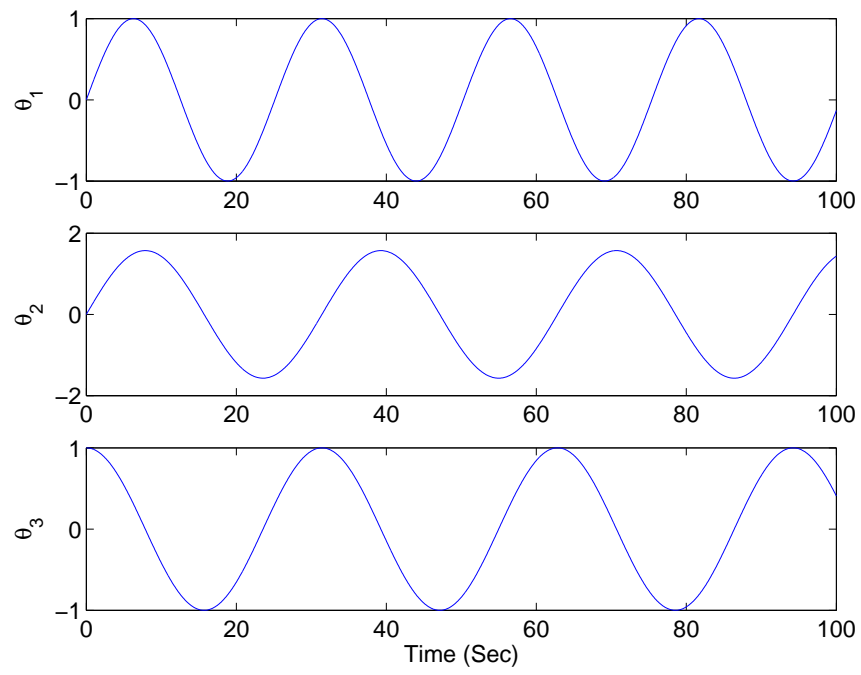
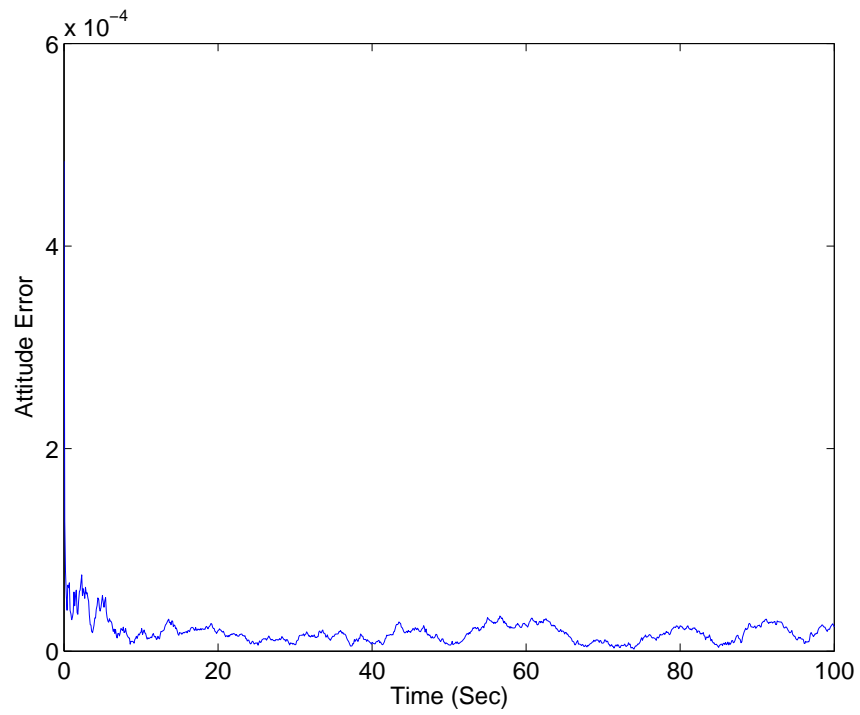Figure 3: True 3-2-1 Euler Angles Motion

Figure 4: Estimated Attitude Error

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8   Conclusion

An efficient algorithm based upon switching between different Euler angle sets is presented to avoid the geometric singularity associated with different Euler angle sets. The switching algorithm presented in this paper is based upon some well known observations regarding Euler angle sets and is of practical significance mostly in the field of robotics and spacecraft attitude control. Beside this an efficient algorithm is presented to find the optimal solution to the Wahba problem in terms of non-singular Euler angle set. The algorithm presented in paper is based upon some basic algebraic concepts and can be used as a starting algorithm for the nonlinear filter algorithms like "Extended Kalman Filter" for real time applications. However, the switching between different Euler angle set in Kalman filter to avoid the singularity is still an issue. Finally, we mention that although we present a mechanism to avoid the singularity in the Euler angle set but lot more work is required to make this procedure attractive for the spacecraft attitude estimation problem.

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