Exercise from Chapter 1: Rate

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1 You should read this before you start

This book is a special book. This book is firstly written in 1920s and some modern notation weren't introduced or used widely. For example, the notation of limit, $\lim x \to a$ was officially introduced by G.H.Hardy in 1908, and it's not used in this *Elementary Calculus*. However, the main idea of this book is very friendly to a beginner. I especially like Woods's unique way of generally leading to differential of a function — starting from rates of change, and at the same time presenting both numerical and algebraic methods. Although it seems a bit long and boring from a modern perspective, this approach make it feel as if the reader is rediscovering calculus on their own. To make more connection with the modern society and exams, I will use the modern notation instead of the old ones and add some other exercise. In this chapter, I add some exercise on limits in order to fill the empty of this book that it doesn't contain any exercise on it.

What you should've known: *The definition of the limit,* , $\epsilon - \delta$ *definition of limit*, *Properties and the basic arithmetic operations of limit*, *Rate*

2 Average Speed

1). A man runs a half mile in 2 min and 3 sec. What is his average speed in feet per second?

*note: unit conversion: 1 mile=5280 feet

Solution: $\bar{v} = \dfrac{\text{distance}}{\text{time}} = \dfrac{2640 \text{ feet}}{123 \text{ s}} = 21.46 \text{ feet/s}$

2). A man walks a mile in 25 min. What is his average speed in yards per second?

*note: unit conversion: 1 yard= 1/1760 mile

Solution: $\bar{v} = \dfrac{\text{distance}}{\text{time}} = \dfrac{1760 \text{ yards}}{1500 \text{ s}} = 1.173 \text{ feet/s}$

3). A 600 ft. train takes 10 sec. to pass a given milepost. What is its average speed in miles per hour?

Solution: The train passes a point: Starting from the time when the head of the train reaches the point and ending when the tail of the train passes the point. Therefore, the total distance is the train's length: $d = 600 \text{ feet} = 0.1136 \text{ miles}$

$\bar{v} = \dfrac{\text{d}}{\text{t}} = \dfrac{0.1136 \text{ miles}}{10 \text{ s}} = 0.01136 \text{ miles/s} = 40.9 \text{ miles/h}$

4). A stone is thrown directly downward from the edge of a vertical cliff. Two seconds later it passes a point 84 feet down the side of the cliff, and 4 seconds after it is thrown it passes a point 296 feet down the side of the cliff. What is the average speed of the stone falling between the two mentioned points?

Solution: $\bar{v} = \dfrac{\Delta d}{\Delta t} = \dfrac{d_2 - d_1}{t_2 - t_1} = \dfrac{(296 - 84) \text{ feet}}{(4 - 2) \text{ s}} = 106 \text{ feet/s}$

5). A railroad train runs on the following schedule:

Find the average speed between each two consecutive stations and for the entire trip.

Solution:

Boston-Worcester: $\bar{v} = \dfrac{\Delta d}{\Delta t} = \dfrac{d_2 - d_1}{t_2 - t_1} = \dfrac{(45 - 0) \text{ miles}}{(70) \text{ mins}} = 0.64 \text{ miles/min}$

Table 1: Schedule

Boston	*start	10:00 a.m.
Worcester	45 miles	11:10 a.m.
Springfield	99 miles	12:35 p.m.
Pittsfield	151 miles	2:25 p.m.
Albany	201 miles	3:55 p.m.

Worcester-Springfield: $\bar{v} = \dfrac{\Delta d}{\Delta t} = \dfrac{d_3 - d_2}{t_3 - t_2} = \dfrac{(99 - 45)\text{ miles}}{(85)\text{ miles}} = 0.64$ miles/min

Springfield-Pittsfield: $\bar{v} = \dfrac{\Delta d}{\Delta t} = \dfrac{d_4 - d_3}{t_4 - t_3} = \dfrac{(151 - 99)\text{ miles}}{(110)\text{ mins}} = 0.47$ miles/min

......same pattern

Entire trip=Boston to Albany: $\bar{v} = \dfrac{\Delta d}{\Delta t} = \dfrac{(201 - 0)\text{ miles}}{(355)\text{ mins}} = 0.57$ miles/min

6). A body moves four times around a circle of diameter 6 ft in 1 min. What is its average speed in feet per second?

Solution: $s = 2\pi r = 18.85$ feet

$d = 4s = 75.4$ feet

$\bar{v} = \dfrac{d}{t} = \dfrac{75.4\text{ feet}}{60\text{ s}} = 1.26$ feet/s

7). A block slides from the top to the bottom of an inclined plane which makes an angle of 30° with the horizontal. If the top is 50 feet higher than the bottom and it requires 2/3 min for the block to slide down, what is its average speed in feet per second?

Solution: $\mathrm{d} = \dfrac{50\text{ feet}}{\sin 30°} = 100\text{ feet}$; $\mathrm{t} = \frac{2}{3}\text{ min} = 40\text{ s}$

$\bar{v} = \dfrac{\mathrm{d}}{\mathrm{t}} = \dfrac{100\text{ feet}}{40\text{ s}} = 2.5\text{ feet/s}$

8). Two roads intersect at a point C. B starts along one road towards C from a point 5 miles distant from C and walks at an average speed of 3 miles per hour. Twenty minutes later A starts along the other road towards C from a point 2 miles away from C. At what average speed must A walk if he is to reach C at the same instant that B arrives?

Solution: From the condition we notice that the time that take A to reach C should equal to the time that takes B to reach C after B reached where he should after that 20 minutes walk.

Assume the average speed must A walk is $\bar{v_A}$

Therefore we have: $\dfrac{2\text{ miles}}{\bar{v_A}} = \dfrac{5\text{ miles} - 3\text{ mph} \times \frac{1}{3}\text{ h}}{3\text{ mph}}$

Solve for $\bar{v_A} = 1.5$ mph

Or, break this process down. In the first part, B travels for 20 minutes, which is 1/3 hours, that is $d = v \times t = 1 mile$. The distance between B and C is $5 - 1 = 4 miles$ now.

Therefore, the questions is converted to: B starts along one road towards C from a point 4 miles distant from C and walks at an average speed of 3 miles per hour. A starts along the other road towards C from a point 2 miles away from C. At what average speed must A walk if he is to reach C at the same instant that B arrives? The following procedure is obvious.

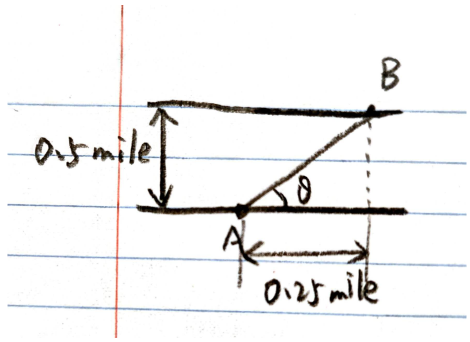

Figure 1: Q9

9). A man rows across a river 0.5 miles wide and lands at a point 0.25 miles farther down the river. If the banks of the river are parallel straight lines and he takes 0.5 hours to cross, what is his average speed in feet per minutes if his course is a straight line?

Solution: From the figure above we can easily see the total distance traveled by the man is the length of straight line AB. And AB is the hypotenuse of the right triangle. By Pythagorean Theorem, $\bar{AB} = \sqrt{0.25^2 + 0.5^2} = 0.559$ miles $=$ 2951 feet

$$\bar{v} = \frac{2951 \text{ feet}}{30 \text{ min}} = 98.37 \text{ feet/min}$$

10). Same method. Since it also requires a graph for illustrating my understanding, and it's not a hard one, I will just skip it. Please ask your school math teacher or contact me through mingzhuog2008@gmail.com for any help.

3 True Speed (with numerical method)

1). Estimate the speed of a falling body at the end of the third second, given that $s = 16t^2$. Exhibiting the work in a table.

Solution: See the table on the next page

$t_1 = 3$ s; $s_1 = 16 \cdot (3)^2 = 144$

It's fairly evident from the above arithmetical work that as the time $t_2 - t_1$ and the corresponding distance $s_2 - s_1$ become smaller, the more nearly is the average speed equal to 96. Therefore we are led to infer, in accordance with the definition of limit, that the speed at which the body passes at the end of the third second is 84 feet per second.

Table 2: Estimation

t_2	s_2	$t_2 - t_1$	$s_2 - s_1$	$\frac{s_2 - s_1}{t_2 - t_1}$
3.1	153.76	0.1	9.76	97.6
3.01	144.9616	0.01	0.9616	96.16
3.001	144.0960	0.001	0.0960	96.016
3.0001	144.0096	0.0001	0.0096	96.0016

2).- 6). Same method as above

4 More Limits

1).

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Solution:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

note: For these "good" functions, you can directly substitute the number that x approaches into the expression. The "good" here represents "the function is continuous at this point", which will be strictly defined in later chapter.

2).

$$\lim_{x \to 0} \left(\frac{\sin x}{x} + \frac{x}{1 + x} \right)$$

Solution:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}+\frac{x}{1+x}\right)=\lim_{x\to 0}\frac{\sin x}{x}+\lim_{x\to 0}\frac{x}{1+x}=1+0=1$$

Note: The limit $\lim_{x\to 0}\frac{\sin x}{x}=1$: This is an important limit. A common way to deal with this limit is by using the Squeeze theorem, and you can easily find it on the Internet. However, this common method is not correct — it leads to circular reasoning.

To prove this limit using the Squeeze Theorem is a standard approach in engineering-oriented calculus textbooks. Engineering math emphasizes practicality over full mathematical rigor. In this method, the sine function is treated according to its classical geometric definition we learned in high school: the input of the sine function is an angle measured in radians, interpreted as the arc length on the unit circle starting from the positive x-axis. The output is the y-coordinate of the corresponding point on the unit circle.

However, this definition inherently depends on the concept of **arc length**, whose rigorous constructions requires the use of definite integrals. since the theory of definite integrals relies on the limit process — particularly on the continuity and properties of the sine function — this result is circular reasoning.

Therefore, to rigorously justify the limit using this method, one must first construct the sine function from a more fundamental standpoint, typically via power series.

$$def. \sin:\mathbb{C}\to\mathbb{C}$$

$$\forall z\in\mathbb{C}:\sin z=z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\frac{1}{7!}z^7...+(-1)^n\frac{z^{2n-1}}{(2n+1)!}+...$$

It is easy to show that the radius of convergence of this power series is ∞. Therefore, the above def. is well-defined. Based on this def. , one can directly prove $\lim_{x\to 0}\frac{\sin x}{x}=1$

3).

$$\lim_{x\to 1}\left(\frac{1}{x}-\frac{1}{x^2}\right)$$

Solution:

$$\lim_{x\to 1}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim_{x\to 1}\frac{1}{x}-\lim_{x\to 1}\frac{1}{x^2}=1-1=0$$

4).

$$\left(\lim_{x\to 0}\frac{1}{x+1}\right)\cdot\lim_{x\to 0}x$$

Solution:

$$\left(\lim_{x\to 0}\frac{1}{x+1}\right)\cdot\lim_{x\to 0}x=\lim_{x\to 0}\left(\frac{x}{x+1}\right)=\frac{0}{1}=0$$

5).

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}-x\right)$$

Solution:

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}-x\right)=\lim_{x\to 3}(x+3-x)=3$$

6).

$$\lim_{x\to 1}\frac{x^3-1}{x-1}$$

Solution:

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}=\lim_{x\to 1}x^2+\lim_{x\to 1}x+\lim_{x\to 1}1=1+1+1=3$$

7).

$$\lim_{x\to +\infty}\left(\sqrt{x^2+1}-x\right)$$

Solution: (Rational)

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{(\sqrt{x^2+1}-x)\cdot(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0$$

(Imagine the denominator is a super super huge number)

5 Algebraic Method

Example: Find the speed in the question (6) above by algebraic method.

Q6; The distance of a falling body from a fixed point at any time is given by the equation $s = 50 + 20t + 16t^2$. Estimate its speed at $t = 1$s

Solution: Instead of taking a definite numerical value for t_1 at first, we keep the algebraic symbol t_1. Then

$s_1 = 16t_1^2 + 20t_1 + 50$

Also, instead of adding successive small quantities to t_1 to got t_2, we represent the amount added by the algebraic symbol Δt. That is,

$t_2 = t_1 + \Delta t$ and $s_2 = 16(t_1 + \Delta t)^2 + 20(t_1 + \Delta t) + 50$

Hence $s_2 - s_1 = [16(t_1 + \Delta t)^2 + 20(t_1 + \Delta t) + 50] - [16t_1^2 + 20t_1 + 50] = 32t_1 \cdot \Delta t + 20\Delta t + 16\Delta t^2$

$\bar{v} = \dfrac{32t_1 \cdot \Delta t + 20\Delta t + 16\Delta t^2}{\Delta t} = 32t_1 + 20 + 16\Delta t$

From the above equation, the quantity $32t_1 + 20$ here satisfies exactly the $\epsilon - \delta$ definition of limit. That is, $\lim_{\Delta t \to 0^+} 32t_1 + 20 + 16\Delta t = 32t_1 + 20$

Then we have the speed of the body at any time is given by the formula $speed = 32t + 20$

At t=1, speed=52

If you check by numerical method, you are supposed to get the same answer.

6 Acceleration

1). If $s = 4t^3$, find the speed and the acceleration when $t = t_1$

Solution: By the algebraic method in the last section, we can easily find the general expression of speed is $v = 12t^2$; so $v(t_1) = 12 \cdot t_1^2$.

By the definition of acceleration, $\bar{a} = \dfrac{\Delta v}{\Delta t}$, the general pattern here is obvious. Thus, by the same algebraic method as above, $a = 24t$; and $a(t_1) = 24t_1$

2). Same method as above, the answer: $v(2) = 16; a(2) = 14$

3). If $s = 3t^2 + 2t + 5$, how far has the body moved at the end of the fifth second? With what speed does it reach that point, and how fast is the speed increasing?

Solution: To find the distance traveled when t=5, just straightly substitute t=5 in the s(t) function, s(5)=90;

To find the speed at this point, apply the same method as above, v(5)=32;

The rate of the speed increasing is the acceleration at this point, apply the same method, a(5)=6

4). Same method as above, the answer: $d(2) = 46; v(2) = 57$

5). If $s = \frac{1}{3}t^3 + t + 10$, find the speed and the acceleration when $t = 2$. Compare the average speed and the average acceleration during this second with the speed and the acceleration at the beginning and the end of the second. (At t=3, same method)

Solution:Speed: v(2)=5; acceleration: a(2)=2

Average speed during t=2: select t=1 and t=3, $\bar{v} = \dfrac{22 - 11.333}{3 - 1} = 5.33$

Average acceleration during t=2: select t=1 and t=3, apply $a = \dfrac{\Delta v}{\Delta t}$, $\bar{a} = \dfrac{10 - 2}{3 - 1} = 4$

Notice the speed is very close to the one we calculated, but acceleration's value has a large bias, that's due to the rate of the speed increased is fast, and we can see this by find the general expression of it (It's a parabola)

6). If $s = at + b$, show that the speed is constant.

Solution: By the algebraic method we can obtain v(t)=a, which is constant.

7). If $s = at^2 + bt + c$, show that the acceleration is constant.

Solution: By the algebraic method we can obtain v(t)=2at+b, which is not constant. And a(t)=2a, which is a constant. (Notice the different meaning of two "a" here)

8). If $s = at^3 + bt^2 + ct + f$, find the formulas for the speed and the acceleration.

Solution: By the algebraic method we can obtain $v(t) = 3at^2 + 2bt + c$, which is not constant. And a(t)=6at+2b, which is a not constant. These are two functions. (Notice the different meaning of two "a" here)

As we move to next chapter, we will get into a quicker way of this algebraic method.

7 Rate of change

I will skip Q1 and Q2 here because they are related to the example in the text. The following questions are using the similar method as these two, so I believe it's easy to figure out Q1 and Q2.

3). A soap bubble is expanding, always remaining spherical. If the radius of the bubble is increasing at the rate of 2 inch per second, how fast is the volume increasing?

Solution: $\Delta r = 2$ inch per second, $\quad V(r) = \frac{4}{3}\pi r^3, \quad r(t + \Delta t) = r + 2\Delta t$

$$\Delta V = \frac{4}{3}\pi(r + 2\Delta t)^3 - \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi \left(r^3 + 6r^2\Delta t + 16r(\Delta t)^2 + 8(\Delta t)^3 - r^3\right)$$

$$= \frac{4}{3}\pi\left(6r^2\Delta t + 16r(\Delta t)^2 + 8(\Delta t)^3\right)$$

$$\lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \lim_{\Delta t \to 0} \frac{4}{3}\pi\left(6r^2 + 12r\Delta t + 8(\Delta t)^2\right) = \frac{4}{3}\pi \cdot 6r^2 = 8\pi r^2$$

4). In Q3 find the general expression for the rate of change of the volume with respect to the radius.

Solution:

$$\lim_{\Delta r \to 0} \frac{\Delta V}{\Delta r} = \lim_{\Delta r \to 0} \frac{V(r+\Delta r) - V(r)}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{\frac{4}{3}\pi((r+\Delta r)^3 - r^3)}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{\frac{4}{3}\pi(3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3)}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{4}{3}\pi(3r^2 + 3r\Delta r + (\Delta r)^2) = 4\pi r^2$$

5). If a soap bubble is expanding as in Q3, how fast is the area of the surface increasing?

Solution:

$$r(t+\Delta t) = r + 2\Delta t, \quad S = 4\pi r^2, \quad r(t) = r$$

$$\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \lim_{\Delta t \to 0} \frac{4\pi(r+2\Delta t)^2 - 4\pi r^2}{\Delta t} = \lim_{\Delta t \to 0} 4\pi(4r + 4\Delta t) = 16\pi r$$

6). In Q5 find the general expression for the rate of change of the surface with respect to the radius.

Solution:

$$\lim_{\Delta r \to 0} \frac{\Delta S}{\Delta r} = \lim_{\Delta r \to 0} \frac{4\pi(r+\Delta r)^2 - 4\pi r^2}{\Delta r} = 4\pi \cdot 2r = 8\pi r$$

7). A cube of metal is expanding under the influence of heat. Assuming that the metal retains the form of a cube, find the rate of change at which the volume is increasing with respect of the edge.

Solution:

$$V = l^3, \quad \lim_{\Delta l \to 0} \frac{\Delta V}{\Delta l} = \lim_{\Delta l \to 0} \frac{(l+\Delta l)^3 - l^3}{\Delta l} = 3l^2$$

8). The altitude of a right circular cylinder is always equal to the diameter of the base. If the cylinder is assumed to expand, always retaining its form and proportions, what is the rate of change of the volume with respect to the radius of the base?

Solution: $V = Sh = \pi r^2 \cdot r = \pi r^3$

$$\lim_{\Delta r \to 0} \frac{\Delta V}{\Delta r} = \lim_{\Delta r \to 0} \frac{V(r+\Delta r) - V(r)}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{\pi(r+\Delta r)^3 - \pi r^3}{\Delta r} = \lim_{\Delta r \to 0} \frac{\pi(3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3)}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \pi(3r^2 + 3r\Delta r + (\Delta r)^2) = 3\pi r^2$$

9). Find the rate of change of the area of a sector of a circle of radius 6 feet with respect of the angle at the center of the circle.

Solution:

$$A = \frac{1}{2}r^2\theta, \quad r = 6$$

$$\lim_{\Delta\theta \to 0} \frac{\Delta A}{\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{A(\theta + \Delta\theta) - A(\theta)}{\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{\frac{1}{2}r^2\Delta\theta}{\Delta\theta} = \frac{1}{2}r^2$$

Substitute r=6, the rate is 18

10). Find the rate of change of the area of a sector of a circle with respect to the radius of the circle if the angle at the center of the circle is always $\frac{\pi}{4}$. What is the value of the rate when the radius is 8 inch?

Solution:

$$A = \frac{1}{2}r^2\theta$$

$$\lim_{\Delta r \to 0} \frac{\Delta A}{\Delta r} = \lim_{\Delta r \to 0} \frac{A(r + \Delta r) - A(r)}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{(2r \cdot \Delta r + (\Delta r)^2) \cdot \frac{1}{2}\theta}{\Delta r} = \lim_{\Delta r \to 0} (2r + \Delta r) \cdot \frac{1}{2}\theta = r\theta$$

At r=8 and $\theta = \frac{\pi}{4}$, the rate is 2π

Note: I didn't take in the units, but you have to be careful of them in your exams.