# ELEMENTARY CALCULUS

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# PREFACE

This book is adapted to the use of students in the first year in technical school or college, and is based upon the experience of the authors in teaching calculus to students in the Massachusetts Institute of Technology immediately upon entrance. It is accordingly assumed that the student has had college-entrance algebra, including graphs, and an elementary course in trigonometry, but that he has not studied analytic geometry.

The first three chapters form an introductory course in which the fundamental ideas of the calculus are introduced, including derivative, differential, and the definite integral, but the formal work is restricted to that involving only the polynomial. These chapters alone are well fitted for a short course of about a term.

The definition of the derivative is obtained through the concept of speed, using familiar illustrations, and the idea of a derivative as measuring the rate of change of related quantities is emphasized. The slope of a curve is introduced later. This is designed to prevent the student from acquiring the notion that the derivative is fundamentally a geometric concept. For the same reason, problems from mechanics are prominent throughout the book.

With Chapter IV a more formal development of the subject begins, and certain portions of analytic geometry are introduced as needed. These include, among other things, the straight line, the conic sections, the cycloid, and polar coördinates.

The book contains a large number of well-graded exercises for the student. Drill exercises are placed at the end of most sections, and a miscellaneous set of exercises, for review or further work, is found at the end of each chapter except the first.

Throughout the book, the authors believe, the matter is presented in a manner which is well within the capacity of a first-year student to understand. They have endeavored to teach the calculus from a common-sense standpoint as a very useful tool. They have used as much mathematical rigor as the student is able to understand, but have refrained from raising the more difficult questions which the student in his first course is able neither to appreciate nor to master.

Students who have completed this text and wish to continue their study of mathematics may next take a brief course in differential equations and then a course in advanced calculus, or they may take a course in advanced calculus which includes differential equations. It would also be desirable for such students to have a brief course in analytic geometry, which may either follow this text directly or come later.

This arrangement of work the authors consider preferable to the one — for a long time common in American colleges — by which courses in higher algebra and analytic geometry precede the calculus. However, the teacher who prefers to follow the older arrangement will find this text adapted to such a program.

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# ELEMENTARY CALCULUS

## CHAPTER I

### RATES

**1. Limits.** Since the calculus is based upon the idea of a limit it is necessary to have a clear understanding of the word. Two examples already familiar to the student will be sufficient.

In finding the area of a circle in plane geometry it is usual to begin by inscribing a regular polygon in the circle. The area of the polygon differs from that of the circle by a certain amount. As the number of sides of the polygon is increased, this difference becomes less and less. Moreover, if we take any small number $e$, we can find an inscribed polygon whose area differs from that of the circle by less than $e$; and if one such polygon has been found, any polygon with a larger number of sides will still differ in area from the circle by less than $e$. The area of the circle is said to be the *limit* of the area of the inscribed polygon.

As another example of a limit consider the geometric progression with an unlimited number of terms

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\cdots.$$

The sum of the first two terms of this series is $1\tfrac{1}{2}$, the sum of the first three terms is $1\tfrac{3}{4}$, the sum of the first four terms is $1\tfrac{7}{8}$, and so on. It may be found by trial and is proved in the algebras that the sum of the terms becomes more nearly equal to 2 as the number of terms which are taken becomes greater. Moreover, it may be shown that if any small number $e$ is assumed, it is possible to take a number of terms $n$ so that the sum of these terms differs from 2 by less than $e$. If a value of $n$ has thus been found, then the sum of a number of terms greater than $n$ will still differ from 2 by less than $e$. The number 2 is said to be the *limit* of the sum of the first $n$ terms of the series.

In each of these two examples there is a certain variable — namely, the area of the inscribed polygon of $n$ sides in one case and the sum of the first $n$ terms of the series in the other case — and a certain constant, the area of the circle and the number 2 respectively. In each case the difference between the constant and the variable may be made less than any small number $e$ by taking $n$ sufficiently large, and this difference then continues to be less than $e$ for any larger value of $n$.

This is the essential property of a limit, which may be defined as follows:

*A constant $A$ is said to be the limit of a variable $X$ if, as the variable changes its value according to some law, the difference between the variable and the constant becomes and remains less than any small quantity which may be assigned.*

The definition does not say that the variable never reaches its limit. In most cases in this book, however, the variable fails to do so, as in the two examples already given. For the polygon is never exactly a circle, nor is the sum of the terms of the series exactly 2. Examples may be given, however, of a variable's becoming equal to its limit, as in the case of a swinging pendulum finally coming to rest. But the fact that a variable may never reach its limit does not make the limit inexact. There is nothing inexact about the area of a circle or about the number 2.

The student should notice the significance of the word "remains" in the definition. If a railroad train approaches a station, the difference between the position of the train and a point on the track opposite the station becomes less than any number which may be named; but if the train keeps on by the station, that difference does not remain small. Hence there is no limit approached in this case.

If $X$ is a variable and $A$ a constant which $X$ approaches as a limit, it follows from the definition that we may write

$$X = A + e, \tag{1}$$

where $e$ is a quantity (not necessarily positive) which may be made, and then will remain, as small as we please.

Conversely, if as the result of any reasoning we arrive at a formula of the form (1) where $X$ is a variable and $A$ a constant, and if we see that we can make $e$ as small as we please and that it will then remain just as small or smaller as $X$ varies, we can say that $A$ is the limit of $X$. It is in this way that we shall determine limits in the following pages.

**2. Average speed.** Let us suppose a body (for example, an automobile) moving from a point $A$ to a point $B$ (Fig. 1), a distance of 100 mi If the automobile takes 5 hr. for the trip, we are accustomed to say that it has traveled at the rate of 20 mi. an hour. Everybody knows that this does not mean that the automobile went exactly 20 mi. in each hour of the trip, exactly 10 mi. in each half hour, exactly 5 mi. in each quarter hour, and so on. Probably no automobile ever ran in such a way as that. The expression "20 mi. an hour" may be understood as meaning that a fictitious automobile traveling in the steady manner just described would actually cover the 100 mi. in just 5 hr.; but for the actual automobile which made the trip, "20 mi. an hour" gives only a certain average speed.

FIG. 1

So if a man walks 9 mi. in 3 hr., he has an average speed of 3 mi. an hour. If a stone falls 144 ft. in 3 sec., it has an average speed of 48 ft. per second. In neither of these cases, however, does the average speed give us any information as to the actual speed of the moving object at a given instant of its motion.

The point we are making is so important, and it is so often overlooked, that we repeat it in the following statement:

*If a body traverses a distance in a certain time, the average speed of the body in that time is given by the formula*

$$\text{average speed} = \frac{\text{distance}}{\text{time}},$$

*but this formula does not in general give the true speed at any given time.*

## EXERCISES

1. A man runs a half mile in 2 min and 3 sec. What is his average speed in feet per second?

2. A man walks a mile in 25 min. What is his average speed in yards per second?

3. A train 600 ft. long takes 10 sec. to pass a given milepost. What is its average speed in miles per hour?

4. A stone is thrown directly downward from the edge of a vertical cliff Two seconds afterwards it passes a point 84 ft down the side of the cliff, and 4 sec after it is thrown it passes a point 296 ft. down the side of the cliff What is the average speed of the stone in falling between the two mentioned points?

5. A railroad train runs on the following schedule:

| | | |
|---|---|---|
| Boston | | 10 00 A M. |
| Worcester | (45 mi.) | 11.10 |
| Springfield | (99 mi) | 12.35 P M. |
| Pittsfield | (151 mi.) | 2 25 |
| Albany | (201 mi.) | 3 55 |

Find the average speed between each two consecutive stations and for the entire trip.

6. A body moves four times around a circle of diameter 6 ft in 1 min. What is its average speed in feet per second?

7. A block slides from the top to the bottom of an inclined plane which makes an angle of 30° with the horizontal. If the top is 50 ft. higher than the bottom and it requires $\frac{2}{3}$ min. for the block to slide down, what is its average speed in feet per second?

8. Two roads intersect at a point *C* B starts along one road toward *C* from a point 5 mi. distant from *C* and walks at an average speed of 3 mi an hour. Twenty minutes later A starts along the other road toward *C* from a point 2 mi. away from *C* At what average speed must A walk if he is to reach *C* at the same instant that B arrives?

9. A man rows across a river $\frac{1}{6}$ mi. wide and lands at a point $\frac{1}{4}$ mi. farther down the river. If the banks of the river are parallel straight lines and he takes $\frac{1}{2}$ hr. to cross, what is his average speed in feet per minute if his course is a straight line?

**10.** A trolley car is running along a straight street at an average speed of 12 mi. per hour. A house is 50 yd. back from the car track and 100 yd. up the street from a car station. A man comes out of the house when a car is 200 yd away from the station What must be the average speed of the man in yards per minute if he goes in a straight line to the station and arrives at the same instant as the car?

**3. True speed.** How then shall we determine the speed at which a moving body passes any given fixed point $P$ in its motion (Fig. 1)? In answering this question the mathematician begins exactly as does the policeman in setting a trap for speeding. He takes a point $Q$ near to $P$ and determines the distance $PQ$ and the time it takes to pass over that distance. Suppose, for example, that the distance $PQ$ is $\frac{1}{2}$ mi. and the time is 1 min. Then, by § 2, the average speed with which the distance is traversed is

$$\frac{\frac{1}{2}\text{ mi.}}{1\text{ min.}} = \frac{\frac{1}{2}\text{ mi.}}{\frac{1}{60}\text{ hr.}} = 30\text{ mi. per hour.}$$

This is merely the average speed, however, and can no more be taken for the true speed at the point $P$ than could the 20 mi. an hour which we obtained by considering the entire distance $AB$. It is true that the 30 mi. an hour obtained from the interval $PQ$ is likely to be nearer the true speed at $B$ than was the 20 mi. an hour obtained from $AB$, because the interval $PQ$ is shorter.

The last statement suggests a method for obtaining a still better measure of the speed at $P$; namely, by taking the interval $PQ$ still smaller. Suppose, for example, that $PQ$ is taken as $\frac{1}{16}$ mi. and that the time is $6\frac{1}{4}$ sec. A calculation shows that the average speed at which this distance was traversed was 36 mi. an hour. This is a better value for the speed at $P$.

Now, having seen that we get a better value for the speed at $P$ each time that we decrease the size of the interval $PQ$, we can find no end to the process except by means of the idea of a limit defined in § 1. We say, in fact, that *the speed of a moving body at any point of its path is the limit approached by the average* *speed computed for a small distance beginning at that point, the limit to be determined by taking this distance smaller and smaller.*

This definition may seem to the student a little intricate, and we shall proceed to explain it further.

In the case of the automobile, which we have been using for an illustration, there are practical difficulties in taking a very small distance, because neither the measurement of the distance nor that of the time can be exact. This does not alter the fact, however, that theoretically to determine the speed of the car we ought to find the time it takes to go an extremely minute distance, and the more minute the distance the better the result. For example, if it were possible to discover that an automobile ran $\frac{1}{10}$ in. in $\frac{1}{5280}$ sec., we should be pretty safe in saying that it was moving at a speed of 30 mi. an hour.

FIG. 2

Such fineness of measurement is, of course, impossible; but if an algebraic formula connecting the distance and the time is known, the calculation can be made as fine as this and finer. We will therefore take a familiar case in which such a formula is known; namely, that of a falling body.

Let us take the formula from physics that if $s$ is the distance through which a body falls from rest, and $t$ is the time it takes to fall the distance $s$, then

$$s = 16\,t^2, \tag{1}$$

and let us ask what is the speed of the body at the instant when $t = 2$. In Fig. 2 let $O$ be the point from which the body falls, $P_1$ its position when $t = 2$, and $P_2$ its position a short time later. The average speed with which the body falls through the distance $P_1P_2$ is, by § 2, that distance divided by the time it takes to traverse it. We shall proceed to make several successive calculations of this average speed, assuming $P_1P_2$ and the corresponding time smaller and smaller.

In so doing it will be convenient to introduce a notation as follows: Let $t_1$ represent the time at which the body reaches $P_1$, and $t_2$ the time at which it reaches $P_2$. Also let $s_1$ equal the distance $OP_1$, and $s_2$ the distance $OP_2$. Then $s_2 - s_1 = P_1P_2$, and

$t_2 - t_1$ is the time it takes to traverse the distance $P_1P_2$. Then the average speed at which the body traverses $P_1P_2$ is

$$\frac{s_2 - s_1}{t_2 - t_1}. \tag{2}$$

Now, by the statement of our particular problem,

$$t_1 = 2.$$

Therefore, from (1), $\qquad s_1 = 16\,(2)^2 = 64.$

We shall assume a value of $t_2$ a little larger than 2, compute $s_2$ from (1), and the average speed from (2). That having been done, we shall take $t_2$ a little nearer to 2 than it was at first, and again compute the average speed This we shall do repeatedly, each time taking $t_2$ nearer to 2.

Our results can best be exhibited in the form of a table, as follows:

| $t_2$ | $s_2$ | $t_2 - t_1$ | $s_2 - s_1$ | $\frac{s_2 - s_1}{t_2 - t_1}$ |
|---|---|---|---|---|
| 2.1 | 70.56 | .1 | 6 56 | 65 6 |
| 2 01 | 64 6416 | 01 | .6416 | 64.16 |
| 2 001 | 64.064016 | .001 | 064016 | 64 016 |
| 2.0001 | 64 00640016 | .0001 | .00640016 | 64 0016 |

It is fairly evident from the above arithmetical work that as the time $t_2 - t_1$ and the corresponding distance $s_2 - s_1$ become smaller, the more nearly is the average speed equal to 64. Therefore we are led to infer, in accordance with § 1, that the speed at which the body passes the point $P_1$ is 64 ft. per second.

In the same manner the speed of the body may be computed at any point of its path by a purely arithmetical calculation. In the next section we shall go farther with the same problem and employ algebra.

## EXERCISES

1. Estimate the speed of a falling body at the end of the third second, given that $s = 16\,t^2$, exhibiting the work in a table.

2. Estimate the speed of the body in Ex. 1 at the end of the fourth second, exhibiting the work in a table.

3. The distance of a falling body from a fixed point at any tim is given by the equation $s = 100 + 16t^2$. Estimate the speed of t body at the end of the fourth second, exhibiting the work in a tab

4. A body is falling so that the distance traversed in the time is given by the equation $s = 16t^2 + 10t$ Estimate the speed of t body when $t = 2$ sec, exhibiting the work in a table.

5. A body is thrown upward with such a speed that at any tim its distance from the surface of the earth is given by the equatio $s = 100t - 16t^2$. Estimate its speed at the end of a second, exhibi ing the work in a table.

6. The distance of a falling body from a fixed point at any tim is given by the equation $s = 50 + 20t + 16t^2$. Estimate its spee at the end of the first second, exhibiting the work in a table.

**4. Algebraic method.** In this section we shall show how it i possible to derive an algebraic formula for the speed, still con fining ourselves to the special example of the falling body whose equation of motion is

$$s = 16t^2. \qquad (1)$$

Instead of taking a definite numerical value for $t_1$, we shal keep the algebraic symbol $t_1$. Then

$$s_1 = 16t_1^2.$$

Also, instead of adding successive small quantities to $t_1$ to get $t_2$, we shall represent the amount added by the algebraic symbol $h$. That is,

$$t_2 = t_1 + h,$$

and, from (1),

$$s_2 = 16(t_1 + h)^2.$$

Hence $\quad s_2 - s_1 = 16(t_1 + h)^2 - 16t_1^2 = 32t_1h + 16h^2.$

This is a general expression for the distance $P_1P_2$ in Fig. 2. Now $t_2 - t_1 = h$, and therefore the average speed with which the body traverses $P_1P_2$ is represented by the expression

$$\frac{32t_1h + 16h^2}{h} = 32t_1 + 16h.$$

It is obvious that if $h$ is taken smaller and smaller, the average speed approaches $32t_1$ as a limit. In fact, the quantity $32t_1$

satisfies exactly the definition of limit given in § 1. For if $e$ is any number, no matter how small, we have simply to take $16h < e$ in order that the average speed should differ from $32t_1$ by less than $e$; and after that, for still smaller values of $h$, this difference remains less than $e$.

We have, then, the result that if the space traversed by a falling body is given by the formula

$$s = 16t^2,$$

the speed of the body at any time is given by the formula

$$\text{speed} = 32t.$$

It may be well to emphasize that this is not the result which would be obtained by dividing $s$ by $t$.

### EXERCISE

Find the speed in each of the problems in § 3 by the method explained in this section.

**5. Acceleration.** Let us consider the case of a body which is supposed to move so that if $s$ is the distance in feet and $t$ is the time in seconds,

$$s = t^3. \tag{1}$$

Then, by the method of § 4, we find that if $v$ is the speed in feet per second,

$$v = 3t^2. \tag{2}$$

We see that when $t = 1$, $v = 3$; when $t = 2$, $v = 12$; when $t = 3$, $v = 27$; and so on. That is, the body is gaining speed with each second. We wish to find how fast it is gaining speed. To find this out, let us take a specific time

$$t_1 = 4.$$

The speed at this time we call $v_1$, so that, by (2),

$$v_1 = 3(4)^2 = 48 \text{ ft. per second.}$$

Take $$t_2 = 5;$$

then $$v_2 = 3(5)^2 = 75 \text{ ft. per second.}$$

Therefore the body has gained $75-48=27$ units of speed in 1 sec. This number, then, represents the average rate at which the body is gaining speed during the particular second considered. It does not give exactly the rate at which the speed is increasing at the beginning of the second, because the rate is constantly changing.

To find how fast the body is gaining speed when $t_1=4$, we must proceed exactly as we did in finding the speed itself. That is, we must compute the gain of speed in a very small interval of time and compare that with the time.

Let us take $t_2=4.1.$

Then $v_2=50.43$

and $v_2-v_1=2.43.$

Then the body has gained 2.43 units of speed in .1 sec., which is at the rate of $\frac{2.43}{.1}=24.3$ units per second.

Again, take $t_2=4.01.$

Then $v_2=48.2403$

and $v_2-v_1=.2403.$

A gain of .2403 units of speed in .01 sec. is at the rate of $\frac{.2403}{.01}=24.03$ units per second. We exhibit these results, and one other obtained in the same way, in a table:

| $t_2$ | $v_2$ | $t_2-t_1$ | $v_2-v_1$ | $\frac{v_2-v_1}{t_2-t_1}$ |
|---|---|---|---|---|
| 4.1 | 50.43 | .1 | 2.43 | 24.3 |
| 4.01 | 48.2403 | .01 | .2403 | 24.03 |
| 4.001 | 48.024003 | 001 | .024003 | 24.003 |

The rate at which a body is gaining speed is called its *acceleration*. Our discussion suggests that in the example before us the acceleration is 24 units of speed per second. But the unit of speed is expressed in feet per second, and so we say that the acceleration is 24 ft. per second per second.

By the method used in determining speed, we may get a general formula to determine the acceleration from equation (2). We take

$$t_2 = t_1 + h.$$

Then
$$v_2 = 3(t_1 + h)^2$$
and
$$v_2 - v_1 = 6t_1h + 3h^2.$$

The average rate at which the speed is gained is then

$$\frac{6t_1h + 3h^2}{h} = 6t_1 + 3h,$$

and the limit of this, as $h$ becomes smaller and smaller, is obviously $6t_1$.

This is, of course, a result which is valid only for the special example that we are considering. A general statement of the meaning of acceleration is as follows:

$$\text{Acceleration} = \text{limit of } \frac{\text{change in speed}}{\text{change in time}}.$$

**EXERCISES**

1. If $s = 4t^3$, find the speed and the acceleration when $t = t_1$.

2. If $s = t^3 + t^2$, find the speed and the acceleration when $t = 2$.

3. If $s = 3t^2 + 2t + 5$, how far has the body moved at the end of the fifth second? With what speed does it reach that point, and how fast is the speed increasing?

4. If $s = 4t^3 + 2t^2 + t + 4$, find the distance traveled and the speed when $t = 2$.

5. If $s = \frac{1}{3}t^3 + t + 10$, find the speed and the acceleration when $t = 2$ and when $t = 3$. Compare the average speed and the average acceleration during this second with the speed and the acceleration at the beginning and the end of the second.

6. If $s = at + b$, show that the speed is constant.

7. If $s = at^2 + bt + c$, show that the acceleration is constant.

8. If $s = at^3 + bt^2 + ct + f$, find the formulas for the speed and the acceleration.

**6. Rate of change.** Let us consider another example which may be solved by processes similar to those used for determining speed and acceleration.

A stone is thrown into still water, forming ripples which travel from the center of disturbance in the form of circles (Fig. 3). Let $r$ be the radius of a circle and $A$ its area. Then

$$A = \pi r^2. \tag{1}$$

We wish to compare changes in the area with changes in the radius. If we take $r_1 = 3$, then $A_1 = 9\pi$; and if we take $r_2 = 4$, then $A_2 = 16\pi$. That is, a change of 1 unit in $r$, when $r = 3$, causes a change of $7\pi$ units in $A$. We are tempted to say that $A$ is increasing $7\pi$ times as fast as $r$. But before making such a statement it is well to see whether this law holds for all changes made in $r$, starting from $r_1 = 3$, and especially for small changes in $r$.

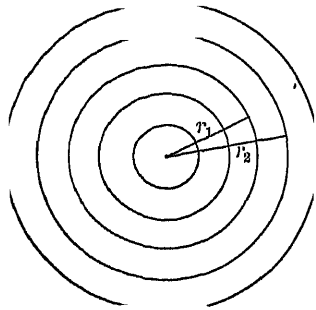


FIG. 3

We will again exhibit the calculation in the form of a table. Here $r_1 = 3$, $A_1 = 9\pi$, and $r_2$ is a variously assumed value of $r$ not much different from 3.

| $r_2$ | $A_2$ | $r_2 - r_1$ | $A_2 - A_1$ | $\frac{A_2 - A_1}{r_2 - r_1}$ |
|---|---|---|---|---|
| 3.1 | $9.61\pi$ | .1 | $.61\pi$ | $6.1\pi$ |
| 3.01 | $9.0601\pi$ | .01 | $.0601\pi$ | $6.01\pi$ |
| 3.001 | $9.006001\pi$ | .001 | $.006001\pi$ | $6.001\pi$ |

The number in the last column changes with the number $r_2 - r_1$. Therefore, if we wish to measure the rate at which $A$ is changing as compared with $r$ at the instant when $r = 3$, we must take the limit of the numbers in the last column. That limit is obviously $6\pi$.

We say that at the instant when $r = 3$, *the area of the circle is changing* $6\pi$ *times as fast as the radius.* Hence, if the radius is changing at the rate of 2 ft. per second, for example, the area is changing at the rate of $12\pi$ sq. ft. per second. Another way of expressing the same idea is to say that when $r = 3$, *the rate* *of change of A with respect to r is* $6\pi$. Whichever form of expression is used, we mean that the change in the area divided by the change in the radius approaches a limit $6\pi$.

The number $6\pi$ was, of course, dependent upon the value $r = 3$, with which we started. Another value of $r_1$ assumed at the start would produce another result. For example, we may compute that when $r_1 = 4$, the rate of change of $A$ with respect to $r$ is $8\pi$; and when $r_1 = 5$, the rate is $10\pi$. Better still, we may derive a general formula which will give us the required rate for any value of $r_1$.

To do this take
$$r_2 = r_1 + h.$$

Then
$$A_2 = \pi(r_1^2 + 2r_1h + h^2)$$

and
$$A_2 - A_1 = \pi(2r_1h + h^2);$$

so that
$$\frac{A_2 - A_1}{r_2 - r_1} = 2\pi r_1 + h\pi.$$

The limit of this quantity, as $h$ is taken smaller and smaller, is
$$2\pi r_1.$$

Hence we see that from formula (1) we may derive the fact that the rate of change of $A$ with respect to $r$ is $2\pi r$.

## EXERCISES

1. In the example of the text, if the circumference of the circle which bounds the disturbed area is 10 ft and the circumference is increasing at the rate of 3 ft. per second, how fast is the area increasing?

2. In the example of the text find a general expression for the rate of change of the area with respect to the circumference.

3. A soap bubble is expanding, always remaining spherical. If the radius of the bubble is increasing at the rate of 2 in. per second, how fast is the volume increasing?

4. In Ex 3 find the general expression for the rate of change of the volume with respect to the radius.

5. If a soap bubble is expanding as in Ex. 3, how fast is the area of its surface increasing?

6. In Ex. 5 find the general expression for the rate of change of the surface with respect to the radius

7. A cube of metal is expanding under the influence of heat. Assuming that the metal retains the form of a cube, find the rate of change at which the volume is increasing with respect to an edge.

8. The altitude of a right circular cylinder is always equal to the diameter of the base. If the cylinder is assumed to expand, always retaining its form and proportions, what is the rate of change of the volume with respect to the radius of the base?

9. Find the rate of change of the area of a sector of a circle of radius 6 ft with respect to the angle at the center of the circle.

10. Find the rate of change of the area of a sector of a circle with respect to the radius of the circle if the angle at the center of the circle is always $\frac{\pi}{4}$. What is the value of the rate when the radius is 8 in.?

# CHAPTER II

## DIFFERENTIATION

**7. The derivative.** The examples we have been considering in the foregoing sections of the book are alike in the methods used to solve them. We shall proceed now to examine this method so as to bring out its general character.

In the first place, we notice that we have to do with two quantities so related that the value of one depends upon the value of the other. Thus the distance traveled by a moving body depends upon the time, and the area of a circle depends upon the radius. In such a case one quantity is said to be a *function* of the other. That is, *a quantity $y$ is said to be a function of another quantity, $x$, if the value of $y$ is determined by the value of $x$.*

The fact that $y$ is a function of $x$ is expressed by the equation

$$y = f(x),$$

and the particular value of the function when $x$ has a definite value $a$ is then expressed as $f(a)$. Thus, if

$$f(x) = x^3 - 3x^2 + 4x + 1,$$
$$f(2) = 2^3 - 3(2)^2 + 4(2) + 1 = 5,$$
$$f(0) = 0 - 3(0) + 4(0) + 1 = 1.$$

It is in general true that a change in $x$ causes a change in the function $y$, and that if the change in $x$ is sufficiently small, the change in $y$ is small also. Some exceptions to this may be noticed later, but this is the general rule. A change in $x$ is called an *increment* of $x$ and is denoted by the symbol $\Delta x$ (read "delta $x$"). Similarly, a change in $y$ is called an increment of $y$ and is denoted by $\Delta y$. For example, consider

$$y = x^2 + 3x + 2.$$

When $x=2$, $y=12$. When $x=2.1$, $y=12.71$. The change in $x$ is .1, and the change in $y$ is .71, and we write

$$\Delta x=.1, \quad \Delta y=.71.$$

So, in general, if $x_1$ is one value of $x$, and $x_2$ a second value of $x$, then

$$\Delta x=x_2-x_1, \quad \text{or} \quad x_2=x_1+\Delta x; \tag{1}$$

and if $y_1$ and $y_2$ are the corresponding values of $y$, then

$$\Delta y=y_2-y_1, \quad \text{or} \quad y_2=y_1+\Delta y. \tag{2}$$

The word *increment* really means "increase," but as we are dealing with algebraic quantities, the increment may be negative when it means a decrease. For example, if a man invests \$1000 and at the end of a year has \$1200, the increment of his wealth is \$200. If he has \$800 at the end of the year, the increment is −\$200. So, if a thermometer registers 65° in the morning and 57° at night, the increment is −8°. The increment is always the second value of the quantity considered minus the first value.

Now, having determined increments of $x$ and of $y$, the next step is to compare them by dividing the increment of $y$ by the increment of $x$. This is what we did in each of the three problems we have worked in §§ 3–6. In finding speed we began by dividing an increment of distance by an increment of time, in finding acceleration we began by dividing an increment of speed by an increment of time, and in discussing the ripples in the water we began by dividing an increment of area by an increment of radius.

The quotient thus obtained is $\frac{\Delta y}{\Delta x}$. That is,

$$\frac{\Delta y}{\Delta x}=\frac{\text{increment of } y}{\text{increment of } x}=\frac{\text{change in } y}{\text{change in } x}.$$

An examination of the tables of numerical values in §§ 3, 5, 6 shows that the quotient $\frac{\Delta y}{\Delta x}$ depends upon the magnitude of $\Delta x$, and that in each problem it was necessary to determine its limit as $\Delta x$ approached zero. This limit is called the *derivative of y with respect to x*, and is denoted by the symbol $\frac{dy}{dx}$. We have then

$$\frac{dy}{dx} = \text{limit of } \frac{\Delta y}{\Delta x} = \text{limit of } \frac{\text{change in } y}{\text{change in } x}.$$

At present the student is to take the symbol $\frac{dy}{dx}$ not as a fraction but as one undivided symbol to represent the derivative. Later we shall consider what meaning may be given to $dx$ and $dy$ separately. At this stage the form $\frac{dy}{dx}$ suggests simply the fraction $\frac{\Delta y}{\Delta x}$, which has approached a definite limiting value.

The process of finding the derivative is called *differentiation*, and we are said to *differentiate* $y$ with respect to $x$. From the definition and from the examples with which we began the book, the process is seen to involve the following four steps:

1. The assumption at pleasure of $\Delta x$.
2. The determination of the corresponding $\Delta y$.
3. The division of $\Delta y$ by $\Delta x$ to form $\frac{\Delta y}{\Delta x}$.
4. The determination of the limit approached by the quotient in step 3 as the increment assumed in step 1 approaches zero.

Let us apply this method to finding $\frac{dy}{dx}$ when $y = \frac{1}{x}$. Let $x_1$ be a definite value of $x$, and $y_1 = \frac{1}{x_1}$ the corresponding value of $y$.

1. Take $$\Delta x = h.$$

Then, by (1), $$x_2 = x_1 + h.$$

2. Then $$y_2 = \frac{1}{x_2} = \frac{1}{x_1 + h};$$

whence, by (2), $$\Delta y = \frac{1}{x_1 + h} - \frac{1}{x_1} = -\frac{h}{x_1^2 + hx_1}.$$

3. By division, $$\frac{\Delta y}{\Delta x} = -\frac{1}{x_1^2 + h_1 x}.$$

4. By inspection it is evident that the limit, as $h$ approaches zero, is $-\frac{1}{x_1^2}$, which is the value of the derivative when $x = x_1$. But $x_1$ may be any value of $x$; so we may drop the subscript 1 and write as a general formula

$$\frac{dy}{dx} = -\frac{1}{x^2}.$$

**EXERCISES**

Find from the definition the derivatives of the following expressions:

1. $y = 4(x^2 + 1)$.
2. $y = x^3 + 2x^2 + 1$.
3. $y = x^4 - x^2$.
4. $y = \frac{1}{x^3}$
5. $y = x^2 + \frac{1}{x^2}$.
6. $y = \frac{2}{2 + x}$.
7. $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x - 5$.
8. $y = \frac{3x^2 + 1}{x}$.

**8. Differentiation of a polynomial.** We shall now obtain formulas by means of which the derivative of a polynomial may be written down quickly. In the first place we have the theorem:

*The derivative of a polynomial is the sum of the derivatives of its separate terms.*

This follows from the definition of a derivative if we reflect that the change in a polynomial is the sum of the changes in its terms. A more formal proof will be given later.

We have then to consider the terms of a polynomial, which have in general the form $ax^n$. Since we wish to have general formulas, we shall omit the subscript 1 in denoting the first values of $x$ and $y$. We have then the theorem:

*If $y = ax^n$, where $n$ is a positive integer and $a$ is a constant, then*

$$\frac{dy}{dx} = anx^{n-1}. \qquad (1)$$

To prove this, apply the method of § 7:

1. Take $\Delta x = h$;

whence $x_2 = x + h$.

2. Then $$y_2 = ax_2^n = a(x+h)^n;$$
whence $\Delta y = a(x+h)^n - ax^n$

$$= a\left(nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n\right).$$

3. By division, $\dfrac{\Delta y}{\Delta x} = a\left(nx^{n-1} + \dfrac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}\right).$

4. By inspection, the limit approached by $\dfrac{\Delta y}{\Delta x}$, as $h$ approaches zero, is seen to be $anx^{n-1}$.

Therefore $\dfrac{dy}{dx} = anx^{n-1}$, as was to be proved.

The polynomial may also have a term of the form $ax$. This is only a special case of (1) with $n=1$, but for clearness we say explicitly:

*If $y = ax$, where $a$ is a constant, then*

$$\frac{dy}{dx} = a. \tag{2}$$

Finally, a polynomial may have a constant term $c$. For this we have the theorem:

*If $y = c$, where $c$ is a constant, then*

$$\frac{dy}{dx} = 0. \tag{3}$$

The proof of this is that as $c$ is constant, $\Delta c$ is always zero, no matter what the value of $\Delta x$ is. Hence

$$\frac{\Delta c}{\Delta x} = 0,$$

and therefore $$\frac{dc}{dx} = 0.$$

As an example of the use of the theorems, consider

$$y = 6x^4 + 4x^3 - 2x + 7.$$

We write at once

$$\frac{dy}{dx} = 24x^3 + 12x^2 - 2.$$

### EXERCISES

Find the derivative of each of the following polynomials:

1. $x^2 + x - 3$.
2. $\frac{1}{3}x^3 + 2x + 1$.
3. $x^4 + 4x^3 + 6x^2 + 4x + 1$.
4. $\frac{1}{3}x^6 + \frac{1}{4}x^4 + 2x^2 + 3$.
5. $x^5 - 4x^4 + x^2 - 4x$
6. $x^7 + 7x^5 + 21x^3 - 14x$
7. $x^4 - x^2 + 4x - 1$
8. $3 + 2x^2 + 7x^4 - 11x^5$.
9. $ax^3 + bx^2 + cx + f$
10. $a + bx^2 + cx^4 + ex^6$.

**9. Sign of the derivative.** *If* $\frac{dy}{dx}$ *is positive, an increase in the value of* $x$ *causes an increase in the value of* $y$. *If* $\frac{dy}{dx}$ *is negative, an increase in the value of* $x$ *causes a decrease in the value of* $y$.

To prove this theorem, let us consider that $\frac{dy}{dx}$ is positive. Then, since $\frac{dy}{dx}$ is the limit of $\frac{\Delta y}{\Delta x}$, it follows that $\frac{\Delta y}{\Delta x}$ is positive for sufficiently small values of $\Delta x$; that is, if $\Delta x$ is assumed positive, $\Delta y$ is also positive, and therefore an increase of $x$ causes an increase of $y$. Similarly, if $\frac{dy}{dx}$ is negative, it follows that $\frac{\Delta y}{\Delta x}$ is negative for sufficiently small values of $\Delta x$; that is, if $\Delta x$ is positive, $\Delta y$ must be negative, so that an increase of $x$ causes a decrease of $y$.

In applying this theorem it is necessary to determine the sign of a derivative. In case the derivative is a polynomial, this may be conveniently done by breaking it up into factors and considering the sign of each factor. It is obvious that a factor of the form $x - a$ is positive when $x$ is greater than $a$, and negative when $x$ is less than $a$.

Suppose, then, we wish to determine the sign of

$$(x + 3)(x - 1)(x - 6).$$

There are three factors to consider, and three numbers are important; namely, those which make one of the factors equal to zero. These numbers arranged in order of size are $-3$, $1$, and $6$. We have the four cases:

1. $x < -3$. All factors are negative and the product is ;ative.

2. $-3 < x < 1$. The first factor is positive and the others are negative. Therefore the product is positive.

3. $1 < x < 6$. The first two factors are positive and the last is negative. Therefore the product is negative.

4. $x > 6$. All factors are positive and the product is positive.

As an example of the use of the theorem, suppose we have

$$y = x^3 - 3x^2 - 9x + 27,$$

and ask for what values of $x$ an increase in $x$ will cause an increase in $y$. We form the derivative and factor it. Thus,

$$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x+1)(x-3).$$

Proceeding as above, we have the three cases:

1. $x < -1$. $\frac{dy}{dx}$ is positive, and an increase in $x$ therefore increases $y$.

2. $-1 < x < 3$. $\frac{dy}{dx}$ is negative, and therefore an increase in $x$ decreases $y$.

3. $x > 3$. $\frac{dy}{dx}$ is positive, and therefore an increase in $x$ increases $y$.

These results may be checked by substituting values of $x$ in the derivative.

**EXERCISES**

Find for what values of $x$ each of the following expressions will increase if $x$ is increased, and for what values of $x$ they will decrease if $x$ is increased:

1. $x^2 - 4x + 5$.
2. $3x^2 + 10x + 7$.
3. $1 + 5x - x^2$.
4. $7 - 3x - 3x^2$
5. $2x^3 + 3x^2 - 12x + 17$.
6. $\frac{1}{3}x^3 + x^2 - 15x + 11$.
7. $x^3 - x^2 - 5x + 5$
8. $1 + 6x + 12x^2 + 8x^3$.
9. $6 + 6x + 6x^2 - 2x^3 - 3x^4$.
10. $12 - 12x - 6x^2 + 4x^3 + 3x^4$.

**10. Velocity and acceleration (continued).** The method by which the speed of a body was determined in § 4 was in reality a method of differentiation, and the speed was the derivative of the distance with respect to the time. In that discussion, however, we so arranged each problem that the result was positive and gave a numerical measure (feet per second, miles per hour, etc.) for the rate at which the body was moving. Since we may now expect, on occasion, negative signs, we will replace the word *speed* by the word *velocity*, which we denote by the letter $v$. In accordance with the previous work, we have

$$v = \frac{ds}{dt}. \tag{1}$$

The distinction between speed and velocity, as we use the words, is simply one of algebraic sign. The speed is the numerical measure of the velocity and is always positive, but the velocity may be either positive or negative.

From § 9 the velocity is positive when the body so moves that $s$ increases with the time. This happens when the body moves in the direction in which $s$ is measured. On the other hand, the velocity is negative when the body so moves that $s$ decreases with the time. This happens when the body moves in the direction opposite to that in which $s$ is measured.

For example, suppose a body moves from $A$ to $B$ (Fig. 1), a distance of 100 mi., and let $P$ be the position of the body at a time $t$, and let us assume that we know that $AP = 4t$. If we measure $s$ from $A$, we have

$$s = AP = 4t;$$

whence
$$v = \frac{ds}{dt} = 4.$$

On the other hand, if we measure $s$ from $B$, we have

$$s = BP = 100 - 4t;$$

whence
$$v = \frac{ds}{dt} = -4.$$

We will now define acceleration by the formula

$$a = \frac{dv}{dt},$$

in full accord with § 5; or, since $v$ is found by differentiating $s$, we may write

$$a = \frac{d^2s}{dt^2},$$

where the symbol on the right indicates that $s$ is to be differentiated twice in succession. The result is called a second derivative.

A positive acceleration means that the velocity is increasing, but it must be remembered that the word *increase* is used in the algebraic sense. Thus, if a number changes from $-8$ to $-5$, it algebraically increases, although numerically it decreases. Hence, if a negative velocity is increased, the speed is less. Similarly, if the acceleration is negative, the velocity is decreasing, but if the velocity is negative, that means an increasing speed.

There are four cases of combinations of signs which may occur:

1. $v$ positive, $a$ positive. The body is moving in the direction in which $s$ is measured and with increasing speed.

2. $v$ positive, $a$ negative. The body is moving in the direction in which $s$ is measured and with decreasing speed.

3. $v$ negative, $a$ positive. The body is moving in the direction opposite to that in which $s$ is measured and with decreasing speed.

4. $v$ negative, $a$ negative. The body is moving in the direction opposite to that in which $s$ is measured and with increasing speed.

As an example, suppose a body thrown vertically into the air with a velocity of 96 ft. per second. From physics, if $s$ is measured up from the earth, we have

$$s = 96\,t - 16\,t^2.$$

From this equation we compute

$$v = 96 - 32\,t,$$

$$a = -32.$$

When $t < 3$, $v$ is positive and $a$ is negative. The body is going up with decreasing speed. When $t > 3$, $v$ is negative and $a$ is negative. The body is coming down with increasing speed.

On the other hand, suppose a body is thrown down from a height with a velocity of 96 ft. per second. Then, if $s$ is measured down from the point from which the body is thrown, we have, from physics,

$$s = 96\,t + 16\,t^2,$$

from which we compute

$$v = 96 + 32\,t,$$

$$a = 32.$$

Here $v$ is always positive and $a$ is always positive. Therefore the body is always going down (until it strikes) with an increasing speed.

**EXERCISES**

In the following examples find the expression for the velocity and determine when the body is moving in the direction in which $s$ is measured and when in the opposite direction.

1. $s = t^3 - 3t + 6.$
2. $s = 10t - t^2$
3. $s = t^3 - 9t^2 + 24t + 3.$
4. $s = 8 + 12t - 6t^2 + t^3.$
5. $s = t^4 - t^3 + 2$

In the following examples find the expressions for the velocity and the acceleration, and determine the periods of time during which the velocity is increasing and those during which it is decreasing.

6. $s = 3t^2 - 4t + 4.$
7. $s = 1 + 5t - t^2.$
8. $s = \frac{1}{3}t^3 - 2t^2 + 3t + 2.$
9. $s = t^3 - 5t^2 + 3t + 1.$
10. $s = 1 + 4t + 2t^2 - t^3.$

**11. Rate of change (continued).** In § 6 we have solved a problem in which we are finally led to find the rate of increase of the area of a circle with respect to its radius. This problem is typical of a good many others.

Let $x$ be an independent variable and $y$ a function of $x$. A change $\Delta x$ made in $x$ causes a change $\Delta y$ in $y$. The fraction $\frac{\Delta y}{\Delta x}$ compares the change in $y$ with the change in $x$. For example, if $\Delta x = .001$, and $\Delta y = .009061$, then we may say that the change in $y$ is at the average rate of $\frac{.009061}{.001} = 9.061$ per unit change in $x$. This does not mean that a unit change in $x$ would actually make a change of 9.061 units in $y$, any more than the statement that an automobile is moving at the rate of 40 mi. an hour means that it actually goes 40 mi. in an hour's time.

The fraction then gives a measure for the average rate at which $y$ is changing compared with the change in $x$. But this measure depends upon the value of $\Delta x$, as has been shown in the numerical calculations of § 6. To obtain a measure of the instantaneous rate of change of $y$ with respect to $x$ which shall not depend upon the magnitude of $\Delta x$, we must take the limit of $\frac{\Delta y}{\Delta x}$, as we did in § 6.

We have, therefore, the following definition:

*The derivative $\frac{dy}{dx}$ measures the rate of change of $y$ with respect to $x$.*

Another way of putting the same thing is to say that *if $\frac{dy}{dx}$ has the value $m$, then $y$ is changing $m$ times as fast as $x$.*

Still another way of expressing the same idea is to say that the rate of change of $y$ with respect to $x$ is defined as meaning the limit of the ratio of a small change in $y$ to a small change in $x$.

We will illustrate the above general discussion, and at the same time show how it may be practically applied, by the following example, which we will first solve arithmetically and then by calculus.

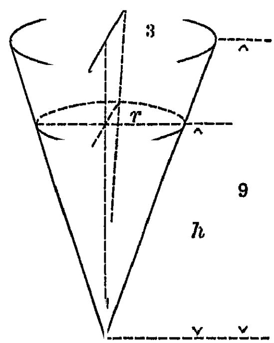


FIG. 4

Suppose we have a vessel in the shape of a cone (Fig 4) of radius 3 in. and altitude 9 in. into which water is being poured at the rate of 100 cu. in. per second. Required the rate at which the depth of the water is increasing when the depth is 6 in.

From similar triangles in the figure, if $h$ is the depth of the water and $r$ the radius of its surface, $r = \frac{h}{3}$. If $V$ is the volume of water,

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{27}\pi h^3. \qquad (1)$$

We are asked to find the rate at which the depth is increasing when $h$ is 6 in. Let us call that depth $h_1$, so that $h_1 = 6$. Then

$V_1 = 8\pi$. Now we will increase $h_1$ by successive small amounts and see how great an increase in $V_1$ is necessary to cause that change in $h_1$; that is, how much water must be poured in to raise the depth by that amount. The calculation may be tabulated as follows.

| $\Delta h$ | $\Delta V$ | $\frac{\Delta V}{\Delta h}$ |
|---|---|---|
| .1 | $.407\pi$ | $4\ 07\pi$ |
| 01 | $04007\pi$ | $4.007\pi$ |
| .001 | $0040007\pi$ | $4.0007\pi$ |

The limit of the numbers in the last column is evidently $4\pi$. Therefore the volume is increasing $4\pi$ times as fast as the depth. But, by hypothesis, the volume is increasing at the rate of 100 cu. in. per second, so that the depth is increasing at the rate of $\frac{100}{4\pi} = 7.96$ in. per second.

We have solved the problem by arithmetic to exhibit again the meaning of the derivative. The solution by calculus is much quicker. We begin by finding

$$\frac{dV}{dh} = \frac{1}{9}\pi h^2.$$

This is the general expression for the rate of change of $V$ with respect to $h$, or, in other words, it tells us that $V$ is instantaneously increasing $\frac{1}{9}\pi h^2$ times as fast as $h$ for any given $h$. Therefore, when $h = 6$, $V$ is increasing $4\pi$ times as fast as $h$, and as $V$ is increasing at the rate of 100 cu. in. per second, $h$ is increasing at the rate of $\frac{100}{4\pi} = 7.96$ in. per second.

**EXERCISES**

1. An icicle, which is melting, is always in the form of a right circular cone of which the vertical angle is 60° Find the rate of change of the volume of the icicle with respect to its length.

2. A series of right sections is made in a right circular cone of which the vertical angle is 90° How fast will the areas of the sections be increasing if the cutting plane recedes from the vertex at the rate of 3 ft. per second?

3. A solution is being poured into a conical filter at the rate of 5 cc per second and is running out at the rate of 1 cc. per second The radius of the top of the filter is 10 cm. and the depth of the filter is 30 cm Find the rate at which the level of the solution is rising in the filter when it is one fourth of the way to the top.

4. A peg in the form of a right circular cone of which the vertical angle is 60° is being driven into the sand at the rate of 1 in. per second, the axis of the cone being perpendicular to the surface of the sand, which is a plane. How fast is the lateral surface of the peg disappearing in the sand when the vertex of the peg is 5 in. below the surface of the sand?

5. A trough is in the form of a right prism with its ends equilateral triangles placed vertically The length of the trough is 10 ft It contains water which leaks out at the rate of $\frac{1}{2}$ cu ft per minute. Find the rate, in inches per minute, at which the level of the water is sinking in the trough when the depth is 2 ft.

6. A trough is 10 ft. long, and its cross section, which is vertical, is a regular trapezoid with its top side 4 ft. in length, its bottom side 2 ft, and its altitude 5 ft. It contains water to the depth of 3 ft, and water is running in so that the depth is increasing at the rate of 2 ft. per second. How fast is the water running in?

7. A balloon is in the form of a right circular cone with a hemispherical top. The radius of the largest cross section is equal to the altitude of the cone. The shape and proportions of the balloon are assumed to be unaltered as the balloon is inflated. Find the rate of increase of the volume with respect to the total height of the balloon.

8. A spherical shell of ice surrounds a spherical iron ball concentric with it. The radius of the iron ball is 6 in. As the ice melts, how fast is the mass of the ice decreasing with respect to its thickness?

**12. Graphs.** The relation between a variable $x$ and a function $y$ may be pictured to the eye by a *graph.* It is expected that students will have acquired some knowledge of the graph in the study of algebra, and the following brief discussion is given for a review.

Take two lines $OX$ and $OY$ (Fig. 5), intersecting at right angles at $O$, which is called the *origin of coördinates.* The line $OX$ is called the axis of $x$, and the line $OY$ the axis of $y$; together they are called the *coordinate axes*, or *axes of reference*. On $OX$ we lay off a distance $OM$ equal to any given value of $x$, measuring to the right if $x$ is positive and to the left if $x$ is negative. From $M$ we erect a perpendicular $MP$, equal in length to the value of $y$, measured up if $y$ is positive and down if $y$ is negative.

The point $P$ thus determined is said to have the coördinates $x$ and $y$ and is denoted by $(x, y)$. It follows that the numerical value of $x$ measures the distance of the point $P$ from $OY$, and the numerical value of $y$ measures the distance of $P$ from $OX$. The coördinate $x$ is called the *abscissa*, and the coördinate $y$ the *ordinate*. It is evident that any pair of coordinates $(x, y)$ fix a single point $P$, and that any point $P$ has a single pair of coordinates. The point $P$ is said to be plotted when its position is fixed in this way, and the plotting is conveniently carried out on paper ruled for that purpose into squares.

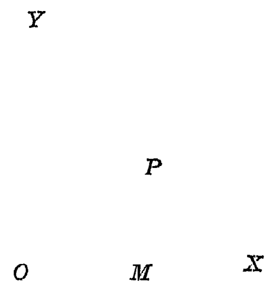


Fig 5

If $y$ is a function of $x$, values of $x$ may be assumed at pleasure and the corresponding values of $y$ computed. Then each pair of values $(x, y)$ may be plotted and a series of points found. The locus of these points is a curve called the *graph* of the function.

It may happen that the locus consists of distinct portions not connected in the graph. In this case it is still customary to say that these portions together form a single curve.

For example, let

$$y = 5x - x^2. \tag{1}$$

We assume values of $x$ and compute values of $y$. The results are exhibited in the following table:

| $x$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| $y$ | $-6$ | 0 | 4 | 6 | 6 | 4 | 0 | $-6$ |

These points are plotted and connected by a smooth curve, giving the result shown in Fig. 6. This curve should have the property that the coördinates of any point on it satisfy equation (1) and that any point whose coördinates satisfy (1) lies on the curve. It is called the graph both of the function $y$ and of the equation (1), and equation (1) is called the *equation of the curve.*

Of course we are absolutely sure of only those points whose coördinates we have actually computed. If greater accuracy is desired, more points must be found by assuming fractional values of $x$. For instance, there is doubt as to the shape of the curve between the points (2, 6) and (3, 6). We take, therefore, $x = 2\frac{1}{2}$ and find $y = 6\frac{1}{4}$. This gives us another point to aid us in drawing the graph. Later, by use of the calculus, we can show that this last point is really the highest point of the curve.

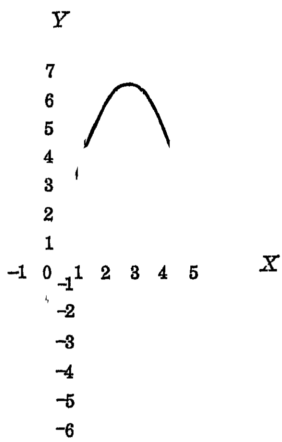



FIG. 6

The curve (Fig. 6) gives us a graphical representation of the way in which $y$ varies with $x$. We see, for example, that when $x$ varies from $-1$ to 2, $y$ is increasing; that when $x$ varies from 3 to 6, $y$ is decreasing; and that at some point between (2, 6) and (3, 6), not yet exactly determined, $y$ has its largest value.

It is also evident that the steepness of the curve indicates in some way the rate at which $y$ is increasing with respect to $x$. For example, when $x = -1$, an increase of 1 unit in $x$ causes an increase of 6 units in $y$; while when $x = 1$, an increase of 1 unit in $x$ causes an increase of only 2 units in $y$. The curve is therefore steeper when $x = -1$ than it is when $x = 1$.

Now we have seen that the derivative $\frac{dy}{dx}$ measures the rate of change of $y$ with respect to $x$. Hence we expect the derivative to be connected in some way with the steepness of the curve. We shall therefore discuss this connection in §§ 14 and 15.

## EXERCISES

Plot the graphs of the following equations:

1. $y = 2x + 3$.
2. $y = -2x + 4$.
3. $y = 5$.
4. $y = x^2 - 5x + 6$.
5. $y = x^2 + 4x + 8$
6. $y = 9 - 3x - x^2$.
7. $y = x^3$.
8. $y = x^3 - 4x^2$
9. $y = x^3 - 1$.

10. What is the effect on the graph of $y = mx + 3$ if different values are assigned to $m$? How are the graphs related? What does this indicate as to the meaning of $m$?

11. What is the effect on the graph of $y = 2x + b$ if different values are assigned to $b$? What is the meaning of $b$?

12. Show by similar triangles that $y = mx$ is always a straight line passing through $O$.

13. By the use of Exs. 11 and 12 show that $y = mx + b$ is always a straight line

**13. Real roots of an equation.** It is evident that the real roots of the equation $f(x) = 0$ determine points on the axis of $x$ at which the curve $y = f(x)$ crosses or touches that axis. Moreover, if $x_1$ and $x_2$ $(x_1 < x_2)$ are two values of $x$ such that $f(x_1)$ and $f(x_2)$ are of opposite algebraic sign, the graph is on one side of the axis when $x = x_1$, and on the other side when $x = x_2$. Therefore it must have crossed the axis an odd number of times between the points $x = x_1$ and $x = x_2$. Of course it may have touched the axis at any number of intermediate points. Now, if $f(x)$ has a factor of the form $(x - a)^k$, the curve $y = f(x)$ crosses the axis of $x$ at the point $x = a$ when $k$ is odd, and touches the axis of $x$ when $k$ is even. In each case the equation $f(x) = 0$ is said to have $k$ equal roots, $x = a$. Since, then, a point of crossing corresponds to an odd number of equal roots of an equation, and a point of touching corresponds to an even number of equal roots, it follows that the equation $f(x) = 0$ has an odd number of real roots between $x_1$ and $x_2$ if $f(x_1)$ and $f(x_2)$ have opposite signs.

The above gives a ready means of locating the real roots of an equation in the form $f(x) = 0$, for we have only to find two values of $x$, as $x_1$ and $x_2$, for which $f(x)$ has different signs. We then know that the equation has an odd number of real roots between these values, and the nearer together $x_1$ and $x_2$, the more nearly do we know the values of the intermediate roots. In locating the roots in this manner it is not necessary to construct the corresponding graph, though it may be helpful.

**Ex.** Find a real root of the equation $x^3 + 2x - 17 = 0$, accurate to two decimal places.

Denoting $x^3 + 2x - 17$ by $f(x)$ and assigning successive integral values to $x$, we find $f(2) = -5$ and $f(3) = 16$ Hence there is a real root of the equation between 2 and 3.

We now assign values to $x$ between 2 and 3, at intervals of one tenth, as 2.1, 2.2, 2.3, etc., and we begin with the values nearer 2; since $f(2)$ is nearer zero than is $f(3)$. Proceeding in this way we find $f(2.3) = -.233$ and $f(2.4) = 1.624$, hence the root is between 2.3 and 2.4.

Now, assigning values to $x$ between 2.3 and 2.4 at intervals of one hundredth, we find $f(2.31) = -.054$ and $f(2.32) = .127$, hence the root is between 2.31 and 2.32.

To determine the last decimal place accurately, we let $x = 2.315$ and find $f(2.315) = .037$. Hence the root is between 2.31 and 2.315 and is 2.31, accurate to two decimal places

If $f(2.315)$ had been negative, we should have known the root to be between 2.315 and 2.32 and to be 2.32, accurate to two decimal places.

#### EXERCISES

Find the real roots, accurate to two decimal places, of the following equations:

1. $x^3 + 2x - 6 = 0.$
2. $x^3 + x + 11 = 0$
3. $x^4 - 11x + 5 = 0.$
4. $x^4 - 4x^3 + 4 = 0.$
5. $x^3 - 3x^2 + 6x - 11 = 0.$
6. $x^3 + 3x^2 + 4x + 7 = 0.$

**14. Slope of a straight line.** Let $LK$ (Figs. 7 and 8) be any straight line not parallel to $OX$ or $OY$, and let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points on it. If we imagine a point to move on the line from $P_1$ to $P_2$, the increment of $x$ is $x_2 - x_1$ and the increment of $y$ is $y_2 - y_1$. We shall define the *slope* as *the ratio of the increment of $y$ to the increment of $x$* and denote it by $m$.

We have then, by definition,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}. \tag{1}$$

A geometric interpretation of the slope is easily given. For if we draw through $P_1$ a line parallel to $OX$, and through $P_2$ a line parallel to $OY$, and call $R$ the intersection of these lines, then $x_2 - x_1 = P_1R$ and $y_2 - y_1 = RP_2$. Also, if $\phi$ is the angle which the line makes with $OX$ measured as in the figure, then

$$m = \frac{RP_2}{P_1R} = \tan \phi. \tag{2}$$

It is clear from the figures as well as from formula (2) that the value of $m$ is independent of the two points chosen to define it, provided only that these are on the given line. We may therefore always choose the two points so that $y_2 - y_1$ is positive.

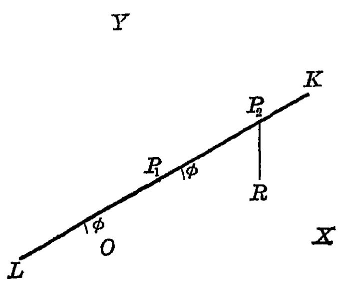



FIG. 7

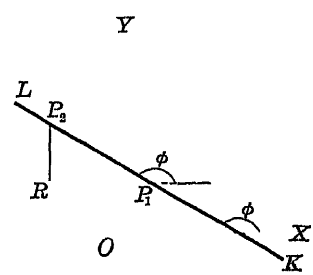



FIG. 8

Then if the line runs up to the right, as in Fig. 7, $x_2 - x_1$ is positive and the slope is positive. If the line runs down to the right, as in Fig. 8, $x_2 - x_1$ is negative and $m$ is negative. Therefore the algebraic sign of $m$ determines the general direction in which the line runs, while the magnitude of $m$ determines the steepness of the line.

Formula (1) may be used to obtain the equation of the line. Let $m$ be given a fixed value and the point $P_1(x_1, y_1)$ be held fixed, but let $P_2$ be allowed to wander over the line, taking on, therefore, variable coördinates $(x, y)$. Equation (1) may then be written

$$y - y_1 = m(x - x_1). \tag{3}$$

This is the equation of a line through a fixed point $(x_1, y_1)$ with a fixed slope $m$, since it is satisfied by the coördinates of any point on the line and by those of no other point.

In particular, $P_1(x_1, y_1)$ may be taken as the point with coördinates $(0, b)$ in which the line cuts $OY$. Then equation (3) becomes

$$y = mx + b. \tag{4}$$

Since any straight line not parallel to $OX$ or to $OY$ intersects $OY$ somewhere and has a definite slope, the equation of any such line may be written in the form (4).

It remains to examine lines parallel either to $OX$ or to $OY$. If the line is parallel to $OX$, we have no triangle as in Figs. 7 and 8, but the numerator of the fraction in (1) is zero, and we therefore say such a line has the slope 0. Its equation is of the form

$$y = b, \tag{5}$$

since it consists of all points for which this equation is true.

If the line is parallel to $OY$, again we have no triangle as in Figs. 7 and 8, but the denominator of the fraction in (1) is zero, and in accordance with established usage we say that the slope of the line is infinite, or that $m = \infty$. This means that as the position of the line approaches parallelism with $OY$ the value of the fraction (1) increases without limit. The equation of such a line is

$$x = a. \tag{6}$$

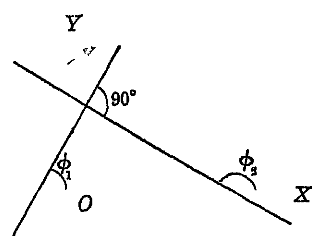


FIG. 9

Finally we notice that *any equation of the form*

$$Ax + By + C = 0 \tag{7}$$

*always represents a straight line.* This follows from the fact that the equation may be written either as (4), (5), or (6).

The line (7) may be plotted by locating two points and drawing a straight line through them. Its slope may be found by writing the equation in the form (4) when possible. The coefficient of $x$ is then the slope.

If two lines are parallel they make equal angles with $OX$. Therefore, if $m_1$ and $m_2$ are the slopes of the lines, we have, from (2),

$$m_2 = m_1. \tag{8}$$

If two lines are perpendicular and make angles $\phi_1$ and $\phi_2$ respectively with $OX$, it is evident from Fig. 9 that $\phi_2 = 90° + \phi_1$;

whence $\tan\phi_2 = -\cot\phi_1 = -\frac{1}{\tan\phi_1}$. Hence, if $m_1$ and $m_2$ are th slopes of the lines, we have

$$m_2 = -\frac{1}{m_1}. \tag{9}$$

It is easy to show, conversely, that if equation (8) is satis fied by two lines, they are parallel, and that if equation (9) i satisfied, they are perpendicular. Therefore equations (8) an (9) are the conditions for parallelism and perpendicularit respectively.

**Ex. 1.** Find the equation of a straight line passing through the poin (1, 2) and parallel to the straight line determined by the two points (4, 2 and (2, − 3)

By (1) the slope of the line determined by the two points (4, 2) an (2, − 3) is $\frac{-3-2}{2-4} = \frac{5}{2}$ Therefore, by (3), the equation of the require line is

$$y - 2 = \tfrac{5}{2}(x - 1),$$

which reduces to $\quad 5x - 2y - 1 = 0.$

**Ex. 2.** Find the equation of a straight line through the point (2, − 3 and perpendicular to the line $2x - 3y + 7 = 0$

The equation of the given straight line may be written in the forn $y = \frac{2}{3}x + \frac{7}{3}$, which is form (4). Therefore $m = \frac{2}{3}$. Accordingly, by (9) the slope of the required line is $-\frac{3}{2}$ By (3) the equation of the require line is

$$y + 3 = -\tfrac{3}{2}(x - 2),$$

which reduces to $\quad 3x + 2y = 0.$

**Ex. 3.** Find the equation of the straight line passing through the poin (− 3, 3) and the point of intersection of the two lines $2x - y - 3 = 0$ an $3x + 2y - 1 = 0$

The coordinates of the point of intersection of the two given lines mus satisfy the equation of each line. Therefore the coördinates of the poin of intersection are found by solving the two equations simultaneously. The result is (1, − 1)

We now have the problem to pass a straight line through the points (− 3, 3) and (1, − 1). By (1) the slope of the required line is $\frac{3+1}{-3-1} = -1$. Therefore, by (3), the equation of the line is

$$y + 1 = -(x - 1),$$

which reduces to $\quad x + y = 0.$

**EXERCISES**

1. Find the equation of the straight line which passes through $(2, -3)$ with the slope 3

2. Find the equation of the straight line which passes through $(-3, 1)$ with the slope $-\frac{2}{3}$

3. Find the equation of the straight line passing through the points $(1, 4)$ and $(\frac{2}{3}, \frac{1}{2})$.

4. Find the equation of the straight line passing through the points $(2, -3)$ and $(-3, -3)$.

5. Find the equation of the straight line passing through the point $(2, -2)$ and making an angle of $60°$ with $OX$.

6. Find the equation of the straight line passing through the point $(\frac{1}{2}, -\frac{3}{2})$ and making an angle of $135°$ with $OX$.

7. Find the equation of the straight line passing through the point $(-2, -3)$ and parallel to the line $x + 2y + 1 = 0$.

8. Find the equation of the straight line passing through the point $(-2, -3)$ and perpendicular to the line $3x + 4y - 12 = 0$.

9. Find the equation of the straight line passing through the point $(\frac{1}{2}, \frac{1}{4})$ and parallel to the straight line determined by the two points $(\frac{2}{3}, \frac{3}{4})$ and $(\frac{1}{3}, -\frac{1}{4})$

10. Find the equation of the straight line passing through $(\frac{1}{2}, -\frac{1}{4})$ and perpendicular to the straight line determined by the points $(2, -1)$ and $(-3, 5)$.

11. If $\beta$ is the angle between two straight lines which make angles $\phi_1$ and $\phi_2 (\phi_2 > \phi_1)$ respectively with $OX$, prove from a diagram similar to Fig 9 that $\beta = \phi_2 - \phi_1$. If $\tan\phi_1 = m_1$ and $\tan\phi_2 = m_2$, prove by trigonometry that

$$\tan\beta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$

12. Find the angle between the lines $x - 2y + 1 = 0$ and $2x - 3y + 7 = 0$.

13. Find the angle between the lines $2x - 4y + 5 = 0$ and $5x + 2y - 6 = 0$.

14. Find the angle between the lines $y = 3x + 4$ and $x + 3y + 7 = 0$.

15. The vertex of a right angle is at $(2, -4)$ and one of its sides passes through the point $(-2, 2)$. Find the equation of the other side.

16. Find the foot of the perpendicular from the origin to the line $2x - 3y + 10 = 0$.

**15. Slope of a curve.** Let $AB$ (Fig. 10) be any curve serving as the graphical representation of a function of $x$. Let $P_1$ be any point on the curve with coordinates $x_1 = OM_1$, $y_1 = M_1P_1$. Take $\Delta x = M_1M_2$ and draw the perpendicular $M_2P_2$, fixing the point $P_2$ on the curve with the coordinates $x_2 = OM_2$, $y_2 = M_2P_2$. Draw $P_1R$ parallel to $OX$. Then

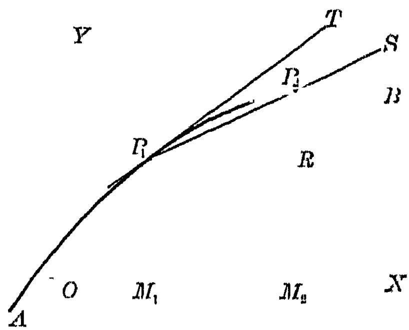



FIG. 10

$$P_1R = M_1M_2 = \Delta x,$$

$$RP_2 = M_2P_2 - M_1P_1 = \Delta y,$$

and $$\frac{\Delta y}{\Delta x} = \frac{RP_2}{P_1R}.$$

Draw the straight line $P_1P_2$, prolonging it to form a secant $P_1S$. Then, by § 14, $\dfrac{\Delta y}{\Delta x}$ is the slope of the secant $P_1S$, and may be called the average slope of the curve between the points $P_1$ and $P_2$.

To obtain a number which may be used for the actual slope of the curve at the point $P_1$, it is necessary to use the limit process (with which the student should now be familiar), by which we allow $\Delta x$ to become smaller and smaller and the point $P_2$ to approach $P_1$ along the curve. The result is the derivative of $y$ with respect to $x$, and we have the following result:

*The slope of a curve at any point is given by the value of the derivative $\dfrac{dy}{dx}$ at that point.*

As this limit process takes place, the point $P_2$ approaching the point $P_1$, it appears from the figure that the secant $P_1S$ approaches a limiting position $P_1T$. The line $P_1T$ is called a *tangent* to the curve, *a tangent being then by definition the line approached as a limit by a secant through two points of the curve as the two points approach coincidence.* It follows that the slope of the tangent is the limit of the slope of the secant. Therefore,

*The slope of a curve at any point is the same as the slope of the tangent at that point.*

From this and § 9 we may at once deduce the theorem:

*If the derivative is positive, the curve runs up to the right. If the derivative is negative, the curve runs down to the right. If the derivative is zero, the tangent to the curve is parallel to OX. If the derivative is infinite, the tangent to the curve is perpendicular to OX.*

The values of $x$ which make $\frac{dy}{dx}$ zero or infinite are of particular interest in the plotting of a curve. If the derivative changes its sign at such a point, the curve will change its direction from down to up or from up to down. Such a point will be called a *turning-point.* If $y$ is an algebraic polynomial, its derivative cannot be infinite; so we shall be concerned in this chapter only with turning-points for which

$$\frac{dy}{dx} = 0.$$

They are illustrated in the two following examples:

**Ex. 1.** Consider equation (1) of § 12,

$$y = 5x - x^2.$$

Here
$$\frac{dy}{dx} = 5 - 2x = 2\left(\frac{5}{2} - x\right)$$

Equating $\frac{dy}{dx}$ to zero and solving, we have $x = \frac{5}{2}$ as a possible turning-point It is evident that when $x < \frac{5}{2}$, $\frac{dy}{dx}$ is positive, and when $x > \frac{5}{2}$, $\frac{dy}{dx}$ is negative Therefore $x = \frac{5}{2}$ corresponds to a turning-point of the curve at which the latter changes its direction from up to down It may be called a *high point* of the curve

**Ex. 2.** Consider

$$y = \tfrac{1}{8}(x^3 - 3x^2 - 9x + 32)$$

Here
$$\frac{dy}{dx} = \frac{3}{8}(x^2 - 2x - 3) = \frac{3}{8}(x - 3)(x + 1).$$

Equating $\frac{dy}{dx}$ to zero and solving, we have $x = -1$ and $x = 3$ as possible turning-points From the factored form of $\frac{dy}{dx}$, and reasoning as in § 9, we see that when $x < -1$, $\frac{dy}{dx}$ is positive; when $-1 < x < 3$, $\frac{dy}{dx}$ is negative;

when $x > 3$, $\frac{dy}{dx}$ is positive. Therefore both $x = -1$ and $x = 3$ give turning-points, the former giving a high point, and the latter a low point. Substituting these values of $x$ in the equation of the curve, we find the high point to be $(-1, 4\frac{5}{8})$ and the low point to be $(3, \frac{5}{8})$. The graph is shown in Fig 11

It is to be noticed that the solutions of the equation $\frac{dy}{dx} = 0$ do not always give turning-points as illustrated in the next example.

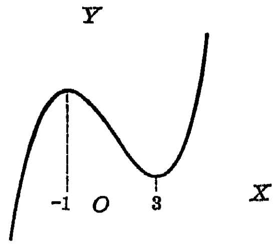



FIG. 11

**Ex. 3.** Consider $y = \frac{1}{8}(x^3 - 9x^2 + 27x - 19)$

Here $$\frac{dy}{dx} = x^2 - 6x + 9 = (x-3)^2.$$

Solving $\frac{dy}{dx} = 0$, we have $x = 3$; but since the derivative is a perfect square, it is never negative Therefore $x = 3$ does not give a turning-point, although when $x = 3$ the tangent to the curve is parallel to $OX$. The curve is shown in Fig. 12.

The equation of the tangent to a curve at a point $(x_1, y_1)$ is easily written down We let $\left(\frac{dy}{dx}\right)_1$ represent the value of $\frac{dy}{dx}$ at the point $(x_1, y_1)$. Then $m = \left(\frac{dy}{dx}\right)_1$, and, from (3), § 14, the equation of the tangent is

$$y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1). \qquad (1)$$

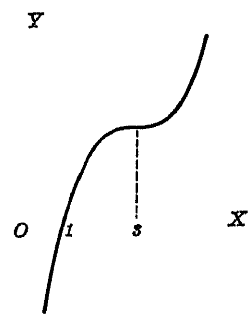



FIG. 12

**Ex. 4.** Find the equation of the tangent at $(1, -1)$ to the curve

$$y = x^2 - 4x + 2$$

We have $$\frac{dy}{dx} = 2x - 4,$$

and $$x_1 = 1,\ y_1 = -1,\ \left(\frac{dy}{dx}\right)_1 = -2.$$

Therefore the equation of the tangent is

$$y + 1 = -2(x - 1),$$

which reduces to $$2x + y - 1 = 0.$$

From (2), § 14, it also follows that if $\phi$ is the angle which the tangent at any point of a curve makes with $OX$, then

$$\frac{dy}{dx} = \tan\phi. \tag{2}$$

**EXERCISES**

Locate the turning-points, and then plot the following curves:

1. $y = 3x^2 + 4x + 4$.
2. $y = 3 + 3x - 2x^2$.
3. $y = x^3 - 3x^2 + 4$.
4. $y = x^3 - 6x^2 + 9x + 3$.
5. $y = \frac{1}{3}(2x^3 + 3x^2 - 12x - 20)$
6. $y = 2 + 9x + 3x^2 - x^3$

7. Find the equation of the tangent to the curve $y = 3 - 2x + x^2$ at the point for which $x = 2$

8. Find the equation of the tangent to the curve $y = 1 + 3x - x^2 - 3x^3$ at the point for which $x = -1$.

9. Find the points on the curve $y = x^3 + 3x^2 - 3x + 1$ at which the tangents to the curve have the slope 6

10. Find the equations of the tangents to the curve

$$y = x^3 + 2x^2 - x + 2$$

which make an angle 135° with $OX$

11. Find the equations of the tangents to the curve $y = x^3 + x^2 - 2x$ which are perpendicular to the line $3x + 2y + 4 = 0$

12. Find the angle of intersection of the tangents to the curve $y = x^3 + x^2 - 2$ at the points for which $x = -1$ and $x = 1$ respectively.

**16. The second derivative.** The derivative of the derivative is called the *second derivative* and is indicated by the symbol $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$. We have met an illustration of this in the case of the acceleration. We wish to see now what the second derivative means for the graph.

Since $\frac{dy}{dx}$ is equal to the slope of the graph, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\text{slope}).$$

From this and § 9 we have the following theorem:

*If the second derivative is positive, the slope is increasing as $x$ increases; and if the second derivative is negative, the slope is decreasing as $x$ increases.*

We may accordingly use the second derivative to distinguish between the high turning-points and the low turning-points of a curve, as follows:

If, when $x = a$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is positive, it is evident that $\frac{dy}{dx}$ is increasing through zero; hence, when $x < a$, $\frac{dy}{dx}$ is negative, and when $x > a$, $\frac{dy}{dx}$ is positive. The point for which $x = a$ is therefore a low turning-point, by § 9.

Similarly, if, when $x = a$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is negative, it is evident that $\frac{dy}{dx}$ is decreasing through zero; hence, when $x < a$, $\frac{dy}{dx}$ is positive, and when $x > a$, $\frac{dy}{dx}$ is negative. The point for which $x = a$ is therefore a high turning-point of the curve, by § 9.

These conclusions may be stated as follows:

*If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is positive at a point of a curve, that point is a low point of the curve. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ is negative at a point of a curve, that point is a high point of the curve.*

In addition to the second derivative, we may also have third, fourth, and higher derivatives indicated by the symbols $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, etc. These have no simple geometric meaning.

## EXERCISES

Plot the following curves after determining their high and low points by the use of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$:

1. $y = 3x^3 - x - 2$
2. $y = 3 + 8x - x^2 - x^3$.
3. $y = 7 - 18x - 3x^2 + 4x^3$.
4. $y = 6 - 15x + 18x^2 - 4x^3$.

**17. Maxima and minima.** If $f(a)$ is a value of $f(x)$ which is greater than the values obtained either by increasing or by decreasing $x$ by a small amount, $f(a)$ is called a *maximum* value of $f(x)$. If $f(a)$ is a value of $f(x)$ which is smaller than the values of $f(x)$ found either by increasing or by decreasing $x$ by a small amount, $f(a)$ is called a *minimum* value of $f(x)$.

It is evident that if we place

$$y = f(x)$$

and make the graph of this equation, a maximum value of $f(x)$ occurs at a high point of the curve and a minimum value at a low point. From the previous sections we have, accordingly, the following rule for finding maxima and minima:

*To find the values of $x$ which give maximum or minimum values of $y$, solve the equation*

$$\frac{dy}{dx} = 0.$$

If $x = a$ is a root of this equation, it must be tested to see whether it gives a maximum or minimum, and which. We have two tests:

TEST I. *If the sign of $\frac{dy}{dx}$ changes from $+$ to $-$ as $x$ increases through $a$, then $x = a$ gives a maximum value of $y$. If the sign of $\frac{dy}{dx}$ changes from $-$ to $+$ as $x$ increases through $a$, then $x = a$ gives a minimum value of $y$.*

TEST II. *If $x = a$ makes $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ negative, then $x = a$ gives a maximum value of $y$. If $x = a$ makes $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ positive, then $x = a$ gives a minimum value of $y$.*

Either of these tests may be applied according to convenience. It may be noticed that Test I always works, while Test II fails to give information if $\frac{d^2y}{dx^2} = 0$ when $x = a$. It is also frequently possible by the application of common sense to a problem to determine whether the result is a maximum or minimum, and neither of the formal tests need then be applied.

**Ex. 1.** A rectangular box is to be formed by cutting a square from each corner of a rectangular piece of cardboard and bending the resulting figure. The dimensions of the piece of cardboard being 20 in by 30 in., required the largest box which can be made.

Let $x$ be the side of the square cut out. Then, if the cardboard is bent along the dotted lines of Fig 13, the dimensions of the box are $30 - 2x$, $20 - 2x$, $x$. Let $V$ be the volume of the box.

Then
$$V = x(20 - 2x)(30 - 2x)$$
$$= 600x - 100x^2 + 4x^3.$$
$$\frac{dV}{dx} = 600 - 200x + 12x^2.$$

Equating $\frac{dV}{dx}$ to zero, we have

$$3x^2 - 50x + 150 = 0;$$

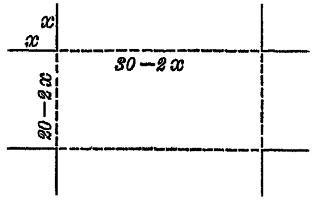


Fig. 13

whence
$$x = \frac{25 \pm 5\sqrt{7}}{3} = 3.9 \text{ or } 12\ 7.$$

The result 12.7 is impossible, since that amount cannot be cut twice from the side of 20 in The result 3.9 corresponds to a possible maximum, and the tests are to be applied.

To apply Test I we write $\frac{dV}{dx}$ in the factored form

$$\frac{dV}{dx} = 12(x - 3\ 9)(x - 12.7),$$

when it appears that $\frac{dV}{dx}$ changes from + to −, as $x$ increases through 3.9 Hence $x = 3.9$ gives a maximum value of $V$.

To apply Test II we find $\frac{d^2V}{dx^2} = -200 + 24x$ and substitute $x = 3.9$. The result is negative. Therefore $x = 3.9$ gives a maximum value of $V$.

The maximum value of $V$ is 1056 + cu. in., found by substituting $x = 3.9$ in the equation for $V$.

**Ex. 2.** A piece of wood is in the form of a right circular cone, the altitude and the radius of the base of which are each equal to 12 in. What is the volume of the largest right circular cylinder that can be cut from this piece of wood, the axis of the cylinder to coincide with the axis of the cone?

Let $x$ be the radius of the base of the required cylinder, $y$ its altitude, and $V$ its volume. Then
$$V = \pi x^2 y. \tag{1}$$

We cannot, however, apply our method directly to this value of $V$, since it involves two variables $x$ and $y$. It is necessary to find a connection between $x$ and $y$ and eliminate one of them. To do so, consider Fig. 14, which is a cross section of cone and cylinder. From similar triangles we have

$$\frac{FE}{EC} = \frac{AD}{DC};$$

that is,
$$\frac{y}{12 - x} = \frac{12}{12};$$

whence
$$y = 12 - x.$$

Substituting in (1), we have

$$V = 12\,\pi x^2 - \pi x^3;$$

whence
$$\frac{dV}{dx} = 24\,\pi x - 3\,\pi x^2.$$

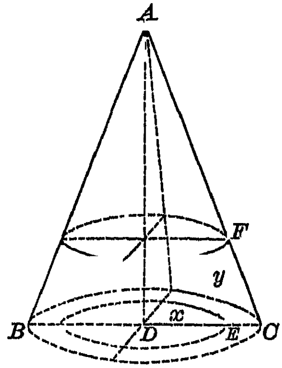


FIG. 14

Equating $\frac{dV}{dx}$ to zero and solving, we find $x = 0$ or 8. The value $x = 0$ is evidently not a solution of the problem, but $x = 8$ is a possible solution.

Applying Test I, we find that as $x$ increases through the value 8, $\frac{dV}{dx}$ changes its sign from $+$ to $-$. Applying Test II, we find that $\frac{d^2V}{dx^2} = 24\,\pi - 6\,\pi x$ is negative when $x = 8$. Either test shows that $x = 8$ corresponds to a maximum value of $V$. To find $V$ substitute $x = 8$ in the expression for $V$. We have $V = 256\,\pi$ cu. in.

## EXERCISES

**1.** A piece of wire of length 20 in. is bent into a rectangle one side of which is $x$. Find the maximum area.

**2.** A gardener has a certain length of wire fencing with which to fence three sides of a rectangular plot of land, the fourth side being made by a wall already constructed. Required the dimensions of the plot which contains the maximum area.

**3.** A gardener is to lay out a flower bed in the form of a sector of a circle. If he has 20 ft. of wire with which to inclose it, what radius will he take for the circle to have his garden as large as possible?

**4.** In a given isosceles triangle of base 20 and altitude 10 a rectangle of base $x$ is inscribed. Find the rectangle of maximum area.

**5.** A right circular cylinder with altitude $2x$ is inscribed in a sphere of radius $a$. Find the cylinder of maximum volume.

6. A rectangular box with a square base and open at the top is to be made out of a given amount of material. If no allowance is made for the thickness of the material or for waste in construction, what are the dimensions of the largest box that can be made?

7. A piece of wire 12 ft. in length is cut into six portions, two of one length and four of another Each of the two former portions is bent into the form of a square, and the corners of the two squares are fastened together by the remaining portions of wire, so that the completed figure is a rectangular parallelepiped Find the lengths into which the wire must be divided so as to produce a figure of maximum volume

8. The strength of a rectangular beam varies as the product of its breadth and the square of its depth Find the dimensions of the strongest rectangular beam that can be cut from a circular cylindrical log of radius $a$ inches

9. An isosceles triangle of constant perimeter is revolved about its base to form a solid of revolution. What are the altitude and the base of the triangle when the volume of the solid generated is a maximum?

10 The combined length and girth of a postal parcel is 60 in. Find the maximum volume (1) when the parcel is rectangular with square cross section, (2) when it is cylindrical

11. A piece of galvanized iron $b$ feet long and $a$ feet wide is to be bent into a U-shaped water drain $b$ feet long. If we assume that the cross section of the drain is exactly represented by a rectangle on top of a semicircle, what must be the dimensions of the rectangle and the semicircle in order that the drain may have the greatest capacity (1) when the drain is closed on top? (2) when it is open on top?

12. A circular filter paper 10 in. in diameter is folded into a right circular cone. Find the height of the cone when it has the greatest volume

**18. Integration.** It is often desirable to reverse the process of differentiation. For example, if the velocity or the acceleration of a moving body is given, we may wish to find the distance traversed; or if the slope of a curve is given, we may wish to find the curve.

The inverse operation to differentiation is called *integration*, and the result of the operation is called an *integral*. In the case of a polynomial it may be performed by simply working the formulas of differentiation backwards. Thus, if $n$ is a positive integer and

$$\frac{dy}{dx} = ax^n,$$

then
$$y = \frac{ax^{n+1}}{n+1} + C. \qquad (1)$$

The first term of this formula is justified by the fact that if it is differentiated, the result is exactly $ax^n$. The second term is justified by the fact that the derivative of a constant is zero. The constant $C$ may have any value whatever and cannot be determined by the process of integration. It is called the *constant of integration* and can only be determined in a given problem by special information given in the problem. The examples will show how this is to be done.

Again, if
$$\frac{dy}{dx} = a,$$

then
$$y = ax + C. \qquad (2)$$

This is only a special case of (1) with $n = 0$.

Finally, if
$$\frac{dy}{dx} = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$
$$y = \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \cdots + \frac{a_{n-1}x^2}{2} + a_nx + C. \qquad (3)$$

**Ex. 1.** The velocity $v$ with which a body is moving along a straight line $AB$ (Fig. 15) is given by the equation

$$v = 16t + 5$$

How far will the body move in the time from $t = 2$ to $t = 4$?

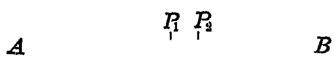


FIG. 15

If when $t = 2$ the body is at $P_1$, and if when $t = 4$ it is at $P_2$, we are to find $P_1P_2$.

By hypothesis,
$$\frac{ds}{dt} = 16t + 5.$$

Therefore
$$s = 8t^2 + 5t + C. \qquad (1)$$

We have first to determine $C$. If $s$ is measured from $P_1$, it follows that when

$$t = 2, \qquad s = 0.$$

Therefore, substituting in (1), we have

$$0 = 8(2)^2 + 5(2) + C;$$

whence $$C = -42,$$

and (1) becomes $$s = 8t^2 + 5t - 42. \tag{2}$$

This is the distance of the body from $P_1$ at any time $t$. Accordingly, it remains for us to substitute $t = 4$ in (2) to find the required distance $P_1P_2$. There results

$$P_1P_2 = 8(4)^2 + 5(4) - 42 = 106.$$

If the velocity is in feet per second, the required distance is in feet.

**Ex. 2.** Required the curve the slope of which at any point is twice the abscissa of the point.

By hypothesis, $\dfrac{dy}{dx} = 2x$

Therefore $$y = x^2 + C \tag{1}$$

Any curve whose equation can be derived from (1) by giving $C$ a definite value satisfies the condition of the problem (Fig. 16). If it is required that the curve should pass through the point (2, 3), we have, from (1),

$$3 = 4 + C;$$

whence $$C = -1,$$

and therefore the equation of the curve is

$$y = x^2 - 1.$$

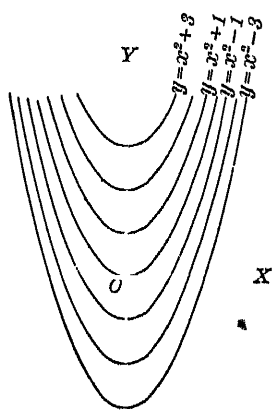


FIG. 16

But if it is required that the curve should pass through $(-3, 10)$, we have, from (1),

$$10 = 9 + C;$$

whence $$C = 1,$$

and the equation is $$y = x^2 + 1$$

## EXERCISES

In the following problems $v$ is the velocity, in feet per second, of a moving body at any time $t$.

**1.** If $v = 32t + 30$, how far will the body move in the time from $t = 2$ to $t = 5$?

2. If $v = 3t^2 + 4t + 2$, how far will the body move in the time from $t = 1$ to $t = 3$?

3. If $v = 20t + 25$, how far will the body move in the fourth second?

4. If $v = t^2 - 2t + 4$, how far will the body move in the fifth and sixth seconds?

5. If $v = 192 - 32t$, how far will the body move before $v = 0$?

6. A curve passes through the point $(1, -1)$, and its slope at any point $(x, y)$ is 3 more than twice the abscissa of the point. What is its equation?

7. The slope of a curve at any point $(x, y)$ is $6x^2 + 2x - 4$, and the curve passes through the point $(0, 6)$. What is its equation?

8. The slope of a curve at any point $(x, y)$ is $4 - 3x - x^2$, and the curve passes through the point $(-6, 1)$. What is its equation?

9. A curve passes through the point $(5, -2)$, and its slope at any point $(x, y)$ is one half the abscissa of the point. What is its equation?

10. A curve passes through the point $(-2, -4)$, and its slope at any point $(x, y)$ is $x^2 - x + 1$. What is its equation?

**19. Area.** An important application of integration occurs in the problem of finding an area bounded as follows:

Let $RS$ (Fig. 17) be any curve with the equation $y = f(x)$, and let $ED$ and $BC$ be any two ordinates. It is required to find the area bounded by the curve $RS$, the two ordinates $ED$ and $BC$, and the axis of $x$.

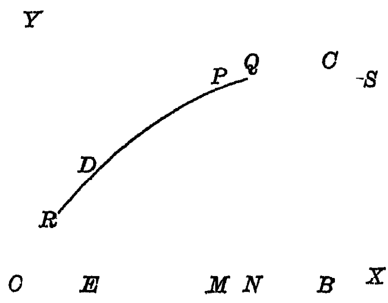


FIG. 17

Take $MP$, any variable ordinate between $ED$ and $BC$, and let us denote by $A$ the area $EMPD$ bounded by the curve, the axis of $x$, the fixed ordinate $ED$, and the variable ordinate $MP$.

It is evident that as values are assigned to $x = OM$, different positions of $MP$ and corresponding values of $A$ are determined. Hence $A$ is a function of $x$ for which we will find $\frac{dA}{dx}$.

Take $MN = \Delta x$ and draw the corresponding ordinate $NQ$. Then the area $MNQP = \Delta A$. If $L$ is the length of the longest ordinate of the curve between $MP$ and $NQ$, and $s$ is the length of the shortest ordinate in the same region, it is evident that

$$s\Delta x < \Delta A < L\Delta x,$$

for $L\Delta x$ is the area of a rectangle entirely surrounding $\Delta A$, and $s\Delta x$ is the area of a rectangle entirely included in $\Delta A$.

Dividing by $\Delta x$, we have

$$s < \frac{\Delta A}{\Delta x} < L.$$

As $\Delta x$ approaches zero, $NQ$ approaches coincidence with $MP$, and hence $s$ and $L$, which are always between $NQ$ and $MP$, approach coincidence with $MP$. Hence at the limit we have

$$\frac{dA}{dx} = MP = y = f(x). \tag{1}$$

Therefore, by integrating,

$$A = F(x) + C, \tag{2}$$

where $F(x)$ is used simply as a symbol for any function whose derivative is $f(x)$.

We must now find $C$. Let $OE = a$. When $MP$ coincides with $ED$, the area is zero. That is, when

$$x = a,\ A = 0.$$

Substituting in (2), we have

$$0 = F(a) + C;$$

whence $$C = -F(a),$$

and therefore (2) becomes

$$A = F(x) - F(a). \tag{3}$$

Finally, let us obtain the required area $EBCD$. If $OB = b$, this will be obtained by placing $x = b$ in (3). Therefore we have, finally,

$$A = F(b) - F(a). \tag{4}$$

In solving problems the student is advised to begin with formula (1) and follow the method of the text, as shown in the following example:

**Ex.** Find the area bounded by the axis of $x$, the curve $y = \frac{1}{3}x^2$, and the ordinates $x = 1$ and $x = 3$.

In Fig 18, $BE$ is the line $x = 1$, $CD$ is the line $x = 3$, and the required area is the area $BCDE$ Then, by (1),

$$\frac{dA}{dx} = \frac{1}{3}x^2,$$

whence $\quad A = \frac{1}{9}x^3 + C$

When $x = 1$, $A = 0$, and therefore

$$0 = \tfrac{1}{9} + C,$$

whence $\quad C = -\frac{1}{9}$,

and $\quad A = \frac{1}{9}x^3 - \frac{1}{9}$

Finally, when

$$x = 3,$$
$$A = \tfrac{1}{9}(3)^3 - \tfrac{1}{9} = 2\tfrac{8}{9}$$

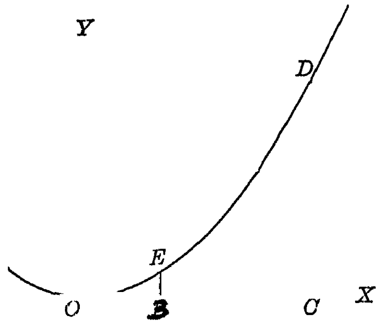



FIG 18

**EXERCISES**

1. Find the area bounded by the curve $y = 4x - x^2$, the axis of $x$, and the lines $x = 1$ and $x = 3$.

2. Find the area bounded by the curve $y = x^2 + 8x + 18$, the axis of $x$, and the lines $x = -6$ and $x = -2$.

3. Find the area bounded by the curve $y = 16 + 12x - x^3$, the axis of $x$, and the lines $x = -1$ and $x = 2$

4. Find the area bounded by the curve $y + x^2 - 9 = 0$ and the axis of $x$.

5. Find the area bounded by the axis of $x$ and the curve $y = 2x - x^2$.

6. Find the smaller of the two areas bounded by the curve $y = 5x^2 - x^3$, the axis of $x$, and the line $x = 1$.

7. Find the area bounded by the axis of $x$, the axis of $y$, and the curve $4y = x^2 - 6x + 9$.

8. Find the area bounded by the curve $y = x^3 - 2x^2 - 4x + 8$ and the axis of $x$.

9. If $A$ denotes the area bounded by the axis of $y$, the curve $x=f(y)$, a fixed line $y=b$, and any variable line parallel to $OX$, prove that

$$\frac{dA}{dy}=x=f(y).$$

10. If $A$ denotes an area bounded above by the curve $y=f(x)$, below by the curve $y=F(x)$, at the left by the fixed line $x=a$, and at the right by a variable ordinate, prove that

$$\frac{dA}{dx}=f(x)-F(x)$$

**20. Differentials.** The derivative has been defined as the limit of $\frac{\Delta y}{\Delta x}$ and has been denoted by the symbol $\frac{dy}{dx}$. This symbol is in the fractional form to suggest that it is the *limit* of a fraction, but thus far we have made no attempt to treat it as a fraction.

It is, however, desirable in many cases to treat the derivative as a fraction and to consider $dx$ and $dy$ as separate quantities. To do this it is necessary to define $dx$ and $dy$ in such a manner that their quotient shall be the derivative. We shall begin by defining $dx$, when $x$ is the independent variable; that is, the variable whose values can be assumed independently of any other quantity.

We shall call $dx$ the *differential* of $x$ and define it as a change in $x$ which may have any magnitude, but which is generally regarded as small and may be made to approach zero as a limit. In other words, *the differential of the independent variable $x$ is identical with the increment of $x$*; that is,

$$dx=\Delta x. \tag{1}$$

After $dx$ has been defined, it is necessary to define $dy$ so that its quotient by $dx$ is the derivative. Therefore, if $y=f(x)$ and $\frac{dy}{dx}=f'(x)$, we have

$$dy=f'(x)\,dx. \tag{2}$$

That is, *the differential of the function $y$ is equal to the derivative times the differential of the independent variable $x$.*

In equation (2) the derivative appears as the coefficient of $dx$. For this reason it is sometimes called the *differential coefficient*.

It is important to notice the distinction between $dy$ and $\Delta y$. The differential $dy$ is not the limit of the increment $\Delta y$, since both $dy$ and $\Delta y$ have the same limit, zero. Neither is $dy$ equal to a very small increment $\Delta y$, since it generally differs in value from $\Delta y$. It is true, however, that when $dy$ and $\Delta y$ both become small, they differ by a quantity which is small compared with each of them. These statements may best be understood from the following examples:

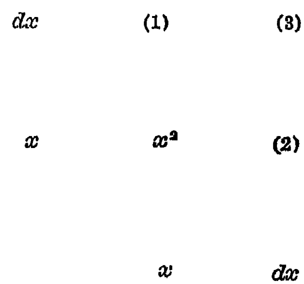


FIG. 19

**Ex. 1.** Let $A$ be the area of a square with the side $x$ so that

$$A = x^2$$

If $x$ is increased by $\Delta x = dx$, $A$ is increased by $\Delta A$, where

$$\Delta A = (x + dx)^2 - x^2 = 2\,x\,dx + (dx)^2.$$

Now, by (2), $\quad dA = 2\,x\,dx,$

so that $\Delta A$ and $dA$ differ by $(dx)^2$

Referring to Fig 19, we see that $dA$ is represented by the rectangles (1) and (2), while $\Delta A$ is represented by the rectangles (1) and (2) together with the square (3), and it is obvious from the figure that the square (3) is very small compared with the rectangles (1) and (2), provided $dx$ is taken small. For example, if $x = 5$ and $dx = .001$, the rectangles (1) and (2) have together the area $2\,x\,dx = .01$ and the square (3) has the area .000001.

**Ex. 2.** Let $$s = 16\,t^2,$$

where $s$ is the distance traversed by a moving body in the time $t$

If $t$ is increased by $\Delta t = dt$, we have

$$\Delta s = 16\,(t + dt)^2 - 16\,t^2 = 32\,t\,dt + 16\,(dt)^2,$$

and, from (2), $$ds = 32\,t\,dt\,;$$

so that $\Delta s$ and $ds$ differ by $16\,(dt)^2$ The term $16\,(dt)^2$ is very small compared with the term $32\,t\,dt$, if $dt$ is small. For example, if $t = 4$ and $dt = .001$, then $32\,t\,dt = .128$, while $16\,(dt)^2 = .000016$

In this problem $\Delta s$ is the actual distance traversed in the time $dt$, and $ds$ is the distance which would have been traversed if the body had moved throughout the time $dt$ with the same velocity which it had at the beginning of the time $dt$.

In general, if $y=f(x)$ and we make a graphical representation, we may have two cases as shown in Figs. 20 and 21.

In each figure, $MN=PR=\Delta x=dx$ and $RQ=\Delta y$, since $RQ$ is the total change in $y$ caused by a change of $dx=MN$ in $x$. If $PT$ is the tangent to the curve at $P$, then, by § 15,

$$\frac{dy}{dx}=f'(x)=\tan RPT;$$

so that, by (2), $\qquad dy=(\tan RPT)(PR)=RT.$

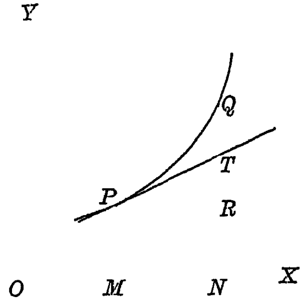



Fig. 20

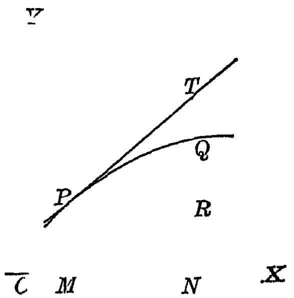



Fig 21

In Fig. 20, $dy<\Delta y$, and in Fig. 21 $dy>\Delta y$; but in each case the difference between $dy$ and $\Delta y$ is represented in magnitude by the length of $QT$.

This shows that $RQ=\Delta y$ is the change in $y$ as the point $P$ is supposed to move along the curve $y=f(x)$, while $RT=dy$ is the change in the value of $y$ as the point $P$ is supposed to move along the tangent to that curve. Now, as a very small arc does not deviate much from its tangent, it is not hard to see graphically that if the point $Q$ is taken close to $P$, the difference between $RQ$ and $RT$, namely, $QT$, is very small compared with $RT$.

A more rigorous examination of the difference between the increment and the differential lies outside the range of this book.

**EXERCISES**

1. If $y=x^3-3x^2+4x+1$, find $dy$.
2. If $y=x^4+4x^3-x^2+6x$, find $dy$.
3. If $V$ is the volume of a cube of edge $x$, find both $\Delta V$ and $dV$ and interpret geometrically.

4. If $A$ is the area of a circle of radius $r$, find both $\Delta A$ and $dA$. Show that $\Delta A$ is the exact area of a ring of width $dr$, and that $dA$ is the product of the inner circumference of the ring by its width.

5. If $V$ is the volume of a sphere of radius $r$, find $\Delta V$ and $dV$. Show that $\Delta V$ is the exact volume of a spherical shell of thickness $dr$, and that $dV$ is the product of the area of the inner surface of the shell by its thickness.

6. If $A$ is the area described in § 19, show that $dA = y\,dx$ Show geometrically how this differs from $\Delta A$

7. If $s$ is the distance traversed by a moving body, $t$ the time, and $v$ the velocity, show that $ds = v\,dt$. How does $ds$ differ from $\Delta s$?

8. If $y = x^2$ and $x = 5$, find the numerical difference between $dy$ and $\Delta y$, with successive assumptions of $dx =$ 01, $dx = .001$, and $dx = .0001$.

9. If $y = x^3$ and $x = 3$, find the numerical difference between $dy$ and $\Delta y$ for $dx = .001$ and for $dx =$ 0001

10. For a circle of radius 4 in. compute the numerical difference between $dA$ and $\Delta A$ corresponding to an increase of $r$ by .001 in.

11. For a sphere of radius 3 ft. find the numerical difference between $dV$ and $\Delta V$ when $r$ is increased by 1 in.

**21. Approximations.** The previous section brings out the fact that the differential of $y$ differs from the increment of $y$ by a very small amount, which becomes less the smaller the increment of $x$ is taken. The differential may be used, therefore, to make certain approximate calculations, especially when the question is to determine the effect upon a function caused by small changes in the independent variable. This is illustrated in the following examples:

**Ex. 1.** Find approximately the change in the area of a square of side 2 in. caused by an increase of .002 in. in the side.

Let $x$ be the side of the square, $A$ its area. Then

$$A = x^2 \quad \text{and} \quad dA = 2\,x\,dx.$$

Placing $x = 2$ and $dx = .002$, we find $dA = .008$, which is approximately the required change in the area.

If we wish to know how nearly correct the approximation is, we may compute $\Delta A = (2.002)^2 - (2)^2 = .008004$, which is the exact change in $A$. Our approximate change is therefore in error by .000004, a very small amount.

**Ex. 2.** Find approximately the volume of a sphere of radius 1.9 in.

The volume of a sphere of radius 2 in. is $\frac{32}{3}\pi$, and the volume of the required sphere may be found by computing the change in the volume of a sphere of radius 2 caused by decreasing its radius by .1.

If $r$ is the radius of the sphere and $V$ its volume, we have

$$V = \tfrac{4}{3}\pi r^3 \quad \text{and} \quad dV = 4\pi r^2 dr.$$

Placing $r = 2$ and $dr = -.1$, we find $dV = -1.6\pi$. Hence the volume of the required sphere is approximately

$$\tfrac{32}{3}\pi - 1.6\pi = 9.0667\pi.$$

To find how much this is in error we may compute exactly the volume of the required sphere by the formula

$$V = \tfrac{4}{3}\pi(1.9)^3 = 9.1453\pi.$$

The approximate volume is therefore in error by $.0786\pi$, which is less than 1 per cent of the true volume.

## EXERCISES

1. The side of a square is measured as 3 ft. long. If this length is in error by 1 in., find approximately the resulting error in the area of the square.

2. The diameter of a spherical ball is measured as $2\frac{1}{2}$ in., and the volume and the surface are computed. If an error of $\frac{1}{10}$ in. has been made in measuring the diameter, what is the approximate error in the volume and the surface?

3. The radius and the altitude of a right circular cone are measured as 3 in. and 5 in. respectively. What is the approximate error in the volume if an error of $\frac{1}{25}$ in. is made in the radius? What is the error in the volume if an error of $\frac{1}{25}$ in. is made in the altitude?

4. Find approximately the volume of a cube with 3.0002 in. on each edge.

5. The altitude of a certain right circular cone is the same as the radius of the base. Find approximately the volume of the cone if the altitude is 3.00002 in.

6. The distance $s$ of a moving body from a fixed point of its path, at any time $t$, is given by the equation $s = 16t^2 + 100t - 50$. Find approximately the distance when $t = 4.0004$.

7. Find the approximate value of $x^3 + x - 2$ when $x = 1.0003$.

8. Find approximately the value of $x^4 + x^2 + 4$ when $x = .99989$.

9. Show that the volume of a thin cylindrical shell is approximately equal to the area of its inner surface times its thickness.

10. If $V$ is the volume and $S$ the curved area of a right circular cone with radius of its base $r$ and its vertical angle $2\alpha$, show that $V = \frac{1}{3}\pi r^3 \operatorname{ctn}\alpha$ and $S = \pi r^2 \csc\alpha$ Thence show that the volume of a thin conical shell is approximately equal to the area of its inner surface multiplied by its thickness.

## GENERAL EXERCISES

Find the derivatives of the following functions from the definition

1. $\dfrac{3+2x}{1-x}$.

2. $\dfrac{a+x}{a-x}$.

3. $\dfrac{1}{x^2+1}$.

4. $\dfrac{x^2-1}{x^2+1}$.

5. $\sqrt{x}$.*

6. $\dfrac{1}{\sqrt{x}}$.*

7. $\sqrt{x^2+1}$.*

8. Prove from the definition that the derivative of $\dfrac{a}{x^n}$ is $\dfrac{-na}{x^{n+1}}$.

9. By expanding and differentiating, prove that the derivative of $(2x+5)^3$ is $6(2x+5)^2$.

10. By expanding and differentiating, prove that the derivative of $(x^2+1)^3$ is $6x(x^2+1)^2$.

11. By expanding and differentiating, prove that the derivative of $(x+a)^n$ is $n(x+a)^{n-1}$, where $n$ is a positive integer.

12. By expanding and differentiating, prove that the derivative of $(x^2+a^2)^n$ is $2nx(x^2+a^2)^{n-1}$, where $n$ is a positive integer.

13. Find when $x^4+8x^3+24x^2+32x+16$ is increasing and when decreasing, as $x$ increases.

14. Find when $9x^4-24x^3-8x^2+32x+11$ is increasing and when decreasing, as $x$ increases.

15. Find a general rule for the values of $x$ for which $ax^2+bx+c$ is increasing or decreasing, as $x$ increases.

16. Find a general rule for the values of $x$ for which $x^3-a^2x+b$ is increasing or decreasing, as $x$ increases.

17. A right circular cone of altitude $x$ is inscribed in a sphere of radius $a$. Find when an increase in the altitude of the cone will cause an increase in its volume and when it will cause a decrease.

* Hint. In these examples make use of the relation $\sqrt{A}-\sqrt{B} = \dfrac{A-B}{\sqrt{A}+\sqrt{B}}$.

18. A particle is moving in a straight line in such a manner that its distance $x$ from a fixed point $A$ of the straight line, at any time $t$, is given by the equation $x = t^3 - 9t^2 + 15t + 100$. When will the particle be approaching $A$?

19. The velocity of a certain moving body is given by the equation $v = t^2 - 7t + 10$. During what time will it be moving in a direction opposite to that in which $s$ is measured, and how far will it move?

20. If a stone is thrown up from the surface of the earth with a velocity of 200 ft. per second, the distance traversed in $t$ seconds is given by the equation $s = 200t - 16t^2$. Find when the stone moves up and when down

21. The velocity of a certain moving body, at any time $t$, is given by the equation $v = t^2 - 8t + 12$ Find when the velocity of the body is increasing and when decreasing.

22. At any time $t$, the distance $s$ of a certain moving body from a fixed point in its path is given by the equation $s = 16 - 24t + 9t^2 - t^3$. When is its velocity increasing and when decreasing? When is its speed increasing and when decreasing?

23. At any time $t$, the distance of a certain moving body from a fixed point in its path is given by the equation $s = t^3 - 6t^2 + 9t + 1$ When is its speed increasing and when decreasing?

24. A sphere of ice is melting at such a rate that its volume is decreasing at the rate of 10 cu in. per minute. At what rate is the radius of the sphere decreasing when the sphere is 2 ft. in diameter?

25. Water is running at the rate of 1 cu. ft. per second into a basin in the form of a frustum of a right circular cone, the radii of the upper and the lower base being 10 in and 6 in. respectively, and the depth being 6 in. How fast is the water rising in the basin when it is at the depth of 3 in.?

26. A vessel is in the form of an inverted right circular cone the vertical angle of which is 60°. The vessel is originally filled with liquid which flows out at the bottom at the rate of 3 cu. in. per minute. At what rate is the inner surface of the vessel being exposed when the liquid is at a depth of 1 ft. in the vessel?

27. Find the equation of the straight line which passes through the point $(4, -5)$ with the slope $-\frac{1}{2}$.

28. Find the equation of the straight line through the points $(2, -3)$ and $(-3, 4)$.

29. Find the equation of the straight line determined by the points $(2, -4)$ and $(2, 4)$.

30. Find the equation of the straight line through the point $(1, -3)$ and parallel to the line $x - 2y + 7 = 0$

31. Find the equation of the straight line through the point $(2, 7)$ and perpendicular to the line $2x + 4y + 9 = 0$

32. Find the angle between the straight lines $2x + 3y + 5 = 0$ and $y + 3x + 1 = 0$.

Find the turning-points of the following curves and draw the graphs

33. $y = 3 - x - x^2$

34. $y = 16x^2 - 40x + 25$

35. $y = \frac{1}{3}(x^2 - 4x - 2)$.

36. $y = x^3 - 6x^2 - 15x + 5$.

37. $y = x^3 + 3x^2 - 9x - 14$.

38. Find the point of intersection of the tangents to the curve $y = x + 2x^2 - x^3$ at the points for which $x = -1$ and $x = 2$ respectively

39. Show that the equation of the tangent to the curve $y = ax^2 + 2bx + c$ at the point $(x_1, y_1)$ is $y = 2(ax_1 + b)x - ax_1^2 + c$.

40. Show that the equation of the tangent to the curve $y = x^3 + ax + b$ at the point $(x_1, y_1)$ is $y = (3x_1^2 + a)x - 2x_1^3 + b$.

41. Find the area of the triangle included between the coordinate axes and the tangent to the curve $y = x^3$ at the point $(2, 8)$.

42. Find the angle between the tangents to the curve $y = 2x^3 + 4x^2 - x$ at the points the abscissas of which are $-1$ and $1$ respectively.

43. Find the equation of the tangent to the curve $y = x^3 - 3x^2 + 4x - 12$ which has the slope 1.

44. Find the points on the curve $y = 3x^3 - 4x^2$ at which the tangents are parallel to the line $x - y = 0$.

45. A length $l$ of wire is to be cut into two portions which are to be bent into the forms of a circle and a square respectively. Show that the sum of the areas of these figures will be least when the wire is cut in the ratio $\pi : 4$.

46. A log in the form of a frustum of a cone is 10 ft. long, the diameters of the bases being 4 ft. and 2 ft. A beam with a square cross section is cut from it so that the axis of the beam coincides with the axis of the log. Find the beam of greatest volume that can be so cut

47. Required the right circular cone of greatest volume which can be inscribed in a given sphere.

48. The total surface of a regular triangular prism is to be $k$. Find its altitude and the side of its base when its volume is a maximum.

49. A piece of wire 9 in. long is cut into five pieces, two of one length and three of another Each of the two equal pieces is bent into an equilateral triangle, and the vertices of the two triangles are connected by the remaining three pieces so as to form a regular triangular prism. How is the wire cut when the prism has the largest volume ?

50. If $t$ is time in seconds, $v$ the velocity of a moving body in feet per second, and $v = 200 - 32\,t$, how far will the body move in the first 5 sec.?

51. If $v = 200 - 32\,t$, where $v$ is the velocity of a moving body in feet per second and $t$ is time in seconds, how far will the body move in the fifth second ?

52. A curve passes through the point $(2, -3)$, and its slope at any point is equal to 3 more than twice the abscissa of the point. Find the equation of the curve.

53. A curve passes through the point $(0, 0)$, and its slope at any point is $x^2 - 2\,x + 7$. Find its equation.

54. Find the area bounded by the curve $y + x^2 - 16 = 0$ and the axis of $x$.

55. Find the area bounded by the curve $y = 2\,x^3 - 15\,x^2 + 36\,x + 1$, the ordinates through the turning-points of the curve, and $OX$.

56. Find the area between the curve $y = x^2$ and the straight line $y = x + 6$.

57. Find the area between the curves $y = x^2$ and $y = 18 - x^2$.

58. The curve $y = ax^2$ is known to pass through the point $(h, k)$. Prove that the area bounded by the curve, the axis of $x$, and the line $x = h$ is $\frac{1}{3}\,hk$.

59. Compute the difference between $\Delta A$ and $dA$ for the area $A$ of a circle of radius 5, corresponding to an increase of .01 in the radius.

**60.** Compute the difference between $\Delta V$ and $dV$ for the volume $V$ of a sphere of radius 5, corresponding to an increase of 01 in the radius

**61.** If a cubical shell is formed by increasing each edge of a cube by $dx$, show that the volume of the shell is approximately equal to its inside surface multiplied by its thickness.

**62.** If the diameter of a sphere is measured and found to be 2 ft, and the volume is calculated, what is the approximate error in the calculated volume if an error of $\frac{1}{2}$ in has been made in obtaining the radius?

**63.** A box in the form of a right circular cylinder is 6 in deep and 6 in. across the bottom. Find the approximate capacity of the box when it is lined so as to be 5 9 in deep and 5 9 in across the bottom.

**64.** A rough wooden model is in the form of a regular quadrangular pyramid 3 in. tall and 3 in. on each side of the base. After it is smoothed down, its dimensions are all decreased by .01. What is the approximate volume of the material removed?

**65.** By use of the differential find approximately the area of a circle of radius 1.99. What is the error made in this approximation?

**66.** Find approximately the value of $x^5 + 4x^3 + x$ when $x = 3.0002$ and when $x = 2.9998$.

**67.** The edge of a cube is 2.0001 in. Find approximately its surface.

**68.** The motion of a certain body is defined by the equation $s = t^3 + 3t^2 + 9t - 27$. Find approximately the distance traversed in the interval of time from $t = 3$ to $t = 3.0087$.

# CHAPTER III

## SUMMATION

**22. Area by summation.** Let us consider the problem to find the area bounded by the curve $y = \frac{1}{5}x^2$, the axis of $x$, and the ordinates $x = 2$ and $x = 3$ (Fig. 22). This may be solved by the method of § 19; but we wish to show that it may also be considered as a problem in summation, since the area is approximately equal to the sum of a number of rectangles constructed as follows:

We divide the axis of $x$ between $x = 2$ and $x = 3$ into 10 parts, each of which we call $\Delta x$, so that $\Delta x = \frac{3-2}{10} = .1$. If $x_1$ is the first point of division, $x_2$ the second point, and so on, and rectangles are constructed as shown in the figure, then the altitude of the first rectangle is $\frac{1}{5}(2)^2$, that of the second rectangle is $\frac{1}{5}x_1^2 = \frac{1}{5}(2.1)^2 = .882$, and so on. The area of the first rectangle is $\frac{1}{5}(2)^2\Delta x = .08$, that of the second rectangle is $\frac{1}{5}x_1^2\Delta x = \frac{1}{5}(2.1)^2\Delta x = .0882$, and so on.

Accordingly, we make the following calculation:

$$\begin{array}{ll r}
x = 2, & \frac{1}{5}(2)^2\Delta x = & .08 \\
x_1 = 2.1, & \frac{1}{5}(x_1)^2\Delta x = & .0882 \\
x_2 = 2.2, & \frac{1}{5}(x_2)^2\Delta x = & .0968 \\
x_3 = 2.3, & \frac{1}{5}(x_3)^2\Delta x = & .1058 \\
x_4 = 2.4, & \frac{1}{5}(x_4)^2\Delta x = & .1152 \\
x_5 = 2.5, & \frac{1}{5}(x_5)^2\Delta x = & .1250 \\
x_6 = 2.6, & \frac{1}{5}(x_6)^2\Delta x = & .1352 \\
x_7 = 2.7, & \frac{1}{5}(x_7)^2\Delta x = & .1458 \\
x_8 = 2.8, & \frac{1}{5}(x_8)^2\Delta x = & .1568 \\
x_9 = 2.9, & \frac{1}{5}(x_9)^2\Delta x = & .1682 \\
 & & 1.2170
\end{array}$$

This is a first approximation to the area.

For a better approximation the axis of $x$ between $x = 2$ and $x = 3$ may be divided into 20 parts with $\Delta x = .05$. The result is 1.2418.

If the base of the required figure is divided into 100 parts with $\Delta x = .01$, the sum of the areas of the 100 rectangles constructed as above is 1.26167.

The larger the number of parts into which the base of the figure is divided, the more nearly is the required area obtained. In fact, the required area is the limit approached as the number of parts is indefinitely increased and the size of $\Delta x$ approaches zero.

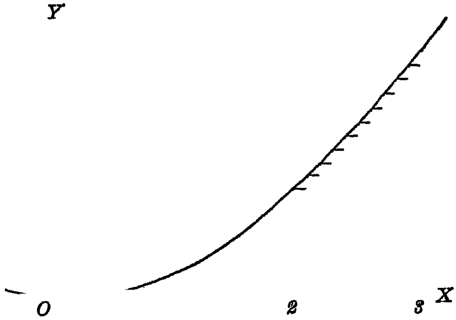



FIG. 22

We shall now proceed to generalize the problem just handled. Let $LK$ (Fig. 23) be a curve with equation $y = f(x)$, and let $OE = a$ and $OB = b$. It is required to find the area bounded by the curve $LK$, the axis of $x$, and the ordinates at $E$ and $B$.

For convenience we assume in the first place that $a < b$ and that $f(x)$ is positive for all values of $x$ between $a$ and $b$. We will divide the line $EB$ into $n$ equal parts by placing $\Delta x = \dfrac{b-a}{n}$ and laying off the lengths $EM_1 = M_1M_2 = M_2M_3 = \cdots = M_{n-1}B = \Delta x$ (in Fig. 23, $n = 9$).

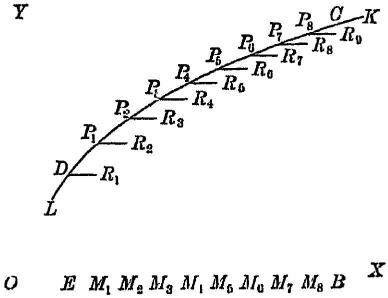


FIG. 23

Let $OM_1 = x_1$, $OM_2 = x_2$, $\cdots$, $OM_{n-1} = x_{n-1}$. Draw $ED = f(a)$, $M_1P_1 = f(x_1)$, $M_2P_2 = f(x_2)$, $\cdots$, $M_{n-1}P_{n-1} = f(x_{n-1})$, and $BC$; also $DR_1$, $P_1R_2$, $P_2R_3$, $\cdots$, $P_{n-1}R_n$, parallel to $OX$. Then

$$f(a)\Delta x = \text{the area of the rectangle } EDR_1M_1,$$
$$f(x_1)\Delta x = \text{the area of the rectangle } M_1P_1R_2M_2,$$
$$f(x_2)\Delta x = \text{the area of the rectangle } M_2P_2R_3M_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f(x_{n-1})\Delta x = \text{the area of the rectangle } M_{n-1}P_{n-1}R_nB.$$

The sum

$$f(a)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x \qquad (1)$$

is then the sum of the areas of these rectangles and equal to the area of the polygon $EDR_1P_1R_2 \cdots R_{n-1}P_{n-1}R_nB$. It is evident that the limit of this sum as $n$ is indefinitely increased is the area bounded by $ED$, $EB$, $BC$, and the arc $DC$.

The sum (1) is expressed concisely by the notation

$$\sum_{i=0}^{i=n-1} f(x_i)\Delta x,$$

where $\Sigma$ (sigma), the Greek form of the letter $S$, stands for the word "sum," and the whole expression indicates that the sum is to be taken of all terms obtained from $f(x_i)\Delta x$ by giving to $i$ in succession the values 0, 1, 2, 3, $\cdots$, $n-1$, where $x_0 = a$.

The limit of this sum is expressed by the symbol

$$\int_a^b f(x)dx,$$

where $\int$ is a modified form of $S$.

Hence $\displaystyle\int_a^b f(x)dx = \lim_{n=\infty} \sum_{i=0}^{i=n-1} f(x_i)\Delta x =$ the area $EBCD$.

It is evident that the result is not vitiated if $ED$ or $BC$ is of length zero.

**23. The definite integral.** We have seen in § 19 that if $A$ is the area $EBCD$ of § 22,

$$A = F(b) - F(a),$$

where $F(x)$ is any function whose derivative is $f(x)$. Comparing this with the result of § 22, we have the important formula

$$\int_a^b f(x)dx = F(b) - F(a). \qquad (1)$$

The limit of the sum (1), § 22, which is denoted by $\int_a^b f(x)\,dx$, is called a *definite integral*, and the numbers $a$ and $b$ are called the *lower limit* and the *upper limit**, respectively, of the definite integral.

On the other hand, the symbol $\int f(x)\,dx$ is called an *indefinite integral* and indicates the process of integration as already defined in § 18.

Thus, from that section, we have

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C,$$

$$\int a\,dx = ax + C,$$

and, in general, $$\int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any function whose derivative is $f(x)$.

We may therefore express formula (1) in the following rule:

*To find the value of* $\int_a^b f(x)\,dx$, *evaluate* $\int f(x)\,dx$, *substitute* $x = b$ *and* $x = a$ *successively, and subtract the latter result from the former.*

It is to be noticed that in evaluating $\int f(x)\,dx$ the constant of integration is to be omitted, because if it is added, it disappears in the subtraction, since

$$[F(b) + C] - [F(a) + C] = F(b) - F(a).$$

In practice it is convenient to express $F(b) - F(a)$ by the symbol $[F(x)]_a^b$, so that

$$\int_a^b f(x)\,dx = [F(x)]_a^b.$$

**Ex. 1.** The example of § 22 may now be completely solved. The required area is

$$\int_2^3 \frac{x^2}{5}dx = \left[\frac{x^3}{15}\right]_2^3 = \frac{27}{15} - \frac{8}{15} = \frac{19}{15} = 1\tfrac{4}{15}.$$

* The student should notice that the word "limit" is here used in a sense quite different from that in which it is used when a variable is said to approach a limit (§ 1).

The expression $f(x)\,dx$ which appears in formula (1) is called the *element of integration*. It is obviously equal to $dF(x)$. In fact, it follows at once from § 19 that

$$dA = y\,dx = f(x)\,dx.$$

In the discussion of § 22 we have assumed that $y$ and $dx$ are positive, so that $dA$ is positive. If $y$ is negative — that is, if the curve in Fig. 23 is below the axis of $x$ — and if $dx$ is positive, the product $y\,dx$ is negative and the area found by formula (1) has a negative sign. Finally, if the area required is partly above the axis of $x$ and partly below, it is necessary to find each part separately, as in the following example:

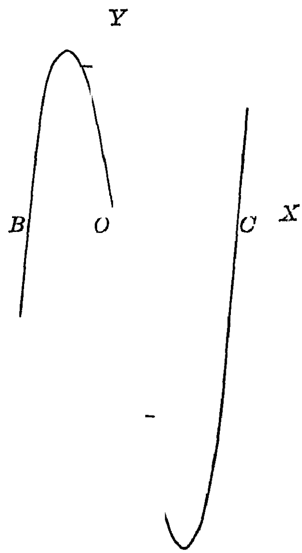


Fig. 24

**Ex. 2.** Find the area bounded by the curve $y = x^3 - x^2 - 6\,x$ and the axis of $x$

Plotting the curve (Fig 24), we see that it crosses the axis of $x$ at the points $B(-2,\ 0)$, $O(0,\ 0)$, and $C(3,\ 0)$. Hence part of the area is above the axis of $x$ and part below. Accordingly, we shall find it necessary to solve the problem in two parts, first finding the area above the axis of $x$ and then finding that below To find the first area we proceed as in § 22, dividing the area up into elementary rectangles for each of which

$$dA = y\,dx = (x^3 - x^2 - 6\,x)\,dx,$$

whence

$$A = \int_{-2}^{0}(x^3 - x^2 - 6\,x)\,dx = \left[\tfrac{1}{4}\,x^4 - \tfrac{1}{3}\,x^3 - 3\,x^2\right]_{-2}^{0}$$

$$= 0 - \left[\tfrac{1}{4}(-2)^4 - \tfrac{1}{3}(-2)^3 - 3(-2)^2\right] = 5\tfrac{1}{3}.$$

Similarly, for the area below the axis of $x$ we find, as before,

$$dA = y\,dx = (x^3 - x^2 - 6\,x)\,dx$$

But in this case $y = x^3 - x^2 - 6\,x$ is negative and hence $dA$ is negative, for we are making $x$ vary from 0 to 3, and therefore $dx$ is positive. Therefore we expect to find the result of the summation negative. In fact, we have

$$A = \int_{0}^{3}(x^3 - x^2 - 6\,x)\,dx = \left[\tfrac{1}{4}\,x^4 - \tfrac{1}{3}\,x^3 - 3\,x^2\right]_{0}^{3}$$

$$= \left[\tfrac{1}{4}(3)^4 - \tfrac{1}{3}(3)^3 - 3(3)^2\right] - 0 = -15\tfrac{3}{4}.$$

As we are asked to compute the total area bounded by the curve and the axis of $x$, we discard the negative sign in the last summation and add $5\frac{1}{3}$ and $15\frac{3}{4}$, thus obtaining $21\frac{1}{12}$ as the required result.

If we had computed the definite integral

$$\int_{-2}^{3}(x^3 - x^2 - 6x)\,dx,$$

we should have obtained the result $-10\frac{5}{12}$, which is the algebraic sum of the two portions of area computed separately.

**Ex. 3.** Find the area bounded by the two curves $y = x^2$ and $y = 8 - x^2$.

We draw the curves (Fig. 25)

$$y = x^2 \qquad (1)$$

and

$$y = 8 - x^2, \qquad (2)$$

and by solving their equations we find that they intersect at the points $P_1(2, 4)$ and $P_2(-2, 4)$.

The required area $OP_1BP_2O$ is evidently twice the area $OP_1BO$, since both curves are symmetrical with respect to $OY$. Accordingly, we shall find the area $OP_1BO$ and multiply it by 2. This latter area may be found by subtracting the area $ON_1P_1O$ from the area $ON_1P_1BO$, each of these areas being found as in the previous example; or we may proceed as follows:

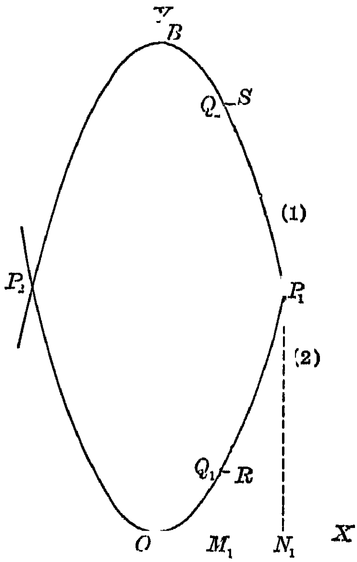



FIG. 25

Divide $ON_1$ into $n$ parts $dx$, and through the points of division draw straight lines parallel to $OY$, intersecting both curves. Let one of these lines be $M_1Q_1Q_2$. Through the points $Q_1$ and $Q_2$ draw straight lines parallel to $OX$ until they meet the next vertical line to the right, thereby forming the rectangle $Q_1RSQ_2$. The area of such a rectangle may be taken as $dA$ and may be computed as follows: its base is $dx$ and its altitude is $Q_1Q_2 = M_1Q_2 - M_1Q_1 = (8 - x^2) - x^2 = 8 - 2x^2$; for $M_1Q_2$ is the ordinate of a point on the curve (2) and $M_1Q_1$ the ordinate of a point on (1).

Therefore

$$dA = (8 - 2x^2)\,dx;$$

whence

$$A = \int_0^2 (8 - 2x^2)\,dx = \left[8x - \tfrac{2}{3}x^3\right]_0^2$$

$$= [16 - \tfrac{1}{3}(16)] - 0 = 10\tfrac{2}{3}.$$

Finally, the required area is $2(10\frac{2}{3}) = 21\frac{1}{3}$.

**EXERCISES**

1. Find the area bounded by the curve $4y - x^2 - 2 = 0$, the axis of $x$, and the lines $x = -2$ and $x = 2$.

2. Find the area bounded by the curve $y = x^3 - 7x^2 + 8x + 16$, the axis of $x$, and the lines $x = 1$ and $x = 3$

3. Find the area bounded by the curve $y = 25x - 10x^2 + x^3$ and the axis of $x$.

4. Find the area bounded by the axis of $x$ and the curve $y = 25 - x^2$.

5. Find the area bounded by the curve $y = 4x^2 - 4x - 3$ and the axis of $x$.

6. Find the area bounded below by the axis of $x$ and above by the curve $y = x^3 - 4x^2 - 4x + 16$.

7. Find the area bounded by the curve $y = 4x^3 - 8x^2 - 9x + 18$ and the axis of $x$.

8. Find the area bounded by the curve $x^2 + 2y - 8 = 0$ and the straight line $x + 2y - 6 = 0$.

9. Find the area bounded by the curve $3y - x^2 = 0$ and the straight line $2x + y - 9 = 0$

10. Find the area of the crescent-shaped figure bounded by the two curves $y = x^2 + 7$ and $y = 2x^2 + 3$

11. Find the area bounded by the curves $4y = x^2 - 4x$ and $x^2 - 4x + 4y - 24 = 0$.

12. Find the area bounded by the curve $x + 3 = y^2 - 2y$ and the axis of $y$.

**24. The general summation problem.** The formula

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{1}$$

has been obtained by the study of an area, but it may be given a much more general application. For if $f(x)$ is any function of $x$ whatever, it may be graphically represented by the curve $y = f(x)$. The rectangles of Fig. 23 are then the graphical representations of the products $f(x)\,dx$, and the symbol $\int_a^b f(x)\,dx$

represents the limit of the sum of these products. We may accordingly say:

*Any problem which requires the determination of the limit of the sum of products of the type $f(x)\,dx$ may be solved by the use of formula* (1).

Let us illustrate this by considering again the problem, already solved in § 18, of determining the distance traveled in the time from $t = t_1$ to $t = t_2$ by a body whose velocity $v$ is known. Since

$$v = \frac{ds}{dt},$$

we have
$$ds = v\,dt,$$

which is approximately the distance traveled in a small interval of time $dt$. Let the whole time from $t = t_1$ to $t = t_2$ be divided into a number of intervals each equal to $dt$. Then the total distance traveled is equal to the sum of the distances traveled in the several intervals $dt$, and hence is equal approximately to the sum of the several terms $v\,dt$. This approximation becomes better as the size of the intervals $dt$ becomes smaller and their number larger, and we conclude that the limit of the sum of the terms $v\,dt$ is the actual distance traveled by the body. Hence we have, if $s$ is the total distance traveled,

$$s = \int_{t_1}^{t_2} v\,dt.$$

If, now, we know $v$ in terms of $t$, we may apply formula (1).

**Ex.** If $v = 16\,t + 5$, find the distance traveled in the time from $t = 2$ to $t = 4$.

We have directly

$$s = \int_2^4 (16\,t + 5)\,dt = \left[8\,t^2 + 5\,t\right]_2^4 = 106.$$

### EXERCISES

**1.** At any time $t$ the velocity of a moving body is $3\,t^2 + 2\,t$ ft. per second. How far will it move in the first 5 sec.?

**2.** How far will the body in Ex. 1 move during the seventh second?

**3.** At any time $t$ the velocity of a moving body is $6 + 5\,t - t^2$ ft. per second. Show that this velocity is positive during the interval from $t = -1$ to $t = 6$, and find how far the body moves during that interval.

4. At any time $t$ the velocity of a moving body is $4t^2 - 24t + 11$ ft. per second. During what interval of time is the velocity negative, and how far will the body move during that interval?

5. The number of foot pounds of work done in lifting a weight is the product of the weight in pounds and the distance in feet through which the weight is lifted. A cubic foot of water weighs $62\frac{1}{2}$ lb. Compute the work done in emptying a cylindrical tank of depth 8 ft. and radius 2 ft., considering it as the limit of the sum of the pieces of work done in lifting each thin layer of water to the top of the tank.

**25. Pressure.** It is shown in physics that the pressure on one side of a plane surface of area $A$, immersed in a liquid at a uniform depth of $h$ units below the surface of the liquid, is equal to $whA$, where $w$ is the weight of a unit volume of the liquid. This may be remembered by noticing that $whA$ is the weight of the column of the liquid which would be supported by the area $A$.

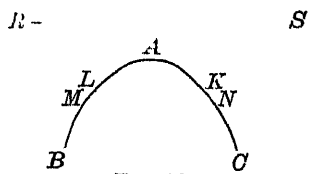



FIG 26

Since the pressure is the same in all directions, we can also determine the pressure on one side of a plane surface which is perpendicular to the surface of the liquid and hence is not at a uniform depth.

Let $ABC$ (Fig. 26) represent such a surface and $RS$ the line of intersection of the plane of $ABC$ with the surface of the liquid. Divide $ABC$ into strips by drawing straight lines parallel to $RS$. Call the area of one of these strips $dA$, as in § 23, and the depth of one edge $h$. Then, since the strip is narrow and horizontal, the depth of every point differs only slightly from $h$, and the pressure on the strip is then approximately $whdA$. Taking $P$ as the total pressure, we write

$$dP = whdA.$$

The total pressure $P$ is the sum of the pressures on the several strips and is therefore the limit of the sum of terms of the form $whdA$, the limit being approached as the number of the strips is indefinitely increased and the width of each indefinitely decreased. Therefore

$$P = \int whdA,$$

where the limits of integration are to be taken so as to include the whole area the pressure on which is to be determined. To evaluate the integral it is necessary to express both $h$ and $dA$ in terms of the same variable.

**Ex. 1.** Find the pressure on one side of a rectangle $BCDE$ (Fig 27), where the sides $BC$ and $ED$ are each 4 ft. long, the sides $BE$ and $CD$ are each 3 ft. long, immersed in water so that the plane of the rectangle is perpendicular to the surface of the water, and the side $BC$ is parallel to the surface of the water and 2 ft. below it

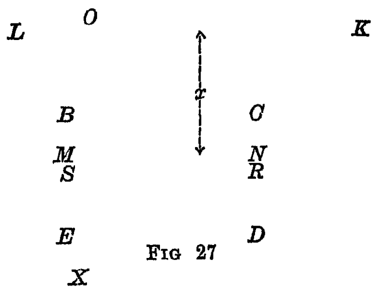


Fig 27

In Fig. 27, $LK$ is the line of intersection of the surface of the water and the plane of the rectangle. Let $O$ be the point of intersection of $LK$ and $BE$ produced Then, if $x$ is measured downward from $O$ along $BE$, $x$ has the value 2 at the point $B$ and $x$ has the value 5 at the point $E$.

We now divide $BE$ into parts $dx$, and through the points of division draw straight lines parallel to $BC$, thus dividing the given rectangle into elementary rectangles such as $MNRS$

Therefore $\quad dA = \text{area of } MNRS = MN \cdot MS = 4\,dx.$

Since $MN$ is at a distance $x$ below $LK$, the pressure on the elementary rectangle $MNRS$ is approximately $wx(4\,dx)$. Accordingly, we have

$$dP = 4\,wx\,dx$$

and
$$P = \int_2^5 4\,wx\,dx = [2\,wx^2]_2^5 = 2\,w(5)^2 - 2\,w(2)^2 = 42\,w$$

For water, $\quad w = 62\tfrac{1}{2}\text{ lb} = \tfrac{1}{32}\text{ T}.$

Hence we have finally
$$P = 2625\text{ lb.} = 1\tfrac{5}{16}\text{ T}$$

**Ex. 2.** The base $CD$ (Fig. 28) of a triangle $BCD$ is 7 ft., and its altitude from $B$ to $CD$ is 5 ft This triangle is immersed in water with its plane perpendicular to the surface of the water and with $CD$ parallel to the surface and 1 ft below it, $B$ being below $CD$ Find the total pressure on one side of this triangle.

Let $LK$ represent the line of intersection of the plane of the triangle and the surface of the water. Then $B$ is 6 ft below $LK$ Let $BX$ be perpendicular to $LK$ and intersect $CD$ at $T$ We will measure distances from $B$ in the direction $BX$ and denote them by $x$. Then, at the point $B$, $x$ has the value 0, and at $T$, $x$ has the value 5.

Divide the distance $BT$ into parts $dx$, and through the points of division draw straight lines parallel to $CD$, and on each of these lines as lower base construct a rectangle such as $MNRS$, where $E$ and $F$ are two consecutive points of division on $BX$.

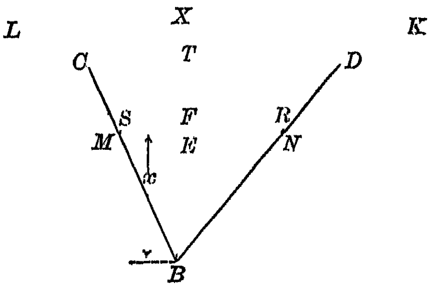


FIG. 28

Then $\quad BE = x,$

$\qquad EF = dx,$

and, by similar triangles,

$$\frac{MN}{CD} = \frac{BE}{BT};$$

whence $\quad \dfrac{MN}{7} = \dfrac{x}{5}$

and $\quad MN = \frac{7}{5}x.$

Then $\quad dA = \text{the area of } MNRS = \frac{7}{5}x\,dx.$

Since $B$ is 6 ft. below $LK$, and $BE = x$, it follows that $E$ is $(6 - x)$ ft. below $LK$.

Hence the pressure on the rectangle is approximately

$$dP = (\tfrac{7}{5}x\,dx)(6 - x)w = (\tfrac{42}{5}wx - \tfrac{7}{5}wx^2)\,dx,$$

and
$$P = \int_0^5 (\tfrac{42}{5}wx - \tfrac{7}{5}wx^2)\,dx = \left[\tfrac{21}{5}wx^2 - \tfrac{7}{15}wx^3\right]_0^5$$
$$= (105w - \tfrac{175}{3}w) - 0 = \tfrac{140}{3}w = 2916\tfrac{2}{3}\text{ lb.} = 1\tfrac{11}{24}\text{ T.}$$

**EXERCISES**

✓ **1.** A gate in the side of a dam is in the form of a square, 4 ft. on a side, the upper side being parallel to and 15 ft. below the surface of the water in the reservoir. What is the pressure on the gate?

✓ **2.** Find the total pressure on one side of a triangle of base 6 ft. and altitude 6 ft., submerged in water so that the altitude is vertical and the vertex is in the surface of the water.

✓ **3.** Find the total pressure on one side of a triangle of base 4 ft. and altitude 6 ft., submerged in water so that the base is horizontal, the altitude vertical, and the vertex above the base and 4 ft. from the surface of the water.

e base of an isosceles triangle is 8 ft. and the equal sides 5 ft. The triangle is completely immersed in water, its base 'allel to and 5 ft. below the surface of the water, its altig perpendicular to the surface of the water, and its vertex being above the base. Find the total pressure on one side of the triangle.

5. Find the pressure on one side of an equilateral triangle, 6 ft. on a side, if it is partly submerged in water so that one vertex is one foot above the surface of the water, the corresponding altitude being perpendicular to the surface of the water.

6. The gate in Ex. 1 is strengthened by a brace which runs diagonally from one corner to another. Find the pressure on each of the two portions of the gate — one above, the other below, the brace.

7. A dam is in the form of a trapezoid, with its two horizontal sides 300 and 100 ft. respectively, the longer side being at the top; and the height is 15 ft. What is the pressure on the dam when the water is level with the top of the dam?

8. What is the pressure on the dam of Ex. 7 when the water reaches halfway to the top of the dam?

9. If it had been necessary to construct the dam of Ex. 7 with the shorter side at the top instead of the longer side, how much greater pressure would the dam have had to sustain when the reservoir is full of water?

10. The center board of a yacht is in the form of a trapezoid in which the two parallel sides are 3 ft and 6 ft., respectively, in length, and the side perpendicular to these two is 4 ft. in length. Assuming that the last-named side is parallel to the surface of the water at a depth of 2 ft., and that the parallel sides are vertical, find the pressure on one side of the board.

11. Where shall a horizontal line be drawn across the gate of Ex. 1 so that the pressure on the portion above the line shall equal the pressure on the portion below?

**26. Volume.** The volume of a solid may be computed by dividing it into $n$ elements of volume, $dV$, and taking the limit of the sum of these elements as $n$ is increased indefinitely, the magnitude of each element at the same time approaching zero. The question in each case is the determination of the form of the element $dV$. We shall discuss a comparatively simple case of a solid such as is shown in Fig. 29.

In this figure let $OH$ be a straight line, and let the distance of any point of it from $O$ be denoted by $h$. At one end the solid is bounded by a plane perpendicular to $OH$ at $C$, where $OC = a$, and at the other end it is bounded by a plane perpendicular to $OH$ at $B$, where $OB = b$, so that it has *parallel bases.*

The solid is assumed to be such that the area $A$ of any plane section made by a plane perpendicular to $OH$ at a point distant $h$ from $O$ can be expressed as a function of $h$.

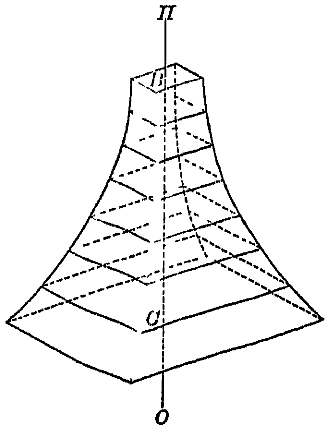


Fig. 29

To find the volume of such a solid we divide the distance $CB$ into $n$ parts $dh$, and through the points of division pass planes perpendicular to $OH$. We have thus divided the solid into slices of which the thickness is $dh$

Since $A$ is the area of the base of a slice, and since the volume of the slice is approximately equal to the volume of a right cylinder of the same base and thickness, we write

$$dV = A\,dh.$$

The volume of the solid is then the limit of the sum of terms of the above type, and therefore

$$V = \int_a^b A\,dh.$$

It is clear that the above discussion is valid even when one or both of the bases corresponding to $h = a$ and $h = b$, respectively, reduces to a point.

**Ex. 1.** Let $OY$ (Fig. 30) be an edge of a solid such that all its sections made by planes perpendicular to $OY$ are rectangles, the sides of a rectangle in a plane distant $y$ from $O$ being respectively $2y$ and $y^2$ We shall find the volume included between the planes $y = 0$ and $y = 2\frac{1}{2}$.

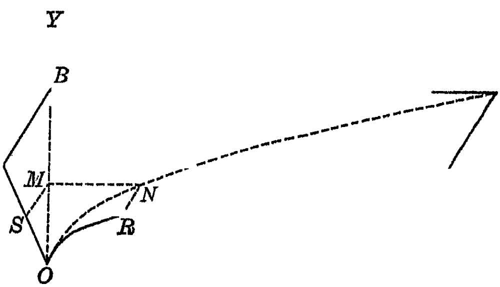


Fig. 30

Dividing the distance from $y = 0$ to $y = 2\frac{1}{2}$ into $n$ parts $dy$, and passing planes perpendicular to $OY$, we form rectangles such as $MNRS$, where, if

$OM = y$, $MN = y^2$ and $MS = 2y$ Hence the area $MNRS = 2y^3$, and the volume of the elementary cylinder standing on $MNRS$ as a base is $2y^3dy$; that is,

$$dV = 2y^3dy.$$

Therefore $$V = \int_0^{2\frac{1}{2}} 2y^3dy = [\tfrac{1}{2}y^4]_0^{2\frac{1}{2}} = 19\tfrac{17}{32}.$$

**Ex. 2.** The axes of two equal right circular cylinders of radius $a$ intersect at right angles. Required the volume common to the two cylinders.

Let $OA$ and $OB$ (Fig. 31) be the axes of the cylinders and $OY$ the common perpendicular to $OA$ and $OB$ at their point of intersection $O$. Then $OAD$ and $OBD$ are quadrants of two equal circles cut from the two cylinders by the planes through $OY$ perpendicular to the axes $OB$ and $OA$, and $OD = a$ Then the figure represents one eighth of the required volume.

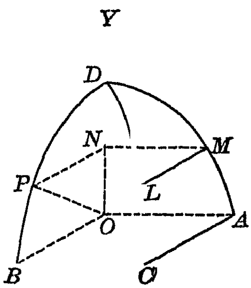



FIG 31

We divide the distance $OD$ into $n$ parts $dy$, and through the points of division pass planes perpendicular to $OY$ Any section, such as $LMNP$, is a square, of which one side $NP$ is equal to $\sqrt{\overline{OP}^2 - \overline{ON}^2}$ $OP = a$, being a radius of one of the cylinders, and hence, as $ON = y$,

$$NP = \sqrt{a^2 - y^2}$$

Accordingly, the area of $LMNP = a^2 - y^2$, and the volume of the elementary cylinder standing on $LMNP$ as a base is

$$dV = (a^2 - y^2)\,dy,$$

whence $$V = \int_0^a (a^2 - y^2)\,dy = [a^2y - \tfrac{1}{3}y^3]_0^a = \tfrac{2}{3}a^3.$$

Hence the total volume is $\frac{16}{3}a^3$.

This method of finding volumes is particularly useful when the sections of the solid made by parallel planes are bounded by circles or by concentric circles. Such a solid may be generated by the revolution of a plane area around an axis in its plane, and is called a *solid of revolution.* We take the following examples of solids of revolution:

**Ex. 3.** Find the volume of the solid generated by revolving about $OX$ the area bounded by the curve $y^2 = 4x$, the axis of $x$, and the line $x = 3$.

The generating area is shown in Fig. 32, where $AB$ is the line $x = 3$. Hence $OA = 3$.

Divide $OA$ into $n$ parts $dx$, and through the points of division pass straight lines parallel to $OY$, meeting the curve. When the area is revolved about $OX$, each of these lines, as $MP$, $NQ$, etc., generates a circle, the plane of which is perpendicular to $OX$. The area of the circle generated by $MP$, for example, is $\pi\overline{MP}^2$, which is equal to $\pi y^2 = \pi(4x)$, if $OM = x$.

Hence the area of any plane section of the solid made by a plane perpendicular to $OX$ can be expressed in terms of its distance from $O$, and we may apply the previous method for finding the volume.

Since the base of any elementary cylinder is $4\pi x$ and its altitude is $dx$, we have

$$dV = 4\pi x\, dx.$$

Hence $V = \int_0^3 4\pi x\, dx = [2\pi x^2]_0^3 = 18\pi.$

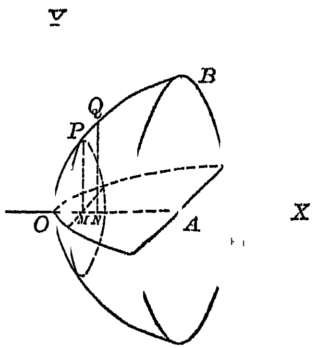



FIG. 32

**Ex. 4.** Find the volume of the ring surface generated by revolving about the axis of $x$ the area bounded by the line $y = 5$ and the curve $y = 9 - x^2$.

The line and the curve (Fig 33) are seen to intersect at the points $P_1(-2, 5)$ and $P_2(2, 5)$, and the ring is generated by the area $P_1BP_2P_1$. Since this area is symmetrical with respect to $OY$, it is evident that the volume of the ring is twice the volume generated by the area $AP_2BA$. Accordingly, we shall find the latter volume and multiply it by 2.

We divide the line $OM_2 = 2$ ($M_2$ being the projection of $P_2$ on $OX$) into $n$ parts $dx$, and through the points of division draw straight lines parallel to $OY$ and intersecting the straight line and the curve. One of these lines, as $MQP$, will, when revolved about $OX$, generate a circular ring, the outer radius of which is $MP = y = 9 - x^2$ and the inner radius of which is $MQ = y = 5$. Hence the area of the ring is

$$\pi\overline{MP}^2 - \pi\overline{MQ}^2 = \pi(9 - x^2)^2 - \pi(5)^2$$
$$= \pi(56 - 18x^2 + x^4).$$

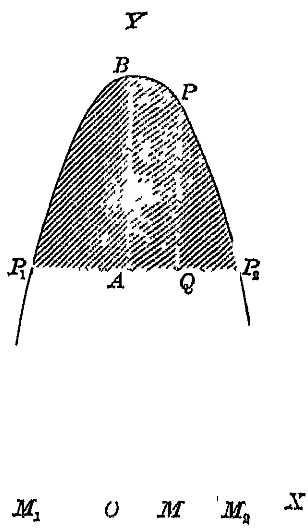



FIG. 33

Accordingly, $dV = \pi(56 - 18x^2 + x^4)\, dx$

and $V = \int_0^2 \pi(56 - 18x^2 + x^4)\, dx = \pi[56x - 6x^3 + \frac{1}{5}x^5]_0^2 = 70\frac{2}{5}\pi.$

Accordingly, the volume of the ring is $2(70\frac{2}{5}\pi) = 140\frac{4}{5}\pi$.

## EXERCISES

1. The section of a certain solid made by any plane perpendicular to a given line $OH$ is a circle with one point in $OH$ and its center on a straight line $OB$ intersecting $OH$ at an angle of 45°. If the height of this solid measured from $O$ along $OH$ is 4 ft., find its volume by integration.

2. A solid is such that any cross section perpendicular to an axis is an equilateral triangle of which each side is equal to the square of the distance of the plane of the triangle from a fixed point on the axis. The total length of the axis from the fixed point is 5. Find the volume.

3. Find the volume of the solid generated by revolving about $OX$ the area bounded by $OX$ and the curve $y = 4x - x^2$.

4. Find the volume of the solid generated by revolving about $OX$ the area included between the axis of $x$ and the curve $y = 2x^2 - x^3$

5. Find the volume of the solid generated by revolving about the line $y = -2$ the area bounded by the axis of $y$, the lines $x = 3$ and $y = -2$, and the curve $y = 3x^2$.

6. On a spherical ball of radius 5 in. two great circles are drawn intersecting at right angles at the points $A$ and $B$. The material of the ball is then cut away so that the sections perpendicular to $AB$ are squares with their vertices on the two great circles. Find the volume left.

7. Find the volume generated by revolving about the line $x = 2$ the area bounded by the curve $y^2 = 8x$, the axis of $x$, and the line $x = 2$.

8. Any plane section of a certain solid made by a plane perpendicular to $OY$ is a square of which the center lies on $OY$ and two opposite vertices lie on the curve $y = 4x^2$. Find the volume of the solid if the extreme distance along $OY$ is 3.

9. Find the volume generated by revolving about $OY$ the area bounded by the curve $y^2 = 8x$ and the line $x = 2$.

10. Find the volume of the solid generated by revolving about $OX$ the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 6x + 10$.

11. The cross section of a certain solid made by any plane perpendicular to $OX$ is a square, the ends of one of whose sides are on the curves $16y = x^2$ and $4y = x^2 - 12$. Find the volume of this solid between the points of intersection of the curves.

**GENERAL EXERCISES**

**1.** The velocity in feet per second of a moving body at any time $t$ is $t^2 - 4t + 4$ Show that the body is always moving in the direction in which $s$ is measured, and find how far it will move during the fifth second

**2.** The velocity in feet per second of a moving body at any time $t$ is $t^2 - 4t$. Show that after $t = 4$ the body will always move in the direction in which $s$ is measured, and find how far it will move in the time from $t = 6$ to $t = 9$.

**3.** At any time $t$ the velocity in feet per second of a moving body is $t^2 - 6t + 5$ How many feet will the body move in the direction opposite to that in which $s$ is measured?

**4.** At any time $t$ the velocity in miles per hour of a moving body is $t^2 - 2t - 3$. If the initial moment of time is 12 o'clock noon, how far will the body move in the time from 11.30 A M. to 2 P M. ?

**5.** Find the area bounded by the curves $9y = 4x^2$ and $45 - 9y = x^2$.

**6.** Find the total area bounded by the curves $y^2 = 4ax$ and $y^2 = 4a^2 - 4ax$.

**7.** Find the total area bounded by the curve $y = x^3$ and the straight line $y = 4x$.

**8.** Find the total area bounded by the curve $y = x(x-1)(x-3)$ and the straight line $y = 4(x-1)$.

**9.** $ABCD$ is a quadrilateral with $A = 90°$, $B = 90°$, $AB = 5$ ft., $BC = 2$ ft., $AD = 4$ ft. It is completely immersed in water with $AB$ in the surface and $AD$ and $BC$ perpendicular to the surface. Find the pressure on one side

**10.** Prove that the pressure on one side of a rectangle completely submerged with its plane vertical is equal to the area of the rectangle multiplied by the depth of its center and by $w$ (consider only the case in which one side of the rectangle is parallel to the surface).

**11.** Prove that the pressure on one side of a triangle completely submerged with its plane vertical is equal to its area multiplied by the depth of its median point and by $w$ (consider only the case in which one side of the triangle is parallel to the surface).

**12.** The end of a trough, full of water, is assumed to be in the form of an equilateral triangle, with its vertex down and its plane vertical. What is the effect upon the pressure on the end if the level of the water sinks halfway to the bottom?

13. A square 2 ft on a side is immersed in water, with one vertex in the surface of the water and with the diagonal through that vertex perpendicular to the surface of the water. How much greater is the pressure on the lower half of the square than that on the upper half?

14. A board is symmetrical with respect to the line $AB$, and is of such a shape that the length of any line across the board perpendicular to $AB$ is twice the cube of the distance of the line from $A$. $AB$ is 2 ft. long The board is totally submerged in water, $AB$ being perpendicular to the surface of the water and $A$ one foot below the surface. Find the pressure on one side of the board.

15. Find the pressure on one side of an area the equations of whose boundary lines are $x = 0$, $y = 4$, and $y^2 = 4x$ respectively, where the axis of $x$ is taken in the surface of the water and where the positive direction of the $y$ axis is downward and vertical.

16. Find the volume generated by revolving about $OX$ the area bounded by $OX$ and the curve $4y = 16 - x^2$.

17. Find the volume generated by revolving about $OX$ the area bounded by the curve $y = x^2 + 2$ and the line $y = 3$.

18. Find the volume generated by revolving about $OX$ the area bounded by $OX$ and the curve $y = 3x - x^3$.

19. Find the volume generated by revolving about the line $y = -1$ the area bounded by the curves $9y = 2x^2$ and $9y = 36 - 2x^2$.

20. An axman makes a wedge-shaped cut in the trunk of a tree. Assuming that the trunk is a right circular cylinder of radius 8 in., that the lower surface of the cut is a horizontal plane, and that the upper surface is a plane inclined at an angle of 45° to the horizontal and intersecting the lower surface of the cut in a diameter, find the amount of wood cut out.

21. On a system of parallel chords of a circle of radius 2 there are constructed equilateral triangles with their planes perpendicular to the plane of the circle and on the same side of that plane, thus forming a solid. Find the volume of the solid.

22. Show that the volume of the solid generated by revolving about $OY$ the area bounded by $OX$ and the curve $y = a - bx^2$ is equal to the area of the base of the solid multiplied by half its altitude.

23. In a sphere of radius $a$ find the volume of a segment of one base and altitude $h$.

**24.** A solid is such that any cross section perpendicular to an axis is a circle, with its radius equal to the square root of the distance of the section from a fixed point of the axis. The total length of the axis from the fixed point is 4. Find the volume of the solid.

**25.** A variable square moves with its plane perpendicular to the axis of $y$ and with the ends of one of its diagonals respectively in the parts of the curves $y^2 = 16x$ and $y^2 = 4x$, which are above the axis of $x$ Find the volume generated by the square as its plane moves a distance 8 from the origin.

**26.** The plane of a variable circle moves so as to be perpendicular to $OX$, and the ends of a diameter are on the curves $y = x^2$ and $y = 3x^2 - 8$ Find the volume of the solid generated as the plane moves from one point of intersection of the curves to the other.

**27.** All sections of a certain solid made by planes perpendicular to $OY$ are isosceles triangles. The base of each triangle is a line drawn perpendicular to $OY$, with its ends in the curve $y = 4x^2$. The altitude of each triangle is equal to its base Find the volume of the solid included between the planes for which $y = 0$ and $y = 6$.

**28.** All sections of a certain solid made by planes perpendicular to $OY$ are right isosceles triangles One leg of each triangle coincides with the line perpendicular to $OY$ with its ends in $OY$ and the curve $y^2 = 4x$. Find the volume of the solid between the sections for which $y = 0$ and $y = 8$.

**29.** Find the work done in pumping all the water from a full cylindrical tank, of height 15 ft. and radius 3 ft., to a height of 20 ft. above the top of the tank.

**30.** Find the work done in emptying of water a full conical receiver of altitude 6 ft and radius 3 ft., the vertex of the cone being down.

# CHAPTER IV

## ALGEBRAIC FUNCTIONS

**27. Distance between two points.** Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ (Fig. 34) be any two points in the plane $XOY$, such that the straight line $P_1P_2$ is not parallel either to $OX$ or to $OY$. Through $P_1$ draw a straight line parallel to $OX$, and through $P_2$ draw a straight line parallel to $OY$, and denote their point of intersection by $R$.

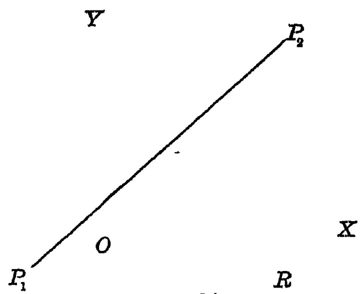



FIG 34

Then $P_1R = \Delta x = x_2 - x_1$

and $RP_2 = \Delta y = y_2 - y_1$.

In the right triangle $P_1RP_2$

$$P_1P_2 = \sqrt{\overline{P_1R}^2 + \overline{RP_2}^2};$$

whence
$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1}$$

If $y_2 = y_1$, $P_1P_2$ is parallel to $OX$, and the formula reduces to

$$P_1P_2 = x_2 - x_1. \tag{2}$$

In like manner, if $x_2 = x_1$, $P_1P_2$ is parallel to $OY$, and the formula reduces to
$$P_1P_2 = y_2 - y_1. \tag{3}$$

**28. Circle.** Since a *circle* is the locus of a point which is always at a constant distance from a fixed point, formula (1) § 27, enables us to write down immediately the equation of a circle.

Let $C(h, k)$ (Fig. 35) be the center of a circle of radius $r$. Then, if $P(x, y)$ is any point of the circle, by (1), § 27, $x$ and $y$ must satisfy the equation

$$(x - h)^2 + (y - k)^2 = r^2. \tag{1}$$

Moreover, any point the coördinates of which satisfy (1), must be at the distance $r$ from $C$ and hence be a point of the circle. Accordingly, (1) is the equation of a circle.

If (1) is expanded, it becomes

$$x^2 + y^2 - 2\,hx - 2\,ky + h^2 + k^2 - r^2 = 0, \tag{2}$$

an equation of the second degree with no term in $xy$ and with the coefficients of $x^2$ and $y^2$ equal.

Conversely, any equation of the second degree with no $xy$ term and with the coefficients of $x^2$ and $y^2$ equal (as

$$Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0, \tag{3}$$

where $A$, $G$, $F$, and $C$ are any constants) may be transformed into the form (1) and represents a circle, unless the number corresponding to $r^2$ is negative (see Ex. 3, page 81), in which case the equation is satisfied by no real values of $x$ and $y$ and accordingly has no corresponding locus.

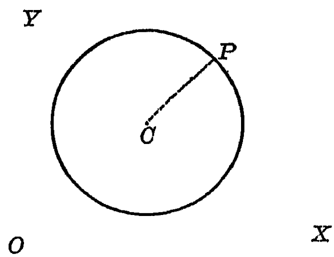



FIG. 35

The circle is most readily drawn by making such transformation, locating the center, and constructing the circle with compasses.

**Ex. 1.** $$x^2 + y^2 - 2\,x - 4\,y = 0.$$

This equation may be written in the form

$$(x^2 - 2\,x \qquad) + (y^2 - 4\,y \qquad) = 0,$$

and the terms in the parentheses may be made perfect squares by adding 1 in the first parenthesis and 4 in the second parenthesis As we have added a total of 5 to the left-hand side of the equation, we must add an equal amount to the right-hand side of the equation The result is

$$(x^2 - 2\,x + 1) + (y^2 - 4\,y + 4) = 5,$$

which may be placed in the form

$$(x - 1)^2 + (y - 2)^2 = 5,$$

the equation of a circle of radius $\sqrt{5}$ with its center at the point (1, 2).

**Ex. 2.** $9x^2 + 9y^2 - 9x + 6y - 8 = 0.$

Placing 8 on the right-hand side of the equation and then dividing by 9, we have

$$x^2 + y^2 - x + \tfrac{2}{3}y = \tfrac{8}{9},$$

which may be treated by the method used in Ex. 1. The result is

$$(x - \tfrac{1}{2})^2 + (y + \tfrac{1}{3})^2 = \tfrac{5}{4},$$

the equation of a circle of radius $\frac{1}{2}\sqrt{5}$, with its center at $(\frac{1}{2}, -\frac{1}{3})$.

**Ex. 3.** $9x^2 + 9y^2 - 6x + 12y + 11 = 0$

Proceeding as in Ex. 2, we have, as the transformed equation,

$$(x - \tfrac{1}{3})^2 + (y + \tfrac{2}{3})^2 = -\tfrac{2}{3},$$

an equation which cannot be satisfied by any real values of $x$ and $y$, since the sum of two positive quantities cannot be negative Hence this equation corresponds to no real curve.

### EXERCISES

**1.** Find the equation of the circle with the center $(4, -2)$ and the radius 3

**2.** Find the equation of the circle with the center $(0, -1)$ and the radius 5.

**3.** Find the center and the radius of the circle

$$x^2 + y^2 + 6x - 10y + 9 = 0.$$

**4.** Find the center and the radius of the circle

$$5x^2 + 5y^2 + 8x - 6y - 15 = 0.$$

**5.** Find the equation of the straight line passing through the center of the circle

$$x^2 + y^2 + 2x - y + 1 = 0$$

and perpendicular to the line

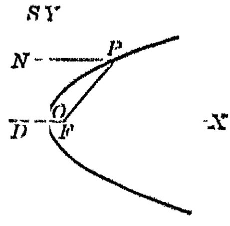
$$2x + 3y - 4 = 0$$

**6.** Prove that two circles are concentric if their equations differ only in the absolute term

**29. Parabola.** *The locus of a point equally distant from a fixed point and a fixed straight line is called a parabola.* The fixed point is called the *focus* and the fixed straight line is called the *directrix.*

Let $F$ (Fig. 36) be the focus and $RS$ the directrix of a parabola. Through $F$ draw a straight line perpendicular to $RS$, intersecting it at $D$, and let this line be the axis of $x$. Let the middle point of $DF$ be taken as $O$, the origin of coördinates, and draw the axis $OY$. Then, if the distance $DF$ is $2c$, the coördinates of $F$ are $(c, 0)$ and the equation of $RS$ is $x = -c$.

Let $P(x, y)$ be any point of the parabola, and draw the straight line $FP$ and the straight line $NP$ perpendicular to $RS$.




Then $\qquad NP = x + c,$

and, by § 27, $\qquad FP = \sqrt{(x-c)^2 + y^2};$

whence, from the definition of the parabola,

$$(x-c)^2 + y^2 = (x+c)^2,$$

which reduces to $\qquad y^2 = 4cx. \qquad (1)$

Fig. 36

Conversely, if the coördinates of any point $P$ satisfy (1), it can be shown that the distances $FP$ and $NP$ are equal, and hence $P$ is a point of the parabola.

Solving (1) for $y$ in terms of $x$, we have

$$y = \pm 2\sqrt{cx}. \qquad (2)$$

We assume that $c$ is positive. Then it is evident that if a negative value is assigned to $x$, $y$ is imaginary, and no corresponding points of the parabola can be located. All positive values may be assigned to $x$, however, and hence the parabola lies entirely on the positive side of the axis $OY$.

Accordingly, we assign positive values to $x$, compute the corresponding values of $y$, and draw a smooth curve through the points thus located.

It is to be noticed that to every value assigned to $x$ there are two corresponding values of $y$, equal in magnitude and opposite in algebraic sign, to which there correspond two points of the parabola on opposite sides of $OX$ and equally distant from it. Hence the parabola is *symmetrical* with respect to $OX$, and accordingly $OX$ is called the *axis* of the parabola.

The point at which its axis intersects a parabola is called the *vertex* of the parabola. Accordingly, $O$ is the vertex of the parabola.

Returning to Fig. 36, if $F$ is taken at the left of $O$ with the coördinates $(-c, 0)$, and $RS$ is taken at the right of $O$ with the equation $x = c$, equation (1) becomes

$$y^2 = -4\,cx \tag{3}$$

and represents a parabola lying on the negative side of $OY$. Hence we conclude that any equation in the form

$$y^2 = kx, \tag{4}$$

where $k$ is a positive or a negative constant, is a parabola, with its vertex at $O$, its axis on $OX$, its focus at the point $\left(\frac{k}{4}, 0\right)$, and its directrix the straight line $x = -\frac{k}{4}$.

Similarly, the equation $$x^2 = ky \tag{5}$$

represents a parabola, with its vertex at $O$ and with its axis coinciding with the positive or the negative part of $OY$, according as $k$ is positive or negative. The focus is always the point $\left(0, \frac{k}{4}\right)$, and the directrix is the line $y = -\frac{k}{4}$, whether $k$ be positive or negative.

**30. Parabolic segment.** An important property of the parabola is contained in the following theorem:

*The square of any two chords of a parabola which are perpendicular to its axis are to each other as their distances from the vertex of the parabola.*

This theorem may be proved as follows:

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points of any parabola $y^2 = kx$ (Fig. 37).

Then $$y_1^2 = kx_1$$

and $$y_2^2 = kx_2;$$

whence $$\frac{y_1^2}{y_2^2} = \frac{x_1}{x_2},$$

whence $$\frac{(2\,y_1)^2}{(2\,y_2)^2} = \frac{x_1}{x_2}. \tag{1}$$

From the symmetry of the parabola, $2y_1 = Q_1P_1$ and $2y_2 = Q_2P_2$. But $x_1 = OM_1$ and $x_2 = OM_2$, and hence (1) becomes

$$\frac{\overline{Q_1P_1}^2}{\overline{Q_2P_2}^2} = \frac{OM_1}{OM_2},$$

and the theorem is proved.

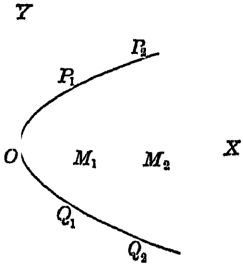


FIG. 37

The figure bounded by the parabola and a chord perpendicular to the axis of the parabola, as $Q_1OP_1$ (Fig. 37), is called a *parabolic segment.* The chord is called the *base* of the segment, the vertex of the parabola is called the *vertex* of the segment, and the distance from the vertex to the base is called the *altitude* of the segment.

### EXERCISES

Plot the following parabolas, determining the focus of each:

1. $y^2 = -8x$
2. $x^2 = 4y$.
3. $y^2 = 6x$.
4. $x^2 = -7y$.

5. The altitude of a parabolic segment is 10 ft., and the length of its base is 16 ft. A straight line drawn across the segment perpendicular to its axis is 10 ft. long. How far is it from the vertex of the segment?

6. An arch in the form of a parabolic curve, the axis being vertical, is 50 ft. across the bottom, and the highest point is 15 ft. above the horizontal. What is the length of a beam placed horizontally across the arch 6 ft. from the top?

7. The cable of a suspension bridge hangs in the form of a parabola. The roadway, which is horizontal and 400 ft. long, is supported by vertical wires attached to the cable, the longest wire being 80 ft. and the shortest being 20 ft. Find the length of a supporting wire attached to the roadway 75 ft. from the middle.

8. Any section of a given parabolic mirror made by a plane passing through the axis of the mirror is a parabolic segment of which the altitude is 6 in. and the length of the base 10 in. Find the circumference of the section of the mirror made by a plane perpendicular to its axis and 4 in. from its vertex

9. Find the equation of the parabola having the line $x = 3$ as its directrix and having its focus at the origin of coördinates.

10. Find the equation of the parabola having the line $y = -2$ as its directrix and having its focus at the point (2, 4).

**31. Ellipse.** *The locus of a point the sum of whose distances from two fixed points is constant is called an ellipse.* The two fixed points are called the *foci.*

Let $F$ and $F'$ (Fig. 38) be the two foci, and let the distance $F'F$ be $2c$. Let the straight line determined by $F'$ and $F$ be taken as the axis of $x$, and the middle point of $F'F$ be taken as $O$, the origin of coördinates, and draw the axis $OY$. Then the coördinates of $F'$ and $F$ are respectively $(-c, 0)$ and $(c, 0)$.

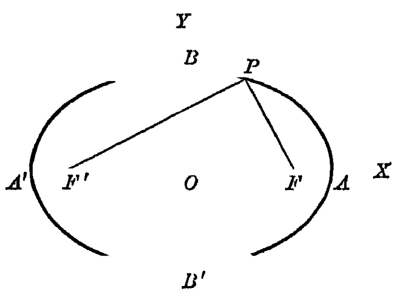


FIG. 38

Let $P(x, y)$ be any point of the ellipse, and $2a$ represent the constant sum of its distances from the foci. Then, from the definition of the ellipse, the sum of the distances $F'P$ and $FP$ is $2a$, and from the triangle $F'PF$ it is evident that $2a > 2c$; whence $a > c$.

By § 27, $$F'P = \sqrt{(x + c)^2 + y^2}$$

and $$FP = \sqrt{(x - c)^2 + y^2};$$

whence, from the definition of the ellipse,

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a. \tag{1}$$

Clearing (1) of radicals, we have

$$(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2. \tag{2}$$

Dividing (2) by $a^4 - a^2c^2$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{3}$$

But since $a > c$, $a^2 - c^2$ is a positive quantity which may be denoted by $b^2$, and (3) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4}$$

Conversely, if the coördinates of any point $P$ satisfy (4), it can be shown that the sum of the distances $F'P$ and $FP$ is $2a$, and hence $P$ is a point of the ellipse.

Solving (4) for $y$ in terms of $x$, we have

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}. \tag{5}$$

It is evident that the only values which can be assigned to $x$ must be numerically less than $a$; for if any numerically larger values are assigned to $x$, the corresponding values of $y$ are imaginary, and no corresponding points can be plotted. Hence the curve lies entirely between the lines $x = -a$ and $x = a$.

We may, then, assign the possible values to $x$, compute the corresponding values of $y$, and, locating the corresponding points, draw a smooth curve through them. As in the case of the parabola, we observe that $OX$ is an axis of symmetry of the ellipse.

We may also solve (4) for $x$ in terms of $y$, with the result

$$x = \pm \frac{a}{b}\sqrt{b^2 - y^2}. \tag{6}$$

From this form of the equation we find that the ellipse lies entirely between the lines $y = -b$ and $y = b$ and is symmetrical with respect to $OY$.

Hence the ellipse has two axes, $A'A$ and $B'B$ (Fig. 38), which are at right angles to each other. But $A'A = 2a$ and $B'B = 2b$; and since $a > b$, it follows that $A'A > B'B$. Hence $A'A$ is called the *major axis* of the ellipse, and $B'B$ is called the *minor axis* of the ellipse.

The ends of the major axis, $A'$ and $A$, are called the *vertices* of the ellipse, and the point midway between the vertices is called the *center* of the ellipse; that is, $O$ is the center of the ellipse, and it can be readily shown that any chord of the ellipse which passes through $O$ is bisected by that point.

From the definition of $b$, $c = \sqrt{a^2 - b^2}$, and the coördinates of the foci are $(\pm \sqrt{a^2 - b^2}, 0)$.

The ratio $\frac{OF}{OA}$ (that is, the ratio of the distance of the focus from the center to the distance of either vertex from the center) is called the *eccentricity* of the ellipse and is denoted by $e$. But

$$OF = \sqrt{a^2 - b^2}, \tag{7}$$

and hence
$$e = \frac{\sqrt{a^2 - b^2}}{a}; \tag{8}$$

whence it follows that the eccentricity of an ellipse is always less than unity.

Similarly, any equation in form (4), in which $b^2 > a^2$, represents an ellipse with its center at $O$, its major axis on $OY$, and its minor axis on $OX$. Then the vertices are the points $(0, \pm b)$, the foci are the points $(0, \pm \sqrt{b^2 - a^2})$, and $e = \frac{\sqrt{b^2 - a^2}}{b}$.

In either case the nearer the foci approach coincidence, the smaller $e$ becomes and the more nearly $b = a$. Hence *a circle may be considered as an ellipse with coincident foci and equal axes.* Its eccentricity is, of course, zero.

### EXERCISES

Plot the following ellipses, finding the vertices, the foci, and the eccentricity of each:

1. $9x^2 + 16y^2 = 144$.
2. $9x^2 + 4y^2 = 36$.
3. $3x^2 + 4y^2 = 2$.
4. $2x^2 + 3y^2 = 1$.

5. Find the equation of the ellipse which has its foci at the points $(-2, 0)$ and $(6, 0)$ and which has the sum of the distances of any point on it from the foci equal to 10.

6. Find the equation of the ellipse having its foci at the points $(0, 0)$ and $(0, 5)$ and having the length of its major axis equal to 7.

**32. Hyperbola.** *The locus of a point the difference of whose distances from two fixed points is constant is called a hyperbola.* The two fixed points are called the *foci.*

Let $F$ and $F'$ (Fig. 39) be the two foci, and let the distance $F'F$ be $2c$. Let the straight line determined by $F'$ and $F$ be taken as the axis of $x$, and the middle point of $F'F$ be taken as $O$, the origin of coordinates, and draw the axis $OY$. Then the coördinates of $F'$ and $F$ are respectively $(-c, 0)$ and $(c, 0)$.

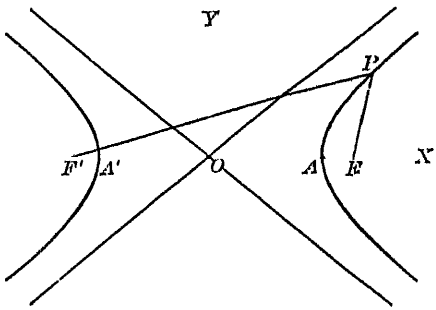



Fig. 39

Let $P(x, y)$ be any point of the hyperbola and $2a$ represent the constant difference of its distances from the foci. Then, from the definition of the hyperbola, the difference of the distances $F'P$ and $FP$ is $2a$, and from the triangle $F'PF$ it is evident that $2a < 2c$, for the difference of any two sides of a triangle is less than the third side; whence $a < c$.

By § 27, $$F'P = \sqrt{(x+c)^2 + y^2}$$

and $$FP = \sqrt{(x-c)^2 + y^2};$$

whence either

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a \quad (1)$$

or $$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a, \quad (2)$$

according as $FP$ or $F'P$ is the greater distance.

Clearing either (1) or (2) of radicals, we obtain the same result:

$$(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2. \quad (3)$$

Dividing (3) by $a^4 - a^2c^2$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \quad (4)$$

But since $a < c$, $a^2 - c^2$ is a negative quantity which may be denoted by $-b^2$, and (4) becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad (5)$$

Conversely, if the coördinates of any point $P$ satisfy (5), it can be shown that the difference of the distances $F'P$ and $FP$ is $2a$, and hence $P$ is a point of the hyperbola.

Solving (5) for $y$ in terms of $x$, we have

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}. \tag{6}$$

In this equation we may assume for $x$ only values that are numerically greater than $a$, as any other values give imaginary values for $y$. Hence there are no points of the hyperbola between the lines $x = -a$ and $x = a$. The hyperbola is symmetrical with respect to $OX$.

As the values assigned to $x$ increase numerically, the corresponding points of the hyperbola recede from the axis $OX$. We may, however, write (6) in the form

$$y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}. \tag{7}$$

Now if $y_1$ and $y_2$ are the ordinates of points of (7) and of the straight lines $y = \pm \frac{b}{a} x$ respectively, then

$$y_2 - y_1 = \pm \frac{b}{a} x \left[1 - \sqrt{1 - \frac{a^2}{x^2}}\right] = \pm \frac{\dfrac{ba}{x}}{1 + \sqrt{1 - \dfrac{a^2}{x^2}}};$$

whence $$\lim_{x \to \infty} (y_2 - y_1) = 0.$$

Hence, by prolonging the straight lines and the curve indefinitely, we can make them come as near together as we please.

Now, when a straight line has such a position with respect to a curve that as the two are indefinitely prolonged the distance between them approaches zero as a limit, the straight line is called an *asymptote* of the curve. It follows that the lines $y = \frac{b}{a} x$ and $y = -\frac{b}{a} x$ are asymptotes of the hyperbola (Fig. 39).

If we had solved (5) for $x$ in terms of $y$, the result would have been

$$x = \pm \frac{a}{b}\sqrt{b^2 + y^2}; \tag{8}$$

from which it appears that all values may be assigned to $y$, and that $OY$ is also an axis of symmetry of the hyperbola.

The points $A'$ and $A$ in which one axis of the hyperbola intersects the hyperbola are called the *vertices*, and the portion of the axis extending from $A'$ to $A$ is called the *transverse axis*. The point midway between the vertices is called the *center*; that is, $O$ is the center of the hyperbola, and it can readily be shown that any chord of the hyperbola which passes through $O$ is bisected by that point. The other axis of the hyperbola, which is perpendicular to the transverse axis, is called the *conjugate axis*. This axis does not intersect the curve, as is evident from the figure, but it is useful in fixing the asymptotes and thus determining the shape of the curve for large values of $x$.

From the definition of $b$, $c = \sqrt{a^2 + b^2}$, and the coördinates of the foci are $(\pm\sqrt{a^2 + b^2}, 0)$. Therefore

$$OF = \sqrt{a^2 + b^2}. \tag{9}$$

If we define the eccentricity of the hyperbola as the ratio $\frac{OF}{OA}$, we have

$$e = \frac{\sqrt{a^2 + b^2}}{a}, \tag{10}$$

a quantity which is evidently always greater than unity.

Similarly, the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \tag{11}$$

is the equation of a hyperbola, with its center at $O$, its transverse axis on $OY$, and its conjugate axis on $OX$. Then the vertices are the points $(0, \pm b)$, the foci are the points $(0, \pm\sqrt{b^2 + a^2})$, the asymptotes are the straight lines $y = \pm \frac{b}{a} x$, and $e = \frac{\sqrt{b^2 + a^2}}{b}$.

If $b = a$, in either (5) or (11), the equation of the hyperbola assumes the form

$$x^2 - y^2 = a^2 \quad \text{or} \quad y^2 - x^2 = a^2, \tag{12}$$

and the hyperbola is called an *equilateral hyperbola*. The equations of the asymptotes become $y = \pm x$; and as these lines are perpendicular to each other, the hyperbola is also called a *rectangular hyperbola*.

**EXERCISES**

Plot the following hyperbolas, finding the vertices, the foci, the asymptotes, and the eccentricity of each.

1. $4x^2 - 9y^2 = 36.$
2. $9x^2 - 4y^2 = 36.$
3. $3y^2 - 2x^2 = 6$
4. $x^2 - y^2 = 8.$
5. $2x^2 - 3y^2 = 1.$
6. $4y^2 - x^2 = 1.$

7. Find the equation of the hyperbola having its foci at the points (0, 0) and (4, 0), and the difference of the distances of any point on it from the foci equal to 2.

8. The foci of a hyperbola are at the points $(-4, 2)$ and $(4, 2)$, and the difference of the distances of any point on it from the foci is 4. Find the equation of the hyperbola, and plot.

**33. Other curves.** In the discussion of the parabola, the ellipse, and the hyperbola, the axes of symmetry and the asymptotes were of considerable assistance in constructing the curves; moreover, the knowledge that there could be no points of the curve in certain parts of the plane decreased the labor of drawing the curves. We shall now plot the loci of a few equations, noting in advance whether the curve is bounded in any direction or has any axes of symmetry or asymptotes. In this way we shall be able to anticipate to a considerable extent the form of the curve.

**Ex. 1.** $(y+3)^2 = (x-2)^2(x+1).$

Solving for $y$, we have

$$y = -3 \pm (x-2)\sqrt{x+1}.$$

In the first place, we see that the only values that may be assigned to $x$ are greater than $-1$, and hence the curve lies entirely on the positive side of the line $x = -1$. Furthermore, corresponding to every value of $x$, there are two values of $y$ which determine two points at equal distances from the line $y = -3$. Hence we conclude that the line $y = -3$ is an axis of symmetry of the curve.

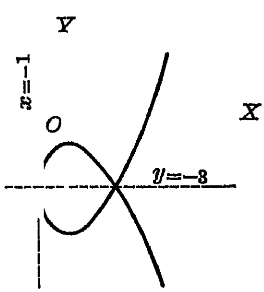


FIG. 40

Assigning values to $x$ and locating the points determined, we draw the curve (Fig. 40).

**Ex. 2.** $$xy = 4.$$

Solving for $y$, we have $$y = \frac{4}{x}.$$

It is evident, then, that we may assign to $x$ any real value except zero, in which case we should be asked to divide 4 by 0, a process that cannot be carried out. Consequently, there can be no point of the curve on the line $x = 0$; that is, on $OY$. We may, however, assume values for $x$ as near to zero as we wish, and the nearer they are to zero, the nearer the corresponding points are to $OY$; but as the points come nearer to $OY$ they recede along the curve. Hence $OY$ is an asymptote of the curve.

If we solve for $x$, we have

$$x = \frac{4}{y};$$

and, reasoning as above, we conclude that the line $y = 0$ (that is, the axis $OX$) is also an asymptote of the curve.

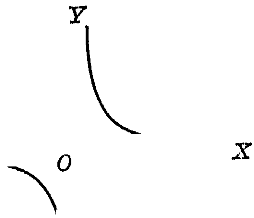


FIG. 41

The curve is drawn in Fig. 41. It is a special case of the curve $xy = k$, where $k$ is a real constant which may be either positive or negative, and is, in fact, a rectangular hyperbola referred to its asymptotes as axes.

It is customary to say that when the denominator of a fraction is zero, the value of the fraction becomes infinite The curve just constructed shows graphically what is meant by such an expression.

**Ex. 3.** $xy + 2x + y - 1 = 0.$

Solving for $y$, we have

$$y = \frac{1 - 2x}{1 + x},$$

from which we conclude that the line $x = -1$ is an asymptote of the curve.

Solving for $x$, we have

$$x = \frac{1 - y}{2 + y},$$

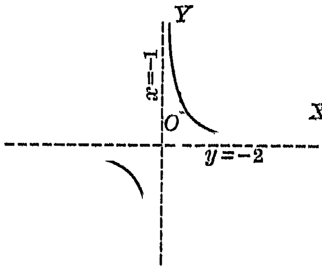


FIG. 42

from which we conclude that the line $y = -2$ is also an asymptote of the curve.

We accordingly draw these two asymptotes (Fig. 42) and the curve through the points determined by assigning values to either $x$ or $y$ and computing the corresponding values of the other variable.

The curve is, in fact, a rectangular hyperbola, with the lines $x = -1$ and $y = -2$ as its asymptotes.

**Ex. 4.** $y^2 = \dfrac{x^3}{2a - x}$.

Solving for $y$, we have

$$y = \pm\sqrt{\frac{x^3}{2a-x}},$$

whence it is evident that the curve is symmetrical with respect to $OX$. The lines $x = 0$ and $x = 2a$, corresponding to the values of $x$ which make the numerator and the denominator of the fraction under the radical sign respectively zero, divide the plane into three strips; and only values between 0 and $2a$ can be substituted for $x$, since all other values make $y$ imaginary. It follows that the curve lies entirely in the strip bounded by the two lines $x = 0$ and $x = 2a$.

By the same reasoning that was used in Exs. 2 and 3, it can be shown that the line $x = 2a$ is an asymptote of the curve.

The curve, which is called a *cissoid*, is drawn in Fig. 43.

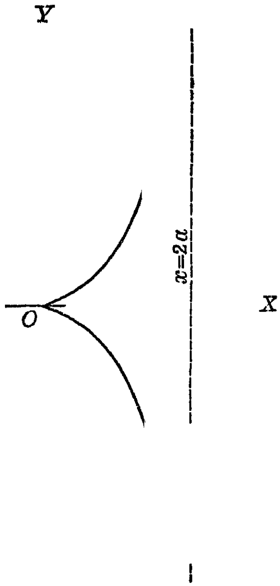



FIG. 43

**EXERCISES**

Plot the following curves:

1. $y^2 = x^3$.
2. $y^2 = x^2(x + 4)$.
3. $y^2 = 4(x - 8)$.
4. $y^2 = x^2 - 5x + 6$.
5. $y^2 = x(x^2 - 4)$.
6. $y^2 = x^3 - 5x^2 + 6x$.
7. $y^2 = 4x^4 - x^6$.
8. $xy^2 = 4 - x$.
9. $xy = -5$.
10. $3y - xy = 12$.
11. $xy - 2x + 4y = 0$.
12. $y = \dfrac{x^3 + 1}{x}$.

**34. Theorems on limits.** In obtaining more general formulas for differentiation, the following theorems on limits will be assumed without formal proof:

1. *The limit of the sum of a finite number of variables is equal to the sum of the limits of the variables.*

2. *The limit of the product of a finite number of variables is equal to the product of the limits of the variables.*

3. *The limit of a constant multiplied by a variable is equal to the constant multiplied by the limit of the variable.*

4. *The limit of the quotient of two variables is equal to the quotient of the limits of the variables, provided the limit of the divisor is not zero.*

**35. Theorems on derivatives.** In order to extend the process of differentiation to functions other than polynomials, we shall need the following theorems:

1. *The derivative of a constant is zero.*

This theorem was proved in § 8.

2. *The derivative of a constant times a function is equal to the constant times the derivative of the function.*

Let $u$ be a function of $x$ which can be differentiated, let $c$ be a constant, and place
$$y = cu.$$

Give $x$ an increment $\Delta x$, and let $\Delta u$ and $\Delta y$ be the corresponding increments of $u$ and $y$. Then
$$\Delta y = c(u + \Delta u) - cu = c\,\Delta u.$$

Hence
$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x},$$
and, by theorem 3, § 34,
$$\text{Lim}\,\frac{\Delta y}{\Delta x} = c\,\text{Lim}\,\frac{\Delta u}{\Delta x}.$$

Therefore
$$\frac{dy}{dx} = c\frac{du}{dx},$$
by the definition of a derivative.

**Ex. 1.** $y = 5(x^3 + 3x^2 + 1)$.
$$\frac{dy}{dx} = 5\frac{d}{dx}(x^3 + 3x^2 + 1) = 5(3x^2 + 6x) = 15(x^2 + 2x).$$

3. *The derivative of the sum of a finite number of functions is equal to the sum of the derivatives of the functions.*

Let $u$, $v$, and $w$ be three functions of $x$ which can be differentiated, and let
$$y = u + v + w.$$

Give $x$ an increment $\Delta x$, and let the corresponding increments of $u$, $v$, $w$, and $y$ be $\Delta u$, $\Delta v$, $\Delta w$, and $\Delta y$. Then

$$\Delta y = (u + \Delta u + v + \Delta v + w + \Delta w) - (u + v + w)$$
$$= \Delta u + \Delta v + \Delta w;$$

whence
$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \frac{\Delta w}{\Delta x}.$$

Now let $\Delta x$ approach zero. By theorem 1, § 34,

$$\text{Lim}\,\frac{\Delta y}{\Delta x} = \text{Lim}\,\frac{\Delta u}{\Delta x} + \text{Lim}\,\frac{\Delta v}{\Delta x} + \text{Lim}\,\frac{\Delta w}{\Delta x};$$

that is, by the definition of a derivative,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}.$$

The proof is evidently applicable to any finite number of functions.

**Ex. 2.** $y = x^4 - 3x^3 + 2x^2 - 7x$.

$$\frac{dy}{dx} = 4x^3 - 9x^2 + 4x - 7.$$

4. *The derivative of the product of a finite number of functions is equal to the sum of the products obtained by multiplying the derivative of each factor by all the other factors.*

Let $u$ and $v$ be two functions of $x$ which can be differentiated, and let
$$y = uv.$$

Give $x$ an increment $\Delta x$, and let the corresponding increments of $u$, $v$, and $y$ be $\Delta u$, $\Delta v$, and $\Delta y$.

Then
$$\Delta y = (u + \Delta u)(v + \Delta v) - uv$$
$$= u\,\Delta v + v\,\Delta u + \Delta u \cdot \Delta v$$

and
$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} \cdot \Delta v.$$

If, now, $\Delta x$ approaches zero, we have, by § 34,

$$\text{Lim}\frac{\Delta y}{\Delta x} = u\,\text{Lim}\frac{\Delta v}{\Delta x} + v\,\text{Lim}\frac{\Delta u}{\Delta x} + \text{Lim}\frac{\Delta u}{\Delta x}\cdot\text{Lim}\,\Delta v.$$

But $$\text{Lim}\,\Delta v = 0,$$

and therefore $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Again, let $$y = uvw.$$

Regarding $uv$ as one function and applying the result already obtained, we have

$$\begin{aligned}\frac{dy}{dx} &= uv\frac{dw}{dx} + w\frac{d(uv)}{dx}\\ &= uv\frac{dw}{dx} + w\left[u\frac{dv}{dx} + v\frac{du}{dx}\right]\\ &= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.\end{aligned}$$

The proof is clearly applicable to any finite numbers of factors.

**Ex. 3.** $y = (3x - 5)(x^2 + 1)x^3$

$$\begin{aligned}\frac{dy}{dx} &= (3x-5)(x^2+1)\frac{d(x^3)}{dx} + (3x-5)x^3\frac{d(x^2+1)}{dx} + (x^2+1)x^3\frac{d(3x-5)}{dx}\\ &= (3x-5)(x^2+1)(3x^2) + (3x-5)x^3(2x) + (x^2+1)x^3(3)\\ &= (18x^3 - 25x^2 + 12x - 15)x^2\end{aligned}$$

5. *The derivative of a fraction is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Let $y = \frac{u}{v}$, where $u$ and $v$ are two functions of $x$ which can be differentiated. Give $x$ an increment $\Delta x$, and let $\Delta u$, $\Delta v$, and $\Delta y$ be the corresponding increments of $u$, $v$, and $y$. Then

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{v^2 + v\,\Delta v}$$

and $$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\,\Delta v}.$$

Now let $\Delta x$ approach zero. By § 34,

$$\text{Lim}\frac{\Delta y}{\Delta x}=\frac{v\,\text{Lim}\,\dfrac{\Delta u}{\Delta x}-u\,\text{Lim}\,\dfrac{\Delta v}{\Delta x}}{v^2+v\,\text{Lim}\,\Delta v};$$

whence
$$\frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}.$$

**Ex. 4.** $y=\dfrac{x^2-1}{x^2+1}$.

$$\frac{dy}{dx}=\frac{(x^2+1)(2x)-(x^2-1)2x}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

6. *The derivative of the nth power of a function is obtained by multiplying n times the (n−1)th power of the function by the derivative of the function.*

Let $y=u^n$, where $u$ is any function of $x$ which can be differentiated and $n$ is a constant. We need to distinguish four cases:

Case I. When $n$ is a positive integer.

Give $x$ an increment $\Delta x$, and let $\Delta u$ and $\Delta y$ be the corresponding increments of $u$ and $y$. Then

$$\Delta y=(u+\Delta u)^n-u^n;$$

whence, by the binomial theorem,

$$\Delta y=nu^{n-1}\Delta u+\frac{n(n-1)}{2}u^{n-2}(\Delta u)^2+\cdots+(\Delta u)^n.$$

$$\frac{\Delta y}{\Delta x}=nu^{n-1}\frac{\Delta u}{\Delta x}+\frac{n(n-1)}{2}u^{n-2}\Delta u\frac{\Delta u}{\Delta x}+\cdots+(\Delta u)^{n-1}\frac{\Delta u}{\Delta x}.$$

Now let $\Delta x$, $\Delta u$, $\Delta y$ approach zero, and apply theorems 1 and 2, § 34. The limit of $\dfrac{\Delta y}{\Delta x}$ is $\dfrac{dy}{dx}$, the limit of $\dfrac{\Delta u}{\Delta x}$ is $\dfrac{du}{dx}$, and the limit of all terms except the first on the right-hand side of the last equation is zero, since each contains the factor $\Delta u$. Therefore

$$\frac{dy}{dx}=nu^{n-1}\frac{du}{dx}.$$

CASE II. When $n$ is a positive rational fraction.

Let $n=\frac{p}{q}$, where $p$ and $q$ are positive integers, and place

$$y=u^{\frac{p}{q}}.$$

By raising both sides of this equation to the $q$th power, we have

$$y^q=u^p.$$

Here we have two functions of $x$ which are equal for all values of $x$.

Taking the derivative of both sides of the last equation, we have, by Case I, since $p$ and $q$ are positive integers,

$$qy^{q-1}\frac{dy}{dx}=pu^{p-1}\frac{du}{dx}.$$

Substituting the value of $y$ and dividing, we have

$$\frac{dy}{dx}=\frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx}.$$

Hence, in this case also,

$$\frac{dy}{dx}=nu^{n-1}\frac{du}{dx}.$$

CASE III. When $n$ is a negative rational number.

Let $n=-m$, where $m$ is a positive number, and place

$$y=u^{-m}=\frac{1}{u^m}.$$

Then

$$\frac{dy}{dx}=\frac{-\frac{d(u^m)}{dx}}{u^{2m}} \qquad \text{(by 5)}$$

$$=-\frac{mu^{m-1}\frac{du}{dx}}{u^{2m}} \qquad \text{(by Cases I and II)}$$

$$=-mu^{-m-1}\frac{du}{dx}.$$

Hence, in this case also,

$$\frac{dy}{dx}=nu^{n-1}\frac{du}{dx}.$$

CASE IV. When $n$ is an irrational number.

The formula is true in this case also, but the proof will not be given.

It appears that the theorem is true for all real values of $n$. It may be restated as a working-rule in the following words:

*To differentiate a power of any quantity, bring down the exponent as a coefficient, write the quantity with an exponent one less, and multiply by the derivative of the quantity.*

**Ex. 5.** $y = (x^3 + 4x^2 - 5x + 7)^3$.

$$\frac{dy}{dx} = 3(x^3 + 4x^2 - 5x + 7)^2 \frac{d}{dx}(x^3 + 4x^2 - 5x + 7)$$
$$= 3(3x^2 + 8x - 5)(x^3 + 4x^2 - 5x + 7)^2.$$

**Ex. 6.** $y = \sqrt[3]{x^2} + \dfrac{1}{x^3} = x^{\frac{2}{3}} + x^{-3}$

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} - 3x^{-4}$$
$$= \frac{2}{3\sqrt[3]{x}} - \frac{3}{x^4}.$$

**Ex. 7.** $y = (x + 1)\sqrt{x^2 + 1}$.

$$\frac{dy}{dx} = (x+1)\frac{d(x^2+1)^{\frac{1}{2}}}{dx} + (x^2+1)^{\frac{1}{2}}\frac{d(x+1)}{dx}$$
$$= (x+1)[\tfrac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x] + (x^2+1)^{\frac{1}{2}}$$
$$= \frac{x(x+1)}{(x^2+1)^{\frac{1}{2}}} + (x^2+1)^{\frac{1}{2}}$$
$$= \frac{2x^2 + x + 1}{\sqrt{x^2+1}}.$$

**Ex. 8.** $y = \sqrt[3]{\dfrac{x}{x^3+1}} = \left(\dfrac{x}{x^3+1}\right)^{\frac{1}{3}}$.

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{x}{x^3+1}\right)^{-\frac{2}{3}} \frac{d}{dx}\left(\frac{x}{x^3+1}\right)$$
$$= \frac{1}{3}\left(\frac{x^3+1}{x}\right)^{\frac{2}{3}} \frac{1-2x^3}{(x^3+1)^2}$$
$$= \frac{1-2x^3}{3x^{\frac{2}{3}}(x^3+1)^{\frac{4}{3}}}.$$

7. *If $y$ is a function of $x$, then $x$ is a function of $y$, and the derivative of $x$ with respect to $y$ is the reciprocal of the derivative of $y$ with respect to $x$.*

Let $\Delta x$ and $\Delta y$ be corresponding increments of $x$ and $y$. It is immaterial whether $\Delta x$ is assumed and $\Delta y$ determined, or $\Delta y$ is assumed and $\Delta x$ determined. In either case

$$\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}},$$

whence
$$\text{Lim}\,\frac{\Delta x}{\Delta y} = \frac{1}{\text{Lim}\,\frac{\Delta y}{\Delta x}};$$

that is,
$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

8. *If $y$ is a function of $u$ and $u$ is a function of $x$, then $y$ is a function of $x$, and the derivative of $y$ with respect to $x$ is equal to the product of the derivative of $y$ with respect to $u$ and the derivative of $u$ with respect to $x$.*

An increment $\Delta x$ determines an increment $\Delta u$, and this in turn determines an increment $\Delta y$. Then, evidently,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x},$$

whence
$$\text{Lim}\,\frac{\Delta y}{\Delta x} = \text{Lim}\,\frac{\Delta y}{\Delta u} \cdot \text{Lim}\,\frac{\Delta u}{\Delta x};$$

that is,
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

**Ex. 9.** $y = u^2 + 3u + 1$, where $u = \frac{1}{x^2}$.

$$\frac{dy}{dx} = (2u + 3)\left(-\frac{2}{x^3}\right) = -\frac{2 + 3x^2}{x^2} \cdot \frac{2}{x^3} = -\frac{4 + 6x^2}{x^5}.$$

The same result is obtained by substituting in the expression for $y$ the value of $u$ in terms of $x$ and then differentiating.

**36. Formulas.** We may now collect our formulas of differentiation in the following table:

$$\frac{dc}{dx} = 0, \tag{1}$$

$$\frac{d(cu)}{dx} = c\frac{du}{dx}, \tag{2}$$

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \tag{3}$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \tag{4}$$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, \tag{5}$$

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}, \tag{6}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \tag{7}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \tag{8}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}. \tag{9}$$

Formula (9) is a combination of (7) and (8).

The first six formulas may be changed to corresponding formulas for differentials by multiplying both sides of each equation by $dx$. They are

$$dc = 0, \tag{10}$$

$$d(cu) = c\,du, \tag{11}$$

$$d(u+v) = du + dv, \tag{12}$$

$$d(uv) = u\,dv + v\,du, \tag{13}$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}, \tag{14}$$

$$d(u^n) = nu^{n-1}du. \tag{15}$$

### EXERCISES

Find $\dfrac{dy}{dx}$ in each of the following cases:

1. $y=(3x+1)(2x^2+3x-2)$.
2. $y=(x^2-2x+3)(x^2+6x+9)$.
3. $y=(x^2+1)^2(x^2-2)$.
4. $y=\dfrac{x^2+9}{x^2-9}$.
5. $y=\dfrac{x+2}{x^2+x+1}$.
6. $y=\sqrt[3]{x^2}-\dfrac{1}{\sqrt[3]{x^2}}$.
7. $y=x^2-x-\dfrac{4}{x}+\dfrac{1}{x^2}$.
8. $y=(4x^2+3x+1)^2$.
9. $y=\sqrt{x^3}+4x^2+1$.
10. $y=(x^2+4)^{\frac{3}{2}}$.
11. $y=(a^2-x^2)^{\frac{3}{2}}$
12. $y=\dfrac{1}{\sqrt{9-x^2}}$.
13. $y=\sqrt[5]{(x^5+10x^2+3)^3}$.
14. $y=\sqrt{\dfrac{x+1}{x-1}}$.
15. $y=\sqrt{\dfrac{x-a}{x-b}}$.
16. $y=x\sqrt{9-x^2}$.
17. $y=(x+1)\sqrt{x^2-1}$.
18. $y=\dfrac{x}{\sqrt{a^2+x^2}}$.
19. $y=\dfrac{x-1}{\sqrt{x^2-1}}$.
20. $y=\dfrac{2x-3}{\sqrt[3]{x^3+3}}$.
21. $y=(x+1)^2(x^2+1)^{\frac{1}{2}}$.
22. $y=\dfrac{x^2+18}{\sqrt{x^2+9}}$.
23. $y=\dfrac{x}{\sqrt[3]{1+x^3}}$.

**37. Differentiation of implicit functions.** Consider any equation containing two variables $x$ and $y$. If one of them, as $x$, is chosen as the independent variable and a value is assigned to it, the values of $y$ are determined. Hence the given equation defines $y$ as a function of $x$. If the equation is solved for $y$ in terms of $x$, $y$ is called an *explicit* function of $x$. If the equation is not solved for $y$, $y$ is called an *implicit* function of $x$. For example,

$$y^2+3x^2+4xy+4x+2y+4=0,$$

which may be written

$$y^2+(4x+2)y+(3x^2+4x+4)=0,$$

defines $y$ as an implicit function of $x$.

If the equation is solved for $y$, the result

$$y=-2x-1\pm\sqrt{x^2-3}$$

expresses $y$ as an explicit function of $x$.

If it is required to find the derivative of an implicit function, the equation may be differentiated as given, the result being an equation which may be solved algebraically for the derivative. This method is illustrated in the following examples:

**Ex. 1.** $x^2 + y^2 = 5$.

If $x$ is the independent variable,

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(5) = 0;$$

that is,
$$2x + 2y\frac{dy}{dx} = 0,$$

whence
$$\frac{dy}{dx} = -\frac{x}{y}.$$

Or the derivative may be found by taking the differential of both sides, as follows:

$$d(x^2 + y^2) = d(5) = 0;$$

that is,
$$2x\,dx + 2y\,dy = 0,$$

whence
$$\frac{dy}{dx} = -\frac{x}{y}.$$

It is also possible first to solve the given equation for $y$, thus.

$$y = \pm\sqrt{5 - x^2};$$

whence
$$\frac{dy}{dx} = \pm\frac{-x}{\sqrt{5 - x^2}},$$

a result evidently equivalent to the result previously found.

The method of finding the second derivative of an implicit function is illustrated in the following example:

**Ex. 2.** Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 5$

We know from Ex. 1 that $\frac{dy}{dx} = -\frac{x}{y}$.

Therefore
$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= -\frac{y - x\left(\frac{dy}{dx}\right)}{y^2}$$

$$= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$= -\frac{y^2 + x^2}{y^3} = -\frac{5}{y^3},$$

since $y^2 + x^2 = 5$, from the given equation

### EXERCISES

Find $\frac{dy}{dx}$ from each of the following equations:

1. $x^3 + y^3 - 3axy = 0$.
2. $x^2y + 4a^2y = 8a^3$.
3. $y^2 = \frac{x-y}{x+y}$.
4. $\sqrt{y+x} + \sqrt{y-x} = a$.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from each of the following equations:

5. $2x^2 + 3y^2 = 6$.
6. $4x^2 - 9y^2 = 36$.
7. $x^3 + y^3 = a^3$.
8. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.
9. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
10. $xy + 2x + 3y = 6$.
11. $x^2 + xy + y^2 = a^2$.

**38. Tangent line.** Let $P_1(x_1, y_1)$ be a chosen point of any curve, and let $\left(\frac{dy}{dx}\right)_1$ be the value of $\frac{dy}{dx}$ when $x = x_1$ and $y = y_1$. Then $\left(\frac{dy}{dx}\right)_1$ is the slope of the curve at the point $P_1$, and also the slope of the tangent line (§ 15) to the curve at that point. Accordingly, the equation of the tangent line at $P_1$ is (§ 15)

$$y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1).$$

**Ex. 1.** Find the equation of the tangent line to the parabola $y^2 = 3x$ at the point (3, 3).

By differentiation we have

$$2y\frac{dy}{dx} = 3;$$

whence

$$\frac{dy}{dx} = \frac{3}{2y}.$$

Hence, at the point (3, 3), the slope of the tangent line is $\frac{1}{2}$, and its equation is

$$y - 3 = \tfrac{1}{2}(x - 3)$$

or

$$x - 2y + 3 = 0.$$

The angle of intersection of two curves is the angle between their respective tangents at the point of intersection. The method of finding the angle of intersection is illustrated in the example on the following page.

**Ex. 2.** Find the angle of intersection of the circle $x^2 + y^2 = 8$ and of the parabola $x^2 = 2y$.

The points of intersection are $P_1(2, 2)$ and $P_2(-2, 2)$ (Fig 44), and from the symmetry of the diagram it is evident that the angles of intersection at $P_1$ and $P_2$ are the same.

Differentiating the equation of the circle, we have $2x + 2y\frac{dy}{dx} = 0$, whence $\frac{dy}{dx} = -\frac{x}{y}$; and differentiating the equation of the parabola, we find $\frac{dy}{dx} = x$.

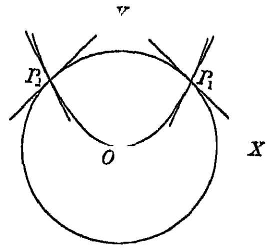



FIG. 44

Hence at $P_1$ the slope of the tangent to the circle is $-1$, and the slope of the tangent to the parabola is 2

Accordingly, if $\beta$ denotes the angle of intersection, by Ex 11, p 35,

$$\tan\beta = \frac{-1-2}{1-2} = 3,$$

or

$$\beta = \tan^{-1} 3.$$

## EXERCISES

1. Find the equation of the tangent line to the curve $x^3 - 8y^2 + 16y - 8 = 0$ at the point (2, 2).

2. Find the equation of the tangent line to the curve $5x^2 - 4x^2y = 4y^3$ at the point (2, 1).

3. Find the point at which the tangent to the curve $8y = x^3$ at $(1, \frac{1}{8})$ intersects the curve again.

4. Find the angle of intersection of the tangents to the curve $y^2 = x^3$ at the points for which $x = 1$

5. Show that the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(x_1, y_1)$ is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$.

6. Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(x_1, y_1)$ is $\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$.

7. Show that the equation of the tangent to the parabola $y^2 = kx$ at the point $(x_1, y_1)$ is $y_1y = \frac{k}{2}(x + x_1)$.

Draw each pair of the following curves in one diagram and determine the angles at which they intersect:

8. $x^2 + y^2 = 5$, $x^2 + y^2 - 2x + 4y - 5 = 0$.
9. $x^2 = 3y$, $9y^2 = 8x$.
10. $y^2 = 4x$, $x^2 + y^2 = 5$
11. $y = 2x$, $xy = 18$.
12. $x - 4y - 1 = 0$, $x^2 - 4x - 4y = 0$.
13. $x^2 + y^2 = 25$, $x^2 = 3y - 3$.

**39. The differentials *dx*, *dy*, *ds*.** On any given curve let the distance from some fixed initial point measured along the curve to any point $P$ be denoted by $s$, where $s$ is positive if $P$ lies in one direction from the initial point and negative if $P$ lies in the opposite direction. The choice of the positive direction is purely arbitrary. We shall take as the positive direction of the tangent that which shows the positive direction of the curve, and shall denote the angle between the positive direction of $OX$ and the positive direction of the tangent by $\phi$.

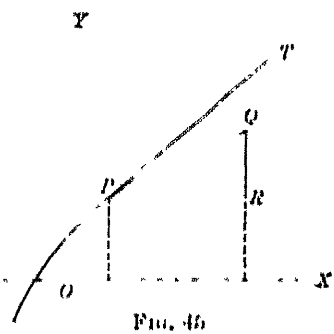


Fig. 45

Now for a fixed curve and a fixed initial point the position of a point $P$ is determined if $s$ is given. Hence $x$ and $y$, the coördinates of $P$, are functions of $s$ which in general are continuous and may be differentiated. We shall now show that

$$\frac{dx}{ds} = \cos\phi, \qquad \frac{dy}{ds} = \sin\phi.$$

Let arc $PQ = \Delta s$ (Fig. 45), where $P$ and $Q$ are so chosen that $\Delta s$ is positive. Then $PR = \Delta x$ and $RQ = \Delta y$, and

$$\frac{\Delta x}{\Delta s} = \frac{PR}{\text{arc } PQ} = \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{PR}{\text{chord } PQ}$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \cos RPQ,$$

$$\frac{\Delta y}{\Delta s} = \frac{RQ}{\text{arc } PQ} = \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{RQ}{\text{chord } PQ}$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \sin RPQ.$$

We shall assume without proof that the ratio of a small chord to its arc is very nearly equal to unity, and that the limit of $\frac{\text{chord}\, PQ}{\text{arc}\, PQ} = 1$ as the point $Q$ approaches the point $P$ along the curve. At the same time the limit of $RPQ = \phi$. Hence, taking limits, we have

$$\frac{dx}{ds} = \cos\phi, \qquad \frac{dy}{ds} = \sin\phi. \tag{1}$$

If the notation of differentials is used, equations (1) become

$$dx = ds \cdot \cos\phi, \qquad dy = ds \cdot \sin\phi;$$

whence, by squaring and adding, we obtain the important equation

$$ds^2 = dx^2 + dy^2. \tag{2}$$

This relation between the differentials of $x$, $y$, and $s$ is often represented by the triangle of Fig. 46. This figure is convenient as a device for memorizing formulas (1) and (2), but it should be borne in mind that $RQ$ is not rigorously equal to $dy$ (§ 20), nor is $PQ$ rigorously equal to $ds$. In fact, $RQ = \Delta y$, and $PQ = \Delta s$; but if this triangle is regarded as a plane right triangle, we recall immediately the values of $\sin\phi$, $\cos\phi$, and $\tan\phi$ which have been previously proved.

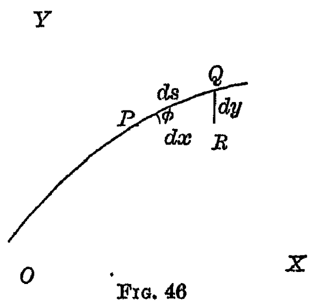


FIG. 46

**40. Motion in a curve.** When a body moves in a curve, the discussion of velocity and acceleration becomes somewhat complicated, as the directions as well as the magnitudes of these quantities need to be considered. We shall not discuss acceleration, but shall notice that the definition for the magnitude of the velocity, or the speed, is the same as before (namely,

$$v = \frac{ds}{dt},$$

where $s$ is distance measured on the curved path) and that the direction of the velocity is that of the tangent to the curve.

Moreover, as the body moves along a curved path through a distance $PQ=\Delta s$ (Fig. 47), $x$ changes by an amount $PR=\Delta x$, and $y$ changes by an amount $RQ=\Delta y$. We have then

$\text{Lim}\frac{\Delta s}{\Delta t}=\frac{ds}{dt}=v=$ velocity of the body in its path,

$\text{Lim}\frac{\Delta x}{\Delta t}=\frac{dx}{dt}=v_x=$ component of velocity parallel to $OX$,

$\text{Lim}\frac{\Delta y}{\Delta t}=\frac{dy}{dt}=v_y=$ component of velocity parallel to $OY$.

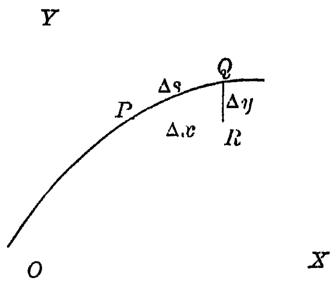


FIG 47

Otherwise expressed, $v$ represents the velocity of $P$, $v_x$ the velocity of the projection of $P$ upon $OX$, and $v_y$ the velocity of the projection of $P$ on $OY$.

Now, by (8), § 36, and by § 39,

$$v_x=\frac{dx}{dt}=\frac{dx}{ds}\cdot\frac{ds}{dt}$$
$$=v\cos\phi, \tag{1}$$

and
$$v_y=\frac{dy}{dt}=\frac{dy}{ds}\cdot\frac{ds}{dt}$$
$$=v\sin\phi. \tag{2}$$

Squaring and adding, we have

$$v^2=v_x^2+v_y^2. \tag{3}$$

Formulas (1), (2), and (3) are of especial value when a particle moves in the plane $XOY$, and the coördinates $x$ and $y$ of its position at any time $t$ are each given as a function of $t$. The path of the moving particle may then be determined as follows:

Assign any value to $t$ and locate the point corresponding to the values of $x$ and $y$ thus determined. This will evidently be the position of the moving particle at that instant of time. In this way, by assigning successive values to $t$ we can locate other points through which the particle is moving at the corresponding instants of time. The locus of the points thus determined is a curve which is evidently the *path* of the particle.

The two equations accordingly represent the curve and are called its *parametric representation*, the variable $t$ being called a *parameter*.* By (9), § 36, the slope of the curve is given by the formula

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{v_y}{v_x}. \tag{4}$$

In case $t$ can be eliminated from the two given equations, the result is the $(x, y)$ equation of the curve, sometimes called the *Cartesian* equation; but such elimination is not essential, and often is not desirable, particularly if the velocity of the particle in its path is to be determined.

**Ex. 1.** A particle moves in the plane $XOY$ so that at any time $t$,

$$x = a + bt, \qquad y = c + dt,$$

where $a$, $b$, $c$, and $d$ are any real constants Determine its path and its velocity in its path

To determine the path we eliminate $t$ from the given equations, with the result

$$y - c = \frac{d}{b}(x - a),$$

the equation of a straight line passing through the point $(a, c)$ with the slope $\frac{d}{b}$

In this case the path may also be determined as follows · From the given equations we find $dx = b\,dt$, and $dy = d\,dt$; whence $\frac{dy}{dx}=\frac{d}{b}$· As the slope of the path is always the same $\left(\text{that is, } \frac{d}{b}\right)$, the path must be a straight line which passes through the point $(a, c)$ — the point determined when $t = 0$.

To determine the velocity of the particle in its path we find, by differentiating the given equations,

$$v_x=\frac{dx}{dt}=b, \qquad v_y=\frac{dy}{dt}=d;$$

whence, by (3),

$$v=\sqrt{b^2+d^2}.$$

Hence the particle moves along the straight line with a constant velocity.

* It may be noted in passing that the parameter in the parametric representation of a curve is not necessarily time, but may be any third variable in terms of which $x$ and $y$ can be expressed.

**Ex. 2.** If a projectile starts with an initial velocity $v_0$ in an initial direction which makes an angle $\alpha$ with the axis of $x$ taken as horizontal, its position at any time $t$ is given by the parametric equations

$$x = v_0 t \cos\alpha, \qquad y = v_0 t \sin\alpha - \tfrac{1}{2} g t^2.$$

Find its velocity in its path.

We have
$$v_x = \frac{dx}{dt} = v_0 \cos\alpha,$$
$$v_y = \frac{dy}{dt} = v_0 \sin\alpha - gt.$$

Hence
$$v = \sqrt{v_0^2 - 2 g v_0 t \sin\alpha + g^2 t^2}.$$

## EXERCISES

1. The coordinates of the position of a moving particle at any time $t$ are given by the equations $x = 2t$, $y = t^3$ Determine the path of the particle and its speed in its path

2. The coordinates of the position of a moving particle at any time $t$ are given by the equations $x = t^2$, $y = t + 1$ Determine the path of the particle and its speed in its path.

3. The coordinates of the position of a moving particle at any time $t$ are given by the equations $x = 2t$, $y = \frac{4}{3}t - \frac{4}{9}t^2$. Determine the path of the particle and its speed in its path.

4. At what point of its path will the particle of Ex. 3 be moving most slowly?

5. The coordinates of the position of a moving particle at any time $t$ are given by the equations $x = t^2 - 3$, $y = t^3 + 2$. Determine the path of the particle and its speed in its path

6. The coordinates of the position of a moving particle at any time $t$ are given by the equations $x = 4t^2$, $y = 4(1 - t)^2$. Determine the path of the particle and its speed in its path.

7. Find the highest point in the path of a projectile.

8. Find the point in its path at which the speed of a projectile is a minimum.

9. Find the range (that is, the distance to the point at which the projectile will fall on $OX$), the velocity at that point, and the angle at which the projectile will meet $OX$.

10. Show that in general the same range may be produced by two different values of $\alpha$, and find the value of $\alpha$ which produces the greatest range.

11. Find the $(x, y)$ equation of the path of a projectile, and plot.

**41. Related velocities and rates.** Another problem of somewhat different type arises when we know the velocity of one point in its path, which may be straight or curved, and wish to find the velocity of another point which is in some way connected with the first but, in general, describes a different path. The method, in general, is to form an equation connecting the distances traveled by the two points and then to differentiate the equation thus formed with respect to the time $t$. The result is an equation connecting the velocities of the two points.

**Ex. 1** A lamp is 60 ft. above the ground. A stone is let drop from a point on the same level as the lamp and 20 ft. away from it. Find the speed of the stone's shadow on the ground at the end of 1 sec., assuming that the distance traversed by a falling body in the time $t$ is $16\,t^2$.

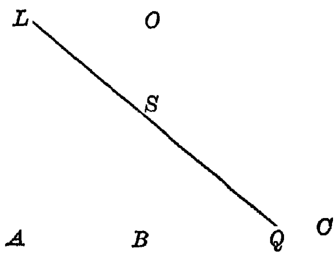


FIG. 48

Let $AC$ (Fig. 48) be the surface of the ground which is assumed to be a horizontal plane, $L$ the position of the lamp, $O$ the point from which the stone was dropped, and $S$ the position of the stone at any time $t$. Then $Q$ is the position of the shadow of $S$ on the ground, $LSQ$ being a straight line. Let $OS = x$ and $BQ = y$. Then $LO = 20$, $BO = 60$, and $BS = 60 - x$. In the similar triangles $LOS$ and $SBQ$,

$$\frac{x}{20} = \frac{60 - x}{y}; \tag{1}$$

whence

$$y = \frac{1200}{x} - 20. \tag{2}$$

We know $x = 16\,t^2$, whence $\frac{dx}{dt} = 32\,t$; and wish to find $\frac{dy}{dt}$, the velocity of $Q$.

Differentiating (2) with respect to $t$, we have

$$\frac{dy}{dt} = -\frac{1200}{x^2}\cdot\frac{dx}{dt}.$$

When $t = 1$ sec., $x = 16$, and $\frac{dx}{dt} = 32$; whence, by substitution, we find

$$\frac{dy}{dt} = -150 \text{ ft. per second.}$$

The result is negative because $y$ is decreasing as time goes on.

In §§ 6 and 11, if the rate of one of two related quantities was known, we were able to find the rate of the other quantity.

This type of problem may also be solved by the same method by which the problem of related velocities has been solved. We shall illustrate by taking the same problem that was used in § 11.

**Ex. 2.** Water is being poured at the rate of 100 cu. in. per second into a vessel in the shape of a right circular cone of radius 3 in. and altitude 9 in. Required the rate at which the depth of the water is increasing when the depth is 6 in.

As in § 11, we have $$V = \frac{1}{27}\pi h^3;$$

whence $$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$$

We have given $\frac{dV}{dt} = 100$, $h = 6$; from which we compute

$$\frac{dh}{dt} = \frac{25}{\pi} = 7\,96.$$

## EXERCISES

**1.** A point is moving on the curve $y^2 = x^3$ The velocity along $OX$ is 2 ft. per second What is the velocity along $OY$ when $x = 2$?

**2.** A ball is swung in a circle at the end of a cord 3 ft long so as to make 40 revolutions per minute. If the cord breaks, allowing the ball to fly off at a tangent, at what rate will it be receding from the center of its previous path 2 sec after the cord breaks, if no allowance is made for the action of any new force?

**3.** The inside of a vessel is in the form of an inverted regular quadrangular pyramid, 4 ft square at the top and 2 ft deep. The vessel is originally filled with water which leaks out at the bottom at the rate of 10 cu. in. per minute. How fast is the level of the water falling when the water is 10 in. deep?

**4.** The top of a ladder 20 ft long slides down the side of a vertical wall at a speed of 3 ft per second. The foot of the ladder slides on horizontal land. Find the path described by the middle point of the ladder, and its speed in its path.

**5.** A boat with the anchor fast on the bottom at a depth of 40 ft. is drifting at the rate of 3 mi per hour, the cable attached to the anchor slipping over the end of the boat At what rate is the cable leaving the boat when 50 ft of cable are out, assuming it forms a straight line from the boat to the anchor?

6. A solution is being poured into a conical filter at the rate of 5 cc per second and is running out at the rate of 2 cc. per second. The radius of the top of the filter is 8 cm and the depth of the filter is 20 cm. Find the rate at which the level of the solution is rising in the filter when it is one third of the way to the top.

7. A trough is in the form of a right prism with its ends isosceles triangles placed vertically. It is 5 ft long, 1 ft across the top, and 8 in. deep It contains water which leaks out at the rate of 1 qt. ($57\frac{3}{4}$ cu. in) per minute Find the rate at which the level of the water is sinking in the trough when the depth is 3 in.

8. The angle between the straight lines $AB$ and $BC$ is 60°, and $AB$ is 40 ft. long. A particle at $A$ begins to move along $AB$ toward $B$ at the rate of 5 ft per second, and at the same time a particle at $B$ begins to move along $BC$ toward $C$ at the rate of 4 ft per second. At what rate are the two particles approaching each other at the end of 1 sec ?

9. The foot of a ladder 50 ft. long rests on horizontal ground, and the top of the ladder rests against the side of a pyramid which makes an angle of 120° with the ground. If the foot of the ladder is drawn directly away from the base of the pyramid at the uniform rate of 2 ft. per second, how fast will the top of the ladder slide down the side of the pyramid ?

## GENERAL EXERCISES

Plot the curves:

1. $3x^2 + 7y^2 = 21$
2. $4y^2 = 9x.$
3. $9x^2 - y^2 = 16.$
4. $y^2 - 2y = x^3 + 2x^2 - 1.$
5. $y = \dfrac{8a^3}{x^2 + 4a^2}.$
6. $a^4y^2 + b^2x^4 = a^2b^2x^2.$
7. $(y - x)^2 = 9 - x^2$
8. $(x + y)^2 = y^2(y + 2).$
9. $y^2(4 + x^2) = x^2(4 - x^2).$
10. $y^2 = x^2\dfrac{a - x}{a + x}.$
11. $y^2(x^2 + a^2) = a^2x^2.$
12. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$
13. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$
14. $(y + 2)^2 = \dfrac{1}{x + 4}.$
15. $x^2y^2 + 36 = 16y^2$

Find the turning-points of the following curves and plot the curves:

16. $y = (2 + x)(4 - x)^2.$
17. $y = (x + 3)^2(x - 2).$
18. $y = \dfrac{(x - 1)^2}{x + 1}.$
19. $y = \dfrac{4}{x^2 - 4}.$

20. Find the equation of the tangent to the curve $y^2 = x^2 \dfrac{a-x}{a+x}$ at the point $\left(-\dfrac{3a}{5}, \dfrac{6a}{5}\right)$.

21. Find the equation of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at the point $(x_1, y_1)$.

22. Prove that if a tangent to a parabola $y^2 = kx$ has the slope $m$, its point of contact is $\left(\dfrac{k}{4m^2}, \dfrac{k}{2m}\right)$ and therefore its equation is $y = mx + \dfrac{k}{4m}$.

23. Prove that if a tangent to an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has the slope $m$, its point of contact is $\left(\pm \dfrac{a^2m}{\sqrt{a^2m^2 + b^2}}, \mp \dfrac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$ and therefore its equation is $y = mx \pm \sqrt{a^2m^2 + b^2}$

24. Show that a tangent to a parabola makes equal angles with the axis and a line from the focus to the point of contact.

25. Show that a tangent to an ellipse makes equal angles with the two lines drawn to the foci from the point of contact

Find the angles of intersection of the following pairs of curves.

26. $y = x,\ y^2 = \dfrac{x^3}{4-x}$.

27. $x^2 + y^2 = 20,\ y = \dfrac{64}{x^2 + 16}$.

28. $x^2 = 4y,\ 2x^2 + 2y^2 = 5x$.

29. $x^2 - 4y - 4 = 0,\ x^2 + 12y - 36 = 0$.

30. $y^2 = x^3,\ y^2 = (2-x)^3$.

31. $y^2 = (x-3)^3,\ y^2 = 16(x-3)$.

32. $2y^2 = x^3,\ x^2 + y^2 - 4x = 0$.

33. The coordinates of a moving particle are given by the equations $x = t^3$, $y = (1-t^2)^{\frac{3}{2}}$. Find its path and its velocity in its path.

34. A particle moves so that its coordinates at the time $t$ are $x = 2t$, $y = \dfrac{2}{t^2+1}$. Find its path and its velocity in its path.

35. A projectile so moves that $x = at$, $y = bt - \frac{1}{2}gt^2$. Find its path and its velocity in its path.

36. A body so moves that $x = -2 + t^{\frac{3}{2}}$, $y = 1 + t$. Find its path and its velocity in its path.

37. A particle is moving along the curve $y^2 = 4x$; and when $x = 4$, its ordinate is increasing at the rate of 10 ft. per second. At what rate is the abscissa then changing, and how fast is the particle moving in the curve? Where will the abscissa be changing ten times as fast as the ordinate?

38. A particle describes the circle $x^2 + y^2 = a^2$ with a constant speed $v_0$. Find the components of its velocity.

39. A particle describes the parabola $y^2 = 4ax$ in such a way that its $x$-component of velocity is equal to $ct$ Find its $y$-component of velocity and its velocity in its path.

40. A particle moves so that $x = 2t$, $y = 2\sqrt{t - t^2}$ Show that it moves around a semicircle in the time from $t = 0$ to $t = 1$, and find its velocity in its path during that time.

41. At 12 o'clock a vessel is sailing due north at the uniform rate of 20 mi. an hour. Another vessel, 40 mi. north of the first, is sailing at the uniform rate of 15 mi an hour on a course 30° north of east. At what rate is the distance between the two vessels diminishing at the end of one hour? What is the shortest distance between the two vessels?

42. The top of a ladder 32 ft. long rests against a vertical wall, and the foot is drawn along a horizontal plane at the rate of 4 ft. per second in a straight line from the wall. Find the path of a point on the ladder one third of the distance from the foot of the ladder, and its velocity in its path.

43. A man standing on a wharf 20 ft. above the water pulls in a rope, attached to a boat, at the uniform rate of 3 ft. per second Find the velocity with which the boat approaches the wharf

44. The volume and the radius of a cylindrical boiler are expanding at the rate of .8 cu. ft. and .002 ft. per minute respectively. How fast is the length of the boiler changing when the boiler contains 40 cu. ft. and has a radius of 2 ft.?

45. The inside of a cistern is in the form of a frustum of a regular quadrangular pyramid. The bottom is 40 ft. square, the top is 60 ft. square, and the depth is 10 ft. If the water leaks out at the bottom at the rate of 5 cu. ft. per minute, how fast is the level of the water falling when the water is 5 ft. deep in the cistern?

**46.** The inside of a cistern is in the form of a frustum of a right circular cone of vertical angle 90°. The cistern is smallest at the base, which is 4 ft in diameter. Water is being poured in at the rate of 5 cu ft. per minute. How fast is the water rising in the cistern when it is $2\frac{1}{2}$ ft deep?

**47.** The inside of a bowl is in the form of a hemispherical surface of radius 10 in If water is running out of it at the rate of 2 cu in. per minute, how fast is the depth of the water decreasing when the water is 3 in deep?

**48.** How fast is the surface of the bowl in Ex 47 being exposed?

**49.** The inside of a bowl 4 in deep and 8 in. across the top is in the form of a surface of revolution formed by revolving a parabolic segment about its axis Water is running into the bowl at the rate of 1 cu in per second How fast is the water rising in the bowl when it is 2 in deep?

**50.** It is required to fence off a rectangular piece of ground to contain 200 sq ft, one side to be bounded by a wall already constructed. Find the dimensions which will require the least amount of fencing.

**51.** The hypotenuse of a right triangle is given. Find the other sides if the area is a maximum

**52.** The stiffness of a rectangular beam varies as the product of the breadth and the cube of the depth. Find the dimensions of the stiffest beam which can be cut from a circular cylindrical log of diameter 18 in.

**53.** A rectangular plot of land to contain 384 sq. ft. is to be inclosed by a fence, and is to be divided into two equal lots by a fence parallel to one of the sides What must be the dimensions of the rectangle that the least amount of fencing may be required?

**54.** An open tank with a square base and vertical sides is to have a capacity of 500 cu ft Find the dimensions so that the cost of lining it may be a minimum

**55.** A rectangular box with a square base and open at the top is to be made out of a given amount of material. If no allowance is made for thickness of material or for waste in construction, what are the dimensions of the largest box which can be made?

**56.** A metal vessel, open at the top, is to be cast in the form of a right circular cylinder. If it is to hold $27\pi$ cu in, and the thickness of the side and that of the bottom are each to be 1 in., what will be the inside dimensions when the least amount of material is used?

57. A gallon oil can (231 cu in.) is to be made in the form of a right circular cylinder. The material used for the top and the bottom costs twice as much per square inch as the material used for the side. What is the radius of the most economical can that can be made if no allowance is made for thickness of material or waste in construction?

58. A tent is to be constructed in the form of a regular quadrangular pyramid. Find the ratio of its height to a side of its base when the air space inside the tent is as great as possible for a given wall surface.

59. It is required to construct from two equal circular plates of radius $a$ a buoy composed of two equal cones having a common base. Find the radius of the base when the volume is the greatest.

60. Two towns, A and B, are situated respectively 12 mi. and 18 mi. back from a straight river from which they are to get their water supply by means of the same pumping-station. At what point on the bank of the river should the station be placed so that the least amount of piping may be required, if the nearest points on the river from A and B respectively are 20 mi. apart and if the piping goes directly from the pumping-station to each of the towns?

61. A man on one side of a river, the banks of which are assumed to be parallel straight lines $\frac{1}{4}$ mi. apart, wishes to reach a point on the opposite side of the river and 5 mi. further along the bank. If he can row 3 mi. an hour and travel on land 5 mi. an hour, find the route he should take to make the trip in the least time.

62. A power house stands upon one side of a river of width $b$ miles, and a manufacturing plant stands upon the opposite side, $a$ miles downstream. Find the most economical way to construct the connecting cable if it costs $m$ dollars per mile on land and $n$ dollars a mile through water, assuming the banks of the river to be parallel straight lines.

63. A vessel A is sailing due east at the uniform rate of 8 mi. per hour when she sights another vessel B directly ahead and 20 mi. away. B is sailing in a straight course S. 30° W. at the uniform rate of 6 mi. per hour. When will the two vessels be nearest to each other?

64. The number of tons of coal consumed per hour by a certain ship is $0.2 + 0.001\,v^3$, where $v$ is the speed in miles per hour. Find an expression for the amount of coal consumed on a voyage of 1000 mi. and the most economical speed at which to make the voyage.

**65.** The fuel consumed by a certain steamship in an hour is proportional to the cube of the velocity which would be given to the steamship in still water. If it is required to steam a certain distance against a current flowing $a$ miles an hour, find the most economical speed.

**66.** An isosceles triangle is inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a>b)$, with its vertex in the upper end of the minor axis of the ellipse and its base parallel to the major axis Determine the length of the base and the altitude of the triangle of greatest area which can be so inscribed.

# CHAPTER V

## TRIGONOMETRIC FUNCTIONS

**42. Circular measure.** The *circular measure* of an angle is the quotient of the length of an arc of a circle, with its center at the vertex of the angle and included between its sides, divided by the radius of the arc. Thus, if $\theta$ is the angle, $a$ the length of the arc, and $r$ the radius, we have

$$\theta = \frac{a}{r}. \tag{1}$$

The unit of angle in this measurement is the *radian*, which is the angle for which $a = r$ in (1), and any angle may be said to contain a certain number of radians. But the quotient $\frac{a}{r}$ in formula (1) is an abstract number, and it is also customary to speak of the angle $\theta$ as having the magnitude $\frac{a}{r}$ without using the word radian. Thus, we speak of the angle 1, the angle $\frac{2}{3}$, the angle $\frac{\pi}{4}$, etc.

In all work involving calculus, and in most theoretical work of any kind, all angles which occur are understood to be expressed in radians. In fact, many of the calculus formulas would be false unless the angles involved were so expressed. The student should carefully note this fact, although the reason for it is not yet apparent.

From this point of view such a trigonometric equation as

$$y = \sin x \tag{2}$$

may be considered as defining a functional relation between two quantities exactly as does the simpler equation $y = x^2$. For we may, in (2), assign any arbitrary value to $x$ and determine the corresponding value of $y$. This may be done by a direct computation (as will be shown in Chapter VII), or it may be done by means of a table of trigonometric functions, in which case we must interpret the value of $x$ as denoting so many radians.

One of the reasons for expressing an angle in circular measure is that it makes true the formula

$$\lim_{h \to 0} \frac{\sin h}{h} = 1, \tag{3}$$

where the left-hand member of the equation is to be read "the limit of $\frac{\sin h}{h}$ as $h$ approaches zero as a limit."

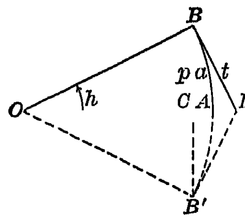


FIG. 49.

To prove this theorem we proceed as follows:

Let $h$ be the angle $AOB$ (Fig. 49), $r$ the radius of the arc $AB$ described from $O$ as a center, $a$ the length of $AB$, $p$ the length of the perpendicular $BC$ from $B$ to $OA$, and $t$ the length of the tangent drawn from $B$ to meet $OA$ produced in $D$.

Revolve the figure on $OA$ as an axis until $B$ takes the position $B'$. Then the chord $BCB' = 2p$, the arc $BAB' = 2a$, and the tangent $B'D =$ the tangent $BD$. Evidently

$$BD + DB' > BAB' > BCB';$$

whence $$t > a > p.$$

Dividing through by $r$, we have

$$\frac{t}{r} > \frac{a}{r} > \frac{p}{r},$$

that is, $$\tan h > h > \sin h.$$

Dividing by $\sin h$, we have

$$\frac{1}{\cos h} > \frac{h}{\sin h} > 1,$$

or, by inverting, $$\cos h < \frac{\sin h}{h} < 1.$$

Now as $h$ approaches zero, $\cos h$ approaches 1. Hence $\frac{\sin h}{h}$, which lies between $\cos h$ and 1, must also approach 1; that is,

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

This result may be used to find the limit of $\frac{1-\cos h}{h}$ as $h$ approaches zero as a limit. For we have

$$\frac{1-\cos h}{h} = \frac{2\sin^2\frac{h}{2}}{h} = \frac{\sin^2\frac{h}{2}}{\frac{h}{2}} = \frac{h}{2}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2.$$

Now as $h$ approaches zero as a limit, $\frac{\sin\frac{h}{2}}{\frac{h}{2}}$ approaches unity, by (3). Therefore

$$\lim_{h \to 0} \frac{1-\cos h}{h} = 0. \tag{4}$$

**43. Graphs of trigonometric functions.** We may plot a trigonometric function by assigning values to $x$ and computing, or taking from a table, the corresponding values of $y$. In so doing, any angle which may occur should be expressed in circular measure, as explained in the previous section. In this connection it is to be remembered that $\pi$ is simply the number 3.1416, and that the angle $\pi$ means an angle with that number of radians and is therefore the angle whose degree measure is 180°.

The manner of plotting can be best explained by examples.

**Ex. 1.** $y = a \sin bx$

It is convenient first to fix the values of $x$ which make $y$ equal to zero Now the sine is zero when the angle is $0$, $\pi$, $2\pi$, $3\pi$, $-\pi$, $-2\pi$, or, in general, $k\pi$, where $k$ is any positive or negative integer. To make $y = 0$, therefore, we have to place $bx = k\pi$; whence

$$x = \cdots, \quad -\frac{2\pi}{b}, \quad -\frac{\pi}{b}, \quad 0, \quad \frac{\pi}{b}, \quad \frac{2\pi}{b}, \quad \frac{3\pi}{b}, \quad \cdots.$$

The sine takes its maximum value $+1$ when the angle has the values $\frac{\pi}{2}$, $\frac{5\pi}{2}$, $\frac{9\pi}{2}$, etc.; that is, in this case, when $x = \frac{\pi}{2b}$, $\frac{5\pi}{2b}$, $\frac{9\pi}{2b}$, etc. For these values of $x$, $y = a$

The sine takes its minimum value $-1$ when the angle is $\frac{3\pi}{2}$, $\frac{7\pi}{2}$, etc., that is, in this case, when $x = \frac{3\pi}{2b}, \frac{7\pi}{2b}$, etc For these values of $x$, $y = -a$.

These values of $x$ for which the sine is $\pm 1$ lie halfway between the values of $x$ for which the sine is 0

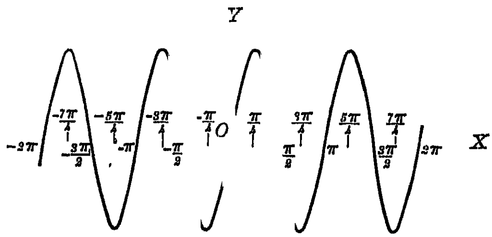



FIG 50

These points on the graph are enough to determine its general shape Other values of $x$ may be used to fix the shape more exactly The graph is shown in Fig 50, with $a = 3$ and $b = 2$ The curve may be said to represent a wave. The distance from peak to peak, $\frac{2\pi}{b}$, is the wave length, and the height $a$ above $OX$ is the amplitude

**Ex 2.** $y = a \cos bx.$

As in Ex 1, we fix first the points for which $y = 0$ Now the cosine of an angle is zero when the angle is $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc ; that is, any odd multiple of $\frac{\pi}{2}$. We have, therefore, $y = 0$ when

$$x = \cdots, \quad -\frac{\pi}{2b}, \quad \frac{\pi}{2b}, \quad \frac{3\pi}{2b}, \quad \frac{5\pi}{2b}, \quad \cdots.$$

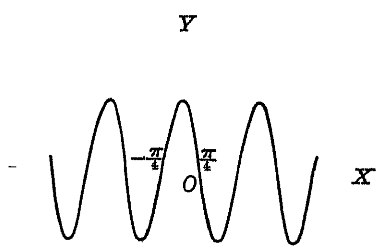



FIG. 51

Halfway between these points the cosine has its maximum value $+1$ or its minimum value $-1$ alternately, and $y = \pm a$. The graph is shown in Fig 51, with $a = 3$ and $b = 2$

**Ex. 3.** $y = a\sin(bx + c)$.

We have $y = 0$ when $bx + c = 0, \pi, 2\pi, 3\pi$, etc.; that is, when

$$x = \cdots, \quad -\frac{c}{b}, \quad -\frac{c}{b} + \frac{\pi}{b}, \quad -\frac{c}{b} + \frac{2\pi}{b}, \cdots.$$

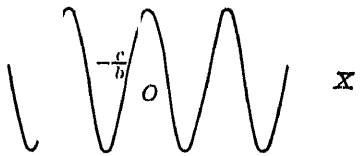



FIG. 52

Halfway between these values of $x$ the sine has its maximum value $+1$ and its minimum value $-1$ alternately, and $y = \pm a$. The curve is the same as in Ex. 1, but is shifted $\frac{c}{b}$ units to the left (Fig. 52).

**Ex. 4.** $y = \sin x + \frac{1}{2}\sin 2x$.

The graph is found by adding the ordinates of the two curves $y = \sin x$ and $y = \frac{1}{2}\sin 2x$, as shown in Fig. 53.

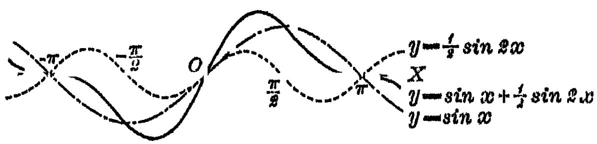



FIG. 53

**EXERCISES**

Plot the graphs of the following equations:

1. $y = 2\sin 3x$.
2. $y = 3\cos\dfrac{x}{2}$.
3. $y = 5\sin\left(x - \dfrac{\pi}{4}\right)$.
4. $y = 2\cos\left(x + \dfrac{\pi}{4}\right)$.
5. $y = 2\sin(x - 2)$.
6. $y = \tan 2x$.
7. $y = \text{ctn}\, 3x$.
8. $y = \sec x$.
9. $y = \csc 2x$.
10. $y = \text{vers}\, x$.
11. $y = 1 + \sin 2x$.
12. $y = \sin\frac{1}{2}x + \sin 2x$.

**44. Differentiation of trigonometric functions.** The formulas for the differentiation of trigonometric functions are as follows, where $u$ represents any function of $x$ which can be differentiated:

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}, \tag{1}$$

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}, \tag{2}$$

$$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}, \tag{3}$$

$$\frac{d}{dx}\operatorname{ctn} u = -\csc^2 u\frac{du}{dx}, \tag{4}$$

$$\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx}, \tag{5}$$

$$\frac{d}{dx}\csc u = -\csc u\operatorname{ctn} u\frac{du}{dx}. \tag{6}$$

These formulas are proved as follows:

1. Let $y = \sin u$, where $u$ is any function of $x$ which may be differentiated. Give $x$ an increment $\Delta x$ and let $\Delta u$ and $\Delta y$ be the corresponding increments of $u$ and $y$. Then

$$\begin{aligned}\Delta y &= \sin(u+\Delta u) - \sin u\\ &= \sin u\cos\Delta u + \cos u\sin\Delta u - \sin u\\ &= \cos u\sin\Delta u - (1-\cos\Delta u)\sin u;\end{aligned}$$

whence $$\frac{\Delta y}{\Delta u} = \cos u\frac{\sin\Delta u}{\Delta u} - \frac{1-\cos\Delta u}{\Delta u}\sin u.$$

Now let $\Delta x$ and therefore $\Delta u$ approach zero. By (3), § 42, $\operatorname{Lim}\frac{\sin\Delta u}{\Delta u} = 1$, and, by (4), § 42, $\operatorname{Lim}\frac{1-\cos\Delta u}{\Delta u} = 0$. Therefore

$$\frac{dy}{du} = \cos u.$$

But by (8), § 36, $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

and therefore $$\frac{dy}{dx} = \cos u\frac{du}{dx}.$$

2. To find $\frac{d}{dx}\cos u$, we write

$$\cos u = \sin\left(\frac{\pi}{2} - u\right).$$

Then
$$\frac{d}{dx}\cos u = \frac{d}{dx}\sin\left(\frac{\pi}{2} - u\right)$$
$$= \cos\left(\frac{\pi}{2} - u\right)\frac{d}{dx}\left(\frac{\pi}{2} - u\right) \qquad \text{(by (1))}$$
$$= -\cos\left(\frac{\pi}{2} - u\right)\frac{du}{dx}$$
$$= -\sin u \frac{du}{dx}.$$

3. To find $\frac{d}{dx}\tan u$, we write

$$\tan u = \frac{\sin u}{\cos u}.$$

Then
$$\frac{d}{dx}\tan u = \frac{d}{dx}\frac{\sin u}{\cos u}$$
$$= \frac{\cos u \frac{d}{dx}\sin u - \sin u \frac{d}{dx}\cos u}{\cos^2 u} \qquad \text{(by (5), § 36)}$$
$$= \frac{(\cos^2 u + \sin^2 u)\frac{du}{dx}}{\cos^2 u} \qquad \text{(by (1) and (2))}$$
$$= \sec^2 u \frac{du}{dx}.$$

4. To find $\frac{d}{dx}\operatorname{ctn} u$, we write

$$\operatorname{ctn} u = \frac{\cos u}{\sin u}.$$

Then
$$\frac{d}{dx}\operatorname{ctn} u = \frac{d}{dx}\frac{\cos u}{\sin u}$$
$$= \frac{\sin u \frac{d}{dx}\cos u - \cos u \frac{d}{dx}\sin u}{\sin^2 u} \qquad \text{(by (5), § 36)}$$
$$= \frac{-\sin^2 u - \cos^2 u}{\sin^2 u}\frac{du}{dx} \qquad \text{(by (1) and (2))}$$
$$= -\csc^2 u \frac{du}{dx}.$$

5. To find $\frac{d}{dx}\sec u$, we write

$$\sec u = \frac{1}{\cos u} = (\cos u)^{-1}.$$

Then
$$\frac{d}{dx}\sec u = -(\cos u)^{-2}\frac{d}{dx}\cos u \qquad \text{(by (6), § 36)}$$
$$= \frac{\sin u}{\cos^2 u}\frac{du}{dx} \qquad \text{(by (2))}$$
$$= \sec u \tan u \frac{du}{dx}.$$

6. To find $\frac{d}{dx}\csc u$, we write

$$\csc u = \frac{1}{\sin u} = (\sin u)^{-1}.$$

Then
$$\frac{d}{dx}\csc u = -(\sin u)^{-2}\frac{d}{dx}\sin u \qquad \text{(by (6), § 36)}$$
$$= -\csc u \operatorname{ctn} u \frac{du}{dx}. \qquad \text{(By (1))}$$

**Ex. 1.** $y = \tan 2x - \tan^2 x = \tan 2x - (\tan x)^2.$

$$\frac{dy}{dx} = \sec^2 2x \frac{d}{dx}(2x) - 2(\tan x)\frac{d}{dx}\tan x$$
$$= 2\sec^2 2x - 2\tan x \sec^2 x.$$

**Ex. 2.** $y = (2\sec^4 x + 3\sec^2 x)\sin x.$

$$\frac{dy}{dx} = \sin x\left[8\sec^3 x\frac{d}{dx}(\sec x) + 6\sec x\frac{d}{dx}(\sec x)\right] + (2\sec^4 x + 3\sec^2 x)\frac{d}{dx}(\sin x)$$
$$= \sin x(8\sec^4 x\tan x + 6\sec^2 x\tan x) + (2\sec^4 x + 3\sec^2 x)\cos x$$
$$= (1-\cos^2 x)(8\sec^5 x + 6\sec^3 x) + (2\sec^3 x + 3\sec x)$$
$$= 8\sec^5 x - 3\sec x.$$

## EXERCISES

Find $\frac{dy}{dx}$ in each of the following cases:

1. $y = 3\sin 5x.$
2. $y = 2\tan\frac{x}{2}.$
3. $y = \frac{1}{3}\sin^3 2x.$
4. $y = \cos^2 5x.$
5. $y = \frac{x}{2} - \frac{1}{4}\sin 2x.$
6. $y = \frac{1}{3}\sin^3 5x - \frac{1}{5}\sin^5 5x.$
7. $y = \sec^2\frac{5x}{2}.$
8. $y = \frac{1}{3}\csc^3 3x.$

9. $y = \frac{2}{3}\cos^3\frac{x}{2} - 2\cos\frac{x}{2}$.

10. $y = \frac{2}{3}\operatorname{ctn}^3\frac{x}{2} + 2\operatorname{ctn}\frac{x}{2}$.

11. $y = 4\sin\frac{x}{2} - 2x\cos\frac{x}{2}$.

12. $y = \frac{\sec x + \tan x}{\sec x - \tan x}$.

13. $y = \sin(2x+1)\cos(2x-1)$.

14. $y = \tan^3 3x - 3\tan 3x + 9x$.

15. $y = \sec 2x \tan 2x$.

16. $y = \frac{1}{30}(3\cos^5 2x - 5\cos^3 2x)$.

17. $\sin 2x + \tan 3y = 0$.

18. $xy + \operatorname{ctn} xy = 0$.

**45. Simple harmonic motion.** Let a particle of mass $m$ move in a straight line so that its distance $s$ measured from a fixed point in the line is given at any time $t$ by the equation

$$s = c\sin bt, \tag{1}$$

where $c$ and $b$ are constants. We have for the velocity $v$ and the acceleration $a$

$$v = cb\cos bt, \tag{2}$$

$$a = -cb^2\sin bt. \tag{3}$$

When $t = 0$, $s = 0$ and the particle is at $O$ (Fig. 54). When $t = \frac{\pi}{2b}$, $s = c$ and the particle is at $A$, where $OA = c$.

When $t$ is between $0$ and $\frac{\pi}{2b}$, $v$ is positive and $a$ is negative, so that the particle is moving from $O$ to $A$ with decreasing speed.


Fig. 54

When $t$ is between $\frac{\pi}{2b}$ and $\frac{\pi}{b}$, $v$ is negative and $a$ is negative, so that the particle moves toward $O$ with increasing speed. When $t = \frac{\pi}{b}$, the particle is at $O$.

As $t$ varies from $\frac{\pi}{b}$ to $\frac{3\pi}{2b}$, the particle moves with decreasing speed from $O$ to $B$, where $OB = -c$.

Finally, as $t$ varies from $\frac{3\pi}{2b}$ to $\frac{2\pi}{b}$, the particle moves back from $B$ to $O$ with increasing speed.

The motion is then repeated, and the particle oscillates between $B$ and $A$, the time required for a complete oscillation being, as we have seen, $\frac{2\pi}{b}$. The motion of the particle is called *simple harmonic motion*. The quantity $c$ is called the *amplitude*, and the interval $\frac{2\pi}{b}$, after which the motion repeats itself, is called the *period*.

Since force is proportional to the mass times the acceleration, the force $F$ acting on the particle is given by the formula

$$F = kma = -kmcb^2 \sin bt = -kmb^2 s.$$

This shows that the force is proportional to the distance $s$ from the point $O$. The negative sign shows that the force produces acceleration with a sign opposite to that of $s$, and therefore slows up the particle when it is moving away from $O$ and increases its speed when it moves toward $O$. The force is therefore always directed toward $O$ and is an attracting force.

If, instead of equation (1), we write the equation

$$s = c \sin b(t - t_0), \qquad (4)$$

the change amounts simply to altering the instant from which the time is measured. For the value of $s$ which corresponds to $t = t_1$ in (1) corresponds to $t = t_1 + t_0$ in (4). Hence (4) represents simple harmonic motion of amplitude $c$ and period $\frac{2\pi}{b}$.

But (4) may be written

$$s = c \cos bt_0 \sin bt - c \sin bt_0 \cos bt,$$

which is the same as

$$s = A \sin bt + B \cos bt, \qquad (5)$$

where $\quad A = c \cos bt_0,\ B = -c \sin bt_0.$

$A$ and $B$ may have any values in (5), for if $A$ and $B$ are given, we have, from the last two equations,

$$c = \sqrt{A^2 + B^2},\ \tan bt_0 = -\frac{B}{A},$$

which determines $c$ and $t_0$ in (4).

Therefore equation (5) also represents simple harmonic motion with amplitude $\sqrt{A^2+B^2}$ and period $\frac{2\pi}{b}$.

In particular, if in (5) $A=O$ and $B=c$, we have

$$s = c \cos bt. \tag{6}$$

If in (4) we place $t_0 = \frac{\pi}{2b} - t_0'$, it becomes

$$s = c \cos b\,(t - t_0'), \tag{7}$$

which differs from (6) only in the instant from which the time is measured.

**EXERCISES**

1. A particle moves with constant speed $v_0$ around a circle. Prove that its projection on any diameter of the circle describes simple harmonic motion

2. A point moves with simple harmonic motion of period 4 sec. and amplitude 3 ft Find the equation of its motion.

3. Given the equation $s = 5 \sin 2t$ Find the time of a complete oscillation and the amplitude of the swing.

4. Find at what time and place the speed is the greatest for the motion defined by the equation $s = c \sin bt$ Do the same for the acceleration.

5. At what point in a simple harmonic motion is the velocity zero, and at what point is the acceleration zero?

6. The motion of a particle in a straight line is expressed by the equation $s = 5 - 2\cos^2 t$. Express the velocity and the acceleration in terms of $s$ and show that the motion is simple harmonic.

7. A particle moving with a simple harmonic motion of amplitude 5 ft has a velocity of 8 ft. per second when at a distance of 3 ft. from its mean position. Find its period.

8. A particle moving with simple harmonic motion has a velocity of 6 ft. per second when at a distance of 8 ft. from its mean position, and a velocity of 8 ft. per second when at a distance of 6 ft. from its mean position. Find its amplitude and its period.

9. A point moves with simple harmonic motion given by the equation $s = a - b \sin ct$. Describe its motion.

**46. Graphs of inverse trigonometric functions.** The equation

$$x = \sin y \tag{1}$$

defines a relation between the quantities $x$ and $y$ which may be stated by saying either that $x$ is the sine of the angle $y$ or that the angle $y$ has the sine $x$. When we wish to use the latter form of expressing the relation, we write in place of equation (1) the equation

$$y = \sin^{-1} x, \tag{2}$$

where $-1$ is not to be understood as a negative exponent but as part of a new symbol $\sin^{-1}$. To avoid the possible ambiguity formula (2) is sometimes written

$$y = \text{arc} \sin x.$$

Equations (1) and (2) have exactly the same meaning, and the student should accustom himself to pass from one to the other without difficulty. In equation (1) $y$ is considered the independent variable, while in (2) $x$ is the independent variable. Equation (2) then defines a function of $x$ which is called the *anti-sine* of $x$ or the *inverse sine* of $x$. It will add to the clearness of the student's thinking, however, if he will read equation (2) as "$y$ is the angle whose sine is $x$."

Similarly, if $x = \cos y$, then $y = \cos^{-1} x$; if $x = \tan y$, then $y = \tan^{-1} x$; and so on for the other trigonometric functions. We get in this way the whole class of *inverse trigonometric functions.*

It is to be noticed that, from equation (2), $y$ is not completely determined when $x$ is given, since there is an infinite number of angles with the same sine. For example, if $x = \frac{1}{2}$, $y = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$, etc. This causes a certain amount of ambiguity in using inverse trigonometric functions, but the ambiguity is removed if the quadrant is known in which the angle $y$ lies. We have the same sort of ambiguity when we pass from the equation $x = y^2$ to the equation $y = \pm\sqrt{x}$, for if $x$ is given, there are two values of $y$.

To obtain the graph of the function expressed in (2) we may change (2) into the equivalent form (1) and proceed as in § 43. In this way it is evident that the graphs of the inverse trigonometric functions are the same as those of the direct functions but differently placed with reference to the coordinate axes. It is to be noticed particularly that to any value of $x$ corresponds an infinite number of values of $y$.

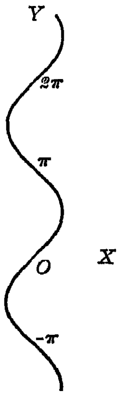



Fig. 55

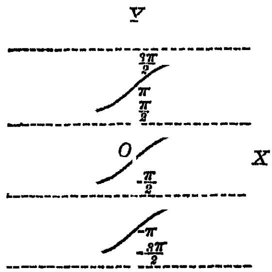



Fig 56

**Ex. 1.** $y = \sin^{-1} x$

From this, $x = \sin y$, and we may plot the graph by assuming values of $y$ and computing those of $x$ (Fig. 55).

**Ex. 2.** $y = \tan^{-1} x$.

Then $x = \tan y$, and the graph is as in Fig. 56.

**EXERCISES**

Plot the graphs of the following equations:

1. $y = \tan^{-1} 2x$.
2. $y = \text{ctn}^{-1} 3x$.
3. $y = \sin^{-1}(x-1)$.
4. $y = \tan^{-1}(x+1)$.
5. $y = 1 + \cos^{-1} x$.
6. $y = \frac{1}{2}\tan^{-1} x$.
7. $y = \cos^{-1}(x-2)$.
8. $y = \sin^{-1}(2x+1) - \frac{\pi}{2}$.

**47. Differentiation of inverse trigonometric functions.** The formulas for the differentiation of the inverse trigonometric functions are as follows:

1. $\dfrac{d}{dx}\sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ when $\sin^{-1} u$ is in the first or the fourth quadrant;

$= -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ when $\sin^{-1} u$ is in the second or the third quadrant.

2. $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ when $\cos^{-1}u$ is in the first or the second quadrant;

$= \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ when $\cos^{-1}u$ is in the third or the fourth quadrant.

3. $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}.$

4. $\frac{d}{dx}\text{ctn}^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}.$

5. $\frac{d}{dx}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$ when $\sec^{-1}u$ is in the first or the third quadrant;

$= -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$ when $\sec^{-1}u$ is in the second or the fourth quadrant.

6 $\frac{d}{dx}\csc^{-1}u = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$ when $\csc^{-1}u$ is in the first or the third quadrant;

$= \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$ when $\csc^{-1}u$ is in the second or the fourth quadrant.

The proofs of these formulas are as follows:

1. If $$y = \sin^{-1}u,$$
then $$\sin y = u.$$
Hence, by § 44, $$\cos y\frac{dy}{dx} = \frac{du}{dx};$$
whence $$\frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx}.$$

But $\cos y = \sqrt{1-u^2}$ when $y$ is in the first or the fourth quadrant, and $\cos y = -\sqrt{1-u^2}$ when $y$ is in the second or the third quadrant.

2. If $$y = \cos^{-1}u,$$
then $$\cos y = u.$$
Hence $$-\sin y\frac{dy}{dx} = \frac{du}{dx};$$
whence $$\frac{dy}{dx} = -\frac{1}{\sin y}\frac{du}{dx}.$$

But $\sin y = \sqrt{1-u^2}$ when $y$ is in the first or the second quadrant, and $\sin y = -\sqrt{1-u^2}$ when $y$ is in the third or the fourth quadrant.

3. If $$y = \tan^{-1} u,$$

then $$\tan y = u.$$

Hence $$\sec^2 y \frac{dy}{dx} = \frac{du}{dx};$$

whence $$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}.$$

4. If $$y = \text{ctn}^{-1} u,$$

then $$\text{ctn}\, y = u.$$

Hence $$-\csc^2 y \frac{dy}{dx} = \frac{du}{dx};$$

whence $$\frac{dy}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}.$$

5. If $$y = \sec^{-1} u,$$

then $$\sec y = u.$$

Hence $$\sec y \tan y \frac{dy}{dx} = \frac{du}{dx};$$

whence $$\frac{dy}{dx} = \frac{1}{\sec y \tan y}\frac{du}{dx}.$$

But $\sec y = u$, and $\tan y = \sqrt{u^2-1}$ when $y$ is in the first or the third quadrant, and $\tan y = -\sqrt{u^2-1}$ when $y$ is in the second or the fourth quadrant.

6. If $$y = \csc^{-1} u,$$

then $$\csc y = u.$$

Hence $$-\csc y\, \text{ctn}\, y \frac{dy}{dx} = \frac{du}{dx};$$

whence $$\frac{dy}{dx} = -\frac{1}{\csc y\, \text{ctn}\, y}\frac{du}{dx}.$$

But $\csc y = u$, and $\operatorname{ctn} y = \sqrt{u^2-1}$ when $y$ is in the first or the third quadrant, and $\operatorname{ctn} y = -\sqrt{u^2-1}$ when $y$ is in the second or the fourth quadrant.

If the quadrant in which an angle lies is not material in a problem, it will be assumed to be in the first quadrant. This applies particularly to formal exercises in differentiation.

**Ex. 1.** $y = \sin^{-1}\sqrt{1-x^2}$, where $y$ is an acute angle

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(1-x^2)}} \cdot \frac{d}{dx}(1-x^2)^{\frac{1}{2}} = -\frac{1}{\sqrt{1-x^2}}.$$

This result may also be obtained by placing $\sin^{-1}\sqrt{1-x^2} = \cos^{-1}x$.

**Ex. 2.** $y = \sec^{-1}\sqrt{4x^2+4x+2}$.

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\sqrt{4x^2+4x+2}}{\sqrt{4x^2+4x+2}\sqrt{(4x^2+4x+2)-1}}$$

$$= \frac{4x+2}{(4x^2+4x+2)(2x+1)} = \frac{1}{2x^2+2x+1}.$$

**EXERCISES**

Find $\frac{dy}{dx}$ in each of the following cases:

1. $y = \sin^{-1}3x$.
2. $y = \sin^{-1}\frac{1}{x}$.
3. $y = \sin^{-1}\frac{x-3}{3}$.
4. $y = \cos^{-1}\frac{3x-2}{2}$.
5. $y = \tan^{-1}(x+1)$.
6. $y = \tan^{-1}\sqrt{x^2-2x}$.
7. $y = \operatorname{ctn}^{-1}\frac{1}{x^2}$.
8. $y = \sec^{-1}5x$.
9. $y = \csc^{-1}2x$.
10. $y = \tan^{-1}\frac{x+6}{3x-2}$.
11. $y = \cos^{-1}\frac{\sqrt{x^2+2x}}{x+1}$.
12. $y = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$.
13. $y = \tan^{-1}\sqrt{x^2-1} + \cos^{-1}\frac{1}{x}$.
14. $y = x\sqrt{1-x^2} + \sin^{-1}x$.
15. $y = \frac{2x}{x^2+4} + \tan^{-1}\frac{x}{2}$.
16. $y = \sqrt{x^2-4} - 2\sec^{-1}\frac{x}{2}$.
17. $y = \tan^{-1}\frac{1}{2}\left(\frac{x}{a} - \frac{a}{x}\right)$.
18. $y = \sqrt{1-x^2} + x\cos^{-1}\sqrt{1-x^2}$.

**48. Angular velocity.** If a line $OP$ (Fig. 57) is revolving in a plane about one of its ends $O$, and in a time $t$ the line $OP$ has moved from an initial position $OM$ to the position $OP$, the angle $MOP = \theta$ denotes the amount of rotation. The rate of change of $\theta$ with respect to $t$ is called the *angular velocity* of $OP$. The angular velocity is commonly denoted by the Greek letter $\omega$; so we have the formula

$$\omega = \frac{d\theta}{dt}. \tag{1}$$

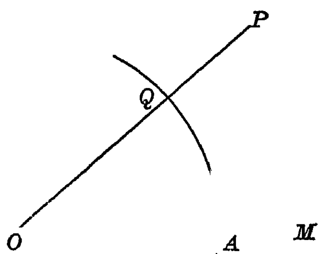


FIG. 57

In accordance with § 42 the angle $\theta$ is taken in radians; so that if $t$ is in seconds, the angular velocity is in radians per second. By dividing by $2\pi$, the angular velocity may be reduced to revolutions per second, since one revolution is equivalent to $2\pi$ radians.

A point $Q$ on the line $OP$ at a distance $r$ from $O$ describes a circle of radius $r$ which intersects $OM$ at $A$. If $s$ is the length of the arc of the circle $AQ$ measured from $A$, then, by § 42,

$$s = r\theta. \tag{2}$$

Now $\frac{ds}{dt}$ is called the *linear velocity* of the point $Q$, since it measures the rate at which $s$ is described; and from (2) and (1),

$$\frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega, \tag{3}$$

showing that the farther the point $Q$ is from $O$ the greater is its linear velocity.

Similarly, the angular acceleration, which is denoted by $\alpha$, is defined by the relation

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{4}$$

This is connected with the linear acceleration $\frac{d^2s}{dt^2}$ by the formula

$$\frac{d^2s}{dt^2} = r\frac{d^2\theta}{dt^2} = r\alpha. \tag{5}$$

**Ex 1.** If a wheel revolves so that the angular velocity is given by the formula $\omega = 8t$, how many revolutions will it make in the time from $t = 2$ to $t = 5$?

We take a spoke of the wheel as the line $OP$ Then we have

$$d\theta = 8\,t\,dt$$

Hence the angle through which the wheel revolves in the given time is

$$\theta = \int_2^5 8\,t\,dt = [4\,t^2]_2^5 = 100 - 16 = 84.$$

The result is in radians. It may be reduced to revolutions by dividing by $2\pi$. The answer is 13.4 revolutions

**Ex. 2.** A particle traverses a circle at a uniform rate of $n$ revolutions a second Determine the motion of the projection of the particle on a diameter of the circle

Let $P$ (Fig 58) be the particle, $OX$ the diameter of the circle, and $M$ the projection of $P$ on $OX$ Then

$$x = a\cos\theta,$$

where $a$ is the radius of the circle. By hypothesis the angular velocity of $OP$ is $2\,n\pi$ radians per second Therefore

$$\omega = \frac{d\theta}{dt} = 2\,n\pi;$$

whence $\theta = 2\,n\pi t + C$

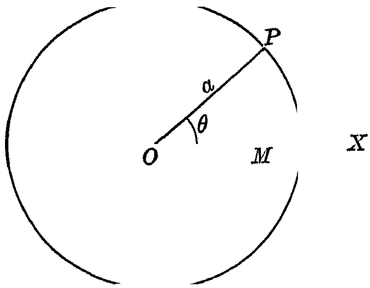


FIG. 58

If we consider that when $t = 0$, the particle is on $OX$, then $C = 0$. Therefore

$$x = a\cos\theta = a\cos 2\,n\pi t = a\cos\omega t.$$

The point $M$ therefore describes a simple harmonic motion In fact, simple harmonic motion is often defined in this way

## EXERCISES

1. A flywheel 4 ft in diameter makes 3 revolutions a second. Find the components of velocity in feet per second of a point on the rim when it is 6 in above the level of the center of the wheel.

2. A point on the rim of a flywheel of radius 5 ft which is 3 ft. above the level of the center of the wheel has a horizontal component of velocity of 100 ft per second Find the number of revolutions per second of the wheel.

3. If the horizontal and vertical projections of a point descr simple harmonic motions given by the equations

$$x = 5\cos 3t, \qquad y = 5\sin 3t,$$

show that the point moves in a circle and find its angular velocit

**49. The cycloid.** If a wheel rolls upon a straight line, ea point of the rim describes a curve called a *cycloid.*

Let a wheel of radius $a$ roll upon the axis of $x$, and let (Fig. 59) be its center at any time of its motion, $N$ its point

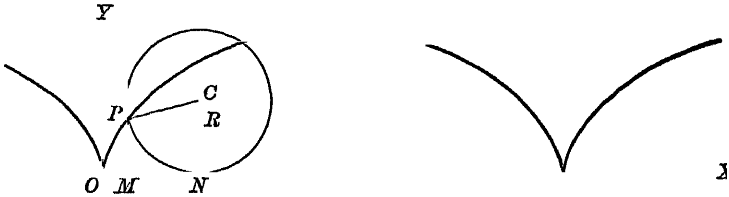



FIG. 59

contact with $OX$, and $P$ the point which describes the cycloi Take as the origin of coordinates, $O$, the point found by rolli the wheel to the left until $P$ meets $OX$.

Then $$ON = \text{arc } PN.$$

Draw $MP$ and $CN$, each perpendicular to $OX$, $PR$ parallel $OX$, and connect $C$ and $P$. Let

$$\text{angle } NCP = \phi.$$

Then
$$\begin{aligned} x &= OM = ON - MN \\ &= \text{arc } PN - PR \\ &= a\phi - a\sin\phi. \\ y &= MP = NC - RC \\ &= a - a\cos\phi. \end{aligned}$$

Hence the parametric representation (§ 40) of the cycloid

$$x = a(\phi - \sin\phi),$$
$$y = a(1 - \cos\phi).$$

E

If the wheel revolves with a constant angular velocity $\omega = \frac{d\phi}{dt}$, we have, by § 40,

$$v_x = a(1 - \cos\phi)\frac{d\phi}{dt} = a\omega(1 - \cos\phi),$$

$$v_y = a\sin\phi\frac{d\phi}{dt} = a\omega\sin\phi;$$

whence $$v^2 = a^2\omega^2(2 - 2\cos\phi) = 4\,a^2\omega^2\sin^2\frac{\phi}{2},$$

and $$v = 2\,a\omega\sin\frac{\phi}{2},$$

as an expression for the velocity in its path of a point on the rim of the wheel.

**EXERCISES**

1. Prove that the slope of the cycloid at any point is $\operatorname{ctn}\frac{\phi}{2}$.

2. Show that the straight line drawn from any point on the rim of a rolling wheel perpendicular to the cycloid which that point is describing goes through the lowest point of the rolling wheel.

3. Show that any point on the rim of the wheel has a horizontal component of velocity which is proportional to the vertical height of the point

4. Show that the highest point of the rolling wheel moves twice as fast as either of the two points whose distance from the ground is half the radius of the wheel

5. Show that the vertical component of velocity is a maximum when the point which describes the cycloid is on the level of the center of the rolling wheel.

6. Show that a point on the spoke of a rolling wheel at a distance $b$ from the center describes a curve given by the equations

$$x = a\phi - b\sin\phi, \qquad y = a - b\cos\phi,$$

and find the velocity of the point in its path. The curve is called a *trochoid*.

7. Find the slope of the trochoid and find the point at which the curve is steepest.

8. Show that when a point on a spoke of a wheel describes a trochoid, the average of the velocities of the point when in its highest and lowest positions is equal to the linear velocity of the wheel.

**50. Curvature.** If a point describes a curve, the change of direction of its motion may be measured by the change of the angle $\phi$ (§ 15).

For example, in the curve of Fig. 60, if $AP_1 = s$ and $P_1P_2 = \Delta s$, and if $\phi_1$ and $\phi_2$ are the values of $\phi$ for the points $P_1$ and $P_2$ respectively, then $\phi_2 - \phi_1$ is the total change of direction of the curve between $P_1$ and $P_2$. If $\phi_2 - \phi_1 = \Delta\phi$, expressed in circular measure, the ratio $\frac{\Delta\phi}{\Delta s}$ is the average change of direction per linear unit of the arc $P_1P_2$. Regarding $\phi$ as a function of $s$, and taking the limit of $\frac{\Delta\phi}{\Delta s}$ as $\Delta s$ approaches zero as a limit, we have $\frac{d\phi}{ds}$, which is called the *curvature* of the curve at the point $P$. Hence the curvature of a curve is the rate of change of the direction of the curve with respect to the length of the arc.

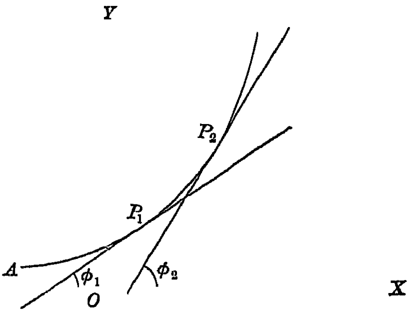


FIG 60

If $\frac{d\phi}{ds}$ is constant, the curvature is constant or *uniform*; otherwise the curvature is variable. Applying this definition to the circle of Fig. 61, of which the center is $C$ and the radius is $a$, we have $\Delta\phi = P_1CP_2$, and hence $\Delta s = a\,\Delta\phi$. Therefore $\frac{\Delta\phi}{\Delta s} = \frac{1}{a}$. Hence $\frac{d\phi}{ds} = \frac{1}{a}$, and *the circle is a curve of constant curvature equal to the reciprocal of its radius.*

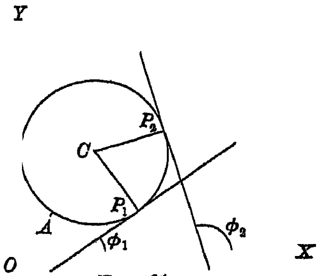


FIG. 61

The reciprocal of the curvature is called the *radius of curvature* and will be denoted by $\rho$. Through every point of a curve we may pass a circle with its radius equal to $\rho$, which shall have the same tangent as the curve at the point and shall lie on the same side of the tangent. Since the curvature of a circle is uniform and equal to the reciprocal of its radius, the curvatures of the curve and of the circle are the same, and the circle shows the curvature of the curve in a manner similar to that in which the tangent shows the direction of the curve. The circle is called the *circle of curvature.*

From the definition of curvature it follows that

$$\rho = \frac{ds}{d\phi}.$$

If the equation of the curve is in rectangular coördinates,

by (9), § 36,
$$\rho = \frac{\dfrac{ds}{dx}}{\dfrac{d\phi}{dx}}.$$

To transform this expression further, we note that

$$ds^2 = dx^2 + dy^2;$$

whence, dividing by $\overline{dx}^2$ and taking the square root, we have

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Since
$$\phi = \tan^{-1}\left(\frac{dy}{dx}\right), \qquad \text{(by § 15)}$$

$$\frac{d\phi}{dx} = \frac{\dfrac{d^2y}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2}.$$

Substituting, we have
$$\rho = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}.$$

In the above expression for $\rho$ there is an apparent ambiguity of sign, on account of the radical sign. If only the numerical value of $\rho$ is required, a negative sign may be disregarded.

**Ex. 1.** Find the radius of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Here
$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$
and
$$\frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}.$$
Therefore
$$\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

**Ex. 2.** Find the radius of curvature of the cycloid (§ 49).

We have
$$\frac{dx}{d\phi}=a(1-\cos\phi)=2a\sin^2\frac{\phi}{2},$$
$$\frac{dy}{d\phi}=a\sin\phi=2a\sin\frac{\phi}{2}\cos\frac{\phi}{2}.$$

Therefore, by (9), § 36,
$$\frac{dy}{dx}=\operatorname{ctn}\frac{\phi}{2}.$$
Hence
$$\frac{d^2y}{dx^2}=-\frac{1}{2}\csc^2\frac{\phi}{2}\frac{d\phi}{dx}=-\frac{1}{4a}\csc^4\frac{\phi}{2},$$
and
$$\rho=\frac{\left(1+\operatorname{ctn}^2\frac{\phi}{2}\right)^{\frac{3}{2}}}{\frac{1}{4a}\csc^4\frac{\phi}{2}}=4a\sin\frac{\phi}{2}.$$

**EXERCISES**

1. Find the radius of curvature of the curve $y^2=\frac{4}{27}x^3$.

2. Find the radius of curvature of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

3. Find the radius of curvature of the curve $y=\tan^{-1}(x-1)$ at the point for which $x=2$

4. Show that the circle $\left(x-\frac{\pi}{2}\right)^2+y^2=1$ is tangent to the curve $y=\sin x$ at the point for which $x=\frac{\pi}{2}$, and has the same radius of curvature at that point.

5. Find the radius of curvature of the curve $x=\cos t$, $y=\cos 2t$, at the point for which $t=0$.

6. Find the radius of curvature of the curve $x=a\cos\phi+a\phi\sin\phi$, $y=a\sin\phi-a\phi\cos\phi$.

7. Prove that the radius of curvature of the curve $x=a\cos^3\phi$, $y=a\sin^3\phi$ has its greatest value when $\phi=\frac{\pi}{4}$.

**51. Polar coordinates.** So far we have determined the position of a point in the plane by two distances, $x$ and $y$. We may, however, use a distance and a direction, as follows:

Let $O$ (Fig. 62), called the *origin*, or *pole*, be a fixed point, and let $OM$, called the *initial line*, be a fixed line. Take $P$ any point in the plane, and draw $OP$. Denote $OP$ by $r$, and the angle $MOP$ by $\theta$. Then $r$ and $\theta$ are called the *polar coordinates* of the point $P(r, \theta)$, and when given will completely determine $P$.

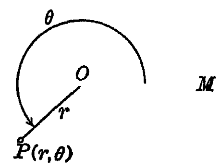


FIG 62

For example, the point $(2, 15°)$ is plotted by laying off the angle $MOP = 15°$ and measuring $OP = 2$.

$OP$, or $r$, is called the *radius vector*, and $\theta$ the *vectorial angle*, of $P$. These quantities may be either positive or negative. A negative value of $\theta$ is laid off in the direction of the motion of the hands of a clock, a positive angle in the opposite direction. After the angle $\theta$ has been constructed, positive values of $r$ are measured from $O$ along the terminal line of $\theta$, and negative values of $r$ from $O$ along the backward extension of the terminal line. It follows that the same point may have more than one pair of coordinates. Thus $(2, 195°)$, $(2, -165°)$, $(-2, 15°)$, and $(-2, -345°)$ refer to the same point. In practice it is usually convenient to restrict $\theta$ to positive values.

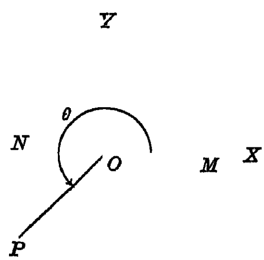


FIG. 63

Plotting in polar coördinates is facilitated by using paper ruled as in Figs. 64 and 65. The angle $\theta$ is determined from the numbers at the ends of the straight lines, and the value of $r$ is counted off on the concentric circles, either toward or away from the number which indicates $\theta$, according as $r$ is positive or negative.

The relation between $(r, \theta)$ and $(x, y)$ is found as follows:

Let the pole $O$ and the initial line $OM$ of a system of polar coördinates be at the same time the origin and the axis of $x$ of a system of rectangular coördinates. Let $P$ (Fig. 63) be any point of the plane, $(x, y)$ its rectangular coördinates, and $(r, \theta)$ its polar coördinates. Then, by the definition of the trigonometric functions,

$$\cos\theta = \frac{x}{r},$$

$$\sin\theta = \frac{y}{r}.$$

Whence follows, on the one hand,

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta; \end{aligned} \qquad (1)$$

and, on the other hand,

$$r = \sqrt{x^2 + y^2}, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}. \qquad (2)$$

By means of (1) a transformation can be made from rectangular to polar coördinates, and by means of (2) from polar to rectangular coördinates

When an equation is given in polar coördinates, the corresponding curve may be plotted by giving to $\theta$ convenient values, computing the corresponding values of $r$, plotting the resulting points, and drawing a curve through them.

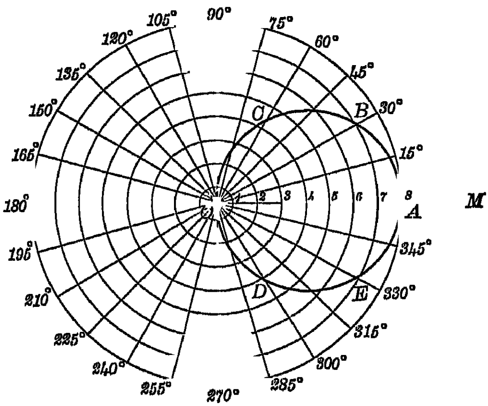



FIG 64

**Ex. 1.** $r = a\cos\theta$

$a$ is a constant which may be given any convenient value We may then find from a table of natural cosines the value of $r$ which corresponds to any value of $\theta$. By plotting the points corresponding to values of $\theta$ from 0° to 90°, we obtain the arc $ABCO$ (Fig. 64). Values of $\theta$ from 90° to 180° give the arc $ODEA$. Values of $\theta$ from 180° to 270° give again the arc $ABCO$, and those from 270° to 360° give again the arc $ODEA$. Values of $\theta$ greater than 360° can clearly give no points not already found. The curve is a *circle*.

**Ex. 2.** $r = a \sin 3\theta$.

As $\theta$ increases from 0° to 30°, $r$ increases from 0 to $a$; as $\theta$ increases from 30° to 60°, $r$ decreases from $a$ to 0, the point $(r, \theta)$ traces out the loop $OAO$ (Fig. 65), which is evidently symmetrical with respect to the radius $OA$. As $\theta$ increases from 60° to 90°, $r$ is negative and decreases from 0 to $-a$; as $\theta$ increases from 90° to 120°, $r$ increases from $-a$ to 0; the point $(r, \theta)$ traces out the loop $OBO$ As $\theta$ increases from 120° to 180°, the point $(r, \theta)$ traces out the loop $OCO$. Larger values of $\theta$ give points already found, since $\sin 3(180° + \theta) = -\sin 3\theta$. The three loops are congruent, because $\sin 3(60° + \theta) = -\sin 3\theta$ This curve is called a *rose of three leaves.*

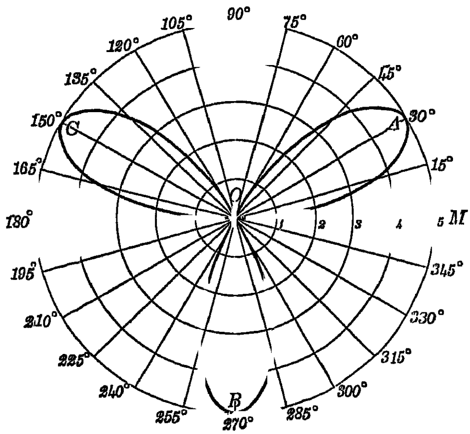



FIG 65

**Ex. 3.** $r^2 = 2a^2 \cos 2\theta$.

Solving for $r$, we have $r = \pm a\sqrt{2 \cos 2\theta}$.

Hence, corresponding to any values of $\theta$ which make $\cos 2\theta$ positive, there will be two values of $r$ numerically equal and opposite in sign, and two corresponding points of the curve symmetrically situated with respect to the pole If values are assigned to $\theta$ which make $\cos 2\theta$ negative, the corresponding values of $r$ will be imaginary and there will be no points on the curve.

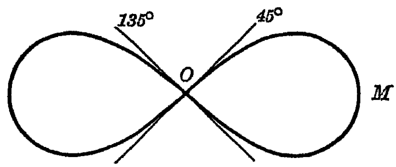



FIG 66

Accordingly, as $\theta$ increases from 0° to 45°, $r$ decreases numerically from $a\sqrt{2}$ to 0, and the portions of the curve in the first and the third quadrant are constructed (Fig 66); as $\theta$ increases from 45° to 135°, $\cos 2\theta$ is negative, and there is no portion of the curve between the lines $\theta = 45°$ and $\theta = 135°$, finally, as $\theta$ increases from 135° to 180°, $r$ increases numerically from 0 to $a\sqrt{2}$, and the portions of the curve in the second and the fourth quadrant are constructed The curve is now complete, as we should only repeat the curve already found if we assigned further values to $\theta$, it is called the *lemniscate.*

**Ex. 4.** *The spiral of Archimedes,*

$$r = a\theta.$$

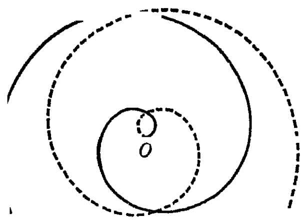


FIG. 67

In plotting, $\theta$ is usually considered in circular measure When $\theta = 0, r = 0$, and as $\theta$ increases, $r$ increases, so that the curve winds an infinite number of times around the origin while receding from it (Fig 67) In the figure the heavy line represents the portion of the spiral corresponding to positive values of $\theta$, and the dotted line the portion corresponding to negative values of $\theta$

### EXERCISES

Plot the graphs of the following curves

1. $r = a \sin \theta$
2. $r = a \sin 2\theta$
3. $r = a \cos 3\theta$
4. $r = a \sin \frac{\theta}{3}$.
5. $r = a \cos \frac{\theta}{2}$
6. $r = 3 \cos \theta + 5$
7. $r = 3 \cos \theta + 3$.*
8. $r = 3 \cos \theta + 2$
9. $r = a \sin^3 \frac{\theta}{3}$.
10. $r^2 = a^2 \sin \theta$
11. $r^2 = a^2 \sin 3\theta$.
12. $r = a(1 - \cos 2\theta)$.
13. $r = a(1 + 2\cos 2\theta)$
14. $r = a \tan \theta$
15. $r = a \tan 2\theta$
16. $r = \dfrac{1}{1 + \cos \theta}$ †

Find the points of intersection of the following pairs of curves:

17. $r = 2 \sin \theta, \quad r = 2\sqrt{3} \cos \theta$.
18. $r^2 = a^2 \cos \theta, \quad r^2 = a^2 \sin 2\theta$
19. $r = 1 + \sin \theta, \quad r = 2 \sin \theta$
20. $r^2 = a^2 \sin \theta, \quad r^2 = a^2 \sin 3\theta$

Transform the following equations to polar coordinates.

21. $xy = 4$.
22. $x^2 + y^2 - 4ax - 4ay = 0$
23. $x^2 + y^2 - 2ay = 0$.
24. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$

Transform the following equations to rectangular coordinates:

25. $r = a \sec \theta$
26. $r = 2a \cos \theta$
27. $r = a \tan \theta$.
28. $r = a \cos 2\theta$

* The curve is called a *cardioid*

† The curve is a parabola with the origin at the focus.

**52. The differentials $dr$, $d\theta$, $ds$, in polar coördinates.** We have seen, in § 39, that the differential of arc in rectangular coördinates is given by the equation

$$ds^2 = dx^2 + dy^2. \qquad (1)$$

If we wish to change this to polar coördinates, we have to place

$$x = r\cos\theta, \qquad y = r\sin\theta;$$

whence
$$dx = \cos\theta\, dr - r\sin\theta\, d\theta,$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta.$$

Substituting in (1), we have

$$ds^2 = dr^2 + r^2 d\theta^2. \qquad (2)$$

This formula may be remembered by means of an "elementary triangle" (Fig. 68), constructed as follows:

Let $P$ be a point on a curve $r = f(\theta)$, the coördinates of $P$ being $(r, \theta)$, where $OP = r$ and $MOP = \theta$. Let $\theta$ be increased by an amount $d\theta$, thus determining another point $Q$ on the curve. From $O$ as a center and with a radius equal to $r$, describe an arc of a circle intersecting $OQ$ in $R$ so that $OR = OP = r$. Then, by § 42, $PR = rd\theta$. Now $RQ$ is equal to $\Delta r$, and $PQ$ is equal to $\Delta s$. We shall mark them, however, as $dr$ and $ds$ respectively, and the formula (2) is then correctly obtained by treating the triangle $PQR$ as a right triangle with straight-line sides. The fact is that the smaller the triangle becomes as $Q$ approaches $P$, the more nearly does it behave as a straight-line triangle; and in the limit, formula (2) is exactly true.

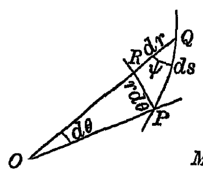


FIG. 68

Other formulas may be read out of the triangle $PQR$. Let us denote by $\psi$ the angle $PQR$, which is the angle made by the curve with any radius vector. Then, if we treat the triangle $PQR$ as a straight-line right-angle triangle, we have the formulas:

$$\sin\psi = \frac{rd\theta}{ds}, \quad \cos\psi = \frac{dr}{ds}, \quad \tan\psi = \frac{rd\theta}{dr}. \qquad (3)$$

The above is not a proof of the formulas. To supply the proof we need to go through a limit process, as follows:

We connect the points $P$ and $Q$ by a straight line (Fig. 69) and draw a straight line from $P$ perpendicular to $OQ$ meeting $OQ$ at $S$. Then the triangle $PQS$ is a straight-line right-angle triangle, and therefore

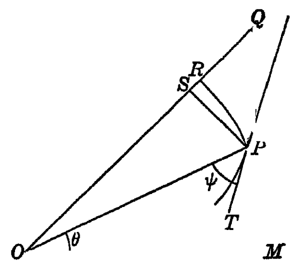



FIG. 69

$$\sin SQP = \frac{SP}{\text{chord } PQ}$$

$$= \frac{SP}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{\text{chord } PQ}.$$

Now angle $POQ = \Delta\theta$, arc $PQ = \Delta s$, and, from the right triangle $OSP$, $SP = OP \sin POQ = r \sin \Delta\theta$. Therefore

$$\sin SQP = \frac{r \sin \Delta\theta}{\Delta s} \cdot \frac{\text{arc } PQ}{\text{chord } PQ} = r \frac{\sin \Delta\theta}{\Delta\theta} \cdot \frac{\Delta\theta}{\Delta s} \cdot \frac{\text{arc } PQ}{\text{chord } PQ}. \quad (4)$$

Now let $\Delta\theta$ approach zero as a limit, so that $Q$ approaches $P$ along the curve. The angle $SQP$ approaches the angle $OPT$, where $PT$ is the tangent at $P$. At the same time $\frac{\sin \Delta\theta}{\Delta\theta}$ approaches 1, by § 42; $\frac{\Delta\theta}{\Delta s}$ approaches $\frac{d\theta}{ds}$, by definition; and $\frac{\text{arc } PQ}{\text{chord } PQ}$ approaches 1, by § 39. In this figure we denote the angle $OPT$ by $\psi$ and have, from (4),

$$\sin \psi = r \frac{d\theta}{ds}, \quad (5)$$

which is the first of formulas (3). It is true that in Fig. 69 we have denoted $OPT$ by $\psi$ and that in Fig. 68 $\psi$ denotes $OQP$. But if we remember that the angle $OQP$ approaches $OPT$ as a limit when $Q$ approaches $P$, and that in using Fig. 68 to read off the formulas (3) we are really anticipating this limit process, the difference appears unessential.

The other formulas (3) may be obtained by a limit process similar to the one just used, or they may be obtained more quickly by combining (5) and (2). For, from (2) and (5), we have

$$1=\left(\frac{dr}{ds}\right)^2+\left(\frac{rd\theta}{ds}\right)^2=\left(\frac{dr}{ds}\right)^2+\sin^2\psi;$$

whence

$$\cos\psi=\frac{dr}{ds}. \tag{6}$$

By dividing (5) by (6) we have

$$\tan\psi=\frac{rd\theta}{dr}. \tag{7}$$

In using (7) it may be convenient to write it in the form

$$\tan\psi=\frac{r}{\dfrac{dr}{d\theta}}, \tag{8}$$

since the equation of the curve is usually given in the form $r=f(\theta)$, and $\dfrac{dr}{d\theta}$ is found by direct differentiation.

**Ex.** Find the angle which the curve $r=a\sin 4\,\theta$ makes with the radius vector $\theta=30°$

Here $\dfrac{dr}{d\theta}=4\,a\cos 4\,\theta$. Therefore, from (8), $\tan\psi=\dfrac{a\sin 4\,\theta}{4\,a\cos 4\,\theta}=\dfrac{1}{4}\tan 4\,\theta$
Substituting $\theta=30°$, we have $\tan\psi=\frac{1}{4}\tan 120°=-\frac{1}{4}\sqrt{3}=-.4330$
Therefore $\psi=156°\,35'$.

**EXERCISES**

1. Find the angle which the curve $r=a\cos 3\,\theta$ makes with the radius vector $\theta=45°$

2. Find the angle which the curve $r=2+3\cos\theta$ makes with the radius vector $\theta=90°$

3. Find the angle which the curve $r=a^2\sin^2\dfrac{\theta}{2}$ makes with the initial line.

4. Show that for the curve $r=a\sin^3\dfrac{\theta}{3}$, $\psi=\dfrac{\theta}{3}$.

5. Show that the angle between the cardioid $r=a(1-\cos\theta)$ and any radius vector is always half the angle between the radius vector and the initial line.

6. Show that the angle between the lemniscate $r^2 = 2a^2 \cos 2\theta$ and any radius vector is always $\frac{\pi}{2}$ plus twice the angle between the radius vector and the initial line

7. Show that the curves $r^2 = a^2 \sin 2\theta$ and $r^2 = a^2 \cos 2\theta$ intersect at right angles

## GENERAL EXERCISES

Find the graphs of the following equations:

1. $y = 4 \sin \frac{x+1}{2}$
2. $y = \cos(2x - 3)$
3. $y = \tan \frac{x}{2}$
4. $y = \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x.$
5. $y = 3\sin\left(x + \frac{\pi}{3}\right).$
6. $y^2 = \tan x.$
7. $y = 2\cos 2(x - 2)$
8. $y = 3\cos 3\left(x + \frac{\pi}{4}\right)$

Find $\frac{dy}{dx}$ in the following cases:

9. $y = 2x + \tan 2x$
10. $y = \frac{1}{3}\tan(3x + 2) + \frac{1}{9}\tan^3(3x + 2).$
11. $y = \frac{1}{2}x - \frac{1}{12}\sin(4 - 6x)$
12. $\tan(x + y) + \tan(x - y) = 0.$
13. $y = 3\,\text{ctn}^5\frac{x}{5} + 5\,\text{ctn}^3\frac{x}{5}.$
14. $y = \csc^2 4x + 2\,\text{ctn}\, 4x$
15. $y = \frac{1}{2}\tan^2 ax$
16. $y = \sin^2 4x \cos^4 2x$
17. $y = \frac{2}{3}\cos^3\frac{x}{4} - 2\cos\frac{x}{4}$
18. $y = \frac{1}{6}\tan^3 2x - \frac{1}{2}\tan 2x + x.$
19. $y = \sin^{-1}\frac{x-1}{x+1}.$
20. $y = \cos^{-1}\frac{2x-1}{3}.$
21. $y = \text{ctn}^{-1}\sqrt{\frac{1}{x}}.$
22. $y = \sec^{-1} 7x.$
23. $y = \cos^{-1}\frac{x^2-4}{x^2+4}.$
24. $y = \text{ctn}^{-1}\sqrt{x^2 - 2x}.$
25. $y = \csc^{-1}\frac{2}{2x+1}$
26. $y = 3\sqrt{9 - x^2} + 2\sin^{-1}\frac{x}{3}.$
27. $y = \tan^{-1} x\sqrt{x^2 - 2}.$

28. A particle moves in a straight line so that $s = 5 - 4\sin^2\frac{\pi t}{2}.$ Show that the motion is simple harmonic and find the center about which the particle oscillates and the amplitude of the motion.

**29.** A particle moves on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ so that its projection upon $OX$ describes simple harmonic motion given by $x=a\cos kt$. Show that its projection upon $OY$ also describes simple harmonic motion and find the velocity of the particle in its path.

**30.** A particle moving with simple harmonic motion of period $\frac{\pi}{3}$ has a velocity of 9 ft per second when at a distance of 2 ft. from its mean position. Find the amplitude of the motion.

**31.** A particle moves according to the equation $s=4\sin\frac{1}{2}t+5\cos\frac{1}{2}t$. Show that the motion is simple harmonic and find the amplitude of the swing and the time at which the particle passes through its mean position.

**32.** Find the radius of curvature of the curve $y=x\sin\frac{1}{x}$ at the point for which $x=\frac{2}{\pi}$

**33.** Find the radius of curvature of the curve $y=\frac{\sin x}{x}$ at the point for which $x=\pi$

**34.** Find the radius of curvature of the curve $y=a\sin^{-1}\frac{x}{a}-\sqrt{a^2-x^2}$ at the point for which $x=\frac{a}{2}$

**35.** Find the radius of curvature of the curve $x=a\cos\phi$, $y=b\sin\phi$, at the point for which $\phi=\frac{\pi}{4}$.

Plot the graphs of the following curves.

**36.** $r^2=a^2\sin\frac{\theta}{2}$.

**37.** $r^2=a^2\sin 4\theta$

**38.** $r=a(1-\sin\theta)$.

**39.** $r=a(1+\cos 2\theta)$.

**40.** $r=a(1+2\sin\theta)$.

**41.** $r=1-2\theta$.

**42.** $r^2=4\sec 2\theta$.

**43.** $r^2=a^2\tan\theta$

**44.** $r=1+\sin\frac{\theta}{2}$.

**45.** $r=1+\sin\frac{3\theta}{2}$.

Find the points of intersection of the following pairs of curves:

**46.** $r^2=3\cos 2\theta$, $r^2=2\cos^2\theta$.

**47.** $r=a\cos\theta$, $r^2=a^2\sin 2\theta$.

**48.** $r=2\sin\theta$, $r^2=4\sin 2\theta$.

**49.** $r=a(1+\sin 2\theta)$, $r^2=4a^2\sin 2\theta$.

Transform the following curves to polar coordinates:

50. $y^2 = \frac{x^3}{2a - x}$

51. $y^4 + y^2x^2 - a^2x^2 = 0.$

Transform the following curves to $xy$-coordinates

52 $r^2 = 2a^2 \sin 2\theta.$

53. $r = a(1 - \cos\theta)$

54. Find the angle at which the curve $r = 3 + \sin 2\theta$ meets the circle $r = 3$

55. Find the angle of intersection of the two curves $r = 2\sin\theta$ and $r^2 = 4\sin 2\theta$

56. Find the angle of intersection of the curves $r = a\cos\theta$ and $r = a\sin 2\theta$.

57. If a ball is fired from a gun with the initial velocity $v_0$, it describes a path the equation of which is $y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$, where $\alpha$ is the angle of elevation of the gun and $OX$ is horizontal What is the value of $\alpha$ when the horizontal range is greatest?

58. In measuring an electric current by means of a tangent galvanometer, the percentage of error due to a small error in reading is proportional to $\tan x + \operatorname{ctn} x$. For what value of $x$ will this percentage of error be least?

59. A tablet 8 ft high is placed on a wall so that the bottom of the tablet is 29 ft. from the ground. How far from the wall should a person stand in order that he may see the tablet to best advantage (that is, that the angle between the lines from his eye to the top and to the bottom of the tablet should be the greatest), assuming that his eye is 5 ft. from the ground?

60. One side of a triangle is 12 ft. and the opposite angle is 36° Find the other angles of the triangle when its area is a maximum

61. Above the center of a round table of radius 2 ft is a hanging lamp. How far should the lamp be above the table in order that the edge of the table may be most brilliantly lighted, given that the illumination varies inversely as the square of the distance and directly as the cosine of the angle of incidence?

62. A weight $P$ is dragged along the ground by a force $F$. If the coefficient of friction is $k$, in what direction should the force be applied to produce the best result?

**63.** An open gutter is to be constructed of boards in such a way that the bottom and sides, measured on the inside, are to be each 5 in wide and both sides are to have the same slope How wide should the gutter be across the top in order that its capacity may be as great as possible?

**64.** A steel girder 27 ft long is to be moved on rollers along a passageway and into a corridor 8 ft. in width at right angles to the passageway. If the horizontal width of the girder is neglected, how wide must the passageway be in order that the girder may go around the corner?

**65.** Two particles are moving in the same straight line so that their distances from a fixed point $O$ are respectively $x = a \cos kt$ and $x' = a \cos\left(kt + \frac{\pi}{3}\right)$, $k$ and $a$ being constants Find the greatest distance between them

**66.** Show that for any curve in polar coordinates the maximum and the minimum values of $r$ occur in general when the radius vector is perpendicular to the curve.

**67.** Two men are at one end of the diameter of a circle of 40 yd radius. One goes directly toward the center of the circle at the uniform rate of 6 ft. per second, and the other goes around the circumference at the rate of $2\pi$ ft per second How fast are they separating at the end of 10 sec.?

**68.** Given that two sides and the included angle of a triangle are 5 ft, 10 ft, and 30° respectively, and are changing at the rates of 1 ft, $-3$ ft, and 12° per second respectively, what is the area of the triangle and how fast is it changing?

**69.** A revolving light in a lighthouse $\frac{1}{4}$ mi offshore makes one revolution a minute If the line of the shore is a straight line, how fast is the ray of light moving along the shore when it passes a point one mile from the point nearest to the lighthouse?

**70.** $BC$ is a rod $a$ feet long, connected with a piston rod at $C$, and at $B$ with a crank $AB$, $b$ feet long, revolving about $A$. Find $C$'s velocity in terms of $AB$'s angular velocity.

**71.** At any time $t$ the coordinates of a point moving in the $xy$-plane are $x = 2 - 3 \cos t$, $y = 3 + 2 \sin t$ Find its path and its velocity in its path. At what points will it have a maximum speed?

**72.** At any time $t$ the coördinates of a moving point are $x = 2 \sec 3t$, $y = 4 \tan 3t$. Find the equation of its path and its velocity in its path.

73. The parametric equations of the path of a moving particle are $x = 2\cos^3\phi$, $y = 2\sin^3\phi$ If the angle $\phi$ increases at the rate of 2 radians per second, find the velocity of the particle in its path

74. A particle moves along the curve $y = \sin x$ so that the $x$-component of its velocity has always the constant value $a$ Find the velocity of the particle along the curve and determine the points of the curve at which the particle is moving fastest and those at which it is moving most slowly

75. Find the angle of intersection of the curves $y = \sin x$ and $y = \cos x$

76. Find the angle of intersection of the curves $y = \sin x$ and $y = \sin\left(x + \frac{\pi}{3}\right)$

77. Find the angle of intersection of the curves $y = \sin x$ and $y = \cos 2x$ between the lines $x = 0$ and $x = 2\pi$.

78. Find the points of intersection of the curves $y = \sin x$ and $y = \sin 3x$ between the lines $x = 0$ and $x = \pi$ Determine the angles at the points of intersection.

79. Find all the points of intersection of the curves $y = \cos x$ and $y = \sin 2x$ which lie between the lines $x = 0$ and $x = 2\pi$, and determine the angles of intersection at each of the points found.

# CHAPTER VI

## EXPONENTIAL AND LOGARITHMIC FUNCTIONS

**53. The exponential function.** The equation

$$y = a^x,$$

where $a$ is any constant, defines $y$ as a function of $x$ called the *exponential function.*

If $x = n$, an integer, $y$ is determined by raising $a$ to the $n$th power by multiplication.

If $x = \frac{p}{q}$, a positive fraction, $y$ is the $q$th root of the $p$th power of $a$.

If $x$ is a positive irrational number, the approximate value of $y$ may be obtained by expressing $x$ approximately as a fraction.

If $x = 0$, $y = a^0 = 1$. If $x = -m$, $y = a^{-m} = \frac{1}{a^m}$.

The graph of the function is readily found.

**Ex.** Find the graph of $y = (1.5)^x$. By giving convenient values to $x$ we obtain the curve shown in Fig 70. To determine the shape of the curve at the extreme left, we place $x$ equal to a large negative number, say $x = -100$. Then $y = (1.5)^{-100} = \frac{1}{(1.5)^{100}}$, which is very small. It is obvious that the larger numerically the negative value of $x$ becomes, the smaller $y$ becomes, so that the curve approaches asymptotically the negative portion of the $x$-axis.

On the other hand, if $x$ is a large positive number, $y$ is also large.

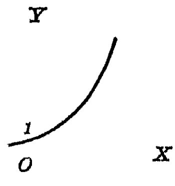



Fig. 70

**54. The logarithm.** If a number $N$ may be obtained by placing an exponent $L$ on another number $a$ and computing the result, then $L$ is said to be the logarithm of $N$ to the base $a$. That is, if

$$N = a^L, \quad (1)$$

then

$$L = \log_a N. \quad (2)$$

Formulas (1) and (2) are simply two different ways of expressing the same fact as to the relation of $N$ and $L$, and the student should accustom himself to pass from one to the other as convenience may demand.

From these formulas follow easily the fundamental properties of logarithms; namely,

$$\begin{aligned} \log_a N + \log_a M &= \log_a MN, \\ \log_a N - \log_a M &= \log_a \frac{N}{M}, \\ n \log_a N &= \log_a N^n, \\ \log_a 1 &= 0, \\ \log_a \frac{1}{N} &= -\log_a N. \end{aligned} \tag{3}$$

Theoretically any number, except 0 or 1, may be used as the base of a system of logarithms. Practically there are only two numbers so used. The first is the number 10, the use of which as a base gives the common system of logarithms, which are the most convenient for calculations and are used almost exclusively in trigonometry.

Another number, however, is more convenient in theoretical discussions, since it gives simpler formulas. This number is denoted by the letter $e$ and is expressed by the infinite series

$$e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots,$$

where $2! = 1 \times 2$, $3! = 1 \times 2 \times 3$, $4! = 1 \times 2 \times 3 \times 4$, etc.

Computing the above series to seven decimal places, we have

$$e = 2.7182818 \cdots.$$

An important property of this number, which is necessary in finding the derivative of a logarithm, is that

$$\lim_{h \to 0} (1 + h)^{\frac{1}{h}} = e.$$

To check this arithmetically we may take successive small values of $h$ and make the following computation:

When $h=.1$, $\quad (1+h)^{\frac{1}{h}}=(1.1)^{10}=2.59374.$

When $h=.01$, $\quad (1+h)^{\frac{1}{h}}=(1.01)^{100}=2.70481.$

When $h=.001$, $\quad (1+h)^{\frac{1}{h}}=(1.001)^{1000}=2.71692.$

When $h=.0001$, $\quad (1+h)^{\frac{1}{h}}=(1.0001)^{10000}=2.71815.$

Working algebraically, we expand $(1+h)^{\frac{1}{h}}$ by the binomial theorem, obtaining

$$(1+h)^{\frac{1}{h}}=1+\frac{1}{h}h+\frac{\frac{1}{h}\left(\frac{1}{h}-1\right)}{2\,!}h^2+\frac{\frac{1}{h}\left(\frac{1}{h}-1\right)\left(\frac{1}{h}-2\right)}{3\,!}h^3+\cdots$$

$$=1+\frac{1}{1}+\frac{(1-h)}{2\,!}+\frac{(1-h)(1-2h)}{3\,!}+\cdots$$

$$=1+\frac{1}{1}+\frac{1}{2\,!}+\frac{1}{3\,!}+\cdots+R,$$

where $R$ represents the sum of all terms involving $h$, $h^2$, $h^3$, etc. Now it may be shown by advanced methods that as $h$ approaches zero, $R$ also approaches zero; so that

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}}=1+\frac{1}{1}+\frac{1}{2\,!}+\frac{1}{3\,!}+\cdots=e.$$

When the number $e$ is used as the base of a system of logarithms, the logarithms are called *natural logarithms*, or *Napierian logarithms*. We shall denote a natural logarithm by the symbol ln*; thus,

$$\text{if} \qquad N=e^L, \qquad\qquad (4)$$
$$\text{then} \qquad L=\ln N.$$

Tables of natural logarithms exist, and should be used if possible. In case such a table is not available, the student

* This notation is generally used by engineers. The student should know that the abbreviation "log" is used by many authors to denote the natural logarithm. In this book "log" is used for the logarithm to the base 10.

may find the natural logarithm by use of a table of common logarithms, as follows:

Let it be required to find ln 213.

If $$x = \ln 213,$$

then, by (4), $$213 = e^x;$$

whence, by (3), $\log 213 = x \log e$,

or $$x = \frac{\log 213}{\log 2.7183} = \frac{2.3284}{0.4343} = 5.361.$$

Certain graphs involving the number $e$ are important and are shown in the examples.

**Ex. 1.** $y = \ln x$.

Giving $x$ positive values and finding $y$, we obtain Fig 71.

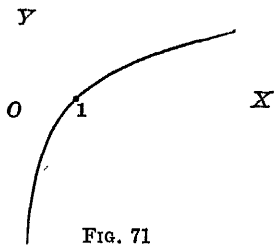



FIG. 71

**Ex. 2.** $y = e^{-x^2}$

The curve (Fig 72) is symmetrical with respect to $OY$ and is always above $OX$. When $x = 0$, $y = 1$ As $x$ increases numerically, $y$ decreases, approaching zero. Hence $OX$ is an asymptote

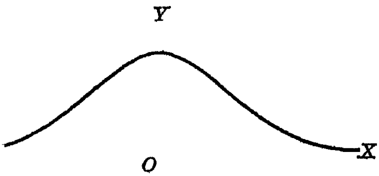



FIG. 72

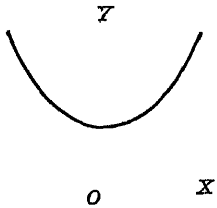



FIG. 73

**Ex. 3.** $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$

This is the curve (Fig. 73) made by a cord or a chain held at the ends and allowed to hang freely. It is called the *catenary*.

**Ex. 4.** $y = e^{-ax} \sin bx$.

The values of $y$ may be computed by multiplying the ordinates of the curve $y = e^{-ax}$ by the values of $\sin bx$ for the corresponding abscissas. Since the value of $\sin bx$ oscillates between 1 and $-1$, the values of $e^{-ax} \sin bx$

cannot exceed those of $e^{-ax}$ Hence the graph lies in the portion of the plane between the curves $y = e^{-ax}$ and $y = -e^{-ax}$. When $x$ is a multiple of $\frac{\pi}{b}$, $y$ is zero. The graph therefore crosses the axis of $x$ an infinite number of times Fig. 74 shows the graph when $a = 1, b = 2\pi$

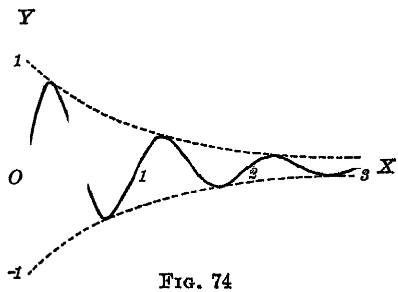



FIG. 74

**Ex. 5.** $y = e^{\frac{1}{x}}$

When $x$ approaches zero, being positive, $y$ increases without limit When $x$ approaches zero, being negative, $y$ approaches zero, for example, when $x = \frac{1}{1000}$, $y = e^{1000}$, and when $x = -\frac{1}{1000}$, $y = e^{-1000} = \frac{1}{e^{1000}}$. The function is therefore discontinuous for $x = 0$

The line $y = 1$ is an asymptote (Fig. 75), for as $x$ increases without limit, being positive or negative, $\frac{1}{x}$ approaches 0, and $y$ approaches 1.

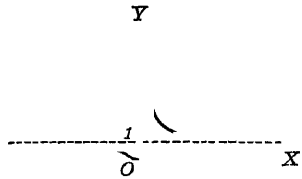



FIG 75

**Ex. 6.** $r = e^{a\theta}$.

The use of $r$ and $\theta$ indicates that we are using polar coordinates.

When $\theta = 0$, $r = 1$. As $\theta$ increases, $r$ increases, and the curve winds around the origin at increasing distances from it (Fig 76). When $\theta$ is negative and increasing numerically without limit, $r$ approaches zero. Hence the curve winds an infinite number of times around the origin, continually approaching it The dotted line in the figure corresponds to negative values of $\theta$

The curve is called the *logarithmic spiral*

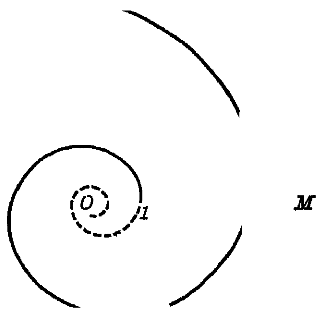



FIG. 76

## EXERCISES

Plot the graphs of the following equations:

1. $y = (\frac{1}{2})^x$.
2. $y = (\frac{1}{2})^{\frac{1}{x}}$
3. $y = e^{x+1}$.
4. $y = e^{-\frac{1}{x}}$.
5. $y = xe^x$.
6. $y = \frac{1}{2}(e^x - e^{-x})$.
7. $y = \log 2x$.
8. $y = \log \frac{1}{x}$.
9. $y = \log \sin x$.
10. $y = \log \tan x$.
11. $y = e^{-2x} \sin 4x$.
12. $y = e^{-x} \cos 3x$.
13. $r = e^{-2\theta}$.

**55. Certain empirical equations.** If $x$ and $y$ are two related quantities which are connected by a given equation, we may plot the corresponding curve on a system of $xy$-coördinates, and every point of this curve determines corresponding values of $x$ and $y$.

Conversely, let $x$ and $y$ be two related quantities of which some corresponding pairs of values have been determined, and let it be desired to find by means of these data an equation connecting $x$ and $y$ in general. On this basis alone the problem cannot be solved exactly. The best we can do is to assume that the desired equation is of a certain form and then endeavor to adjust the constants in the equation in such a way that it fits the data as nearly as possible. We may proceed as follows:

Plot the points corresponding to the known values of $x$ and $y$. The simplest case is that in which the plotted points appear to lie on a straight line or nearly so. In that case it is assumed that the required relation may be put in the form

$$y = mx + b, \tag{1}$$

where $m$ and $b$ are constants to be determined to fit the data. The next step is to draw a straight line so that the plotted points either lie on it or are close to it and about evenly distributed on both sides of it. The equation of this line may be found by means of two points on it, which may be either two points determined by the original data or any other two points on the line.

The resulting equation is called an *empirical equation* and expresses approximately the general relation between $x$ and $y$. In fact, more than one such equation may be derived from the same data, and the choice of the best equation depends on the judgment and experience of the worker.

**Ex. 1.** Corresponding values of two related quantities $x$ and $y$ are given by the following table:

| $x$ | 1 | 2 | 4 | 6 | 10 |
|---|---|---|---|---|---|
| $y$ | 1.3 | 2.2 | 2.9 | 3.9 | 6.1 |

Find the empirical equation connecting them.

We plot the points $(x, y)$ and draw the straight line, as shown in Fig. 77. The straight line is seen to pass through the points $(0, 1)$ and $(2, 2)$. Its equation is therefore

$$y = 5x + 1,$$

which is the required equation

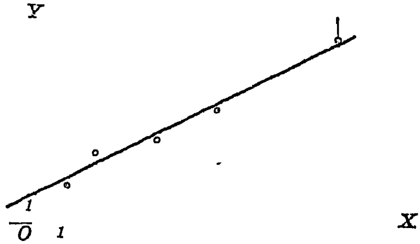



FIG 77

In many cases, however, the plotted points will not appear to lie on or near a straight line. We shall consider here only two of these cases, which are closely connected with the case just considered. They are the cases in which it may be anticipated from previous experience that the required relation is either of the form

$$y = ab^x, \tag{2}$$

where $a$ and $b$ are constants, or of the form

$$y = ax^n, \tag{3}$$

where $a$ and $n$ are constants.

Both of these cases may be brought directly under the first case by taking the logarithm of the equation as written. Equation (2) then becomes

$$\log y = \log a + x \log b. \tag{4}$$

As $\log a$ and $\log b$ are constants, if we denote $\log y$ by $y'$, (4) assumes the form (1) in $x$ and $y'$, and we have only to plot the points $(x, y')$ on an $xy'$-system of axes and determine a straight line by means of them. The transformation from (4) back to (2) is easy, as shown in Ex. 2.

Taking the logarithm of (3), we have

$$\log y = \log a + n \log x. \tag{5}$$

If we denote $\log y$ by $y'$ and $\log x$ by $x'$, (5) assumes the form (1) in $x'$ and $y'$, since $\log a$ and $n$ are constants. Accordingly we plot the points $(x', y')$ on an $x'y'$-system of axes, determine the corresponding straight line, and then transform back to (3), as shown in Ex. 3.

**Ex. 2.** Corresponding values of two related quantities $x$ and $y$ are given by the following table.

| $x$ | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|
| $y$ | 3.2 | 4.6 | 7.3 | 9.8 | 15.2 | 24.6 | 36.4 |

Find an empirical equation of the form $y = ab^x$

Taking the logarithm of the equation $y = ab^x$, and denoting $\log y$ by $y'$, we have

$$y' = \log a + x \log b.$$

Determining the logarithm of each of the given values of $y$, we form a table of corresponding values of $x$ and $y'$, as follows:

| $x$ | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|
| $y' = \log y$ | .5051 | .6628 | .8633 | .9912 | 1.1818 | 1.3909 | 1.5611 |

We choose a large-scale plotting-paper, assume on the $y'$-axis a scale four times as large as that on the $x$-axis, plot the points $(x, y')$, and draw the straight line (Fig 78) through the first and the sixth point. Its equation is

$$y' = .08858\,x - .20354.$$

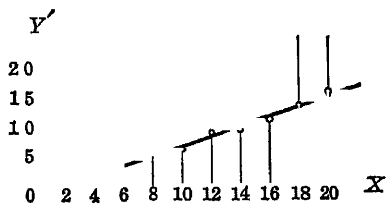



FIG 78

Therefore $\log a = -.20354 = 9.7965 - 10$, whence $a = .626$; and $\log b = .08858$, whence $b = 1.22$. Substituting these values in the assumed equation, we have

$$y = .626\,(1.22)^x$$

as the required empirical equation. The result may be tested by substituting the given values of $x$ in the equation. The computed values of $y$ will be found to agree fairly well with the given values.

**Ex. 3.** Corresponding values of pressure and volume taken from an indicator card of an air-compressor are as follows:

| $p$ | 18 | 21 | 26.5 | 33.5 | 44 | 62 |
|---|---|---|---|---|---|---|
| $v$ | .635 | .556 | .475 | .397 | .321 | .243 |

Find the relation between them in the form $pv^n = c$.

Writing the assumed relation in the form $p = cv^{-n}$ and taking the logarithms of both sides of the equation, we have

$$\log p = -n \log v + \log c,$$

or
$$y = -nx + b,$$

where
$$y = \log p,\ x = \log v, \text{ and } b = \log c.$$

The corresponding values of $x$ and $y$ are

| | | | | | | |
|---|---|---|---|---|---|---|
| $x = \log v$ | – 1972 | – 2549 | –.3233 | – 4012 | –.4935 | – 6144 |
| $y = \log p$ | 1 2553 | 1 3222 | 1 4232 | 1 5250 | 1 6435 | 1 7924 |

For convenience we assume on the $x$-axis a scale twice as large as that on the $y$-axis, plot the points $(x, y)$, and draw the straight line as shown in Fig 79 The construction should be made on large-scale plotting-paper. The line is seen to pass through the points $(-05, 1\ 075)$ and $(-.46, 1.6)$ Its equation is therefore

$$y = -1\ 28\, x + 1\ 01$$

Hence $n = 1\ 28$, $\log c = 1\ 01$, $c = 10\ 2$, and the required relation between $p$ and $v$ is

$$pv^{1\ 28} = 10\ 2$$

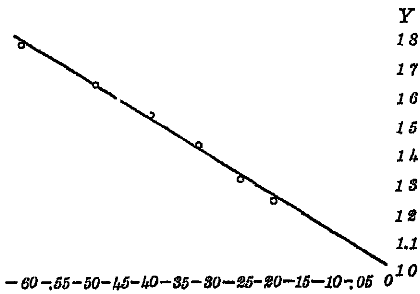



FIG. 79

EXERCISES

1. Show that the following points lie approximately on a straight line, and find its equation:

| $x$ | 4 | 9 | 13 | 20 | 22 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| $y$ | 2 1 | 4 6 | 7 | 12 | 12 9 | 14.5 | 18.2 |

2. For a galvanometer the deflection $D$, measured in millimeters on a proper scale, and the current $I$, measured in microamperes, are determined in a series of readings as follows:

| $D$ | 29 1 | 48.2 | 72.7 | 92.0 | 118.0 | 140.0 | 165.0 | 199.0 |
|---|---|---|---|---|---|---|---|---|
| $I$ | 0.0493 | 0.0821 | 0 123 | 0.154 | 0 197 | 0 234 | 0.274 | 0 328 |

Find an empirical law connecting $D$ and $I$.

3. Corresponding values of two related quantities $x$ and $y$ are given in the following table:

| $x$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 1.1 | 1.3 | 1.5 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 0.3316 | 0.4050 | 0.4946 | 0.6041 | 0.7379 | 0.9013 | 1.1008 | 1.3445 |

Find an empirical equation connecting $x$ and $y$ in the form $y = ab^x$.

4. In a certain chemical reaction the concentration $c$ of sodium acetate produced at the end of the stated number of minutes $t$ is as follows:

| $t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $c$ | 0.00837 | 0.00700 | 0.00586 | 0.00492 | 0.00410 |

Find an empirical equation connecting $c$ and $t$ in the form $c = ab^t$

5. The deflection $a$ of a loaded beam with a constant load is found for various lengths $l$ as follows

| $l$ | 1000 | 900 | 800 | 700 | 600 |
|---|---|---|---|---|---|
| $a$ | 7.14 | 5.22 | 3.64 | 2.42 | 1.50 |

Find an empirical equation connecting $a$ and $l$ in the form $a = nl^m$

6. The relation between the pressure $p$ and the volume $v$ of a gas is found experimentally as follows:

| $p$ | 20 | 23.5 | 31 | 42 | 59 | 78 |
|---|---|---|---|---|---|---|
| $v$ | 0.619 | 0.540 | 0.442 | 0.358 | 0.277 | 0.219 |

Find an empirical equation connecting $p$ and $v$ in the form $pv^n = c$.

**56. Differentiation.** The formulas for the differentiation of the exponential and the logarithmic functions are as follows, where, as usual, $u$ represents any function which can be differentiated with respect to $x$, ln means the Napierian logarithm, and $a$ is any constant:

$$\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx}, \tag{1}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \tag{2}$$

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}, \tag{3}$$

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}. \tag{4}$$

The proofs of these formulas are as follows:

1. By (8), § 36, $\frac{d}{dx}\log_a u = \frac{d}{du}\log_a u \cdot \frac{du}{dx}$.

To find $\frac{d}{du}\log_a u$ place $y = \log_a u$.

Then, if $u$ is given an increment $\Delta u$, $y$ receives an increment $\Delta y$, where

$$\begin{aligned}\Delta y &= \log_a(u + \Delta u) - \log_a u\\ &= \log_a\left(1 + \frac{\Delta u}{u}\right)\\ &= \frac{\Delta u}{u}\log_a\left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}},\end{aligned}$$

the transformations being made by (3), § 54.

Then $$\frac{\Delta y}{\Delta u} = \frac{1}{u}\log_a\left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}.$$

Now, as $\Delta u$ approaches zero the fraction $\frac{\Delta u}{u}$ may be taken as $h$ of § 54.

Hence $$\lim_{\Delta u \to 0}\left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} = e.$$

Therefore $$\frac{dy}{du} = \frac{1}{u}\log_a e$$

and $$\frac{dy}{dx} = \frac{\log_a e}{u}\frac{du}{dx}.$$

2. If $y = \ln u$, the base $a$ of the previous formula is $e$; and since $\log_e e = 1$, we have

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}.$$

3. If $$y = a^u,$$
we have $$\ln y = \ln a^u = u \ln a.$$

Hence, by formula (2),

$$\frac{1}{y}\frac{dy}{dx} = \ln a \frac{du}{dx};$$

whence $$\frac{dy}{dx} = a^u \ln a \frac{du}{dx}.$$

4. If $y = e^u$ the previous formula becomes

$$\frac{dy}{dx} = e^u \frac{du}{dx}.$$

**Ex. 1.** $y = \ln(x^2 - 4x + 5)$.

$$\frac{dy}{dx} = \frac{2x - 4}{x^2 - 4x + 5}.$$

**Ex. 2.** $y = e^{-x^2}$. $\qquad \dfrac{dy}{dx} = -2xe^{-x^2}$.

**Ex. 3.** $y = e^{-ax}\cos bx$.

$$\begin{aligned}\frac{dy}{dx} = \cos bx \frac{d}{dx}(e^{-ax}) + e^{-ax}\frac{d}{dx}(\cos bx) &= -ae^{-ax}\cos bx - be^{-ax}\sin bx \\ &= -e^{-ax}(a\cos bx + b\sin bx).\end{aligned}$$

**EXERCISES**

Find $\dfrac{dy}{dx}$ in each of the following cases:

1. $y = e^{-\frac{1}{x}}$.
2. $y = \frac{1}{2}(e^x + e^{-x})$.
3. $y = a^{x^2-1}$.
4. $y = a^{\sin^{-1}x}$.
5. $y = \ln(x^2 + 4x - 1)$.
6. $y = \ln\sqrt{2x^2 + 6x + 9}$.
7. $y = \frac{1}{6}\ln\dfrac{x-3}{x+3}$.
8. $y = \ln(x + \sqrt{x^2 + 4})$.
9. $y = \ln(3x + \sqrt{9x^2 + 1})$.
10. $y = \ln\dfrac{1 - \sin 2x}{1 + \sin 2x}$.
11. $y = \ln(e^{2x} + e^{-2x})$.
12. $y = e^{-2x}\sin 3x$.
13. $y = \ln\sqrt{1 + x^2} + x\,\text{ctn}^{-1}x$.
14. $y = e^{3x}(9x^2 - 6x + 2)$.
15. $y = e^{2x}(2\sin x - \cos x)$.
16. $y = \sin^{-1}\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.
17. $y = \sec x \tan x + \ln(\sec x + \tan x)$.
18. $y = \ln\dfrac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1}$.

**57. The compound-interest law.** An important use of the exponential function occurs in the problem to determine a function whose rate of change is proportional to the value of the function. If $y$ is such a function of $x$, it must satisfy the equation

$$\frac{dy}{dx} = ky; \tag{1}$$

where $k$ is a constant called the proportionality factor.

We may write equation (1) in the form

$$\frac{1}{y}\frac{dy}{dx} = k;$$

whence, by a very obvious reversal of formula (2), § 56, we have

$$\ln y = kx + C,$$

where $C$ is the constant of integration (§ 18).

From this, by (1) and (2), § 54,

$$y = e^{kx+C} = e^{kx}e^{C}.$$

Finally we place $e^C = A$, where $A$ may be any constant, since $C$ is any constant, and have as a final result

$$y = Ae^{kx}. \tag{2}$$

The constants $A$ and $k$ must be determined by other conditions of a particular problem, as was done in § 18.

The law of change here discussed is often called the *compound-interest law*, because of its occurrence in the following problem:

**Ex.** Let a sum of money $P$ be put at interest at the rate of $r\%$ per annum. The interest gained in a time $\Delta t$ is $P\frac{r}{100}\Delta t$, where $\Delta t$ is expressed in years. But the interest is an increment of the principal $P$, so that we have

$$\Delta P = P\frac{r}{100}\Delta t.$$

In ordinary compound interest the interest is computed for a certain interval (usually one-half year), the principal remaining constant during that interval The interest at the end of the half year is then added to the principal to make a new principal on which interest is computed for the next half year. The principal $P$ therefore changes abruptly at the end of each half year.

Let us now suppose that the principal changes continuously; that is, that any amount of interest theoretically earned, in no matter how small a time, is immediately added to the principal. The average rate of change of the principal in the period $\Delta t$ is, from § 11,

$$\frac{\Delta P}{\Delta t} = \frac{Pr}{100} \tag{1}$$

To obtain the rate of change we must let $\Delta t$ approach zero in equation (1), and have

$$\frac{dP}{dt} = P\frac{r}{100}.$$

From this, as in the text, we have

$$P = Ae^{\frac{r}{100}t}. \tag{2}$$

To make the problem concrete, suppose the original principal were \$100 and the rate 4%, and we ask what would be the principal at the end of 14 yr. We know that when $t = 0$, $P = 100$. Substituting these values in (2), we have $A = 100$, so that (2) becomes

$$P = 100\, e^{\frac{4}{100}t} = 100\, e^{\frac{t}{25}}$$

Placing now $t = 14$, we have to compute $P = 100\, e^{\frac{14}{25}}$ The value of $P$ may be taken from a table if the student has access to tables of powers of $e$ In case a table of common logarithms is alone available, $P$ may be found by first taking the logarithm of both sides of the last equation. Thus

$$\log P = \log 100 + \tfrac{14}{25} \log e = 2.4053;$$

whence $\qquad P = \$254$, approximately

## EXERCISES

**1.** The rate of change of $y$ with respect to $x$ is always equal to $\frac{1}{2} y$, and when $x = 0$, $y = 5$. Find the law connecting $y$ and $x$.

**2.** The rate of change of $y$ with respect to $x$ is always 0.01 times $y$, and when $x = 10$, $y = 50$. Find the law connecting $y$ and $x$.

**3.** The rate of change of $y$ with respect to $x$ is proportional to $y$. When $x = 0$, $y = 7$, and when $x = 2$, $y = 14$ Find the law connecting $y$ and $x$.

**4.** The sum of \$100 is put at interest at the rate of 5% per annum under the condition that the interest shall be compounded at each instant of time. How much will it amount to in 40 yr.?

5. At a certain date the population of a town is 10,000. Forty years later it is 25,000. If the population increases at a rate which is always proportional to the population at the time, find a general expression for the population at any time $t$.

6. In a chemical reaction the rate of change of concentration of a substance is proportional to the concentration at any time. If the concentration is $\frac{1}{100}$ when $t = 0$, and is $\frac{1}{125}$ when $t = 5$, find the law connecting the concentration and the time.

7. A rotating wheel is slowing down in such a manner that the angular acceleration is proportional to the angular velocity. If the angular velocity at the beginning of the slowing down is 100 revolutions per second, and in 1 min. it is cut down to 50 revolutions per second, how long will it take to reduce the velocity to 25 revolutions per second?

## GENERAL EXERCISES

Plot the graphs of the following equations:

1. $y = (\frac{1}{2})^{-x}$.
2. $y = e^{1-x}$.
3. $y = e^{-\frac{x}{2}}\cos x$.
4. $y = e^{\frac{1}{1-x}}$.
5. $y = \frac{1}{2}(e^{x} + e^{-x})$.
6. $y = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$.
7. $y = xe^{\frac{1}{x}}$.
8. $y = xe^{-x}$.
9. $y = x^2e^{-x}$.

10. For a copper-nickel thermocouple the relation between the temperature $t$ in degrees and the thermoelectric power $p$ in microvolts is given by the following table:

| $t$ | 0 | 50 | 100 | 150 | 200 |
|---|---|---|---|---|---|
| $p$ | 24 | 25 | 26 | 26.9 | 27.5 |

Find an empirical law connecting $t$ and $p$.

11. The safe loads in thousands of pounds for beams of the same cross section but of various lengths in feet are found as follows:

| Length | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|
| Load | 123.6 | 121.5 | 111.8 | 107.2 | 101.8 | 90.4 |

Find an empirical equation connecting the data.

12. In the following table $s$ denotes the distance of a moving body from a fixed point in its path at time $t$

| $t$ | 1 | 2 | 4 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $s$ | 10 | 4 | 0 6400 | 0.1024 | 0 0410 | 0 0164 |

Find an empirical equation connecting $s$ and $t$ in the form $s = ab^t$.

13. In the following table $c$ denotes the chemical concentration of a substance at the time $t$.

| $t$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| $c$ | 0 0069 | 0 0048 | 0.0033 | 0.0023 | 0 0016 |

Find an empirical equation connecting $c$ and $t$ in the form $c = ab^t$.

14. The relation between the length $l$ (in millimeters) and the time $t$ (in seconds) of a swinging pendulum is found as follows:

| $l$ | 63 4 | 80 5 | 90.4 | 101 3 | 107.3 | 140 6 |
|---|---|---|---|---|---|---|
| $t$ | 0.806 | 0 892 | 0.960 | 1 010 | 1 038 | 1.198 |

Find an empirical equation connecting $l$ and $t$ in the form $t = kl^n$.

15. For a dynamometer the relation between the deflection $\theta$, when the unit $\theta = \frac{2\pi}{400}$, and the current $I$, measured in amperes, is as follows:

| $\theta$ | 40 | 80 | 120 | 160 | 201 | 240 | 280 | 320 | 362 |
|---|---|---|---|---|---|---|---|---|---|
| $I$ | 0.147 | 0 215 | 0.252 | 0 293 | 0.329 | 0 360 | 0.390 | 0.417 | 0 442 |

Find an empirical equation connecting $I$ and $\theta$ in the form $I = k\theta^n$

16. In a chemical experiment the relation between the concentration $y$ of undissociated hydrochloric acid and the concentration $x$ of hydrogen ions is shown in the table·

| $x$ | 1.68 | 1.22 | 0 784 | 0 426 | 0.092 | 0 047 | 0 0096 | 0.0049 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 1 82 | 0 676 | 0 216 | 0 074 | 0 0085 | 0.00315 | 0.00036 | 0 00014 |

Find an empirical equation connecting the two quantities in the form $y = kx^n$.

**17.** Assuming Boyle's law, $pv = c$, determine $c$ graphically from the following pairs of observed values:

| $p$ | 39.92 | 42.17 | 45.80 | 48.52 | 51.89 | 60.47 | 65.97 |
|---|---|---|---|---|---|---|---|
| $v$ | 40.37 | 38.32 | 35.32 | 33.29 | 31.22 | 26.80 | 24.58 |

Find $\dfrac{dy}{dx}$ in each of the following cases:

**18.** $y = \frac{1}{12} \ln \dfrac{3x - 2}{3x + 2}$.

**19.** $y = \ln \sin x$.

**20.** $y = \tan^{-1} \dfrac{e^x - e^{-x}}{2}$.

**21.** $y = \ln (2x + \sqrt{4x^2 - 1}) + 2x \csc^{-1} 2x$.

**22.** $y = x^2 e^{-\frac{2}{x}}$.

**23.** $y = \ln \sqrt{\dfrac{1 + e^{2x}}{1 - e^{2x}}} - e^{-2x}$.

**24.** $y = \frac{1}{2} \tan^2 ax + \ln \cos ax$.

**25.** $y = x \tan^{-1} x - \frac{1}{2} \ln (1 + x^2)$.

**26.** A substance of amount $x$ is being decomposed at a rate which is proportional to $x$. If $x = 3.12$ when $t = 0$, and $x = 1.36$ when $t = 40$ min., find the value of $x$ when $t = 1$ hr.

**27.** A substance is being transformed into another at a rate which is proportional to the amount of the substance still untransformed. If the amount is 50 when $t = 0$, and 15.6 when $t = 4$ hr., find how long it will be before $\frac{1}{100}$ of the original substance will remain.

**28.** According to Newton's law the rate at which the temperature of a body cools in air is proportional to the difference between the temperature of the body and that of the air. If the temperature of the air is kept at 60°, and the body cools from 130° to 120° in 300 sec., when will its temperature be 100°?

**29.** Assuming that the rate of change of atmospheric pressure $p$ at a distance $h$ above the surface of the earth is proportional to the pressure, and that the pressure at sea level is 14.7 lb. per square inch and at a distance of 1600 ft. above sea level is 13.8 lb. per square inch, find the law connecting $p$ and $h$.

30. Prove that the curve $y = e^{-2x} \sin 3x$ is tangent to the curve $y = e^{-2x}$ at any point common to the two curves.

31. At any time $t$ the coordinates of a point moving in a plane are $x = e^{-2t} \cos 2t$, $y = e^{-2t} \sin 2t$. Find the velocity of the point at any time $t$. Find the rate at which the distance of the point from the origin is decreasing. Prove that the path of the point is a logarithmic spiral.

32. Show that the logarithmic spiral $r = e^{a\theta}$ cuts all radius vectors at a constant angle.

33. Find the radius of curvature of the curve $y = e^{-2x} \sin 2x$ at the point for which $x = \frac{\pi}{2}$.

34. Show that the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ and the parabola $y = a + \frac{1}{2a}x^2$ have the same slope and the same curvature at their common point.

35. Find the radius of curvature of the curve $x = e^t \sin t$, $y = e^t \cos t$.

36. Show that the product of the radii of curvature of the curve $y = ae^{-\frac{x}{a}}$ at the two points for which $x = \pm a$ is $a^2(e + e^{-1})^3$.

37. Find the radius of curvature of the curve $y = \ln x$ and its least value.

38. Find the radius of curvature of the curve $y = e^x \cos x$ at the point for which $x = \frac{\pi}{2}$.

# CHAPTER VII

## SERIES

**58. Power series.** The expression

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots, \tag{1}$$

where $a_0$, $a_1$, $a_2$, $\cdots$ are constants, is called a *power series* in $x$. The terms of the series may be unlimited in number, in which case we have an infinite series, or the series may terminate after a finite number of terms, in which case it reduces to a polynomial.

If the series (1) is an infinite series, it is said to *converge* for a definite value of $x$ when the sum of the first $n$ terms approaches a limit as $n$ increases indefinitely.

Infinite series may arise through the use of elementary operations. Thus, if we divide 1 by $1-x$ in the ordinary manner, we obtain the quotient

$$1 + x + x^2 + x^3 + \cdots,$$

and we may write

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots. \tag{2}$$

Similarly, if we extract the square root of $1+x$ by the rule taught in elementary algebra, arranging the work as follows:

$$\begin{array}{l|l}
 & 1 + x \quad \big|\; 1 + \frac{x}{2} - \frac{x^2}{8} + \\
 & \underline{1\qquad\;\;} \\
2 + \frac{x}{2} & \qquad x \\
 & \qquad x + \frac{x^2}{4} \\
2 + x - \frac{x^2}{8} & \qquad - \frac{x^2}{4} \\
 & \qquad - \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64},
\end{array}$$

the operation may be continued indefinitely. We may write

$$\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\cdots. \qquad (3)$$

The results (2) and (3) are useful only for values of $x$ for which the series in each case converges. When that happens the more terms we take of the series, the more nearly is their sum equal to the function on the left of the equation, and in that sense the function is equal to the series. For example, the series (2) is a geometric progression which is known to converge when $x$ is a positive or negative number numerically less than 1. If we place $x=\frac{1}{3}$ in (2), we have

$$\tfrac{3}{2}=1+\tfrac{1}{3}+\tfrac{1}{9}+\tfrac{1}{27}+\tfrac{1}{81}+\cdots,$$

which is true in the sense that the limit of the sum of the terms on the right is $\frac{3}{2}$. If, however, we place $x=3$ in (2), we have

$$-\tfrac{1}{2}=1+3+9+27+\cdots,$$

which is false. A reason for this difference may be seen by considering the remainder in the division which produced (2) but which is neglected in writing the series. This remainder is $\dfrac{x^n}{1-x}$ after $n$ terms of the quotient have been obtained; and if $x$ is numerically less than 1, the remainder becomes smaller and smaller as $n$ increases, while if $x$ is numerically greater than 1, the remainder becomes larger. Hence in the former case it may be neglected, but not in the latter case.

The calculus offers a general method for finding such series as those obtained by the special methods which led to (2) and (3). This method will be given in the following section.

**59. Maclaurin's series.** We shall assume that a function can usually be expressed by a power series which is valid for appropriate values of $x$, and that the derivative of the function may be found by differentiating the series term by term. The proof of these assumptions lies outside the scope of this book. Let us proceed to find the expansion of $\sin x$ into a series. We begin by writing

$$\sin x=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\cdots, \qquad (1)$$

where $A$, $B$, $C$, etc. are coefficients to be determined.

By differentiating (1) successively, we have

$$\cos x = B + 2\,Cx + 3\,Dx^2 + 4\,Ex^3 + 5\,Fx^4 + \cdots,$$

$$-\sin x = 2\,C + 3\cdot 2\cdot Dx + 4\cdot 3\cdot Ex^2 + 5\cdot 4\cdot Fx^3 + \cdots,$$

$$-\cos x = 3\cdot 2\cdot D + 4\cdot 3\cdot 2\cdot Ex + 5\cdot 4\cdot 3Fx^2 + \cdots,$$

$$\sin x = 4\cdot 3\cdot 2\cdot E + 5\cdot 4\cdot 3\cdot 2\cdot Fx + \cdots,$$

$$\cos x = 5\cdot 4\cdot 3\cdot 2F + \cdots.$$

By substituting $x = 0$ in equation (1) and each of the following equations, we get

$A=0,\ B=1,\ C=0,\ 3\cdot 2\cdot D=-1,\ E=0,\ 5\cdot 4\cdot 3\cdot 2\cdot F=1;$

whence $A=0,\quad B=1,\quad C=0,\quad D=-\frac{1}{3\,!},\quad E=0,\quad F=\frac{1}{5\,!}.$

Substituting these values in (1), we have

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots, \tag{2}$$

and the law of the following terms is evident.

The above method may obviously be used for any function which may be expanded into a series. We may also obtain a general formula by repeating the above operations for a general function $f(x)$.

We place $f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$ (3)

and, by differentiation, obtain in succession

$$f'(x) = B + 2\,Cx + 3\,Dx^2 + 4\,Ex^3 + \cdots,$$

$$f''(x) = 2\,!\ \ C + 3\cdot 2\,Dx + 4\cdot 3\,Ex^2 + \cdots,$$

$$f'''(x) = 3\,!\ \ D + 4\cdot 3\cdot 2\cdot Ex + \cdots,$$

$$f^{\text{iv}}(x) = 4\,!\ \ E + \cdots,$$

where $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{\text{iv}}(x)$ represent the first, second, third, and fourth derivatives of $f(x)$.

We now place $x = 0$ in these equations, indicating the results of that substitution on the left of the equations by the symbols

$f(0), f'(0), f''(0)$, etc. We thus determine $A, B, C, D, E$, etc., and, substituting in (3), have

$$f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\frac{1}{3!}f'''(0)x^3+\frac{1}{4!}f^{\text{iv}}(0)x^4+\cdots. \quad (4)$$

This is called *Maclaurin's series.*

**Ex. 1.** Find the value of $\sin 10°$ to four decimal places.

We may use series (2), but have to remember (§ 42) that $x$ must be in circular measure. Hence we place $x = \frac{10\,\pi}{180} = .17453$, where we take five significant figures in order to insure accuracy in the fourth significant figure of the result.*

Substituting in (2), we have

$$\sin\frac{\pi}{18} = .17453 - \frac{(.17453)^3}{6} + \cdots$$
$$= .17453 - .00089 = .17364.$$

Hence to four decimal places $\sin 10° = .1736$.

We have used only two terms of the series, since a rough calculation, which may be made with $x = .2$, shows that the third term of the series will not affect the fourth decimal place.

**Ex. 2.** Find the value of $\sin 61°$ to four decimal places.

In radians the angle $61°$ is $\frac{61}{180}\pi = 1.0647$. If this number were substituted in the series (2), a great many terms would have to be taken to include all which affect the first four decimal places. We shall therefore find a series for $\sin\left(\frac{\pi}{3}+x\right)$ and afterwards place $x = \frac{\pi}{180}(=1°)$. We chose the angle $\frac{\pi}{3}(=60°)$ because it is an angle near $61°$ for which we know the sine and cosine. The series may be obtained by the method by which (2) was obtained. For variety we shall use the general formula (4). We have then

$$f(x) = \sin\left(\frac{\pi}{3}+x\right), \quad f(0) = \sin\frac{\pi}{3} = \frac{1}{2}\sqrt{3},$$

$$f'(x) = \cos\left(\frac{\pi}{3}+x\right), \quad f'(0) = \cos\frac{\pi}{3} = \frac{1}{2},$$

$$f''(x) = -\sin\left(\frac{\pi}{3}+x\right), \quad f''(0) = -\sin\frac{\pi}{3} = -\frac{1}{2}\sqrt{3}.$$

* This is not a general rule. In other cases the student may need to carry two or even three more significant figures in the calculation than are needed in the result.

Therefore, substituting in (4), we have

$$\sin\left(\frac{\pi}{3}+x\right)=\frac{1}{2}\sqrt{3}+\frac{1}{2}x-\frac{1}{4}\sqrt{3}\,x^2-\cdots.$$

In this we place $x=\frac{\pi}{180}=.01745$ and perform the arithmetical calculation. We have $\sin 61^\circ=\sin\left(\frac{\pi}{3}+\frac{\pi}{180}\right)=.8746$.

**Ex. 3.** Expand $\ln(1+x)$

The function $\ln x$ is an example of a function which cannot be expanded into a Maclaurin's series, since if we place $f(x)=\ln x$, we find $f(0), f'(0)$, etc to be infinite, and the series (4) cannot be written We can, however, expand $\ln(1+x)$ by series (4) or by using the method employed in obtaining (2). The latter method is more instructive because of an interesting abbreviation of the work We place

$$\ln(1+x)=A+Bx+Cx^2+Dx^3+Ex^4+\cdots.$$

Then, by differentiating,

$$\frac{1}{1+x}=B+2\,Cx+3\,Dx^2+4\,Ex^3+\cdots.$$

But we know, by elementary algebra, that

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots.$$

Hence, by comparing the last two series, we have

$$B=1,\quad C=-\tfrac{1}{2},\quad D=\tfrac{1}{3},\quad E=-\tfrac{1}{4},\quad \text{etc.}$$

By placing $x=0$ in the first series, we find $\ln 1=A$, whence $A=0$. We have, therefore,

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots.$$

## EXERCISES

Expand each of the following functions into a Maclaurin's series:

1. $e^x$. 2. $\cos x$. 3. $\tan x$. 4. $\sin^{-1}x$.

5. $\tan^{-1}x$ 6. $\sin\left(\frac{\pi}{4}+x\right)$. 7. $\ln(2+x)$.

8. Prove the binomial theorem

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{2!}a^{n-2}x^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}x^3+\cdots.$$

9. Compute $\sin 5^\circ$ to four decimal places

10. Compute $\cos 62^\circ$ to four decimal places.

**60. Taylor's series.** In the use of Maclaurin's series, as given in the previous section, it is usually necessary to restrict ourselves to small values of $x$. This is for two reasons. In the first place, the series may not converge for large values of $x$; and in the second place, even if it converges, the number of terms of the series which it is necessary to take to obtain a required degree of accuracy may be inconveniently large. This difficulty may be overcome by an ingenious use of Maclaurin's series as illustrated in Ex. 2 of the previous section. We may, however, obtain another form of series which may be used when Maclaurin's series is inconvenient.

Let $f(x)$ be a given function, and let $a$ be a fixed value of $x$ for which the values of $f(x)$ and its derivatives are known. Let $x$ be a variable, or general, value of $x$ which does not differ much from $a$; that is, let $x-a$ be a small number, positive or negative. We shall then assume that $f(x)$ can be expanded in powers of the binomial $x-a$; that is, we write

$$f(x)=A+B(x-a)+C(x-a)^2+D(x-a)^3+\cdots, \qquad (1)$$

and the problem is to determine the coefficients $A$, $B$, $C$, $\cdots$.

We differentiate equation (1) successively, obtaining

$$f'(x)=B+2C(x-a)+3D(x-a)^2+\cdots,$$
$$f''(x)=2C+3.2D(x-a)+\cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In each of these equations place $x=a$. We have

$$f(a)=A, \quad f'(a)=B, \quad f''(a)=2.1C, \quad \text{etc.};$$

whence $A=f(a), \quad B=f'(a), \quad C=\dfrac{f''(a)}{2!}, \quad D=\dfrac{f'''(a)}{3!}, \quad$ etc.

Substituting in equation (1), we have as the final result

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots. \qquad (2)$$

This is known as *Taylor's series.* Since, as has been said, it is valid for values of $x$ which make $x-a$ a small quantity, the function $f(x)$ is said to be expanded in the neighborhood of $x = a$. It is to be noticed that Taylor's series reduces to Maclaurin's series when $a = 0$. Maclaurin's series is therefore an expansion in the neighborhood of $x = 0$.

**Ex.** Expand $\ln x$ in the neighborhood of $x = 3$.

Here we have to place $a = 3$ in the general formula. The calculation of the coefficients is as follows:

$$\begin{aligned} f(x) &= \ln x, & f(3) &= \ln 3, \\ f'(x) &= \frac{1}{x}, & f'(3) &= \tfrac{1}{3}, \\ f''(x) &= -\frac{1}{x^2}, & f''(3) &= -\tfrac{1}{9}, \\ f'''(x) &= \frac{2}{x^3}, & f''(3) &= \tfrac{2}{27}; \end{aligned}$$

and therefore

$$\ln x = \ln 3 + \tfrac{1}{3}(x-3) - \tfrac{1}{18}(x-3)^2 + \tfrac{1}{81}(x-3)^3 + \cdots.$$

This enables us to calculate the natural logarithm of a number near 3, provided we know the logarithm of 3. For example, let us have given $\ln 3 = 1\ 0986$ and desire $\ln 3\tfrac{1}{2}$. Then $x - 3 = \tfrac{1}{2}$, and the series gives

$$\begin{aligned} \ln 3\tfrac{1}{2} &= 1.0986 + \tfrac{1}{6} - \tfrac{1}{72} + \tfrac{1}{648} - \tfrac{1}{5184} + \cdots \\ &= 1.0986 + 1667 - .0139 + 0015 - .0002 + \cdots \\ &= 1\ 2527. \end{aligned}$$

The last figure cannot be depended upon, since we have used only four decimal places in the calculation.

## EXERCISES

Expand each of the following functions into a Taylor's series, using the value of $a$ given in each case:

1. $\ln x$, $a = 5$.
2. $\dfrac{1}{1+x}$, $a = 2$.
3. $\sin x$, $a = \dfrac{\pi}{4}$.
4. $\cos x$, $a = \dfrac{\pi}{6}$.
5. $e^x$, $a = 3$.
6. $\tan^{-1} x$, $a = 1$.
7. $\sqrt{1+x^2}$, $a = 1$.
8. Compute $\sin 46°$ to four decimal places by Taylor's series.
9. Compute $\cos 32°$ to four decimal places by Taylor's series.
10. Compute $e^{1.1}$ to four decimal places by Taylor's series.

## GENERAL EXERCISES

Expand each of the following functions into series in powers of $x$:

1. $\ln(1-x)$.
2. $\sec x$.
3. $\dfrac{1}{\sqrt{1+x^2}}$.
4. $\ln\dfrac{1+x}{1-x}$.
5. $\cos\left(\dfrac{\pi}{3}+x\right)$.
6. $\sin\left(\dfrac{\pi}{6}+x\right)$.
7. $\dfrac{1}{\sqrt{1-x^2}}$.

8. Verify the expansion of $\tan x$ (Ex 3, § 59) by dividing the series for $\sin x$ by that for $\cos x$.

9. Verify the expansion of $\sec x$ (Ex. 2) by dividing 1 by the series for $\cos x$.

10. Expand $\dfrac{1-x}{1+x}$ by Maclaurin's series and verify by dividing the numerator by the denominator

11. Expand $e^x \cos x$ into a Maclaurin's series, and verify by multiplying the series for $e^x$ by that for $\cos x$.

12. Expand $e^x \sin x$ into a Maclaurin's series, and verify by multiplying the series for $e^x$ by that for $\sin x$.

13. Expand $e^x \ln(1+x)$ into a Maclaurin's series, and verify by multiplying the series for $e^x$ by that for $\ln(1+x)$.

14. Compute $\cos 15°$ to four decimal places

15. Compute $\sin 31°$ to four decimal places

16. Compute $e^{\frac{1}{2}}$ to four decimal places by the series found in Ex. 1, § 59.

17. Using the series for $\ln(1+x)$, compute $\ln\frac{3}{2}$ to five decimal places.

18. Using the series found in Ex. 4, compute $\ln 2$ to five decimal places, and thence, by aid of the result of Ex. 17, find $\ln 3$ to four decimal places.

19. Using the series found in Ex. 4, compute $\ln\frac{5}{4}$ to five decimal places, and thence, by aid of the first result of Ex. 18, find $\ln 5$ to four decimal places.

20. Using the series found in Ex. 4, compute $\ln\frac{7}{5}$ to four decimal places, and thence, by aid of the result of Ex. 18, find $\ln 7$ to three decimal places.

21. Compute the value of $\pi$ to four decimal places, from the expansion of $\sin^{-1}x$ (Ex. 4, § 59) and the relation $\sin^{-1}\dfrac{1}{2}=\dfrac{\pi}{6}$.

22. Compute the value of $\pi$ to four decimal places, from the expansion of $\tan^{-1}x$ (Ex. 5, § 59) and the relation $\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = \frac{\pi}{4}$.

23. Compute $\sqrt[4]{17}$ to four decimal places by the binomial theorem (Ex. 8, § 59), placing $a = 16$, $x = 1$.

24. Compute $\sqrt[3]{26}$ to four decimal places by the binomial theorem (Ex. 8, § 59), placing $a = 27$, $x = -1$.

25. Obtain the integral $\int_0^x \frac{\sin x}{x} dx$ in the form of a series expansion.

26. Obtain the integral $\int_0^x e^{-x^2} dx$ in the form of a series expansion.

27. Obtain the integral $\int_0^x \frac{dx}{1+x}$ in the form of a series expansion.

28. Obtain the integral $\int_0^x \frac{dx}{1-x^2}$ in the form of a series expansion.

# CHAPTER VIII

## PARTIAL DIFFERENTIATION

**61. Partial differentiation.** A quantity is a function of two variables $x$ and $y$ when the values of $x$ and $y$ determine the quantity. Such a function is represented by the symbol $f(x, y)$. For example, the volume $V$ of a right circular cylinder is a function of its radius $r$ and its altitude $h$, and in this case

$$V = f(r, h) = \pi r^2 h.$$

Similarly, we may have a function of three or more variables represented by the symbols $f(x, y, z)$, $f(x, y, z, u)$, etc.

Consider now $f(x, y)$, where $x$ and $y$ are independent variables so that the value of $x$ depends in no way upon the value of $y$ nor does the value of $y$ depend upon that of $x$. Then we may change $x$ without changing $y$, and the change in $x$ causes a change in $f$. The limit of the ratio of these changes is the derivative of $f$ with respect to $x$ when $y$ is constant, and may be represented by the symbol $\left(\dfrac{df}{dx}\right)_y$.

Similarly, the derivative of $f$ with respect to $y$ when $x$ is constant, is represented by the symbol $\left(\dfrac{df}{dy}\right)_x$. These derivatives are called *partial derivatives* of $f$ with respect to $x$ and $y$ respectively. The symbol used indicates by the letter outside the parenthesis the variable held constant in the differentiation. When no ambiguity can arise as to this variable, the partial derivatives are represented by the symbols $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, thus:

$$\frac{\partial f}{\partial x} = \left(\frac{df}{dx}\right)_y = \operatorname*{Lim}_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

$$\frac{\partial f}{\partial y} = \left(\frac{df}{dy}\right)_x = \operatorname*{Lim}_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

So, in general, if we have a function of any number of variables $f(x, y, \cdots, z)$, we may have a partial derivative with respect to each of the variables. These derivatives are expressed by the symbols $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \cdots, \frac{\partial f}{\partial z}$, or sometimes by $f_x(x, y, \cdots, z), f_y(x, y, \cdots, z), \cdots, f_z(x, y, \cdots, z)$.

To compute these derivatives we have to apply the formulas for the derivative of a function of one variable, regarding as constant all the variables except the one with respect to which we differentiate.

**Ex. 1.** Consider a perfect gas obeying the law $v = \frac{ct}{p}$. We may change the temperature while keeping the pressure unchanged. If $\Delta t$ and $\Delta v$ are corresponding increments of $t$ and $v$, then

$$\Delta v = \frac{c(t + \Delta t)}{p} - \frac{ct}{p} = \frac{c\Delta t}{p}$$

and

$$\frac{\partial v}{\partial t} = \frac{c}{p}.$$

Or we may change the pressure while keeping the temperature unchanged If $\Delta p$ and $\Delta v$ are corresponding increments of $p$ and $v$, then

$$\Delta v = \frac{ct}{p + \Delta p} - \frac{ct}{p} = -\frac{ct\Delta p}{p^2 + p\Delta p}$$

and

$$\frac{\partial v}{\partial p} = -\frac{ct}{p^2}.$$

**Ex. 2.** $f = x^3 - 3x^2y + y^3$,

$$\frac{\partial f}{\partial x} = 3x^2 - 6xy,$$

$$\frac{\partial f}{\partial y} = -3x^2 + 3y^2.$$

**Ex. 3.** $f = \sin(x^2 + y^2)$,

$$\frac{\partial f}{\partial x} = 2x\cos(x^2 + y^2),$$

$$\frac{\partial f}{\partial y} = 2y\cos(x^2 + y^2).$$

**Ex. 4.** In differentiating in this way care must be taken to have the functions expressed in terms of the independent variables. Let

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Then

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial y}{\partial r} = \sin\theta,$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \qquad \frac{\partial y}{\partial \theta} = r\cos\theta, \tag{1}$$

where $r$ and $\theta$ are the independent variables.

Moreover, since $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$,

$$\left.\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta, &\quad \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \sin\theta, \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}, &\quad \frac{\partial \theta}{\partial y} &= \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}, \end{aligned}\right\} \qquad (2)$$

where $x$ and $y$ are the independent variables.

It is to be emphasized that $\dfrac{\partial x}{\partial r}$ in (1) is not the reciprocal of $\dfrac{\partial r}{\partial x}$ in (2).

In fact, in (1), $\dfrac{\partial x}{\partial r} = \left(\dfrac{dx}{dr}\right)_\theta$ and, in (2), $\dfrac{\partial r}{\partial x} = \left(\dfrac{dr}{dx}\right)_y$, and because the variable held constant is different in the two cases, there is no reason that one should be the reciprocal of the other. It happens in this case that the two are equal, but this is not a general rule. Graphically (Fig. 80), if $OP = r$ is increased by $PQ = \Delta r$, while $\theta$ is constant, then $PR = \Delta x$ is determined Then $\dfrac{\partial x}{\partial r} = \left(\dfrac{dx}{dr}\right)_\theta = \text{Lim}\,\dfrac{PR}{PQ} = \cos\theta$.

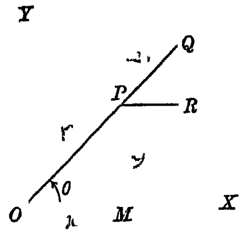


FIG. 80

Moreover (Fig. 81), if $OM = x$ is increased by $MN = PQ = \Delta x$, while $y$ is constant, then $RQ = \Delta r$ is determined Then $\dfrac{\partial r}{\partial x} = \left(\dfrac{dr}{dx}\right)_y = \text{Lim}\,\dfrac{RQ}{PQ} = \cos\theta$. It happens here that $\dfrac{\partial x}{\partial r} = \dfrac{\partial r}{\partial x}$. But $\dfrac{\partial x}{\partial \theta}$, in (1), and $\dfrac{\partial \theta}{\partial x}$, in (2), are neither equal nor reciprocal.

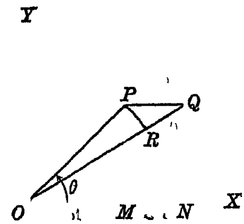


FIG. 81

## EXERCISES

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in each of the following cases.

1. $z = x^3 - 4x^2y + 8xy^2 + 5y^3$.
2. $z = \dfrac{xy}{x^2 + y^2}$.
3. $z = \text{ctn}^{-1}\dfrac{y}{x}$.
4. $z = \sin^{-1} xy$.
5. $z = \ln\dfrac{xy}{x - y}$.
6. $z = \sin\dfrac{2xy}{x + y}$.
7. $z = e^{\frac{x}{y}}$.
8. $z = \ln(x + \sqrt{x^2 + y^2})$
9. If $z = \ln(x^2 - 2xy + y^2 + 3x - 3y)$, prove $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$
10. If $z = \sqrt{x^2 + y^2}\,e^{\frac{y}{x}}$, prove $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

**62. Higher partial derivatives.** The partial derivatives of $f(x, y)$ are themselves functions of $x$ and $y$ which may have partial derivatives, called the *second partial derivatives* of $f(x, y)$. They are $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$. But it may be shown that the order of differentiation with respect to $x$ and $y$ is immaterial when the functions and their derivative fulfill the ordinary conditions as to continuity, so that the second partial derivatives are three in number, expressed by the symbols

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x\,\partial y}=f_{xy},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}.$$

Similarly, the *third partial derivatives of* $f(x, y)$ are four in number; namely,

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial x^3},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^3 f}{\partial x^2\,\partial y},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)=\frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^3 f}{\partial x\,\partial y^2},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial^3 f}{\partial y^3}.$$

So, in general, $\frac{\partial^{p+q} f}{\partial x^p\,\partial y^q}$ signifies the result of differentiating $f(x, y)$ $p$ times with respect to $x$, and $q$ times with respect to $y$, the order of differentiating being immaterial.

In like manner, $\frac{\partial^{p+q+r} f}{\partial x^p \partial y^q \partial z^r}$ signifies the result of differentiating $f(x, y, z)$ $p$ times with respect to $x$, $q$ times with respect to $y$, and $r$ times with respect to $z$, in any order.

**EXERCISES**

1. If $z = (x^2 + y^2)\tan^{-1}\dfrac{y}{x}$, find $\dfrac{\partial^2 z}{\partial x \partial y}$.

2. If $z = e^y \sin(x - y)$, find $\dfrac{\partial^2 z}{\partial x^2}$

3. If $z = \ln(x^2 + y^2)$, find $\dfrac{\partial^2 z}{\partial y^2}$.

Verify $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$ in each of the following cases:

4. $z = xy^2 + 2ye^{\frac{1}{x}}$.

5. $z = \dfrac{x + y}{x - y}$

6. $z = \sin^{-1}\dfrac{x}{y}$.

7. $z = \dfrac{x}{\sqrt{x^2 + y^2}}$.

8. If $z = \tan^{-1}\dfrac{y}{x}$, prove $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

9. If $z = \ln(x^2 - a^2 y^2)$, prove $a^2\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0$.

10. If $V = r^m \cos n\phi$, prove $n^2 r\dfrac{\partial^2 (rV)}{\partial r^2} + m(m+1)\dfrac{\partial^2 V}{\partial \phi^2} = 0$.

**63. Total differential of a function of two variables.** In § 20 the differential of a function of a single variable, $y = f(x)$, is defined by the equation

$$dy = f'(x)\,dx, \tag{1}$$

where $f'(x)$ is the derivative of $y$.

But

$$f'(x) = \operatorname*{Lim}_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right); \tag{2}$$

and hence, according to the definition of a limit (§ 1),

$$\frac{\Delta y}{\Delta x} = f'(x) + \epsilon, \tag{3}$$

where $\epsilon$ denotes the difference between the variable $\dfrac{\Delta y}{\Delta x}$ and its limit $f'(x)$ and approaches zero as a limit as $\Delta x \to 0$.

Multiplying (3) by $\Delta x$, we have

$$\Delta y = f'(x)\,\Delta x + \epsilon\,\Delta x. \tag{4}$$

But $\Delta x = dx$ and $\Delta y = f(x + \Delta x) - f(x)$, so that (4) may be written in the form

$$f(x + \Delta x) - f(x) = f'(x)\,dx + \epsilon\,dx. \tag{5}$$

In the case of a function of two variables, $f(x, y)$, if $x$ alone is changed, we have, by (5),

$$f(x+\Delta x,\ y)-f(x,\ y)=\frac{\partial f}{\partial x}dx+\epsilon_1 dx, \tag{6}$$

the theory being the same as in the case of a function of one variable, since $y$ is held constant. The term $\frac{\partial f}{\partial x}dx$ may be denoted by the symbol $d_x f$.

Similarly, if $x$ is held constant and $y$ alone is changed, we have

$$f(x,\ y+\Delta y)-f(x,\ y)=\frac{\partial f}{\partial y}dy+\epsilon_2 dy, \tag{7}$$

and $\frac{\partial f}{\partial y}dy$ may be denoted by the symbol $d_y f$.

Finally, let $x$ and $y$ both change. Then

$$\begin{aligned}\Delta f&=f(x+\Delta x,\ y+\Delta y)-f(x,\ y)\\ &=f(x+\Delta x,\ y+\Delta y)-f(x+\Delta x,\ y)+f(x+\Delta x, y)-f(x, y).\end{aligned} \tag{8}$$

Then, by (6),

$$f(x+\Delta x,\ y)-f(x,\ y)=\frac{\partial f}{\partial x}dx+\epsilon_1 dx; \tag{9}$$

and similarly, by (7),

$$f(x+\Delta x,\ y+\Delta y)-f(x+\Delta x,\ y)=\frac{\partial f}{\partial y}dy+\epsilon_1' dy, \tag{10}$$

where $\frac{\partial f}{\partial y}$ is to be computed for the value $(x+\Delta x,\ y)$. But if $\frac{\partial f}{\partial y}$ is a continuous function, as we shall assume it is, its value for $(x+\Delta x,\ y)$ differs from its value for $(x,\ y)$ by an amount which approaches zero as $dx$ approaches zero. Hence we may write, from (8), (9), and (10),

$$\Delta f=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\epsilon_1 dx+\epsilon_2 dy, \tag{11}$$

where both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are computed for $(x,\ y)$.

We now write

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy, \tag{12}$$

so that

$$\Delta f=df+\epsilon_1 dx+\epsilon_2 dy, \tag{13}$$

and $df$ is called *the total differential of the function*, the expressions $d_x f$ and $d_y f$ being called the *partial differentials*.

It is evident, by analogy with the case of a function of a single variable, that a partial differential expresses approximately the change in the function caused by a change in one of the independent variables, and that the total differential expresses approximately the change in the function caused by changes in both the independent variables. It is evident from the definition that

$$df = d_x f + d_y f. \tag{14}$$

**Ex.** The period of a simple pendulum with small oscillations is

$$T = 2\pi\sqrt{\frac{l}{g}},$$

whence

$$g = \frac{4\pi^2 l}{T^2}.$$

Let $l = 100$ cm with a possible error of $\frac{1}{2}$ mm in measuring, and $T = 2$ sec. with a possible error of $\frac{1}{100}$ sec. in measuring. Then $dl = \pm\frac{1}{20}$ and $dT = \pm\frac{1}{100}$.

Moreover,

$$dg = \frac{4\pi^2}{T^2}dl - \frac{8\pi^2 l}{T^3}dT,$$

and we obtain the largest possible error in $g$ by taking $dl$ and $dT$ of opposite signs, say $dl = \frac{1}{20}$, $dT = -\frac{1}{100}$.

Then

$$dg = \frac{\pi^2}{20} + \pi^2 = 1.05\,\pi^2 = 10.36.$$

The ratio of error is

$$\frac{dg}{g} = \frac{dl}{l} - 2\frac{dT}{T} = .0005 + .01 = .0105 = 1.05\%.$$

## EXERCISES

1. Calculate the numerical difference between $\Delta z$ and $dz$ when $z = 4xy - x^2 - y^2$, $x = 2$, $y = 3$, $\Delta x = dx = .01$, and $\Delta y = dy = .001$

2. An angle $\phi$ is determined from the formula $\phi = \tan^{-1}\frac{y}{x}$ by measuring the sides $x$ and $y$ of a right triangle. If $x$ and $y$ are found to be 6 ft. and 8 ft. respectively, with a possible error of one tenth of an inch in measuring each, find approximately the greatest possible error in $\phi$.

3. If $C$ is the strength of an electric current due to an electromotive force $E$ along a circuit of resistance $R$, by Ohm's law

$$C = \frac{E}{R}.$$

If errors of 1 per cent are made in measuring $E$ and $R$, find approximately the greatest possible percentage of error in computing $C$

4. If $F$ denotes the focal length of a combination of two lenses in contact, their thickness being neglected, and $f_1$ and $f_2$ denote the respective focal lengths of the lenses, then

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}.$$

If $f_1$ and $f_2$ are said to be 6 in and 10 in respectively, find approximately the greatest possible error in the computation of $F$ from the above formula if errors of .01 in. in $f_1$ and 0.1 in. in $f_2$ are made

5. The eccentricity $e$ of an ellipse of axes $2a$ and $2b$ $(a > b)$ is given by the formula

$$e=\frac{\sqrt{a^2-b^2}}{a}.$$

The axes of an ellipse are said to be 10 ft. and 6 ft. respectively. Find approximately the greatest possible error in the determination of $e$ if there are possible errors of .1 ft. in $a$ and .01 ft. in $b$.

6. The hypotenuse and one side of a right triangle are respectively 13 in and 5 in. If the hypotenuse is increased by .01 in., and the given side is decreased by 01 in, find approximately the change in the other side, the triangle being kept a right triangle

7. The horizontal range $R$ of a bullet having an initial velocity of $v_0$, fired at an elevation $\alpha$, is given by the formula

$$R=\frac{v_0^2 \sin 2\alpha}{g}.$$

Find approximately the greatest possible error in the computation of $R$ if $v_0=10{,}000$ ft per second with a possible error of 10 ft per second, and $\alpha = 60°$ with a possible error of $1'$ (take $g = 32$).

8. The density $D$ of a body is determined by the formula

$$D=\frac{w}{w-w'},$$

where $w$ is the weight of the body in air and $w'$ the weight in water.

If $w = 244{,}000$ gr. and $w' = 220{,}400$ gr., find approximately the largest possible error in $D$ caused by an error of 5 gr. in $w$ and an error of 10 gr. in $w'$.

**64. Rate of change.** The partial derivative $\frac{\partial f}{\partial x}$ gives the rate of change of $f$ with respect to $x$ when $x$ alone varies, and the partial derivative $\frac{\partial f}{\partial y}$ gives the rate of change of $f$ with respect to $y$ when $y$ alone varies. It is sometimes desirable to find the rate of change of $f$ with respect to some other variable, $t$. Obviously, if this rate is to have any meaning, $x$ and $y$ must be functions of $t$, thus making $f$ also a function of $t$. Now, by § 11, the rate of change of $f$ with respect to $t$ is the derivative $\frac{df}{dt}$. To obtain this derivative we have simply to divide $df$, as given by (12), § 63, by $dt$, obtaining in this way

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{1}$$

The same result may be obtained by dividing $\Delta f$, as given by (11), § 63, by $\Delta t$ and taking the limit as $\Delta t$ approaches zero as a limit.

Ex. 1 If the radius of a right circular cylinder is increasing at the rate of 2 in per second, and the altitude is increasing at the rate of 3 in per second, how fast is the volume increasing when the altitude is 15 in. and the radius 5 in ?

Let $V$ be the volume, $r$ the radius, and $h$ the altitude. Then

$$V=\pi r^2 h.$$

By (1),
$$\frac{dV}{dt}=\frac{\partial V}{\partial r}\frac{dr}{dt}+\frac{\partial V}{\partial h}\frac{dh}{dt}$$
$$=2\pi rh\frac{dr}{dt}+\pi r^2\frac{dh}{dt}.$$

By hypothesis, $\frac{dr}{dt}=2$, $\frac{dh}{dt}=3$, $r=5$, $h=15$. Therefore $\frac{dV}{dt}=375\pi$ cu. in. per second.

The same result may be obtained without partial differentiation by expressing $V$ directly in terms of $t$. For, by hypothesis, $r=5+2t$, $h=15+3t$ if we choose $t=0$ when $r=5$ and $h=15$. Therefore

$$V=(375+375t+120t^2+12t^3)\pi;$$

whence
$$\frac{dV}{dt}=(375+240t+36t^2)\pi.$$

When $t=0$,
$$\frac{dV}{dt}=375\pi \text{ cu. in. per second, as before.}$$

**Ex. 2.** The temperature of a point in a plane is given by the formula

$$u = \frac{1}{x^2 + y^2}$$

The rate of change of the temperature in a direction parallel to $OX$ is, accordingly,

$$\frac{\partial u}{\partial x} = -\frac{2x}{(x^2 + y^2)^2},$$

which gives the limit of the change in the temperature compared with a change in $x$ when $x$ alone varies

Similarly, the rate of change of $u$ in a direction parallel to $OY$ is

$$\frac{\partial u}{\partial y} = -\frac{2y}{(x^2 + y^2)^2}$$

Suppose now we wish to find the rate of change of the temperature in a direction which makes an angle $\alpha$ with $OX$ From Fig. 82, if $P_1(x_1, y_1)$ is a fixed point, and $P(x, y)$ a moving point on the line through $P_1$ making an angle $\alpha$ with $OX$, and $s$ is the distance $P_1P$, we have

$$P_1R = x - x_1 = s\cos\alpha,$$
$$RP = y - y_1 = s\sin\alpha,$$

whence

$$x = x_1 + s\cos\alpha,$$
$$y = y_1 + s\sin\alpha,$$

and

$$\frac{dx}{ds} = \cos\alpha, \quad \frac{dy}{ds} = \sin\alpha.$$

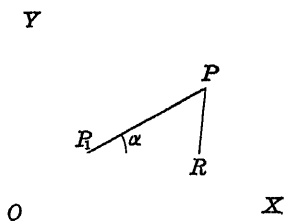


FIG. 82

Replacing $t$ by $s$ in formula (1), and substituting the values of $\frac{dx}{ds}$ and $\frac{dy}{ds}$ which we just found, we have

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha$$
$$= -\frac{2x\cos\alpha + 2y\sin\alpha}{(x^2 + y^2)^2}.$$

Formula (1) has been written on the hypothesis that $x$ and $y$ are functions of $t$ only. If $x$ and $y$ are functions of two variables, $t$ and $s$, and (1) is derived on the assumption that $t$ alone varies, we have simply to use the notation of § 61 to write at once

$$\left(\frac{df}{dt}\right)_s = \frac{\partial f}{\partial x}\left(\frac{dx}{dt}\right)_s + \frac{\partial f}{\partial y}\left(\frac{dy}{dt}\right)_s, \tag{2}$$

which may also be written as

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}. \tag{3}$$

### EXERCISES

1. If $z = e^{\tan^{-1}\frac{y}{x}}$, $x = \sin t$, $y = \cos t$, find the rate of change of $z$ with respect to $t$

2. If $z = \tan^{-1}\dfrac{1+x}{1-y}$, $x = \sin t$, $y = \cos t$, find the rate of change of $z$ with respect to $t$ when $t = \dfrac{\pi}{2}$.

3. If $V = (e^{ax} - e^{-ax})\cos ay$, prove that $V$ and its derivatives in any direction are all equal to zero at the point $\left(0, \dfrac{\pi}{2a}\right)$.

4. If $V = \dfrac{1}{\sqrt{x^2 + y^2}}$, find the rate of change of $V$ at the point (1, 1) in a direction making an angle of 45° with $OX$.

5. If the electric potential $V$ at any point of a plane is given by the formula $V = \ln\sqrt{x^2 + y^2}$, find the rate of change of potential at any point (1) in a direction toward the origin; (2) in a direction at right angles to the direction toward the origin

6. If the electric potential $V$ at any point of the plane is given by the formula $V = \ln\dfrac{\sqrt{(x-a)^2 + y^2}}{\sqrt{(x+a)^2 + y^2}}$, find the rate of change of potential at the point $(0, a)$ in the direction of the axis of $y$, and at the point $(a, a)$ in the direction toward the point $(-a, 0)$.

### GENERAL EXERCISES

1. If $z = \sin\dfrac{xy-1}{xy+1}$, prove $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 0$.

2. If $z = \dfrac{1}{x^2 + y^2}\sin(x^2 + y^2)$, prove $y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$.

3. If $z = y^3 + ye^{\frac{1}{x}}$, prove $x^2\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 3y^3$.

4. If $z = e^{-ay}\cos a(k - x)$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

5. If $z = e^{-(b^2 + a^2k^2)t}\sin kx$, prove that $\dfrac{\partial z}{\partial t} = a^2\dfrac{\partial^2 z}{\partial x^2} - b^2 z$.

6. If $z = e^{-kx}\sin\left(my + x\sqrt{a^2m^2 - k^2}\right)$, prove that $\dfrac{\partial^2 z}{\partial x^2} + 2k\dfrac{\partial z}{\partial x} = a^2\dfrac{\partial^2 z}{\partial y^2}$.

7. If $V = e^{a\phi}\cos(a\ln r)$, prove that $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \phi^2} = 0$.

8. A right circular cylinder has an altitude 8 ft. and a radius 6 ft. Find approximately the change in the volume caused by decreasing the altitude by .1 ft. and the radius by .01 ft.

9. The velocity $v$, with which vibrations travel along a flexible string, is given by the formula

$$v = \sqrt{\frac{t}{m}},$$

where $t$ is the tension of the string and $m$ the mass of a unit length of it. Find approximately the greatest possible error in the computation of $v$ if $t$ is found to be 6,000,000 dynes and $m$ to be .005 gr. per centimeter, the measurement of $t$ being subject to a possible error of 1000 dynes and that of $m$ to a possible error of .0005 gr.

10. The base $AB$ of a triangle is 12 in. long, the side $AC$ is 10 in., and the angle $A$ is 60°. Calculate the change in the area caused by increasing $AC$ by .01 in. and the angle $A$ by 1°. Calculate also the differential of area corresponding to the same increments.

11. The distance between two points $A$ and $B$ on opposite sides of a pond is determined by taking a third point $C$ and measuring $AC = 90$ ft., $BC = 110$ ft., and $BCA = 60°$. Find approximately the greatest possible error in the computed length of $AB$ caused by possible errors of 4 in. in the measurement of both $AC$ and $BC$.

12. The distance of an inaccessible object $A$ from a point $B$ is found by measuring a base line $BC = 100$ ft., the angle $CBA = \alpha = 45°$, and the angle $BCA = \beta = 60°$. Find the greatest possible error in the computed length of $AB$ caused by errors of $1'$ in measuring both $\alpha$ and $\beta$.

13. The equal sides of an isosceles triangle are increasing at the uniform rate of .01 in. per second, and the vertical angle is increasing at the uniform rate of .01 radians per second. How fast is the area of the triangle increasing when the equal sides are each 2 ft. long and the angle at the vertex is 45°?

14. Prove that the rate of change of $z = \ln\left(x + \sqrt{x^2 + y^2}\right)$ in the direction of the line drawn from the origin of coordinates to any point $P(x, y)$ is equal to the reciprocal of the length of $OP$.

15. The altitude of a right circular cone increases at the uniform rate of .1 in. per second, and its radius increases at the uniform rate of .01 in. per second. How fast is the lateral surface of the cone increasing when its altitude is 2 ft. and its radius 1 ft.?

16. Given $z = \tan^{-1}\frac{1-x}{y} + \tan^{-1}\frac{1+x}{y}$. Find the general expression for the derivative of $z$ along the line drawn from the origin of coordinates to any point. Find also the value of this derivative at the point (1, 1).

17. In what direction from the point (3, 4) is the rate of change of the function $z = kxy$ a maximum, and what is the value of that maximum rate?

18. Find a general expression for the rate of change of the function $u = e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x$ at the point $\left(\frac{\pi}{3}, 0\right)$. Find also the maximum value of the rate of change.

# CHAPTER IX

## INTEGRATION

**65. Introduction.** In §§ 18 and 23 the process of *integration* was defined as the determination of a function when its derivative or its differential is known. We denoted the process of integration by the symbol $\int$; that is, if

$$f(x)\,dx = dF(x),$$

then
$$\int f(x)\,dx = F(x) + C,$$

where $C$ is the *constant of integration* (§ 18).

The expression $f(x)\,dx$ is said to be under the sign of integration, and $f(x)$ is called the *integrand*. The expression $F(x)+C$ is called the *indefinite integral* to distinguish it from the *definite integral* defined in § 23.

Since integration appears as the converse of differentiation, it is evident that some formulas of integration may be found by direct reversal of the corresponding formulas of differentiation, possibly with some modifications, and that the correctness of any formula may be verified by differentiation.

In all the formulas which will be derived, the constant $C$ will be omitted, since it is independent of the form of the integrand; but it must be added in all the indefinite integrals found by means of the formulas. However, if the indefinite integral is found in the course of the evaluation of a definite integral, the constant $C$ may be omitted, as it will simply cancel out if it has previously been written in (§ 23).

The two formulas
$$\int c\,du = c\int du \tag{1}$$

and
$$\int (du + dv + dw + \cdots) = \int du + \int dv + \int dw + \cdots \tag{2}$$

are of fundamental importance. Stated in words they are as follows:

(1) *A constant factor may be changed from one side of the sign of integration to the other.*

(2) *The integral of the sum of a finite number of functions is the sum of the integrals of the separate functions.*

To prove (1), we note that since $c\,du = d(cu)$, it follows that

$$\int c\,du = \int d(cu) = cu = c\int du.$$

In like manner, to prove (2), since

$$du + dv + dw + \cdots = d(u + v + w + \cdots),$$

we have

$$\begin{aligned}\int (du + dv + dw + \cdots) &= \int d(u + v + w + \cdots)\\ &= u + v + w + \cdots\\ &= \int du + \int dv + \int dw + \cdots.\end{aligned}$$

The application of these formulas is illustrated in the following articles.

**66. Integral of $u^n$.** Since for all values of $m$ except $m = 0$

$$d(u^m) = mu^{m-1}\,du,$$

or

$$d\left(\frac{u^m}{m}\right) = u^{m-1}\,du,$$

it follows that $$\int u^{m-1}\,du = \frac{u^m}{m}.$$

Placing $m = n + 1$, we have

$$\int u^n\,du = \frac{u^{n+1}}{n+1} \tag{1}$$

*for all values of $n$ except $n = -1$.*

In the case $n = -1$, the expression under the sign of integration in (1) becomes $\dfrac{du}{u}$, which is recognized as $d(\ln u)$.

Therefore $$\int \frac{du}{u} = \ln u. \tag{2}$$

In applying these formulas the problem is to choose for $u$ some function of $x$ which will bring the given integral, if possible, under one of the formulas. The form of the integrand suggests the function of $x$ which should be chosen for $u$.

**Ex. 1.** Find the value of $\int\left(ax^2 + bx + \frac{c}{x} + \frac{k}{x^2}\right)dx$

Applying (2), § 65, and then (1), § 65, we have

$$\int\left(ax^2 + bx + \frac{c}{x} + \frac{k}{x^2}\right)dx = \int ax^2\,dx + \int bx\,dx + \int\frac{c}{x}\,dx + \int\frac{k}{x^2}\,dx$$
$$= a\int x^2\,dx + b\int x\,dx + c\int\frac{dx}{x} + k\int x^{-2}\,dx$$

The first, the second, and the fourth of these integrals may be evaluated by formula (1) and the third by formula (2), where $u = x$, the results being respectively $\frac{1}{3}ax^3$, $\frac{1}{2}bx^2$, $-\frac{k}{x}$, and $c\ln x$.

Therefore $\int\left(ax^2 + bx + \frac{c}{x} + \frac{k}{x^2}\right)dx = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + c\ln x - \frac{k}{x} + C$

**Ex. 2.** Find the value of $\int(x^2 + 2)x\,dx$.

If the factors of the integrand are multiplied together, we have

$$\int(x^2 + 2)x\,dx = \int(x^3 + 2x)\,dx,$$

which may be evaluated by the same method as that used in Ex 1, the result being $\frac{1}{4}x^4 + x^2 + C$.

Or we may let $x^2 + 2 = u$, whence $2x\,dx = du$, so that $x\,dx = \frac{1}{2}du$. Hence

$$\int(x^2 + 2)x\,dx = \int\tfrac{1}{2}u\,du = \tfrac{1}{2}\int u\,du$$
$$= \frac{1}{2}\,\frac{u^2}{2} + C$$
$$= \tfrac{1}{4}(x^2 + 2)^2 + C.$$

Comparing the two values of the integral found by the two methods of integration, we see that they differ only by the constant unity, which may be made a part of the constant of integration.

**Ex. 3.** Find the value of $\int(ax^2 + 2bx)^3(ax + b)\,dx$.

Let $ax^2 + 2bx = u$ Then $(2ax + 2b)\,dx = du$, so that $(ax + b)\,dx = \frac{1}{2}du$.

Hence $\int(ax^2 + 2bx)^3(ax + b)\,dx = \int\frac{1}{2}u^3\,du$

$$= \frac{1}{2}\int u^3\,du = \frac{1}{2}\cdot\frac{u^4}{4} + C$$
$$= \tfrac{1}{8}(ax^2 + 2bx)^4 + C.$$

**Ex. 4.** Find the value of $\int \frac{4(ax+b)\,dx}{ax^2+2bx}$

As in Ex. 3, let $ax^2 + 2bx = u$. Then $(2ax + 2b)\,dx = du$, so that $(ax+b)\,dx = \frac{1}{2}du$.

Hence
$$\int \frac{4(ax+b)\,dx}{ax^2+2bx} = \int \frac{2\,du}{u} = 2\int \frac{du}{u}$$
$$= 2\ln u + C$$
$$= 2\ln(ax^2+2bx) + C$$
$$= \ln(ax^2+2bx)^2 + C.$$

**Ex. 5.** Find the value of $\int (e^{ax}+b)^2 e^{ax}\,dx$.

Let $e^{ax} + b = u$. Then $e^{ax}a\,dx = du$.

Hence
$$\int (e^{ax}+b)^2 e^{ax}\,dx = \int u^2 \frac{du}{a}$$
$$= \frac{1}{a}\int u^2\,du$$
$$= \frac{1}{3a}u^3 + C$$
$$= \frac{1}{3a}(e^{ax}+b)^3 + C.$$

If the integrand is a trigonometric expression it is often possible to carry out the integration by either formula (1) or (2). This may happen when the integrand can be expressed in terms of one of the elementary trigonometric functions, the whole expression being multiplied by the differential of that function. For instance, the expression to be integrated may consist of a function of $\sin x$ multiplied by $\cos x\,dx$, or a function of $\cos x$ multiplied by $(-\sin x\,dx)$, etc.

**Ex. 6.** Find the value of $\int \sqrt{\sin x}\cos^3 x\,dx$.

Since $d(\sin x) = \cos x\,dx$, we will separate out the factor $\cos x\,dx$ and express the rest of the integrand in terms of $\sin x$.

Thus
$$\sqrt{\sin x}\cos^3 x\,dx = \sqrt{\sin x}\,(1-\sin^2 x)(\cos x\,dx).$$

Now place $\sin x = u$, and we have
$$\int \sqrt{\sin x}\cos^3 x\,dx = \int u^{\frac{1}{2}}(1-u^2)\,du$$
$$= \int (u^{\frac{1}{2}} - u^{\frac{5}{2}})\,du$$
$$= \tfrac{2}{3}u^{\frac{3}{2}} - \tfrac{2}{7}u^{\frac{7}{2}} + C$$
$$= \tfrac{2}{21}\sin^{\frac{3}{2}} x\,(7 - 3\sin^2 x) + C.$$

**Ex. 7.** Find the value of $\int \sec^6 2\,x\,dx$.

Since $d(\tan 2\,x) = 2\sec^2 2\,x\,dx$, we separate out the factor $\sec^2 2\,x\,dx$ and express the rest of the integrand in terms of $\tan 2\,x$

Thus
$$\begin{aligned}\sec^6 2\,x\,dx &= \sec^4 2\,x\,(\sec^2 2\,x\,dx)\\ &= (1+\tan^2 2\,x)^2(\sec^2 2\,x\,dx)\\ &= (1+2\tan^2 2\,x+\tan^4 2\,x)(\sec^2 2\,x\,dx).\end{aligned}$$

Now place $\tan 2\,x = u$, and we have
$$\begin{aligned}\int \sec^6 2\,x\,dx &= \tfrac{1}{2}\int (1+2\,u^2+u^4)\,du\\ &= \tfrac{1}{2}(u+\tfrac{2}{3}u^3+\tfrac{1}{5}u^5)+C\\ &= \tfrac{1}{2}\tan 2\,x+\tfrac{1}{3}\tan^3 2\,x+\tfrac{1}{10}\tan^5 2\,x+C.\end{aligned}$$

## EXERCISES

Find the values of the following integrals

1. $\int\left(6\,x^2+4\,x+\frac{6}{x}\right)dx.$
2. $\int\left(\sqrt[4]{x^3}+\frac{1}{\sqrt[4]{x}}\right)dx.$
3. $\int\left(x\sqrt{x}-\frac{1}{x\sqrt{x}}\right)dx.$
4. $\int\frac{(x+1)^2}{x^3}\,dx.$
5. $\int\frac{x^3\,dx}{x^2-1}.$
6. $\int(x^2+1)^3 x\,dx.$
7. $\int\sqrt{x^4+4}\,x^3\,dx.$
8. $\int\frac{e^{3x}\,dx}{e^{3x}+6}.$
9. $\int\frac{1+\cos 2\,x}{2\,x+\sin 2\,x}\,dx.$
10. $\int\frac{1-\cos x}{(x-\sin x)^4}\,dx.$
11. $\int\frac{e^{ax}+\sec^2 ax}{e^{ax}+\tan ax}\,dx.$
12. $\int\frac{x^2+2\,x}{x^3+3\,x^2+1}\,dx.$
13. $\int\frac{\sin ax}{1+\cos ax}\,dx.$
14. $\int\cos^3 2\,x\sin 2\,x\,dx.$
15. $\int\sin^2 3\,x\cos 3\,x\,dx.$
16. $\int\sin(x+2)\cos(x+2)\,dx.$
17. $\int\cos^{\frac{5}{3}} 3\,x\sin 3\,x\,dx.$
18. $\int\sec^4 3\,x\,dx.$
19. $\int\operatorname{ctn}^2(2x+1)\csc^2(2x+1)\,dx.$
20. $\int\sin^3(2\,x-3)\,dx.$

**67. Other algebraic integrands.** From the formulas for the differentiation of $\sin^{-1}u$, $\tan^{-1}u$, and $\sec^{-1}u$, we derive, by reversal, the corresponding formulas of integration:

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u,$$

$$\int \frac{du}{u^2+1} = \tan^{-1}u,$$

and

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}u.$$

These formulas are much more serviceable, however, if $u$ is replaced by $\frac{u}{a}$ $(a > 0)$. Making this substitution and evident reductions, we have as our required formulas

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}, \tag{1}$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}, \tag{2}$$

and

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a}. \tag{3}$$

Referring to 1, § 47, we see that $\sin^{-1}\frac{u}{a}$ must be taken in the first or the fourth quadrant; if, however, it is necessary to have $\sin^{-1}\frac{u}{a}$ in the second or the third quadrant, the minus sign must be prefixed. In like manner, in (3), $\sec^{-1}\frac{u}{a}$ must be taken in the first or the third quadrant or else its sign must be changed.

**Ex. 1** Find the value of $\int \frac{dx}{\sqrt{9-4x^2}}$. Letting $2x = u$, we have $du = 2\,dx$; whence $dx = \frac{1}{2}\,du$, and

$$\begin{aligned}\int \frac{dx}{\sqrt{9-4x^2}} &= \int \frac{\frac{1}{2}\,du}{\sqrt{9-u^2}}\\ &= \frac{1}{2}\int \frac{du}{\sqrt{9-u^2}}\\ &= \frac{1}{2}\sin^{-1}\frac{u}{3} + C\\ &= \frac{1}{2}\sin^{-1}\frac{2x}{3} + C.\end{aligned}$$

**Ex. 2** Find the value of $\int \frac{dx}{x\sqrt{3x^2-4}}$. If we let $\sqrt{3}x = u$, then $du = \sqrt{3}\,dx$; whence $dx = \frac{1}{\sqrt{3}}du$, and

$$\int \frac{dx}{x\sqrt{3x^2-4}} = \int \frac{du}{u\sqrt{u^2-4}}$$

$$= \frac{1}{2}\sec^{-1}\frac{u}{2} + C$$

$$= \frac{1}{2}\sec^{-1}\frac{\sqrt{3}x}{2} + C.$$

**Ex. 3.** Find the value of $\int \frac{dx}{\sqrt{4x-x^2}}$.

Since $\sqrt{4x-x^2} = \sqrt{4-(x-2)^2}$, we may let $u = x-2$; whence $dx = du$, and

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{du}{\sqrt{4-u^2}}$$

$$= \sin^{-1}\frac{u}{2} + C$$

$$= \sin^{-1}\frac{x-2}{2} + C.$$

**Ex. 4.** Find the value of $\int \frac{dx}{2x^2+3x+5}$.

We may first write the integrand in the form

$$\frac{1}{2}\,\frac{1}{x^2+\frac{3}{2}x+\frac{5}{2}} = \frac{1}{2}\cdot\frac{1}{(x+\frac{3}{4})^2+\frac{31}{16}},$$

and let $u = x + \frac{3}{4}$. Then $du = dx$,

and
$$\int \frac{dx}{2x^2+3x+5} = \frac{1}{2}\int \frac{dx}{(x+\frac{3}{4})^2+\frac{31}{16}}$$

$$= \frac{1}{2}\int \frac{du}{u^2+\frac{31}{16}}$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{31}}{4}}\tan^{-1}\frac{u}{\frac{\sqrt{31}}{4}} + C$$

$$= \frac{2}{\sqrt{31}}\tan^{-1}\frac{4u}{\sqrt{31}} + C$$

$$= \frac{2}{\sqrt{31}}\tan^{-1}\frac{4x+3}{\sqrt{31}} + C.$$

**Ex. 5.** Find the value of $\int \frac{5x-2}{2x^2+3}dx$

Separating the integrand into two fractions

$$\frac{5x}{2x^2+3} - \frac{2}{2x^2+3},$$

and using (2), § 65, we have

$$\int \frac{5x-2}{2x^2+3}dx = \int \frac{5x\,dx}{2x^2+3} - \int \frac{2\,dx}{2x^2+3}.$$

If we let $u = 2x^2+3$, then $du = 4x\,dx$

and $$\int \frac{5x\,dx}{2x^2+3} = \frac{5}{4}\int \frac{du}{u} = \frac{5}{4}\ln u = \frac{5}{4}\ln(2x^2+3);$$

and if we let $u = \sqrt{2}\,x$, then $du = \sqrt{2}\,dx$

and $$\int \frac{2\,dx}{2x^2+3} = \sqrt{2}\int \frac{du}{u^2+3} = \sqrt{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} = \frac{\sqrt{6}}{3}\tan^{-1}\frac{x\sqrt{6}}{3}.$$

Therefore $$\int \frac{5x-2}{2x^2+3}dx = \frac{5}{4}\ln(2x^2+3) - \frac{\sqrt{6}}{3}\tan^{-1}\frac{x\sqrt{6}}{3}.$$

**Ex. 6.** Find the value of $\int_{-1}^{\sqrt{3}} \frac{dx}{x^2+1}$.

$$\int_{-1}^{\sqrt{3}} \frac{dx}{x^2+1} = \left[\tan^{-1}x\right]_{-1}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}(-1)$$

There is here a certain ambiguity, since $\tan^{-1}\sqrt{3}$ and $\tan^{-1}(-1)$ have each an infinite number of values. If, however, we remember that the graph of $\tan^{-1}x$ is composed of an infinite number of distinct parts, or *branches* (Fig 56, § 46), the ambiguity is removed by taking the values of $\tan^{-1}\sqrt{3}$ and $\tan^{-1}(-1)$ from the same branch of the graph For if we consider $\int_a^b \frac{dx}{x^2+1} = \tan^{-1}b - \tan^{-1}a$ and select any value of $\tan^{-1}a$, then if $b = a$, $\tan^{-1}b$ must be taken equal to $\tan^{-1}a$, since the value of the integral is then zero. As $b$ varies from equality with $a$ to its final value, $\tan^{-1}b$ will vary from $\tan^{-1}a$ to the nearest value of $\tan^{-1}b$.

The simplest way to choose the proper values of $\tan^{-1}b$ and $\tan^{-1}a$ is to take them both between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Then we have

$$\int_{-1}^{\sqrt{3}} \frac{dx}{x^2+1} = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{7\pi}{12}.$$

The same ambiguity occurs in the determination of a definite integral by (1), but the simplest way to obviate it is to take both values of $\sin^{-1}\frac{u}{a}$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The proof is left to the student.

**EXERCISES**

Find the values of the following integrals:

1. $\int \frac{dx}{\sqrt{16 - 9x^2}}$.
2. $\int \frac{dx}{\sqrt{7 - 3x^2}}$.
3. $\int \frac{dx}{x\sqrt{4x^2 - 9}}$.
4. $\int \frac{dx}{x\sqrt{3x^2 - 1}}$.
5. $\int \frac{dx}{\sqrt{2 + 4x - 3x^2}}$.
6. $\int \frac{dx}{x^2 + 7}$.
7. $\int \frac{dx}{3x^2 + 7}$.
8. $\int \frac{dx}{x^2 + 6x + 13}$.
9. $\int \frac{dx}{\sqrt{6x - x^2}}$.
10. $\int \frac{dx}{\sqrt{6x - 4x^2}}$.
11. $\int \frac{dx}{\sqrt{5x - 3x^2}}$.
12. $\int \frac{dx}{\sqrt{1 - 4x - x^2}}$.
13. $\int \frac{dx}{3x^2 - 4x + 2}$.
14. $\int \frac{3x + 11}{x^2 + 4} dx$.
15. $\int \frac{2x + 5}{\sqrt{4 - x^2}} dx$.
16. $\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{a^2 - x^2}}$.
17. $\int_{-2}^{2} \frac{dx}{x^2 + 4}$.
18. $\int_0^{3} \frac{dx}{\sqrt{3x - x^2}}$.
19. $\int_{-3}^{\sqrt{3}} \frac{dx}{x^2 + 9}$.
20. $\int_{\sqrt{3}}^{3} \frac{dx}{x\sqrt{4x^2 - 9}}$.

**68.** Closely resembling formulas (1) and (2) of the last section in the form of the integrand are the following formulas:

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln(u + \sqrt{u^2 + a^2}), \tag{1}$$

$$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln(u + \sqrt{u^2 - a^2}), \tag{2}$$

and

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \frac{u - a}{u + a}. \tag{3}$$

These formulas can be easily verified by differentiation, and this verification should be made by the student.

**Ex. 1.** Find the value of $\int \frac{dx}{\sqrt{2x^2-3}}$.

Letting $\sqrt{2}\,x = u$, we have $du = \sqrt{2}\,dx$; whence $dx = \frac{1}{\sqrt{2}}\,du$, and

$$\int \frac{dx}{\sqrt{2x^2-3}} = \int \frac{\frac{1}{\sqrt{2}}\,du}{\sqrt{u^2-3}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{du}{\sqrt{u^2-3}}$$

$$= \frac{1}{\sqrt{2}} \ln\left[u + \sqrt{u^2-3}\right] + C$$

$$= \frac{1}{\sqrt{2}} \ln\left[x\sqrt{2} + \sqrt{2x^2-3}\right] + C.$$

**Ex. 2.** Find the value of $\int \frac{dx}{\sqrt{3x^2+4x}}$.

As in Ex. 4, § 67, we may write the integrand in the form

$$\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{x^2+\frac{4}{3}x}} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{(x+\frac{2}{3})^2-\frac{4}{9}}}$$

and let $u = x + \frac{2}{3}$; whence $du = dx$.

Then
$$\int \frac{dx}{\sqrt{3x^2+4x}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{(x+\frac{2}{3})^2-\frac{4}{9}}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{u^2-\frac{4}{9}}}$$

$$= \frac{1}{\sqrt{3}} \ln\left(u + \sqrt{u^2-\frac{4}{9}}\right)$$

$$= \frac{1}{\sqrt{3}} \ln\left(x + \frac{2}{3} + \sqrt{x^2+\frac{4}{3}x}\right) + C,$$

$$= \frac{1}{\sqrt{3}} \ln\left(3x + 2 + \sqrt{9x^2+12x}\right) + K,$$

where $C = \frac{1}{\sqrt{3}} \ln 3 + K$.

**Ex. 3.** Find the value of $\int \frac{dx}{2x^2+x-15}$.

Writing the integrand in the form

$$\frac{1}{2} \cdot \frac{1}{x^2+\frac{1}{2}x-\frac{15}{2}} = \frac{1}{2} \cdot \frac{1}{(x+\frac{1}{4})^2-\frac{121}{16}},$$

we let $u = x + \frac{1}{4}$; whence $du = dx$.

Then
$$\int \frac{dx}{2x^2+x-15} = \frac{1}{2}\int \frac{dx}{(x+\frac{1}{4})^2-\frac{121}{16}}$$
$$= \frac{1}{2}\int \frac{du}{u^2-\frac{121}{16}}$$
$$= \frac{1}{2}\cdot\frac{1}{2\left(\frac{11}{4}\right)}\ln\frac{u-\frac{11}{4}}{u+\frac{11}{4}}+C$$
$$= \frac{1}{11}\ln\frac{x-\frac{5}{2}}{x+3}+C$$
$$= \frac{1}{11}\ln\frac{2x-5}{x+3}+K,$$
where $C = \frac{1}{11}\ln 2 + K$.

## EXERCISES

Find the values of the following integrals:

1. $\int \frac{dx}{\sqrt{x^2+2}}$.
2. $\int \frac{dx}{\sqrt{9x^2-1}}$.
3. $\int \frac{dx}{\sqrt{3x^2-4}}$.
4. $\int \frac{dx}{\sqrt{x^2+2x}}$.
5. $\int \frac{dx}{\sqrt{3x^2+2x+3}}$.
6. $\int \frac{dx}{4x^2-25}$.
7. $\int \frac{dx}{2x^2-1}$.
8. $\int \frac{dx}{3x^2-5}$.
9. $\int \frac{x+3}{x^2-5}\,dx$.
10. $\int \frac{dx}{x^2+4x}$.
11. $\int \frac{dx}{3x^2+5x}$.
12. $\int \frac{dx}{x^2-3x+1}$.
13. $\int \frac{dx}{x^2+5x-2}$.
14. $\int \frac{dx}{4x^2-2x-3}$.
15. $\int_{\frac{5}{2}}^{3} \frac{dx}{\sqrt{x^2-4}}$.
16. $\int_0^1 \frac{dx}{\sqrt{9x^2+1}}$.
17. $\int_{\frac{4}{3}}^{\frac{5}{3}} \frac{dx}{\sqrt{9x^2-6x-3}}$.
18. $\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{2x^2-2x+1}}$.
19. $\int_3^4 \frac{dx}{2x-x^2}$.
20. $\int_2^5 \frac{dx}{2x^2-x-3}$.

**69. Integrals of trigonometric functions.** Of the following formulas for the integration of the trigonometric functions, each of the first six is the direct converse of the corresponding formula of differentiation (§ 44), and the last four can readily be verified by differentiation, which is left to the student.

$$\int \sin u\,du = -\cos u, \tag{1}$$

$$\int \cos u\,du = \sin u, \tag{2}$$

$$\int \sec^2 u\,du = \tan u, \tag{3}$$

$$\int \csc^2 u\,du = -\operatorname{ctn} u, \tag{4}$$

$$\int \sec u \tan u\,du = \sec u, \tag{5}$$

$$\int \csc u \operatorname{ctn} u\,du = -\csc u, \tag{6}$$

$$\int \tan u\,du = \ln \sec u, \tag{7}$$

$$\int \operatorname{ctn} u\,du = \ln \sin u, \tag{8}$$

$$\int \sec u\,du = \ln(\sec u + \tan u), \tag{9}$$

$$\int \csc u\,du = \ln(\csc u - \operatorname{ctn} u). \tag{10}$$

**Ex. 1.** Find the value of $\int \sin 7x\,dx$.

If we let $u = 7x$,

then $du = 7\,dx$;

whence $dx = \frac{1}{7}\,du$,

and
$$\begin{aligned}\int \sin 7x\,dx &= \int \sin u\,(\tfrac{1}{7}\,du)\\ &= \tfrac{1}{7}\int \sin u\,du\\ &= -\tfrac{1}{7}\cos u + C\\ &= -\tfrac{1}{7}\cos 7x + C.\end{aligned}$$

**Ex. 2.** Find the value of $\int \sec(2x+1)\tan(2x+1)\,dx$

If we let $u = 2x+1$, then $du = 2\,dx$,

and
$$\int \sec(2x+1)\tan(2x+1)\,dx = \tfrac{1}{2}\int \sec u \tan u\,du$$
$$= \tfrac{1}{2}\sec u + C$$
$$= \tfrac{1}{2}\sec(2x+1) + C.$$

Often a trigonometric transformation of the integrand facilitates the carrying out of the integration, as shown in the following examples:

**Ex. 3.** Find the value of $\int \cos^2 ax\,dx$

Since
$$\cos^2 ax = \tfrac{1}{2}(1+\cos 2ax),$$
$$\int \cos^2 ax\,dx = \int (\tfrac{1}{2} + \tfrac{1}{2}\cos 2ax)\,dx$$
$$= \tfrac{1}{2}\int dx + \tfrac{1}{2}\int \cos 2ax\,dx$$
$$= \frac{1}{2}x + \frac{1}{4a}\sin 2ax + C,$$

the second integral being evaluated by formula (2) with $u = 2ax$.

**Ex. 4.** Find the value of $\int \sqrt{1+\cos x}\,dx$.

Since
$$\cos x = 2\cos^2\frac{x}{2} - 1,$$
$$\sqrt{1+\cos x} = \sqrt{2}\cos\frac{x}{2},$$

and
$$\int \sqrt{1+\cos x}\,dx = \int \sqrt{2}\cos\frac{x}{2}\,dx$$
$$= \sqrt{2}\int \cos\frac{x}{2}\,dx$$
$$= 2\sqrt{2}\sin\frac{x}{2} + C.$$

**Ex. 5.** Find the value of $\int \tan^2 3x\,dx$.

Since
$$\tan^2 3x = \sec^2 3x - 1,$$
$$\int \tan^2 3x\,dx = \int (\sec^2 3x - 1)\,dx$$
$$= \int \sec^2 3x\,dx - \int dx$$
$$= \tfrac{1}{3}\tan 3x - x,$$

the first integral being evaluated by formula (3) with $u = 3x$.

**EXERCISES**

Find the values of the following integrals:

1. $\int \sin(3x-2)\,dx.$
2. $\int \cos(4-2x)\,dx.$
3. $\int \sec(3x-1)\tan(3x-1)\,dx$
4. $\int \sec^2\frac{x}{4}\,dx.$
5. $\int \tan\frac{3x}{2}\,dx.$
6. $\int \operatorname{ctn} 5x\,dx.$
7. $\int \csc(2x+3)\,dx.$
8. $\int \csc\frac{x}{2}\operatorname{ctn}\frac{x}{2}\,dx$
9. $\int \sec(4x+2)\,dx.$
10. $\int \csc^2(3-2x)\,dx.$
11. $\int \frac{\cos 2x}{\sin x}\,dx.$
12. $\int \sin^2\frac{x}{2}\,dx.$
13. $\int \cos^2\frac{x}{6}\,dx.$
14. $\int \left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 dx.$
15. $\int \left(\sec\frac{2x}{3}-\tan\frac{2x}{3}\right)^2 dx$
16. $\int \sin^2\frac{x}{4}\cos^2\frac{x}{4}\,dx.$
17. $\int \sqrt{1+\cos\frac{3x}{2}}\,dx.$
18. $\int \sqrt{1-\cos 4x}\,dx.$
19. $\int_0^{\pi} \sin 3x\,dx.$
20. $\int_0^{\frac{\pi}{3}} \tan\frac{x}{2}\,dx.$
21. $\int_0^{\frac{\pi}{12}} \tan^2\left(x+\frac{\pi}{4}\right)dx.$
22. $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \sec 2x\,dx.$
23. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$
24. $\int_{\frac{3\pi}{8}}^{\frac{5\pi}{8}} \sec^2 2x\,dx.$

**70. Integrals of exponential functions.** The formulas

$$\int e^u\,du = e^u \tag{1}$$

and

$$\int a^u\,du = \frac{1}{\ln a}a^u \tag{2}$$

are derived immediately from the corresponding formulas of differentiation.

**Ex. 1.** Find the value of $\int e^{3x}\,dx$

If we let $3x = u$, we have

$$\begin{aligned}\int e^{3x}\,dx &= \tfrac{1}{3}\int e^{u}\,du \\ &= \tfrac{1}{3}e^{u} + C \\ &= \tfrac{1}{3}e^{3x} + C.\end{aligned}$$

**Ex. 2.** Find the value of $\int \frac{\sqrt[x]{5}}{x^2}\,dx.$

If we place $\sqrt[x]{5} = 5^{\frac{1}{x}}$ and let $\frac{1}{x} = u$, we have

$$\begin{aligned}\int \frac{\sqrt[x]{5}}{x^2}\,dx &= -\int 5^{u}\,du \\ &= -\frac{1}{\ln 5}5^{u} + C \\ &= -\frac{1}{\ln 5}\cdot\sqrt[x]{5} + C.\end{aligned}$$

### EXERCISES

Find the values of the following integrals:

1. $\int e^{2x+5}\,dx.$
2. $\int e^{x^2}x\,dx.$
3. $\int (e^{x} + x^{e})\,dx$
4. $\int e^{a+bx}c^{a+bx}\,dx.$
5. $\int (e^{x} + e^{-x})^2\,dx.$
6. $\int \frac{e^{2x} - e^{-2x}}{e^{x} + e^{-x}}\,dx.$
7. $\int \frac{e^{2x} + 1}{e^{2x} - 1}\,dx.$
8. $\int_0^1 \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)dx.$
9. $\int_0^2 10^{x}\,dx.$
10. $\int_0^{\pi} 2^{\cos x}\sin x\,dx.$
11. $\int_{\frac{1}{2}}^1 e^{2x-1}\,dx.$
12. $\int_0^1 \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\,dx.$

**71. Substitutions.** In all the integrations that have been made in the previous sections we have substituted a new variable $u$ for some function of $x$, thereby making the given integral identical with one of the formulas. There are other cases in which the choice of the new variable $u$ is not so evident, but in which, nevertheless, it is possible to reduce the given integral to one of the known integrals by an appropriate choice and *substitution* of a new variable. We shall suggest in this section a few of the more common substitutions which it is desirable to try.

I. *Integrand involving powers of* $a + bx$. The substitution of some power of $z$ for $a + bx$ is usually desirable.

**Ex. 1.** Find the value of $\int \frac{x^2\,dx}{(1+2x)^{\frac{1}{3}}}$.

Here we let $1 + 2x = z^3$; then $x = \frac{1}{2}(z^3 - 1)$ and $dx = \frac{3}{2}z^2\,dz$.

Therefore
$$\int \frac{x^2\,dx}{(1+2x)^{\frac{1}{3}}} = \frac{3}{8}\int (z^7 - 2z^4 + z)\,dz$$
$$= \frac{3}{8}\left(\frac{1}{8}z^8 - \frac{2}{5}z^5 + \frac{1}{2}z^2\right) + C$$
$$= \tfrac{3}{320}z^2(5z^6 - 16z^3 + 20) + C.$$

Replacing $z$ by its value $(1+2x)^{\frac{1}{3}}$ and simplifying, we have
$$\int \frac{x^2\,dx}{(1+2x)^{\frac{1}{3}}} = \frac{3}{320}(1+2x)^{\frac{2}{3}}(9 - 12x + 20x^2) + C.$$

II. *Integrand involving powers of* $a + bx^n$. The substitution of some power of $z$ for $a + bx^n$ is desirable if the expression under the integral sign contains $x^{n-1}dx$ as a factor, since $d(a + bx^n) = bnx^{n-1}dx$.

**Ex. 2.** Find the value of $\int \frac{\sqrt{x^2 + a^2}}{x}\,dx$.

We may write the integral in the form
$$\int \frac{\sqrt{x^2+a^2}}{x^2}(x\,dx)$$
and place $x^2 + a^2 = z^2$. Then $x\,dx = z\,dz$, and the integral becomes
$$\int \frac{z^2\,dz}{z^2 - a^2} = \int \left(1 + \frac{a^2}{z^2 - a^2}\right)dz = z + \frac{a}{2}\ln\frac{z-a}{z+a} + C.$$

Replacing $z$ by its value in terms of $x$, we have
$$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} + \frac{a}{2}\ln\frac{\sqrt{x^2+a^2} - a}{\sqrt{x^2+a^2}+a} + C.$$

**Ex. 3.** Find the value of $\int x^5(1+2x^3)^{\frac{1}{2}}dx$.

We may write the integral in the form
$$\int x^3(1+2x^3)^{\frac{1}{2}}(x^2\,dx),$$
and place $1 + 2x^3 = z^2$. Then $x^2dx = \frac{1}{3}z\,dz$, and the new integral in $z$ is
$$\tfrac{1}{6}\int (z^4 - z^2)\,dz = \tfrac{1}{90}z^3(3z^2 - 5) + C.$$

Replacing $z$ by its value, we have
$$\int x^5(1+2x^3)^{\frac{1}{2}}dx = \tfrac{1}{45}(1+2x^3)^{\frac{3}{2}}(3x^3 - 1) + C.$$

**III.** *Integrand involving* $\sqrt{a^2 - x^2}$. If a right triangle is constructed with one leg equal to $x$ and with the hypotenuse equal to $a$ (Fig. 83), the substitution $x = a \sin \phi$ is suggested.

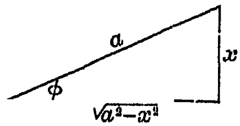



Fig. 83

**Ex. 4.** Find the value of $\int \sqrt{a^2 - x^2}\, dx$.

Let $x = a \sin \phi$. Then $dx = a \cos \phi \, d\phi$ and, from the triangle, $\sqrt{a^2 - x^2} = a \cos \phi$.

Therefore
$$\int \sqrt{a^2 - x^2}\, dx = a^2 \int \cos^2 \phi \, d\phi$$
$$= \tfrac{1}{2} a^2 \int (1 + \cos 2\phi)\, d\phi$$
$$= \tfrac{1}{2} a^2 (\phi + \tfrac{1}{2} \sin 2\phi) + C.$$

But
$$\phi = \sin^{-1}\frac{x}{a},$$

and
$$\sin 2\phi = 2 \sin \phi \cos \phi$$
$$= \frac{2x\sqrt{a^2 - x^2}}{a^2};$$

for, from the triangle, $\sin \phi = \dfrac{x}{a}$ and $\cos \phi = \dfrac{\sqrt{a^2 - x^2}}{a}$.

Finally, by substitution, we have
$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right) + C.$$

**IV.** *Integrand involving* $\sqrt{x^2 + a^2}$. If a right triangle is constructed with the two legs equal to $x$ and $a$ respectively (Fig. 84), the substitution $x = a \tan \phi$ is suggested.

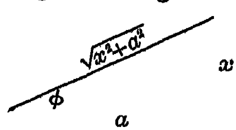



Fig. 84

**Ex. 5.** Find the value of $\displaystyle\int \frac{dx}{(x^2 + a^2)^{\frac{3}{2}}}$.

Let $x = a \tan \phi$. Then $dx = a \sec^2 \phi \, d\phi$ and, from the triangle, $\sqrt{x^2 + a^2} = a \sec \phi$.

Therefore $\displaystyle\int \frac{dx}{(x^2 + a^2)^{\frac{3}{2}}} = \frac{1}{a^2}\int \frac{d\phi}{\sec \phi} = \frac{1}{a^2}\int \cos \phi \, d\phi = \frac{1}{a^2} \sin \phi + C.$

But, from the triangle, $\sin \phi = \dfrac{x}{\sqrt{x^2 + a^2}}$; so that, by substitution,
$$\int \frac{dx}{(x^2 + a^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{x^2 + a^2}} + C.$$

V. *Integrand involving* $\sqrt{x^2-a^2}$. If a right triangle is constructed with the hypotenuse equal to $x$ and with one leg equal to $a$ (Fig. 85), the substitution $x = a\sec\phi$ is suggested.

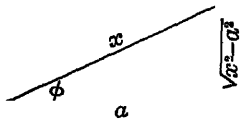



FIG. 85

**Ex. 6.** Find the value of $\int x^3\sqrt{x^2-a^2}\,dx$.

Let $x = a\sec\phi$. Then $dx = a\sec\phi\tan\phi\,d\phi$ and, from the triangle, $\sqrt{x^2-a^2} = a\tan\phi$

Therefore
$$\int x^3\sqrt{x^2-a^2}\,dx = a^5\int \tan^2\phi\sec^4\phi\,d\phi$$
$$= a^5\int(\tan^2\phi+\tan^4\phi)\sec^2\phi\,d\phi$$
$$= a^5(\tfrac{1}{3}\tan^3\phi+\tfrac{1}{5}\tan^5\phi)+C.$$

But, from the triangle, $\tan\phi = \dfrac{\sqrt{x^2-a^2}}{a}$, so that, by substitution, we have
$$\int x^3\sqrt{x^2-a^2}\,dx = \tfrac{1}{15}(2a^2+3x^2)\sqrt{(x^2-a^2)^3}+C$$

We might have written this integral in the form $\int x^2\sqrt{x^2-a^2}(x\,dx)$ and solved by letting $z^2 = x^2-a^2$.

**72.** If the value of the indefinite integral is found by substitution, the evaluation of the definite integral $\int_a^b f(x)\,dx$ may be performed in two ways, differing in the manner in which the limits are substituted. These two ways are shown in the solutions of the following example:

**Ex.** Find $\int_0^a\sqrt{a^2-x^2}\,dx$.

By Ex. 4, § 71,
$$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right)+C.$$

Therefore
$$\int_0^a\sqrt{a^2-x^2}\,dx = \left[\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right)\right]_0^a$$
$$=\frac{1}{2}\left(a\sqrt{a^2-a^2}+a^2\sin^{-1}\frac{a}{a}\right)$$
$$-\frac{1}{2}\left(0\sqrt{a^2-0}+a^2\sin^{-1}\frac{0}{a}\right)$$
$$=\frac{\pi a^2}{4}.$$

Or we may proceed as follows Let $x = a \sin \phi$ When $x = 0$, $\phi = 0$; and when $x = a$, $\phi = \frac{\pi}{2}$, so that $\phi$ varies from 0 to $\frac{\pi}{2}$ as $x$ varies from 0 to $a$. Accordingly,

$$\begin{aligned}\int_0^a \sqrt{a^2 - x^2}\,dx &= a^2 \int_0^{\frac{\pi}{2}} \cos^2 \phi \, d\phi \\ &= \left[\frac{a^2}{2}\left(\phi + \frac{1}{2}\sin 2\phi\right)\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi a^2}{4}\end{aligned}$$

The second method is evidently the better method, as it obviates the necessity of replacing $z$ in the indefinite integral by its value in terms of $x$ before the limits of integration can be substituted.

### EXERCISES

Find the values of the following integrals:

1. $\displaystyle\int \frac{x^3\,dx}{(x+2)^2}$

2. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x-1}}.$

3. $\displaystyle\int x(2x-3)^{\frac{1}{3}}\,dx$

4. $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx.$

5. $\displaystyle\int \frac{x^3\,dx}{(x^2+4)^{\frac{3}{2}}}.$

6. $\displaystyle\int \frac{x^2\,dx}{(4-x^2)^{\frac{3}{2}}}.$

7. $\displaystyle\int \frac{x^4\,dx}{(3-x^2)^{\frac{7}{2}}}$

8. $\displaystyle\int \frac{dx}{x^2\sqrt{1+4x^2}}$

9. $\displaystyle\int \frac{dx}{(x^2-4)^{\frac{3}{2}}}$

10. $\displaystyle\int_{\frac{3}{2}}^{2} x\sqrt{2x-3}\,dx.$

11. $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}}$

12. $\displaystyle\int_{-1}^{1} x^2\sqrt{2-x^2}\,dx.$

13. $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{(x^2+1)^{\frac{5}{2}}}.$

14. $\displaystyle\int_3^{3\sqrt{2}} \frac{(x^2-9)^{\frac{3}{2}}}{x^6}\,dx.$

15. $\displaystyle\int_0^2 x\sqrt{4-x^2}\,dx.$

**73. Integration by parts.** Another method of importance in the reduction of a given integral to a known type is that of *integration by parts*, the formula for which is derived from the formula for the differential of a product,

$$d(uv) = u\,dv + v\,du.$$

From this formula we derive

$$uv = \int u\,dv + \int v\,du,$$

which is usually written in the form

$$\int u\,dv = uv - \int v\,du.$$

In the use of this formula the aim is evidently to make the original integration depend upon the evaluation of a simpler integral.

**Ex. 1.** Find the value of $\int xe^x\,dx$.

If we let $x = u$ and $e^x dx = dv$, we have $du = dx$ and $v = e^x$. Substituting in our formula, we have

$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C \\ &= (x-1)e^x + C.\end{aligned}$$

It is evident that in selecting the expression for $dv$ it is desirable, if possible, to choose an expression that is easily integrated.

**Ex. 2.** Find the value of $\int \sin^{-1} x\,dx$.

Here we may let $\sin^{-1} x = u$ and $dx = dv$, whence $du = \dfrac{dx}{\sqrt{1-x^2}}$ and $v = x$. Substituting in our formula, we have

$$\begin{aligned}\int \sin^{-1} x\,dx &= x\sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}} \\ &= x\sin^{-1} x + \sqrt{1-x^2} + C,\end{aligned}$$

the last integral being evaluated by (1), § 66

Sometimes an integral may be evaluated by successive integration by parts.

**Ex. 3.** Find the value of $\int x^2e^x\,dx$.

Here we let $x^2 = u$ and $e^x dx = dv$. Then $du = 2\,x\,dx$ and $v = e^x$.

Therefore $$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx$$

The integral $\int xe^x\,dx$ may be evaluated by integration by parts (see Ex. 1), so that finally

$$\int x^2e^x\,dx = x^2e^x - 2(x-1)e^x + C = e^x(x^2 - 2\,x + 2) + C.$$

**Ex. 4.** Find the value of $\int e^{ax}\sin bx\,dx$.

Letting $\sin bx = u$ and $e^{ax}dx = dv$, we have

$$\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx.$$

In the integral $\int e^{ax}\cos bx\,dx$ we let $\cos bx = u$ and $e^{ax}dx = dv$, and hav

$$\int e^{ax}\cos bx\,dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx.$$

Substituting this value above, we have

$$\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left(\frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx\right).$$

Now bringing to the left-hand member of the equation all the terms containing the integral, we have

$$\left(1+\frac{b^2}{a^2}\right)\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx,$$

whence $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}.$$

**Ex. 5.** Find the value of $\int\sqrt{x^2+a^2}\,dx$.

Placing $\sqrt{x^2+a^2} = u$ and $dx = dv$, whence $du = \dfrac{x\,dx}{\sqrt{x^2+a^2}}$ and $v = x$, we have

$$\int\sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int\frac{x^2\,dx}{\sqrt{x^2+a^2}}. \qquad (1)$$

Since $x^2 = (x^2+a^2) - a^2$, the second integral of (1) may be written as

$$\int\frac{(x^2+a^2)\,dx}{\sqrt{x^2+a^2}} - a^2\int\frac{dx}{\sqrt{x^2+a^2}},$$

which equals $$\int\sqrt{x^2+a^2}\,dx - a^2\int\frac{dx}{\sqrt{x^2+a^2}}.$$

Evaluating this last integral and substituting in (1), we have

$$\int\sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2\ln\left(x+\sqrt{x^2+a^2}\right),$$

whence $$\int\sqrt{x^2+a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\ln\left(x+\sqrt{x^2+a^2}\right)\right].$$

**74.** If the value of the indefinite integral $\int f(x)\,dx$ is found by integration by parts, the value of the definite integral $\int_a^b f(x)\,dx$ may be found by substituting the limits $a$ and $b$, in the usual manner, in the indefinite integral.

**Ex.** Find the value of $\int_0^{\frac{\pi}{2}} x^2 \sin x\, dx$.

To find the value of the indefinite integral, let $x^2 = u$ and $\sin x\, dx = dv$.

Then $$\int x^2 \sin x\, dx = -x^2 \cos x + 2\int x \cos x\, dx.$$

In $\int x \cos x\, dx$, let $x = u$ and $\cos x\, dx = dv$.

Then $$\int x \cos x\, dx = x \sin x - \int \sin x\, dx$$
$$= x \sin x + \cos x.$$

Finally, we have

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

Hence $$\int_0^{\frac{\pi}{2}} x^2 \sin x\, dx = \Big[-x^2 \cos x + 2x \sin x + 2\cos x\Big]_0^{\frac{\pi}{2}}$$
$$= \pi - 2.$$

The better method, however, is as follows:

If $f(x)\,dx$ is denoted by $u\,dv$, the definite integral $\int_a^b f(x)\,dx$ may be denoted by $\int_a^b u\,dv$, where it is understood that $a$ and $b$ are the values of the independent variable. Then

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du.$$

To prove this, note that it follows at once from the equation

$$[uv]_a^b = \int_a^b d(uv) = \int_a^b (u\,dv + v\,du) = \int_a^b u\,dv + \int_a^b v\,du.$$

Applying this method to the problem just solved, we have

$$\int_0^{\frac{\pi}{2}} x^2 \sin x\, dx = \Big[-x^2 \cos x\Big]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} x \cos x\, dx$$
$$= 2\int_0^{\frac{\pi}{2}} x \cos x\, dx$$
$$= \Big[2x \sin x\Big]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \sin x\, dx$$
$$= \pi + \Big[2\cos x\Big]_0^{\frac{\pi}{2}}$$
$$= \pi - 2.$$

### EXERCISES

Find the values of the following integrals:

1. $\int x e^{3x}\, dx.$
2. $\int x^2 e^{2x}\, dx.$
3. $\int \cos^{-1} x\, dx.$
4. $\int \tan^{-1} 3x\, dx$
5. $\int x \sec^{-1} 2x\, dx.$
6. $\int (\ln \sin x) \cos x\, dx.$
7. $\int e^{3x} \cos x\, dx.$
8. $\int x \cos^2 \frac{x}{2}\, dx.$
9. $\int_0^1 x e^{\frac{x}{2}}\, dx.$
10. $\int_1^e x^2 \ln x\, dx.$
11. $\int_0^{\frac{1}{2}} \sin^{-1} 2x\, dx.$
12. $\int_0^{\frac{\pi}{4}} x \cos 2x\, dx.$

**75. Integration of rational fractions.** A *rational fraction* is a fraction whose numerator and denominator are polynomials. It can often be integrated by expressing it as the sum of *partial fractions* whose denominators are factors of the denominator of the original fraction. We shall illustrate only the case in which the degree of the numerator is less than the degree of the denominator and in which the factors of the denominator are all of the first degree and all different.

Ex. Find the value of $\int \frac{x^2 + 11x + 14}{(x+3)(x^2-4)}\, dx.$

The factors of the denominator are $x + 3$, $x - 2$, and $x + 2$. We assume

$$\frac{x^2 + 11x + 14}{(x+3)(x^2-4)} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{x+2}, \tag{1}$$

where $A$, $B$, and $C$ are constants to be determined.

Clearing (1) of fractions by multiplying by $(x+3)(x^2-4)$, we have

$$x^2 + 11x + 14 = A(x-2)(x+2) + B(x+3)(x+2) + C(x+3)(x-2), \tag{2}$$

or $$x^2 + 11x + 14 = (A + B + C)x^2 + (5B + C)x + (-4A + 6B - 6C). \tag{3}$$

Since $A$, $B$, and $C$ are to be determined so that the right-hand member of (3) shall be identical with the left-hand member, the coefficients of like powers of $x$ on the two sides of the equation must be equal.

Therefore, equating the coefficients of like powers of $x$ in (3), we obtain the equations

$$A + B + C = 1,$$
$$5B + C = 11,$$
$$-4A + 6B - 6C = 14,$$

whence we find $A = -2$, $B = 2$, $C = 1$.

Substituting these values in (1), we have

$$\frac{x^2+11x+14}{(x+3)(x^2-4)}=-\frac{2}{x+3}+\frac{2}{x-2}+\frac{1}{x+2},$$

and $$\int\frac{x^2+11x+14}{(x+3)(x^2-4)}dx=-\int\frac{2\,dx}{x+3}+\int\frac{2\,dx}{x-2}+\int\frac{dx}{x+2}$$
$$=-2\ln(x+3)+2\ln(x-2)+\ln(x+2)+C$$
$$=\ln\frac{(x+2)(x-2)^2}{(x+3)^2}+C.$$

**EXERCISES**

Find the values of the following integrals:

1. $\displaystyle\int\frac{x+10}{x^2-6x+8}dx.$
2. $\displaystyle\int\frac{5x+1}{2x^2+5x-3}dx$
3. $\displaystyle\int\frac{x^2-5x+5}{(x-1)(x-2)(x-3)}dx.$
4. $\displaystyle\int\frac{4x+2}{(x+2)(x^2-1)}dx.$
5. $\displaystyle\int\frac{2x^2-4x-1}{2x^3-x^2-x}dx.$
6. $\displaystyle\int\frac{x^2-x-1}{(x-1)(x^2-x-6)}dx.$

**76. Table of integrals.** The formulas of integration used in this chapter are sufficient for the solution of most of the problems which occur in practice. To these formulas we have added a few others. In some cases they represent an integral which has already been evaluated, and in other cases they are the result of an integration by parts. In all cases they can be verified by differentiating both sides of the equation.

These collected formulas form a brief table of integrals which will aid in the solution of the problems in this book. It will be noticed that some of the formulas express the given integral only in terms of a simpler integral.

### I. Fundamental

1. $\displaystyle\int c\,du=c\int du.$
2. $\displaystyle\int(du+dv+dw\cdots)=\int du+\int dv+\int dw\cdots.$
3. $\displaystyle\int u\,dv=uv-\int v\,du.$

## II. Algebraic

4. $$\int u^n\,du = \frac{u^{n+1}}{n+1}. \quad (n \neq -1)$$

5. $$\int \frac{du}{u} = \ln u.$$

6. $$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}.$$

7. $$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\frac{u-a}{u+a}.$$

8. $$\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(u\sqrt{a^2-u^2}+a^2\sin^{-1}\frac{u}{a}\right).$$

9. $$\int u\sqrt{a^2-u^2}\,du = -\tfrac{1}{3}(a^2-u^2)^{\frac{3}{2}}.$$

10. $$\int u^n\sqrt{a^2-u^2}\,du = -\frac{u^{n-1}(a^2-u^2)^{\frac{3}{2}}}{n+2} + \frac{(n-1)a^2}{n+2}\int u^{n-2}\sqrt{a^2-u^2}\,du. \quad (n+2\neq 0)$$

11. $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}.$$

12. $$\int \frac{u\,du}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2}.$$

13. $$\int \frac{u^n\,du}{\sqrt{a^2-u^2}} = -\frac{u^{n-1}\sqrt{a^2-u^2}}{n} + \frac{(n-1)a^2}{n}\int\frac{u^{n-2}\,du}{\sqrt{a^2-u^2}}. \quad (n\neq 0)$$

14. $$\int \sqrt{u^2\pm a^2}\,du = \tfrac{1}{2}\left[u\sqrt{u^2\pm a^2}\pm a^2\ln\left(u+\sqrt{u^2\pm a^2}\right)\right].$$

15. $$\int u\sqrt{u^2\pm a^2}\,du = \tfrac{1}{3}(u^2\pm a^2)^{\frac{3}{2}}.$$

16. $$\int u^n\sqrt{u^2\pm a^2}\,du = \frac{u^{n-1}(u^2\pm a^2)^{\frac{3}{2}}}{n+2} \mp \frac{(n-1)a^2}{n+2}\int u^{n-2}\sqrt{u^2\pm a^2}\,du. \quad (n+2\neq 0)$$

17. $$\int \frac{du}{\sqrt{u^2\pm a^2}} = \ln\left(u+\sqrt{u^2\pm a^2}\right).$$

18. $$\int \frac{u\,du}{\sqrt{u^2\pm a^2}} = \sqrt{u^2\pm a^2}.$$

19. $$\int \frac{u^n\,du}{\sqrt{u^2\pm a^2}} = \frac{u^{n-1}\sqrt{u^2\pm a^2}}{n} \mp \frac{(n-1)a^2}{n}\int\frac{u^{n-2}\,du}{\sqrt{u^2\pm a^2}}. \quad (n\neq 0)$$

20. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a}.$

21. $\int \sqrt{2au-u^2}\,du = \frac{1}{2}\left[(u-a)\sqrt{2au-u^2}+a^2\sin^{-1}\frac{u-a}{a}\right].$

22. $\int \frac{du}{\sqrt{2au-u^2}} = \sin^{-1}\frac{u-a}{a}.$

## III. Trigonometric

23. $\int \sin u\,du = -\cos u.$

24. $\int \sin^2 u\,du = \frac{u}{2}-\frac{1}{4}\sin 2u.$

25. $\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du.\quad (n \neq 0)$

26. $\int \cos u\,du = \sin u.$

27. $\int \cos^2 u\,du = \frac{u}{2}+\frac{1}{4}\sin 2u.$

28. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du.\quad (n \neq 0)$

29. $\int \tan u\,du = \ln \sec u.$

30. $\int \tan^n u\,du = \frac{\tan^{n-1}u}{n-1} - \int \tan^{n-2}u\,du.\quad (n-1 \neq 0)$

31. $\int \operatorname{ctn} u\,du = \ln \sin u.$

32. $\int \operatorname{ctn}^n u\,du = -\frac{\operatorname{ctn}^{n-1}u}{n-1} - \int \operatorname{ctn}^{n-2}u\,du.\quad (n-1 \neq 0)$

33. $\int \sec u\,du = \ln(\sec u + \tan u).$

34. $\int \sec^2 u\,du = \tan u.$

35. $\int \csc u\,du = \ln(\csc u - \operatorname{ctn} u).$

36. $\int \csc^2 u\,du = -\operatorname{ctn} u.$

37. $\int \sec u \tan u\,du = \sec u.$

38. $\int \csc u \operatorname{ctn} u\,du = -\csc u.$

39. $\int \sin^m u \cos^n u\,du = \frac{\sin^{m+1} u \cos^{n-1} u}{m+n} + \frac{n-1}{m+n}\int \sin^m u \cos^{n-2} u\,du.$ $(m+n \neq 0)$

40. $\int \sin^m u \cos^n u\,du = -\frac{\sin^{m-1} u \cos^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} u \cos^n u\,du.$ $(m+n \neq 0)$

41. $\int \sin^m u \cos^n u\,du = -\frac{\sin^{m+1} u \cos^{n+1} u}{n+1}$

$\qquad + \frac{m+n+2}{n+1}\int \sin^m u \cos^{n+2} u\,du.$ $(n+1 \neq 0)$

42. $\int \sin^m u \cos^n u\,du = \frac{\sin^{m+1} u \cos^{n+1} u}{m+1}$

$\qquad + \frac{m+n+2}{m+1}\int \sin^{m+2} u \cos^n u\,du.$ $(m+1 \neq 0)$

## IV. Exponential

43. $\int e^u\,du = e^u.$

44. $\int a^u\,du = \frac{1}{\ln a}\,a^u.$

45. $\int e^{au} \sin bu\,du = \frac{e^{au}(a \sin bu - b \cos bu)}{a^2+b^2}.$

46. $\int e^{au} \cos bu\,du = \frac{e^{au}(a \cos bu + b \sin bu)}{a^2+b^2}.$

### GENERAL EXERCISES

Find the values of the following integrals:

1. $\int \left(3x^2 + 4x - \frac{1}{x^2} - \frac{1}{x^3}\right) dx.$

2. $\int \frac{2x^2 + \sqrt[3]{x^2} + 2}{\sqrt[3]{x}}\,dx.$

3. $\int \left(\sqrt[5]{x} - \frac{1}{\sqrt[5]{x^3}}\right) dx.$

4. $\int (x+2)^2 x\,dx.$

5. $\int (x^2-4)x\,dx.$

6. $\int \sqrt[3]{x^3+3}\,x^2\,dx.$

7. $\int (2+e^{2x})^2 e^{2x}\,dx.$

8. $\int \frac{1+2x^3}{\sqrt[3]{1+2x+x^4}}\,dx.$

9. $\int \frac{1+x^2}{(3x+x^3)^{\frac{2}{3}}}\,dx.$

10. $\int \frac{x^2\,dx}{x-1}.$

11. $\int \sin^2(2x-1)\cos^3(2x-1)\,dx$

12. $\int \cos^5\frac{x}{5}\,dx.$

13. $\int \csc^6 4x\,dx.$

14. $\int \sec^3(x-2)\tan^3(x-2)\,dx.$

15. $\int \operatorname{ctn}(x-1)\sec^4(x-1)\,dx.$

16. $\int \csc^5 2x \operatorname{ctn}^3 2x\,dx.$

17. $\int \tan^3 3x \sqrt[3]{\sec 3x}\,dx.$

18. $\int \operatorname{ctn} 2x \sqrt{\csc 2x}\,dx.$

19. $\int \csc^4 5x \sqrt[5]{\operatorname{ctn} 5x}\,dx.$

20. $\int \frac{\cos^5 4x}{\sin^{\frac{4}{3}} 4x}\,dx.$

21. $\int \frac{\csc^4 5x}{\sqrt[5]{\tan^3 5x}}\,dx.$

22. $\int \frac{dx}{\sqrt{25-4x^2}}.$

23. $\int \frac{dx}{\sqrt{5-x^2}}.$

24. $\int \frac{dx}{\sqrt{8-9x^2}}.$

25. $\int \frac{dx}{x\sqrt{9x^2-5}}.$

26. $\int \frac{dx}{2x\sqrt{5x^2-4}}.$

27. $\int \frac{dx}{(x+1)\sqrt{x^2+2x-3}}.$

28. $\int \frac{dx}{(3x-1)\sqrt{9x^2-6x-2}}.$

29. $\int \frac{dx}{(2x+3)\sqrt{x^2+3x-4}}.$

30. $\int \frac{dx}{\sqrt{5x-x^2}}.$

31. $\int \frac{dx}{\sqrt{3+2x-x^2}}.$

32. $\int \frac{dx}{\sqrt{7+4x-4x^2}}.$

33. $\int \frac{dx}{\sqrt{-3-6x-2x^2}}.$

34. $\int \frac{dx}{4x^2+5}.$

35. $\int \frac{dx}{5x^2+3}.$

36. $\int \frac{dx}{x^2-4x+10}.$

37. $\int \frac{dx}{4x^2+8x+7}.$

38. $\int \frac{dx}{3x^2+4x+1}.$

39. $\int \frac{dx}{9x^2+5x+1}.$

40. $\int \frac{x\,dx}{x^4+9}.$

41. $\displaystyle\int \frac{x^2\,dx}{3x^6+7}$.

42. $\displaystyle\int \frac{x\,dx}{\sqrt{4x^2-x^4}}$.

43. $\displaystyle\int \frac{dx}{x\sqrt{x^4-6}}$.

44. $\displaystyle\int \frac{dx}{x\sqrt{x^6-4}}$.

45. $\displaystyle\int \frac{x\,dx}{\sqrt{1-12x^2-4x^4}}$.

46. $\displaystyle\int \frac{2x-3}{3x^2+7}\,dx$

47. $\displaystyle\int \frac{4-3x}{\sqrt{9-2x^2}}\,dx$

48. $\displaystyle\int \frac{dx}{\sqrt{x^2-7}}$.

49. $\displaystyle\int \frac{dx}{\sqrt{4x^2+3}}$.

50. $\displaystyle\int \frac{dx}{\sqrt{2x^2+5}}$.

51. $\displaystyle\int \frac{x\,dx}{\sqrt{4x^4-5}}$.

52 $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^6+7}}$.

53. $\displaystyle\int \frac{2x+1}{\sqrt{x^2+4}}\,dx$.

54. $\displaystyle\int \frac{3x-4}{\sqrt{3x^2+1}}\,dx$.

55. $\displaystyle\int \frac{dx}{\sqrt{2x^2-3x}}$.

56. $\displaystyle\int \frac{dx}{\sqrt{4x^2+4x+7}}$.

57. $\displaystyle\int \frac{dx}{\sqrt{5x^2+4x-1}}$.

58. $\displaystyle\int \frac{dx}{\sqrt{9x^2-24x+14}}$.

59. $\displaystyle\int \frac{dx}{9x^2-7}$.

60. $\displaystyle\int \frac{dx}{5x^2-4}$.

61. $\displaystyle\int \frac{dx}{4-6x^2}$.

62. $\displaystyle\int \frac{2x-11}{3x^2-7}\,dx$

63. $\displaystyle\int \frac{dx}{4x^2-6x}$.

64. $\displaystyle\int \frac{dx}{2x^2-5x}$.

65. $\displaystyle\int \frac{dx}{9x^2-6x-3}$.

66. $\displaystyle\int \frac{dx}{4x^2-4x-5}$.

67. $\displaystyle\int \frac{dx}{25x^2-5x-2}$.

68. $\displaystyle\int \frac{dx}{3x^2-4x-4}$.

69. $\displaystyle\int \frac{dx}{5x^2+2x-1}$.

70. $\displaystyle\int \frac{dx}{2x^2+3x-2}$.

71. $\displaystyle\int (\tan 3x+\operatorname{ctn} 3x)^2\,dx$.

72. $\displaystyle\int \sec\left(x-\tfrac{2}{3}\right)dx$.

73. $\displaystyle\int \frac{\sin 4x}{\cos 2x}\,dx$.

74. $\displaystyle\int \left(\frac{\sin 2x}{\sin x}-\frac{\cos 2x}{\cos x}\right)dx$.

75. $\int \sin^4 \frac{x}{3}\, dx.$

76. $\int \frac{1 - \cos 4x}{1 + \cos 4x}\, dx.$

77. $\int \frac{\cos 4x}{\cos 2x - \sin 2x}\, dx.$

78. $\int \frac{\operatorname{ctn}\frac{x}{2}}{\csc\frac{x}{2} + \operatorname{ctn}\frac{x}{2}}\, dx.$

79. $\int (\sec^4 2x - \tan^4 2x)\, dx.$

80. $\int \sqrt{1 + \sin 2x}\, dx.$

81. $\int \sqrt{e^x}\, dx.$

82. $\int x \sqrt[3]{e^{x^2}}\, dx.$

83. $\int \frac{dx}{1 + e^{2x}}.$

84. $\int \frac{x\, dx}{(5x + 1)^{\frac{2}{3}}}.$

85. $\int x^5 \sqrt[3]{x^3 + 2}\, dx.$

86. $\int \frac{x^4\, dx}{\sqrt{x^5 + 3}}.$

87. $\int \frac{x^6\, dx}{2x^3 + 1}.$

88. $\int \frac{x^3\, dx}{(1 - x^2)^{\frac{3}{2}}}.$

89. $\int \frac{x^2\, dx}{(4x^2 + 9)^{\frac{5}{2}}}.$

90. $\int \frac{dx}{x^4 \sqrt{x^2 - 25}}.$

91. $\int x\, 5^{x+2}\, dx.$

92. $\int x^3 e^{2x}\, dx.$

93. $\int x \tan^{-1} \frac{x}{2}\, dx.$

94. $\int x^2 \sin 3x\, dx.$

95. $\int (\ln 2x)^2\, dx.$

96. $\int \ln (3x + \sqrt{9x^2 - 4})\, dx.$

97. $\int \frac{x\, dx}{4x^2 + 4x - 3}.$

98. $\int \frac{x^2 + 6x - 18}{x^3 - 9x}\, dx.$

99. $\int \frac{4x^2 + 8x - 18}{4x^3 - 8x^2 - 9x + 18}\, dx.$

100. $\int \frac{2x^2 - x - 2}{8x^3 + 12x^2 - 2x - 3}\, dx.$

101. $\int \frac{x^3 - 6x^2 - 9x + 24}{x^4 - 13x^2 + 36}\, dx.$

102. $\int_3^{3\sqrt{2}} \frac{dx}{x\sqrt{x^2 - 9}}.$

103. $\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4 - x^2}}.$

104. $\int_1^2 \frac{dx}{6x^2 + 7x - 3}.$

105. $\int_{\frac{7}{2}}^{8} \frac{dx}{2x^2 - 7x + 3}.$

106. $\int_{\frac{\pi}{16}}^{\frac{\pi}{6}} \csc^2 4x\, dx.$

107. $\int_0^{\frac{\pi}{6}} \tan\left(x-\frac{\pi}{6}\right)\sec\left(x-\frac{\pi}{6}\right)dx.$

108. $\int_{\frac{2\pi}{9}}^{\frac{4\pi}{9}} \csc\frac{3x}{2}\operatorname{ctn}\frac{3x}{2}\,dx.$

109. $\int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{e^{1+2x}}\,dx.$

110. $\int_2^3 \frac{x\,dx}{e^{x^2}}.$

111. $\int_1^{\sqrt{3}} \frac{5^{\tan^{-1}x}}{1+x^2}\,dx$

112 $\int_{-2}^{1\frac{1}{2}} \frac{x\,dx}{\sqrt[4]{2x+5}}.$

113. $\int_2^{\sqrt[3]{15}} \frac{x^5\,dx}{\sqrt{x^3+1}}.$

114. $\int_3^6 \frac{dx}{x\sqrt{x^3+9}}.$

115. $\int_{\sqrt[4]{14}}^{2} \frac{dx}{x\sqrt{x^4+2}}.$

116. $\int_0^2 (4-x^2)^{\frac{3}{2}}\,dx.$

117. $\int_1^{\sqrt{3}} \frac{\sqrt{4-x^2}}{x^4}\,dx.$

118. $\int_{\sqrt{2}}^{2} \frac{x^2\,dx}{\sqrt{(x^2-1)^5}}.$

119. $\int_0^{\frac{1}{2}} x\sin^{-1}x\,dx.$

120. $\int_0^{\frac{\pi}{2}} e^x \sin 2x\,dx.$

121. $\int_{\frac{1}{3}}^{1} \frac{\ln 3x}{x^2}\,dx.$

122. $\int_0^{\pi} x\sin^2 x\,dx.$

123. $\int_0^1 x^2\tan^{-1}x\,dx.$

# CHAPTER X

## APPLICATIONS

**77. Review problems.** The methods in Chapter III for determining areas, volumes, and pressures are entirely general, and with our new formulas of integration we can now apply these methods to a still wider range of cases.

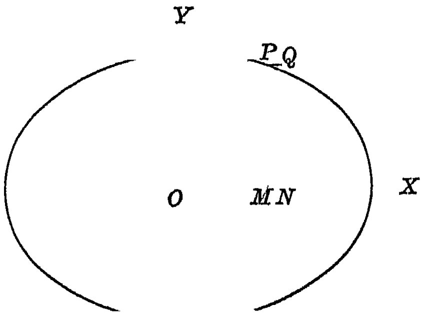



FIG. 86

**Ex. 1.** Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

It is evident from the symmetry of the curve (Fig. 86) that one fourth of the required area is bounded by the axis of $y$, the axis of $x$, and the curve. Constructing the rectangle $MNQP$ as the element of area $dA$, we have

$$dA = y\,dx = \frac{b}{a}\sqrt{a^2-x^2}\,dx.$$

Hence

$$\begin{aligned} A &= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx \\ &= \frac{2b}{a}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right]_0^a \\ &= \pi ab. \end{aligned}$$

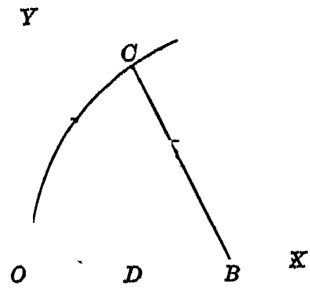



FIG. 87

**Ex. 2.** Find the area bounded by the axis of $x$, the parabola $y^2 = kx$, and the straight line $y + 2x - k = 0$ (Fig. 87).

The straight line and the parabola intersect at the point $C\left(\frac{k}{4}, \frac{k}{2}\right)$, and the straight line intersects $OX$ at $B\left(\frac{k}{2}, 0\right)$. Draw $CD$ perpendicular to $OX$. If we construct the elements of area as in Ex. 1, they will be of different form according as they are to the left or to the right of the line $CD$; fc on the left of $CD$ we shall have

$$dA = y\,dx = k^{\frac{1}{2}}x^{\frac{1}{2}}dx,$$

and on the right of $CD$ we shall have

$$dA = y\,dx = (k - 2x)\,dx$$

It will, accordingly, be necessary to compute separately the areas $OD$ and $DBC$ and take their sum.

$$\text{Area } ODC = \int_0^{\frac{k}{4}} k^{\frac{1}{2}}x^{\frac{1}{2}}dx = \left[\tfrac{2}{3}k^{\frac{1}{2}}x^{\frac{3}{2}}\right]_0^{\frac{k}{4}} = \tfrac{1}{12}k^2.$$

$$\text{Area } DBC = \int_{\frac{k}{4}}^{\frac{k}{2}} (k - 2x)\,dx = \Big[kx - x^2\Big]_{\frac{k}{4}}^{\frac{k}{2}} = \tfrac{1}{16}k^2.$$

Hence the required area is $\frac{7}{48}k^2$ It is to be noted that the area $DBC$ since it is that of a right triangle, could have been found by the formulas of plane geometry; for the altitude $DC = \dfrac{k}{2}$ and the base $DB = \dfrac{k}{2} - \dfrac{k}{4} = \dfrac{k}{4}$, and hence the area $= \dfrac{k^2}{16}$.

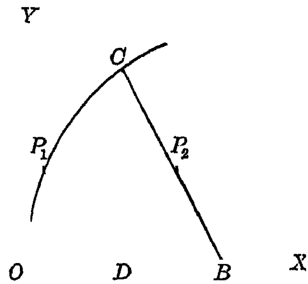



FIG. 88

Or we may construct the element of area as shown in Fig 88

Then, if $x_1$ and $x_2$ are the abscissas respectively of $P_1$ and $P_2$,

$$dA = (x_2 - x_1)\,dy = \left(\frac{k-y}{2} - \frac{y^2}{k}\right)dy.$$

Hence
$$A = \int_0^{\frac{k}{2}}\left(\frac{k-y}{2} - \frac{y^2}{k}\right)dy$$
$$= \left[\frac{ky}{2} - \frac{y^2}{4} - \frac{y^3}{3k}\right]_0^{\frac{k}{2}} = \frac{7}{48}k^2.$$

**Ex. 3.** Let the ellipse of Ex 1 be represented by the equations

$$x = a\cos\phi, \qquad y = b\sin\phi.$$

Using the same element of area, and expressing $y$ and $dx$ in terms of $\phi$ we have

$$dA = (b\sin\phi)(-a\sin\phi\,d\phi)$$
$$= -ab\sin^2\phi\,d\phi$$

As $x$ varies from 0 to $a$, $\phi$ varies from $\dfrac{\pi}{2}$ to 0;

hence
$$A = 4\int_0^a y\,dx = -4\int_{\frac{\pi}{2}}^0 ab\sin^2\phi\,d\phi.$$

It is evident from formula (1), § 23, that the sign of a definite integral is changed by interchanging the limits. Hence

$$A = 4\,ab\int_0^{\frac{\pi}{2}}\sin^2\phi\,d\phi$$
$$= 4\,ab\left[\frac{\phi}{2} - \frac{\sin 2\,\phi}{4}\right]_0^{\frac{\pi}{2}} = \pi ab$$

**Ex. 4.** Find the volume of the ring solid generated by revolving a circle of radius $a$ about an axis in its plane $b$ units from its center ($b > a$)

Take the axis of revolution as $OY$ (Fig. 89) and the line through the center as $OX$. Then the equation of the circle is

$$(x - b)^2 + y^2 = a^2.$$

A straight line parallel to $OX$ meets the circle in two points. $P_1$, where $x = x_1 = b - \sqrt{a^2 - y^2}$, and $P_2$, where $x = x_2 = b + \sqrt{a^2 - y^2}$. A section of the required solid made by a plane through $P_1P_2$ perpendicular to $OY$ is bounded by two concentric circles with radii $MP_1 = x_1$ and $MP_2 = x_2$ respectively. Hence, if $dV$ denotes the element of volume,

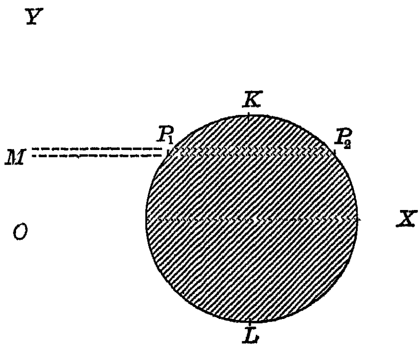



FIG. 89

$$dV = (\pi x_2^2 - \pi x_1^2)\,dy$$
$$= 4\,\pi b\sqrt{a^2 - y^2}\,dy.$$

The summation extends from the point $L$, where $y = -a$, to the point $K$, where $y = a$. On account of symmetry, however, we may take twice the integral from $y = 0$ to $y = a$. Hence

$$V = 2\int_0^a 4\,\pi b\sqrt{a^2 - y^2}\,dy = 2\,\pi^2 a^2 b.$$

**Ex. 5.** Find the pressure on a parabolic segment, with base $2\,b$ and altitude $a$, submerged so that its base is in the surface of the liquid and its axis is vertical

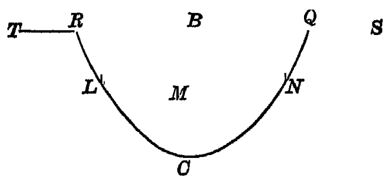



FIG. 90

Let $RQC$ (Fig. 90) be the parabolic segment, and let $CB$ be drawn through the vertex $C$ of the segment perpendicular to $RQ$ in the surface of the liquid. According to the data, $RQ = 2\,b$, $CB = a$. Draw $LN$ parallel to $TS$, and on $LN$ as a base construct an element of area, $dA$. Let

$$CM = x.$$

Then $$dA = (LN)\,dx.$$

But, from § 30, $$\frac{\overline{LN}^2}{\overline{RQ}^2} = \frac{CM}{CB};$$

whence $$\overline{LN}^2 = \frac{4\,b^2x}{a},$$

and therefore $$dA = \frac{2\,b}{a^{\frac{1}{2}}}x^{\frac{1}{2}}\,dx.$$

The depth of $LN$ below the surface of the liquid is $CB - CM = a - x$; hence, if $w$ is the weight of a unit volume of the liquid,

$$dP = \frac{2\,b}{a^{\frac{1}{2}}}x^{\frac{1}{2}}(a - x)\,w dx,$$

and
$$P = \int_0^a \frac{2\,bw}{a^{\frac{1}{2}}}x^{\frac{1}{2}}(a - x)\,dx$$
$$= \tfrac{8}{15}\,wba^2.$$

## EXERCISES

1. Find the area of an arch of the curve $y = \sin x$.

2. Find the area bounded by the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, the axis of $x$, and the lines $x = \pm h$.

3. Find the area included between the curve $y = \frac{8\,a^3}{x^2 + 4\,a^2}$ and its asymptote.

4. Find the area of one of the closed figures bounded by the curves $y^2 = 16\,x$ and $y^2 = x^3$.

5. Find the area bounded by the curve $y^2 = 2(x - 1)$ and the line $2\,x - 3\,y = 0$

6. Find the area between the axis of $x$ and one arch of the cycloid $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$.

7. Find the volume of the solid generated by revolving about $OY$ the plane surface bounded by $OY$ and the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

8. Any section of a certain solid made by a plane perpendicular to $OX$ is an isosceles triangle with the ends of its base resting on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its altitude equal to the distance of the plane from the center of the ellipse. Find the total volume of the solid.

9. Find the volume of the solid formed by revolving about the line $2\,y + a = 0$ the area bounded by one arch of the curve $y = \sin x$ and the axis of $x$.

10. Find the volume of the solid formed by revolving about the line $y + a = 0$ the area bounded by the circle $x^2 + y^2 = a^2$.

11. Find the volume of the solid formed by revolving about the line $x = a$ the area bounded by that line and the curve $ay^2 = x^3$.

12. A right circular cone with vertical angle 60° has its vertex at the center of a sphere of radius $a$ Find the volume of the portion of the sphere included in the cone.

13. A trough 2 ft. deep and 2 ft. broad at the top has semielliptical ends If it is full of water, find the pressure on one end.

14. A parabolic segment with base 18 and altitude 6 is submerged so that its base is horizontal, its axis vertical, and its vertex in the surface of the liquid Find the total pressure.

15. A pond of 15 ft. depth is crossed by a roadway with vertical sides. A culvert, whose cross section is in the form of a parabolic segment with horizontal base on a level with the bottom of the pond, runs under the road. Assuming that the base of the parabolic segment is 4 ft. and its altitude is 3 ft., find the total pressure on the bulkhead which temporarily closes the culvert.

16. Find the pressure on a board whose boundary consists of a straight line and one arch of a sine curve, submerged so that the board is vertical and the straight line is in the surface of the water.

**78. Infinite limits or integrand.** There are cases in which it may seem to be necessary to use infinity for one or both of the limits of a definite integral, or in which the integrand becomes infinite. We shall restrict the discussion of these cases to the solution of the following illustrative examples:

**Ex. 1.** Find the area bounded by the curve $y = \frac{1}{x^2}$ (Fig. 91), the axis of $x$, and the ordinate $x = 1$.

It is seen that the curve has the axis of $x$ as an asymptote; and hence, strictly speaking, the required area is not completely bounded, since the curve and its asymptote do not intersect. Accordingly, in Fig. 91, let $OM = 1$ and $ON = b\ (b > 1)$ and draw the ordinates $MP$ and $NQ$. Then

$$\text{area } MNQP = \int_1^b \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^b = 1 - \frac{1}{b}.$$

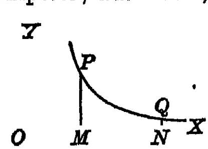



FIG. 91

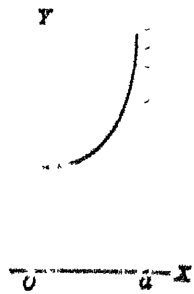
If the value of $b$ is increased, the boundary line $NQ$ moves to the right; and the greater $b$ becomes, the nearer the area approaches unity.

We may, accordingly, define the area bounded by the curve, the axis of $x$, and the ordinate $x = 1$ as the limit of the area $MNQP$ as $b$ increases indefinitely, and denote it by the symbol

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^2} = 1.$$

**Ex. 2.** Find the area bounded by the curve $y = \frac{1}{\sqrt{a^2 - x^2}}$ (Fig. 92), the axis of $x$, and the ordinates $x = 0$ and $x = a$.

Since the line $x = a$ is an asymptote of the curve, $y \longrightarrow \infty$ when $x \longrightarrow a$; furthermore, the area is not, strictly speaking, bounded. We may, however, find the area bounded on the right by the ordinate $x = a - h$, where $h$ is a small quantity, with the result



$$\int_0^{a-h} \frac{dx}{\sqrt{a^2 - x^2}} = \left[\sin^{-1}\frac{x}{a}\right]_0^{a-h} = \sin^{-1}\frac{a-h}{a}.$$

If $\qquad h \longrightarrow 0,\ \sin^{-1}\frac{a-h}{a} \longrightarrow \sin^{-1} 1 = \frac{\pi}{2}.$

Hence we may regard $\frac{\pi}{2}$ as the value of the area required, and express it by the integral

$$\int_0^{a} \frac{dx}{\sqrt{a^2 - x^2}} = \lim_{h \to 0} \int_0^{a-h} \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{2}.$$

Fig. 92

**Ex. 3.** Find the value of $\int_1^{\infty} \frac{dx}{\sqrt{x}}$.

Proceeding as in Ex. 1, we place

$$\int_1^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b \to \infty} \int_1^b \frac{dx}{\sqrt{x}}.$$

But $\qquad \int_1^b \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_1^b = 2\sqrt{b} - 2,$

an expression which increases indefinitely as $b \longrightarrow \infty$; hence the given integral has no finite value.

We accordingly conclude that in each case we must determine a limit, and that the problem has no solution if we cannot find a limit.

**79. Area in polar coördinates.** Let $O$ (Fig. 93) be the pole and $OM$ the initial line of a system of polar coördinates $(r, \theta)$, $OP_1$ and $OP_2$ two fixed radius vectors for which $\theta = \theta_1$ and $\theta = \theta_2$ respectively, and $P_1P_2$ any curve for which the equation is $r = f(\theta)$. Required the area $P_1OP_2$.

To construct the differential of area, $dA$, we divide the angle $P_1OP_2$ into parts, $d\theta$. Let $OP$ and $OQ$ be any two consecutive radius vectors; then the angle $POQ = d\theta$. With $O$ as a center and $OP$ as a radius, we draw the arc of a circle, intersecting $OQ$ at $R$. The area of the sector $POR = \frac{1}{2}(OP)^2 d\theta = \frac{1}{2}r^2\, d\theta$.

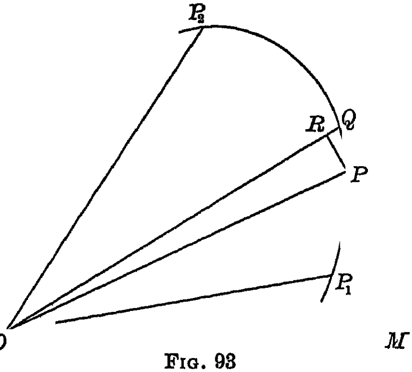



FIG. 93

It is obvious that the required area is the limit of the sum of the sectors as their number is indefinitely increased. Therefore we have

$$dA = \frac{1}{2}r^2 d\theta$$

and

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 d\theta.$$

This result is unchanged if $P_1$ coincides with $O$, but in that case $OP_1$ must be tangent to the curve. So also $P_2$ may coincide with $O$.

**Ex. 1.** Find the area of one loop of the curve $r = a \sin 3\theta$ (Fig 65, § 51)

As the loop is contained between the two tangents $\theta = 0$ and $\theta = \frac{\pi}{3}$, the required area is given by the equation

$$\begin{aligned} A &= \int_0^{\frac{\pi}{3}} \frac{1}{2} a^2 \sin^2 3\theta\, d\theta \\ &= \frac{a^2}{4}\int_0^{\frac{\pi}{3}} (1 - \cos 6\theta)\, d\theta \\ &= \frac{\pi a^2}{12} \end{aligned}$$

**Ex. 2.** Find the area bounded by the lines $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$, the curve $r = 2a\cos\theta$, and the loop of the curve $r = a\cos 2\theta$ which is bisected by the initial line.

Since the loop of the curve $r = a\cos 2\theta$ is tangent to the line $OL$ (Fig. 94), for which $\theta = -\frac{\pi}{4}$, and the line $ON$, for which $\theta = \frac{\pi}{4}$, it is evident that the required area can be found by obtaining the area $OLMNO$, bounded by the lines $OL$ and $ON$ and the curve $r = 2a\cos\theta$, and subtracting from it the area of the loop. The area may also be found as follows:

Let $OP_1P_2$ be any radius vector cutting the loop $r = a\cos 2\theta$ at $P_1$ and the curve $r = 2a\cos\theta$ at $P_2$. Let $OP_1 = r_1$ and $OP_2 = r_2$. Draw the radius vector $OQ_1Q_2$, making an angle $d\theta$ with $OP_1P_2$. With $OP_1$ and $OP_2$ as radii and $O$ as a center, construct arcs of circles intersecting $OQ_1Q_2$ at $R_1$ and $R_2$ respectively. Then the area of the sector $P_1OR_1$ is $\frac{1}{2}r_1^2\,d\theta$ and the area of the sector $P_2OR_2$ is $\frac{1}{2}r_2^2\,d\theta$. We may now take the area $P_1P_2R_2R_1$ as $dA$, and have

$$dA = \tfrac{1}{2}(r_2^2 - r_1^2)\,d\theta$$

Then $A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(r_2^2 - r_1^2)\,d\theta$;

or, since the required area is symmetrical with respect to the line $OM$, we may place

$$A = 2\int_0^{\frac{\pi}{4}} \tfrac{1}{2}(r_2^2 - r_1^2)\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} (r_2^2 - r_1^2)\,d\theta.$$

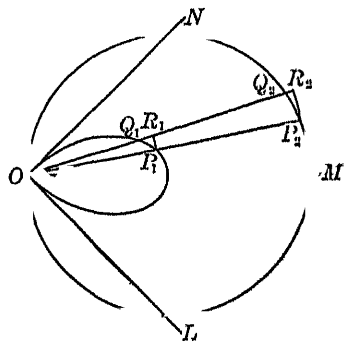



FIG. 94

From the curve $r = 2a\cos\theta$, we have $r_2^2 = 4a^2\cos^2\theta$, and from the curve $r = a\cos 2\theta$, we have $r_1^2 = a^2\cos^2 2\theta$; so that finally

$$A = \int_0^{\frac{\pi}{4}} (4a^2\cos^2\theta - a^2\cos^2 2\theta)\,d\theta$$

$$= a^2\left[2\theta + \sin 2\theta - \frac{\theta}{2} - \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{a^2}{8}(3\pi + 8)$$

**EXERCISES**

1. Find the total area of the lemniscate $r^2 = 2a^2\cos 2\theta$.

2. Find the area of one loop of the curve $r = a\sin n\theta$.

3. Find the total area of the cardioid $r = a(1 + \cos\theta)$.

4. Find the total area bounded by the curve $r = 5 + 3\cos\theta$.

5. Find the area of the loop of the curve $r^2 = a^2\cos 2\theta\cos\theta$ which is bisected by the initial line.

6. Find the area bounded by the curves $r = a\cos 3\theta$ and $r = a$.

7. Find the total area bounded by the curve $r = 3 + 2\cos 4\theta$.

8. Find the area bounded by the curve $r\cos^2\frac{\theta}{2} = 1$ and the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$.

9. Find the area bounded by the curves $r = 6 + 4\cos\theta$ and $r = 4\cos\theta$

10. Find the area bounded by the curves $r = a\cos\theta$ and $r^2 = a^2\cos 2\theta$.

**80. Mean value of a function.** Let $f(x)$ be any function of $x$ and let $y = f(x)$ be represented by the curve $AB$ (Fig. 95), where $OM = a$ and $ON = b$. Take the points $M_1$, $M_2$, $\cdots$, $M_{n-1}$ so as to divide distance $MN$ into $n$ *equal* parts, each equal to $dx$, and at the points $M$, $M_1$, $M_2$, $\cdots$, $M_{n-1}$ erect the ordinates $y_0$, $y_1$, $y_2$, $\cdots$, $y_{n-1}$. Then the average, or mean, value of these $n$ ordinates is

$$\frac{y_0 + y_1 + y_2 + \cdots + y_{n-1}}{n}.$$

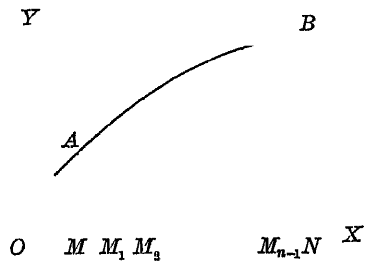


FIG. 95

This fraction is equal to

$$\frac{(y_0 + y_1 + y_2 + \cdots + y_{n-1})\,dx}{n\,dx} = \frac{y_0\,dx + y_1\,dx + y_2\,dx + \cdots + y_{n-1}\,dx}{b - a}.$$

If $n$ is indefinitely increased, this expression approaches as a limit the value

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

This is evidently the mean value of an "infinite number" of values of the function $f(x)$ taken at equal distances between the values $x = a$ and $x = b$. It is called the *mean value* of the function for that interval.

Graphically this value is the altitude of a rectangle with the base $MN$ which has the same area as $MNBA$ which equals

$$\int_a^b f(x)\,dx.$$

We see from the above discussion that the average of the function $y$ depends upon the variable $x$ of which the equal intervals $dx$ were taken, and we say that the function was averaged with respect to $x$. If the function can also be averaged with respect to some other variable which is divided into equal parts the result may be different. This is illustrated in the examples which follow.

**Ex. 1.** Find the mean velocity of a body falling from rest during the time $t_1$ if the velocity is averaged with respect to the time.

Here we imagine the time from 0 to $t_1$ divided into equal intervals $dt$ and the velocities at the beginning of each interval averaged. Proceeding as in the text, we find, since $v = gt$, that the mean velocity equals

$$\frac{1}{t_1 - 0}\int_0^{t_1} gt\,dt = \tfrac{1}{2}gt_1.$$

Since the velocity is $gt_1$ when $t = t_1$, it appears that in this case the mean velocity is half the final velocity.

**Ex. 2** Find the mean velocity of a body falling from rest through a distance $s_1$ if the velocity is averaged with respect to the distance.

Here we imagine the distance from 0 to $s_1$ divided into equal intervals $ds$ and the velocities at the beginning of each interval averaged Proceeding as in the text, we find, since $v = \sqrt{2gs}$, that the mean velocity is

$$\frac{1}{s_1 - 0}\int_0^{s_1} \sqrt{2gs}\,ds = \tfrac{2}{3}\sqrt{2gs_1}.$$

Since the velocity is $\sqrt{2gs_1}$, when $s = s_1$, we see that in this case the mean velocity is two thirds the final velocity.

### EXERCISES

1. Find the mean value of the lengths of the perpendiculars from a diameter of a semicircle to the circumference, assuming the perpendiculars to be drawn at equal distances on the diameter.

2. Find the mean length of the perpendiculars drawn from the circumference of a semicircle to its diameter, assuming the perpendiculars to be drawn at equal distances on the circumference

3. Find the mean value of the ordinates of the curve $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$, assuming that the points at which the ordinates are drawn are at equal distances on the axis of $x$.

4. The range of a projectile fired with an initial velocity $v_0$ and an elevation $\alpha$ is $\frac{v_0^2}{g}\sin 2\alpha$. Find the mean range as $\alpha$ varies from 0 to $\frac{\pi}{2}$, averaging with respect to $\alpha$.

5. Find the mean area of the plane sections of a right circular cone of altitude $h$ and radius $a$ made by planes perpendicular to the axis at equal distances apart.

6. In a sphere of radius $a$ a series of right circular cones is inscribed, the bases of which are perpendicular to a given diameter at equidistant points. Find the mean volume of the cones

7. The angular velocity of a certain revolving wheel varies with the time until at the end of 5 min. it becomes constant and equal to 200 revolutions per minute If the wheel starts from rest, what is its mean angular velocity with respect to the time during the interval in which the angular velocity is variable?

8. The formula connecting the pressure $p$ in pounds per square inch and the volume $v$ in cubic inches of a certain gas is $pv = 20$. Find the average pressure as the gas expands from $2\frac{1}{2}$ cu in. to 5 cu. in.

9. Show that if $y$ is a linear function of $x$, the mean value of $y$ with respect to $x$ is equal to one half the sum of the first and the last value of $y$ in the interval over which the average is taken.

**81. Length of a plane curve.** To find the length of any curve $AB$ (Fig. 96), assume $n-1$ points, $P_1$, $P_2$, $\cdots$, $P_{n-1}$, between $A$ and $B$ and connect each pair of consecutive points by a straight line. The length of $AB$ is then defined as the limit of the sum of the lengths of the $n$ chords $AP_1$, $P_1P_2$, $P_2P_3$, $\cdots$, $P_{n-1}B$ as $n$ is increased without limit and the length of each chord approaches zero as a limit. By means of this definition we have already shown (§§ 39 and 52) that

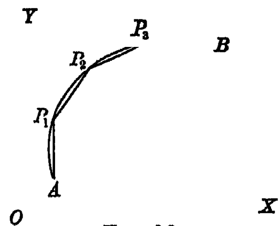


FIG. 96

$$ds = \sqrt{dx^2 + dy^2} \tag{1}$$

in Cartesian coördinates, and

$$ds = \sqrt{dr^2 + r^2 d\theta^2} \tag{2}$$

in polar coördinates.

Hence we have
$$s = \int \sqrt{dx^2 + dy^2} \tag{3}$$

and
$$s = \int \sqrt{dr^2 + r^2 d\theta^2}. \tag{4}$$

To evaluate either (3) or (4) we must express one of the variables involved in terms of the other, or both in terms of a third. The limits of integration may then be determined.

**Ex. 1.** Find the length of the parabola $y^2 = kx$ from the vertex to the point $(a, b)$.

From the equation of the parabola we find $2\,y\,dy = k\,dx$ Hence formula (3) becomes either

$$s = \int_0^a \sqrt{1 + \frac{k^2}{4\,y^2}}\,dx = \int_0^a \sqrt{\frac{4\,x + k}{4\,x}}\,dx,$$

or
$$s = \int_0^b \sqrt{\frac{4\,y^2}{k^2} + 1}\,dy = \frac{1}{k}\int_0^b \sqrt{4\,y^2 + k^2}\,dy.$$

Either integral leads to the result

$$s = \frac{b}{2\,k}\sqrt{4\,b^2 + k^2} + \frac{k}{4}\ln\frac{2\,b + \sqrt{4\,b^2 + k^2}}{k}.$$

**Ex. 2.** Find the length of one arch of the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi).$$

We have $\quad dx = a(1 - \cos\phi)\,d\phi, \qquad dy = a\sin\phi\,d\phi;$

whence, from (1), $\quad ds = a\sqrt{2 - 2\cos\phi}\,d\phi = 2\,a\sin\frac{\phi}{2}\,d\phi.$

Therefore $\quad s = 2\,a\int_0^{2\pi}\sin\frac{\phi}{2}\,d\phi = 8\,a.$

## EXERCISES

1. Find the length of the curve $3\,y^2 = (x - 1)^3$ from its point of intersection with $OX$ to the point $(4, 3)$.

2. Find the length of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ from $x = 0$ to $x = h$.

3. Find the total length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

4. Find the total length of the curve $x = a\cos^3\phi$, $y = a\sin^3\phi$.

5. Find the length of the curve

$$x = a\cos\phi + a\phi\sin\phi, \qquad y = a\sin\phi - a\phi\cos\phi,$$

from $\phi = 0$ to $\phi = 4\,\pi$.

6. Find the length of the curve $x = e^{-t}\cos t$, $y = e^{-t}\sin t$, between the points for which $t = 0$ and $t = \frac{\pi}{2}$.

7. Find the length of the curve $r = a\cos^4\frac{\theta}{4}$ from the point on the curve for which $\theta = 0$ to the pole

8. Find the total length of the curve $r = a(1 + \cos\theta)$.

9. Show that the length of the logarithmic spiral $r = e^{a\theta}$ between any two points is proportional to the difference of the radius vectors of the points.

**82. Work.** By definition the work done in moving a body against a constant force is equal to the force multiplied by the distance through which the body is moved. If the foot is taken as the unit of distance and the pound is taken as the unit of force, the unit of measure of work is called a *foot-pound*. Thus the work done in lifting a weight of 25 lb. through a distance of 50 ft. is 1250 ft.-lb.

Suppose now that a body is moved along $OX$ (Fig. 97) from $A\,(x=a)$ to $B\,(x=b)$ against a force which is not constant but is a function of $x$, expressed by $f(x)$. Let the line $AB$ be divided into intervals each equal to $dx$, and let one of these intervals be $MN$, where $OM=x$. Then the force at the point $M$ is $f(x)$, and if the force were constantly equal to $f(x)$ throughout the interval $MN$, the work done in moving the body through $MN$ would be $f(x)\,dx$. This expression therefore represents approximately the work actually done, and the approximation becomes more and more nearly exact as $MN$ is taken smaller and smaller. The work done in moving from $A$ to $B$ is the limit of the sum of the terms $f(x)\,dx$ computed for all the intervals between $A$ and $B$. Hence we have

Fig. 97

$$dW=f(x)\,dx$$

and
$$W=\int_a^b f(x)\,dx.$$

Ex. The force which resists the stretching of a spring is proportional to the amount the spring has been already stretched. For a certain spring this force is known to be 10 lb. when the spring has been stretched $\frac{1}{2}$ in. Find the work done in stretching the spring 1 in. from its natural (unstretched) length.

If $F$ is the force required to stretch the spring through a distance $x$, we have, from the statement of the problem,

$$F=kx;$$

and since $F=10$ when $x=\frac{1}{2}$, we have $k=20$. Therefore $F=20x$.

Reasoning as in the text, we have

$$W=\int_0^1 20\,x\,dx=10 \text{ in.-lb.}$$

**EXERCISES**

1. A positive charge $m$ of electricity is fixed at $O$. The repulsion on a unit charge at a distance $x$ from $O$ is $\frac{m}{x^2}$. Find the work done in bringing a unit charge from infinity to a distance $a$ from $O$.

2. Assuming that the force required to stretch a wire from the length $a$ to the length $a + x$ is proportional to $\frac{x}{a}$, and that a force of 1 lb stretches a certain wire 36 in in length to a length .03 in. greater, find the work done in stretching that wire from 36 in. to 40 in.

3. A block slides along a straight line from $O$ against a resistance equal to $\frac{ka^2}{x^2 + a^2}$, where $k$ and $a$ are constants and $x$ is the distance of the block from $O$ at any time. Find the work done in moving the block from a distance $a$ to a distance $a\sqrt{3}$ from $O$.

4. Find the foot-pounds of work done in lifting to a height of 20 ft above the top of a tank all the water contained in a full cylindrical tank of radius 2 ft. and altitude 10 ft.

5. A bag containing originally 80 lb. of sand is lifted through a vertical distance of 8 ft. If the sand leaks out at such a rate that while the bag is being lifted, the number of pounds of sand lost is equal to a constant times the square of the number of feet through which the bag has been lifted, and a total of 20 lb. of sand is lost during the lifting, find the number of foot-pounds of work done in lifting the bag.

6. A body moves in a straight line according to the formula $x = ct^2$, where $x$ is the distance traversed in a time $t$. If the resistance of the air is proportional to the square of the velocity, find the work done against the resistance of the air as the body moves from $x = 0$ to $x = a$.

7. Assuming that above the surface of the earth the force of the earth's attraction varies inversely as the square of the distance from the earth's center, find the work done in moving a weight of $w$ pounds from the surface of the earth to a distance $a$ miles above the surface.

8. A wire carrying an electric current of magnitude $C$ is bent into a circle of radius $a$. The force exerted by the current upon a unit magnetic pole at a distance $x$ from the center of the circle in a straight line perpendicular to the plane of the circle is known to be $\frac{2\pi Ca^2}{(a^2 + x^2)^{\frac{3}{2}}}$. Find the work done in bringing a unit magnetic pole from infinity to the center of the circle along the line just mentioned.

9. A piston is free to slide in a cylinder of cross section $S$ The force acting on the piston is $pS$, where $p$ is the pressure of the gas in the cylinder, and is 7.7 lb. per square inch when the volume $v$ is 2 5 cu in. Find the work done as the volume changes from 2 cu. in. to 6 cu. in., according as the law connecting $p$ and $v$ is (1) $pv = k$ or (2) $pv^{1\,4} = k$

## GENERAL EXERCISES

1. Find the area of the sector of the ellipse $4x^2 + 9y^2 = 36$ cut out of the first quadrant by the axis of $x$ and the line $2y = x$.

2. Find the area of each of the two parts into which the area of the circle $x^2 + y^2 = 36$ is divided by the curve $y^2 = x^3$.

3. Find the area bounded by the hyperbola $xy = 12$ and the straight line $x + y - 8 = 0$.

4. Find the area bounded by the parabola $x^2 = 4ay$ and the curve $y = \dfrac{8a^3}{x^2 + 4a^2}$.

5. Find the area of the loop of the curve $ay^2 = (x - a)(x - 2a)^2$.

6. Find the area of the two parts into which the loop of the curve $y^2 = x^2(4 - x)$ is divided by the line $x - y = 0$.

7. Find the area bounded by the curve $x^2y^2 + a^2b^2 = a^2y^2$ and its asymptotes.

8. Find the area bounded by the curve $y^2(x^2 + a^2) = a^2x^2$ and its asymptotes.

9. Find the area bounded by the curve $x = a\cos\theta$, $y = b\sin^3\theta$

10. Find the area inclosed by the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$.

11. Two parabolas have a common vertex and a common axis, but lie in perpendicular planes. An ellipse moves with its plane perpendicular to the axis and with the ends of its axes on the parabolas. Find the volume generated when the ellipse has moved a distance $h$ from the common vertex of the parabolas.

12. Find the volume of the solid formed by revolving about the line $x = 4$ the figure bounded by the parabola $y^2 = 4x$ and the line $x = 1$.

13. A right circular cylinder of radius $a$ is intersected by two planes, the first of which is perpendicular to the axis of the cylinder and the second of which makes an angle $\theta$ with the first. Find the volume of the portion of the cylinder included between these two planes if their line of intersection is tangent to the circle cut from the cylinder by the first plane.

14. On the double ordinate of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ as a base, an isosceles triangle is constructed with its altitude equal to the ordinate and its plane perpendicular to the plane of the curve. Find the volume generated as the triangle moves from $x = -a$ to $x = a$.

15. Find the volume of the solid generated by revolving about the line $OY$ the figure bounded by the curve $y = \dfrac{8a^3}{x^2 + 4a^2}$ and the line $y = a$.

16. Find the volume of the solid formed by revolving about the line $x = -2$ the plane area bounded by that line, the parabola $y^2 = 3x$, and the lines $y = \pm 3$.

17. Find the volume formed by revolving about the line $x = 2$ the plane figure bounded by the curve $y^2 = 4(2 - x)$ and the axis of $y$.

18. The sections of a solid made by planes perpendicular to $OY$ are circles with one diameter extending from the curve $y^2 = 4x$ to the curve $y^2 = 4 - 4x$. Find the volume of the solid between the points of intersection of the curves.

19. The area bounded by the circle $x^2 + y^2 - 2ax = 0$ is revolved about $OX$, forming a solid sphere. Find the volume of the two parts into which the sphere is divided by the surface formed by revolving the curve $y^2 = \dfrac{x^3}{2a - x}$ about $OX$.

20. Find the volume of the two solids formed by revolving about $OY$ the areas bounded by the curves $x^2 + y^2 = 5$ and $y^2 = 4x$.

21. Find the volume of the solid formed by revolving about $OX$ the area bounded by $OX$, the lines $x = 0$ and $x = a$, and the curve $y = x + ae^{\frac{x}{a}}$.

22. The three straight lines $OA$, $OB$, and $OC$ determine two planes which intersect at right angles in $OA$. The angle $AOB$ is 45° and the angle $AOC$ is 60°. The section of a certain solid made by any plane perpendicular to $OA$ is a quadrant of an ellipse, the center of the ellipse being in $OA$, an end of an axis of the ellipse being in $OB$, and an end of the other axis of the ellipse being in $OC$. Find the volume of this solid between the point $O$ and a plane perpendicular to $OA$ at a distance of two units from $O$.

23. The section of a solid made by any plane perpendicular to $OX$ is a rectangle of dimensions $x^2$ and $\sin x$, $x$ being the distance of the plane from $O$. Find the volume of this solid included between the planes for which $x = 0$ and $x = \pi$.

24. An oil tank is in the form of a horizontal cylinder the ends of which are circles 4 ft. in diameter. The tank is full of oil, which weighs 50 lb. per cubic foot Calculate the pressure on one end of the tank.

25. The gasoline tank of an automobile is in the form of a horizontal cylinder the ends of which are plane ellipses 20 in high and 10 in broad. Assuming $w$ as the weight of a cubic inch of gasoline, find the pressure on one end of the tank when the gasoline is 15 in deep.

26. A horizontal gutter is U-shaped, a semicircle of radius 3 in, surmounted by a rectangle 6 in wide by 4 in. deep If the gutter is full of water and a board is placed across the end, how much pressure is exerted on the board?

27. The end of a horizontal gutter is in the form of a semicircle of 3 in. radius, the diameter of the semicircle being at the top and horizontal. The gutter receives water from a roof 50 ft above the top of the gutter. If the pipe leading from the roof to the gutter is full, what is the pressure on a board closing the end of the gutter?

28. A circular water main has a diameter of 5 ft. One end is closed by a bulkhead, and the other is connected with a reservoir in which the surface of the water is 20 ft above the center of the bulkhead Find the total pressure on the bulkhead.

29. Find the area of a loop of the curve $r^2 = a^2 \sin n\theta$.

30. Find the area swept over by a radius vector of the curve $r = a \tan\theta$ as $\theta$ changes from 0 to $\frac{\pi}{4}$.

31. Find the area inclosed by the curve $r = \frac{4}{1 - \cos\theta}$ and the curve $r = \frac{4}{1 + \cos\theta}$.

32. Find the area bounded by the circles $r = a\cos\theta$ and $r = a\sin\theta$.

33. Find the area cut off from one loop of the curve $r^2 = 2a^2 \sin 2\theta$ by the circle $r = a$

34. Find the area of the segment of the cardioid $r = a(1 + \cos\theta)$ cut off by a straight line perpendicular to the initial line at a distance $\frac{3}{4}a$ from the origin $O$.

35. Find the area cut off from a loop of the curve $r = a\sin 3\theta$ by the circle $r = \frac{a\sqrt{3}}{2}$.

36. Find the area cut off from the lemniscate $r^2 = 2a^2 \cos 2\theta$ by the straight line $r \cos \theta = \frac{a\sqrt{3}}{2}$.

37. Find each of the three areas bounded by the curves $r = a$ and $r = a(1 + \sin\theta)$.

38. Find the mean height of the curve $y = \frac{8a^3}{x^2 + 4a^2}$ between the lines $x = -2a$ and $x = 2a$.

39. A particle describes a simple harmonic motion defined by the equation $s = a \sin kt$. Show that the mean kinetic energy $\left(\frac{mv^2}{2}\right)$ during a complete vibration is half the maximum kinetic energy if the average is taken with respect to the time.

40. In the motion defined in Ex 39 what will be the ratio of the mean kinetic energy during a complete vibration to the maximum kinetic energy, if the average is taken with respect to the space traversed?

41. A quantity of steam expands according to the law $pv^{0.8} = 2000$, $p$ being the pressure in pounds absolute per square foot Find the average pressure as the volume $v$ increases from 1 cu. ft. to 5 cu. ft.

42. Find the length of the curve $y = a \ln \frac{a^2}{a^2 - x^2}$ from the origin to the point for which $x = \frac{a}{2}$

43. Find the length of the curve $y = \ln \frac{e^x + 1}{e^x - 1}$ between the points for which $x = 1$ and $x = 2$ respectively.

44. Find the total length of the curve $x = a \cos^3 \phi$, $y = b \sin^3 \phi$.

45. Find the total length of the curve $r = a \sin^3 \frac{\theta}{3}$.

46. Find the length of the spiral $r = a\theta$ from the pole to the end of the first revolution.

47. If a center of force attracts with a magnitude equal to $\frac{k}{x^{\frac{5}{3}}}$, where $x$ is the distance of the body from the center, how much work will be done in moving the body in a straight line away from the center, from a distance $a$ to a distance $8a$ from the center?

48. A body is moved along a straight line toward a center of force which repels with a magnitude equal to $kx$ when the body is at a distance $x$ from the center. How much work will be done in moving the body from a distance $2a$ to a distance $a$ from the center?

**49.** A central force attracts a body at a distance $x$ from the center by an amount $\frac{k}{x^3}$. Find the work done in moving the body directly away from the center from a distance $a$ to the distance $2a$.

**50.** How much work is done against hydrostatic pressure in raising a plate 2 ft. square from a depth of 20 ft. to the surface of the water, if it is kept at all times parallel to the surface of the water?

**51.** A spherical bag of radius 5 in. contains gas at a pressure equal to 15 lb. per square inch. Assuming that the pressure is inversely proportional to the volume occupied by the gas, find the work required to compress the bag into a sphere of radius 4 in.

# CHAPTER XI

## REPEATED INTEGRATION

**83. Double integrals.** The symbol

$$\int_a^b \int_{y_1}^{y_2} f(x, y)\, dx\, dy, \tag{1}$$

in which $a$ and $b$ are constants and $y_1$ and $y_2$ are either constants or functions of $x$, indicates that two integrations are to be carried out in succession. The first integral to be evaluated is

$$\int_{y_1}^{y_2} f(x, y)\, dx\, dy,$$

where $x$ and $dx$ are to be held constant. The result is a function of $x$ only, multiplied by $dx$; let us say, for convenience, $F(x)\, dx$.

The second integral to be evaluated is, then,

$$\int_a^b F(x)\, dx,$$

which is of the familiar type.

Similarly, the symbol

$$\int_a^b \int_{x_1}^{x_2} f(x, y)\, dy\, dx, \tag{2}$$

where $a$ and $b$ are constants and $x_1$ and $x_2$ are either constants or functions of $y$, indicates first the integration

$$\int_{x_1}^{x_2} f(x, y)\, dy\, dx,$$

in which $y$ and $dy$ are handled as constants, and afterwards integration with respect to $y$ between the limits $a$ and $b$.

**Ex. 1.** Evaluate $$\int_0^3\int_0^2 xy\,dx\,dy.$$

The first integral is

$$\int_0^2 xy\,dx\,dy = \left[\tfrac{1}{2}\,xy^2\,dx\right]_0^2 = 2\,x\,dx.$$

The second integration is

$$\int_0^3 2\,x\,dx = \left[x^2\right]_0^3 = 9.$$

**Ex. 2.** Evaluate $$\int_0^1\int_{1-x}^{1-x^2}(x^2+y^2)\,dx\,dy$$

The first integration is

$$\int_{1-x}^{1-x^2}(x^2+y^2)\,dx\,dy = \left[(x^2y+\tfrac{1}{3}y^3)\,dx\right]_{1-x}^{1-x^2} = (x-2\,x^2+\tfrac{4}{3}\,x^3-\tfrac{1}{3}\,x^6)\,dx.$$

The second integration is

$$\int_0^1(x-2\,x^2+\tfrac{4}{3}\,x^3-\tfrac{1}{3}\,x^6)\,dx = \tfrac{5}{42}$$

**Ex. 3.** Evaluate $$\int_0^{2a}\int_0^{\frac{y^2}{4a}} y^2\,dy\,dx.$$

The first integration is

$$\int_0^{\frac{y^2}{4a}} y^2\,dy\,dx = \left[y^2x\,dy\right]_0^{\frac{y^2}{4a}} = \frac{y^4}{4a}\,dy.$$

The second integration is

$$\int_0^{2a}\frac{y^4}{4a}\,dy = \left[\frac{y^5}{20\,a}\right]_0^{2a} = \frac{8}{5}\,a^4.$$

A definite integral in one variable has been shown to be the limit of a sum, from which we infer that formula (1) involves first the determination of the limit of a sum with respect to $y$, followed by the determination of the limit of a sum with respect to $x$. The application of the double integral comes from its interpretation as the limit of a double summation.

How such forms arise in practice will be illustrated in the following sections.

## EXERCISES

Find the values of the following integrals:

1. $\displaystyle\int_3^9\int_1^{y^2}\frac{y}{x^2}\,dy\,dx.$

2. $\displaystyle\int_1^9\int_0^{\frac{1}{x}} xy\,dx\,dy.$

3. $\displaystyle\int_1^3\int_x^{\frac{4}{x}} x^2\,dy\,dx.$

4. $\displaystyle\int_{\frac{\pi}{2}}^{\pi}\int_0^{x^2}\cos\frac{y}{x}\,dx\,dy.$

5. $\displaystyle\int_{\frac{1}{2}}^{1}\int_{y}^{y^2}\frac{dy\,dx}{\sqrt{y^2-x^2}}.$

6. $\displaystyle\int_{2}^{4}\int_{\frac{4}{x}}^{5-x}(x^2+y^2)\,dx\,dy.$

7. $\displaystyle\int_{0}^{\pi}\int_{0}^{a\sin\theta}r\,d\theta\,dr.$

8. $\displaystyle\int_{0}^{\frac{\pi}{4}}\int_{0}^{a\cos 2\theta}\frac{r\,d\theta\,dr}{\sqrt{a^2-r^2}}.$

9. $\displaystyle\int_{0}^{\frac{\pi}{2}}\int_{0}^{a(1+\cos\theta)}r\sin\theta\,d\theta\,dr.$

10. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_{a}^{2a\sin\theta}r^2\,d\theta\,dr.$

11. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{a\cos 2\theta}^{a}\frac{d\theta\,dr}{\sqrt{a+r}}.$

12. $\displaystyle\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sqrt{2}\sin 2\theta}r\cos\theta\,d\theta\,dr$

**84. Area as a double integral.** Let it be required to find an area (such as is shown in Fig. 98) bounded by two curves, with the equations $y_1=f_1(x)$ and $y_2=f_2(x)$ intersecting in points for

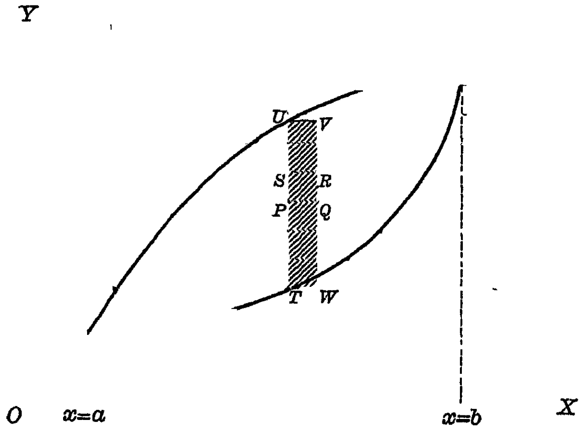



FIG. 98

which $x=a$ and $x=b$ respectively. Let the plane be divided into rectangles by straight lines parallel to $OX$ and $OY$ respectively. Then the area of one such rectangle is

$$dA=dx\,dy, \tag{1}$$

where $dx$ is the distance between two consecutive lines parallel to $OY$, and where $dy$ is the distance between two consecutive lines parallel to $OX$. The sum of the rectangles which are either wholly or partially within the required area will be an approximation to the required area, but only an approximation, because the rectangles will extend partially outside the area. We assume as evident, however, that the sum thus found becomes more nearly equal to the required area as the number of rectangles becomes larger and $dx$ and $dy$ smaller. Hence we say that the required area is the limit of the sum of the terms $dx\,dy$.

The summation must be so carried out as to include every rectangle once and only once. To do this systematically we begin with any rectangle in the interior, such as $PQRS$, and add first those rectangles which lie in the vertical column with it. That is, we take the limit of the sum of $dx\,dy$, with $x$ and $dx$ constant and $y$ varying from $y_1 = f_1(x)$ to $y_2 = f_2(x)$. This is indicated by the symbol

$$\int_{y_1}^{y_2} dx\,dy = (y_2 - y_1)\,dx = [f_2(x) - f_1(x)]\,dx. \tag{2}$$

This is the area of the strip $TUVW$. We are now to take the limit of the sum of all such strips as $dx$ approaches zero and $x$ varies from $a$ to $b$.

We have then

$$A = \int_a^b (y_2 - y_1)\,dx = \int_a^b [f_2(x) - f_1(x)]\,dx. \tag{3}$$

If we put together what we have done, we see that we have

$$A = \int_a^b \int_{y_1}^{y_2} dx\,dy. \tag{4}$$

This discussion enables us to express the area as a double integral. It does not, however, give us any more convenient way to compute the area than that found in Chapter III, for the result (2) is simply what we may write down at once for the area of a vertical strip (see Ex. 3, § 23).

If it should be more convenient first to find the area of a horizontal strip, we may write

$$A = \int_c^d \int_{x_1}^{x_2} dy\,dx. \tag{5}$$

Consider a similar problem in polar coördinates. Let an area, as in Fig. 99, be bounded by two curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, and let the values of $\theta$ corresponding to the points $B$ and $C$ be $\theta_1$ and $\theta_2$ respectively. The plane may be divided into four-sided figures by circles with centers at $O$ and straight lines radiating from $O$. Let the angle between two consecutive radii be $d\theta$ and the distance between two consecutive circles be $dr$. We want first the area of one of the quadrilaterals such as $PQRS$. Here $OP = r$, $PQ = dr$, and the angle $POS = d\theta$. By geometry the area of the sector $POS = \frac{1}{2} r^2 d\theta$ and the area of the sector $QOR = \frac{1}{2}(r + dr)^2 d\theta$;

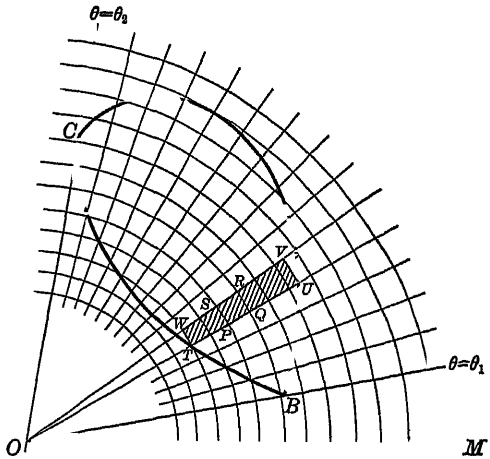



FIG. 99

therefore $PQRS = \frac{1}{2}(r + dr)^2 d\theta - \frac{1}{2} r^2 d\theta = r\,dr\,d\theta + \frac{1}{2}(dr)^2 d\theta.$ Now as $dr$ and $d\theta$ approach zero as a limit the ratio of the second term in this expression to the first term also approaches zero, since this ratio involves the factor $dr$. It may be shown that the second term does not affect the limit of the sum of the expression, and we are therefore justified in writing as the differential of area

$$dA = r\,d\theta\,dr. \tag{6}$$

The required area is the limit of the sum of these differentials. To find it we first take the limit of the sum of the quadrilaterals, such as $PQRS$, which lie in the same sector $UOV$. That is, we integrate $r\,d\theta\,dr$, holding $\theta$ and $d\theta$ constant and allowing $r$ to vary from $r_1$ to $r_2$. We have

$$\int_{r_1}^{r_2} r\,d\theta\,dr = \frac{1}{2}(r_2^2 - r_1^2)\,d\theta, \tag{7}$$

which is the area of the strip $TUVW$.

Finally we take the limit of the sum of the areas of all such strips in the required area and have

$$A=\int_{\theta_1}^{\theta_2}\tfrac{1}{2}(r_2^2-r_1^2)\,d\theta. \tag{8}$$

If we put together what we have done, we may write

$$A=\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}r\,d\theta\,dr. \tag{9}$$

It is clear that this formula leads to nothing which has not been obtained in § 79, but it is convenient sometimes to have the expression (9).

**85. Center of gravity.** It is shown in mechanics that the center of gravity of $n$ particles of masses $m_1, m_2, \cdots, m_n$ lying in a plane at points whose coördinates are $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ respectively is given by the formulas

$$\begin{aligned}\bar{x}&=\frac{m_1x_1+m_2x_2+\cdots+m_nx_n}{m_1+m_2+\cdots+m_n},\\ \bar{y}&=\frac{m_1y_1+m_2y_2+\cdots+m_ny_n}{m_1+m_2+\cdots+m_n}.\end{aligned} \tag{1}$$

This is the point through which the resultant of the weights of the particles always passes, no matter how the particles are placed with respect to the direction of the earth's attraction.

We now wish to extend formulas (1) so that they may be applied to physical bodies in which the number of particles may be said to be infinite. For that purpose we divide the body into $n$ elementary portions such that the mass of each may be considered as concentrated at a point $(x, y)$. Then, if $m$ is the total mass of the body, the mass of each element is $dm$. We have then to replace the $m$'s of formula (1) by $dm$ and to take the limit of the sums involved in (1) as the number $n$ is indefinitely increased and the elements of mass become indefinitely small. There result the general formulas

$$\bar{x}=\frac{\int x\,dm}{\int dm},\qquad \bar{y}=\frac{\int y\,dm}{\int dm}. \tag{2}$$

To apply these formulas we consider first a slender wire so fine and so placed that it may be represented by a plane curve. More strictly speaking, the curve may be taken as the mathematical line which runs through the center of the physical wire. Let the curve be divided into elements of length $ds$. Then, if $c$ is the area of the cross section of the wire and $D$ is its density, the mass of an element of the wire is $Dc\,ds$. For convenience we place $Dc = \rho$ and write

$$dm = \rho\, ds,$$

where $\rho$ is a constant. If this is substituted in (2), the constant $\rho$ may be taken out of the integrals and canceled, and the result may be written in the form

$$s\bar{x} = \int x ds, \qquad s\bar{y} = \int y ds, \tag{3}$$

where $s$ on the left of the equations is the total length of the curve. These formulas give the *center of gravity of a plane curve.*

**Ex. 1.** Find the center of gravity of one fourth of the circumference of a circle of radius $a$.

Here we know that the total length is $\frac{1}{2}\pi a$, so that, from (3), we have

$$\tfrac{1}{2}\pi a\bar{x} = \int x ds, \qquad \tfrac{1}{2}\pi a\bar{y} = \int y ds.$$

To integrate, it is convenient to introduce the central angle $\phi$ (Fig 100), whence $x = a\cos\phi$, $y = a\sin\phi$, $ds = a d\phi$.

Then $\frac{1}{2}\pi a\bar{x} = \int_0^{\frac{\pi}{2}} a^2 \cos\phi\, d\phi$,

$$\tfrac{1}{2}\pi a\bar{y} = \int_0^{\frac{\pi}{2}} a^2 \sin\phi\, d\phi;$$

whence $\bar{x} = \dfrac{2a}{\pi}, \quad \bar{y} = \dfrac{2a}{\pi}.$

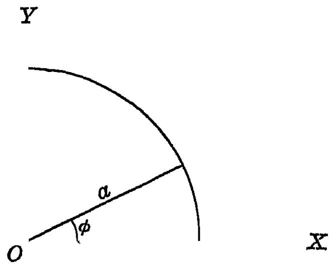


FIG. 100

Consider next a thin plate, which may be represented by a plane area. Strictly speaking, the area is that of a section through the middle of the plate. If $t$ is the thickness of the plate and $D$ its density, the mass of an element of the plate with the area $dA$ is $Dt\,dA$. For convenience we place $Dt = \rho$ and write

$$dm = \rho\, dA,$$

where $\rho$ is a constant. If this is substituted in (2) and the $\rho$'s are canceled, we have

$$A\bar{x} = \int x\,dA, \qquad A\bar{y} = \int y\,dA, \tag{4}$$

where $A$ is the total area. These formulas give the *center of gravity of a plane area.*

**Ex. 2.** Find the center of gravity of the area bounded by the parabola $y^2 = kx$, the axis of $x$, and the ordinate through the point $(a, b)$ of the parabola (Fig 101)

We place $dA = dx\,dy$ in (4) and have

$$A\bar{x} = \int\int x\,dx\,dy, \qquad A\bar{y} = \int\int y\,dx\,dy.$$

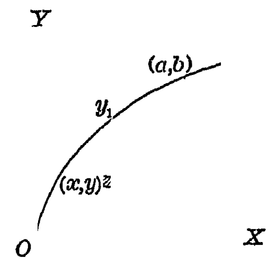



Fig 101

To evaluate, we choose the element $dx\,dy$ inside the area in a general position, and first sum with respect to $y$ along a vertical strip. We shall denote by $y_1$ the value of $y$ on the parabola, to distinguish it from the general values of $y$ inside the area The first integration gives us, therefore, respectively

$$\int_0^{y_1} x\,dx\,dy = xy_1\,dx \quad \text{and} \quad \int_0^{y_1} y\,dx\,dy = \tfrac{1}{2}\,y_1^2\,dx,$$

so that we have $$A\bar{x} = \int x\,y_1\,dx, \qquad A\bar{y} = \int \tfrac{1}{2}\,y_1^2\,dx.$$

On examination of these results we see that each contains the factor $y_1\,dx$ (which is the area (§ 22) of an elementary vertical strip), multiplied respectively by $x$ and $\frac{1}{2}\,y_1$, which are the coordinates of the middle point of the ordinate $y_1$. These results are the same as if we had taken $dA = y_1\,dx$ in the general formula (4), and had taken the point $(x, y)$ at which the mass of $dA$ is concentrated as $(x, \frac{1}{2}\,y_1)$, which is in the limit the middle point of $dA$. In fact this is often done in computing centers of gravity of plane areas, and the first integration is thus avoided.

Now, from the equation of the parabola $y_1^2 = kx$, and to complete the integration, we have to substitute this value for $y_1$ and integrate with respect to $x$ from $x = 0$ to $x = a$ We have

$$A\bar{x} = \int_0^a k^{\frac{1}{2}}x^{\frac{3}{2}}\,dx = \tfrac{2}{5}\,k^{\frac{1}{2}}a^{\frac{5}{2}}, \qquad A\bar{y} = \int_0^a \tfrac{1}{2}\,kx\,dx = \frac{ka^2}{4}.$$

But, from the equation of the curve, $k = \dfrac{b^2}{a}$ and, by § 23, $A = \dfrac{2}{3}\,ab$, Substituting these values and reducing, we have finally

$$\bar{x} = \tfrac{3}{5}\,a, \qquad \bar{y} = \tfrac{3}{8}\,b$$

In solving this problem we have carried out the successive integrations separately, in order to show clearly just what has been done. If now we collect all this into a double integral, we have

$$A\bar{x} = \int_0^a \int_0^{\sqrt{kx}} x\,dx\,dy. \qquad A\bar{y} = \int_0^a \int_0^{\sqrt{kx}} y\,dx\,dy.$$

**Ex. 3.** Find the center of gravity of a sextant of a circle of radius $a$

To solve this problem it is convenient to place the sextant so that the axis of $x$ bisects it (Fig. 102) and to use polar coordinates.

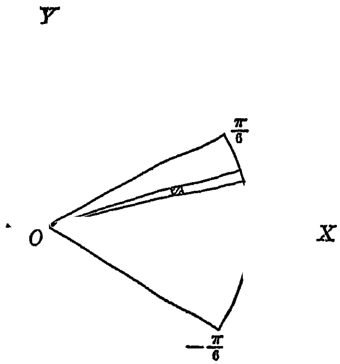



FIG. 102

From the symmetry of the figure the center of gravity lies on $OX$, so that we may write at once $\bar{y} = 0$. To find $\bar{x}$ take an element of area $r\,d\theta\,dr$ in polar coordinates and place $x = r\cos\theta$ We have then, from (4),

$$A\bar{x} = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^a r^2 \cos\theta\,d\theta\,dr,$$

where $A = \frac{1}{6}\pi a^2$, one sixth the area of a circle In the first integration $\theta$ and $d\theta$ are constant, and the summation takes place along a line radiating from $O$ with $r$ varying from 0 to $a$ The angle $\theta$ then varies from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$, and thus the entire area is covered The solution is as follows.

$$\tfrac{1}{6}\pi a^2 \bar{x} = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tfrac{1}{3}\,a^3 \cos\theta\,d\theta = \tfrac{1}{3}\,a^3;$$

whence

$$\bar{x} = \frac{2\,a}{\pi}.$$

Consider now a solid of revolution formed by revolving the plane area (Fig. 103) $ABCD$ about $OY$ as an axis. It is assumed that the equation of the curve $CD$ is given. It is evident from symmetry that the center of gravity of the solid lies on $OY$, so that we have to find only $\bar{y}$.

Let $dV$ be any element of volume. Then $dm = \rho\,dV$, where $\rho$ is the density and is assumed constant. Substituting in (2), we have

$$V\bar{y} = \int y\,dV. \tag{5}$$

Let the solid be divided into thin slices perpendicular to $OY$, as was done in § 26, and let the summation first take place over one of these slices. In this summation $y$ is constant, and the result of the summation is therefore $y$ times the volume of the slice. It is therefore $y(\pi x^2 dy)$. We have now to extend the summation over all the slices. This gives the result

$$V\bar{y} = \int_a^b \pi x^2 y\,dy, \qquad (6)$$

where $OA = a$ and $OB = b$.

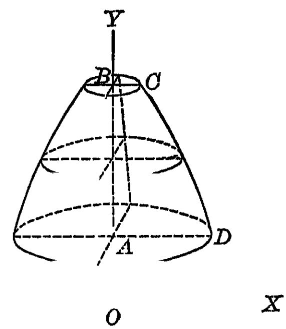


FIG. 103

It is to be noticed that this result is what we obtain if we interpret $dm$ in (2) as the mass of the slice and consider it concentrated at the middle point of one base of the slice.

**Ex. 4.** Find the center of gravity of a right circular cone of altitude $b$ and radius $a$ (Fig. 104)

This is a solid of revolution formed by revolving a right triangle about $OY$. However, the equation of a straight line need not be used, as similar triangles are simpler. We have $\frac{x}{y} = \frac{a}{b}$; whence $x = \frac{a}{b} y$. The volume $V$ is known to be $\frac{1}{3}\pi a^2 b$. Therefore, from (6), we have

$$\tfrac{1}{3}\pi a^2 b\,\bar{y} = \int_0^b \frac{\pi a^2}{b^2} y^3 dy = \frac{\pi a^2 b^2}{4};$$

whence $\qquad \bar{y} = \frac{3}{4} b.$

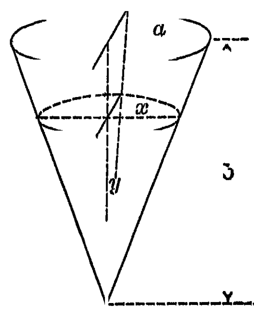


FIG. 104

**EXERCISES**

1. Show that the center of gravity of a semicircumference of radius $a$ lies at a distance of $\frac{2a}{\pi}$ from the center of the circle on the radius which bisects the semicircumference.

2. Show that the center of gravity of a circular arc which subtends an angle $\alpha$ at the center of a circle of radius $a$ lies at a distance $\frac{2a}{\alpha}\sin\frac{\alpha}{2}$ from the center of the circle on the radius which bisects the arc.

3. A wire hangs so as to form the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$. Find the center of gravity of the piece of the curve between the points for which $x = 0$ and $x = a$.

4. Find the center of gravity of the arc of the cycloid $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$, between the first two sharp points.

5. Find the center of gravity of a parabolic segment of base $2b$ and altitude $a$.

6. Find the center of gravity of a quadrant of the area of a circle.

7. Find the center of gravity of a triangle.

8. Find the center of gravity of the area bounded by the curve $y = \sin x$ and the axis of $x$ between $x = 0$ and $x = \pi$.

9. Find the center of gravity of the plane area bounded by the two parabolas $y^2 = 20x$ and $x^2 = 20y$.

10. Find the center of gravity of a figure which is composed of a rectangle of base $2a$ and altitude $b$ surmounted by a semicircle of radius $a$

11. Find the center of gravity of the area bounded by the first arch of the cycloid (Ex. 4) and the axis of $x$.

12. Show that the center of gravity of a sector of a circle lies at a distance $\frac{4a}{3\alpha}\sin\frac{\alpha}{2}$ from the vertex of the sector on a line bisecting the angle of the sector, where $\alpha$ is the angle and $a$ the radius.

13. Find the center of gravity of the area bounded by the cardioid $r = a(1 + \cos\theta)$

14. Find the center of gravity of the area bounded by the curve $r = 2\cos\theta + 3$.

15. Find the center of gravity of a solid hemisphere.

16. Find the center of gravity of a solid formed by revolving about its altitude a parabolic segment of base $2b$ and altitude $a$.

17. Find the center of gravity of the solid formed by revolving about $OY$ the plane figure bounded by the parabola $y^2 = kx$, the axis of $y$, and the line $y = k$.

18. Find the center of gravity of the solid bounded by the surfaces of a right circular cone and a hemisphere of radius $a$, with the base of the cone coinciding with the base of the hemisphere and the vertex of the cone in the surface of the hemisphere.

**86. Center of gravity of a composite area.** In finding the center of gravity of a body which is composed of several parts the following theorem is useful:

*If a body of mass $M$ is composed of several parts of masses $M_1$, $M_2$, $\cdots$, $M_n$, and if the centers of gravity of these parts are respectively $(\bar{x}_1, \bar{y}_1)$, $(\bar{x}_2, \bar{y}_2)$, $\cdots$, $(\bar{x}_n, \bar{y}_n)$, then the center of gravity of the composite body is given by the formulas*

$$\begin{aligned} M\bar{x} &= M_1\bar{x}_1 + M_2\bar{x}_2 + \cdots + M_n\bar{x}_n, \\ M\bar{y} &= M_1\bar{y}_1 + M_2\bar{y}_2 + \cdots + M_n\bar{y}_n. \end{aligned} \tag{1}$$

We shall prove the theorem for the $\bar{x}$ coördinate. The proof for $\bar{y}$ is the same.

By § 85 we have, for the original body,

$$M\bar{x} = \int x\,dm, \tag{2}$$

where the integration is to be taken over all the partial masses $M_1$, $M_2$, $\cdots$, $M_n$ into which the body is divided. But we have also

$$\begin{aligned} M_1\bar{x}_1 &= \int x_1\,dm_1, \\ M_2\bar{x}_2 &= \int x_2\,dm_2, \\ &\cdots\cdots \\ M_n\bar{x}_n &= \int x_n\,dm_n, \end{aligned} \tag{3}$$

where the subscripts indicate that the integration in each case is restricted to one of the several bodies. But formula (2) can be written

$$M\bar{x} = \int x_1\,dm_1 + \int x_2\,dm_2 + \cdots + \int x_n\,dm_n;$$

and, by substitution from (3), the theorem is proved.

**Ex.** Find the center of gravity of an area bounded by two circles one of which is completely inside the other.

Let the two circles be placed as in Fig. 105, where the center of the larger circle of radius $a$ is at the origin, and the center of the smaller circle of radius $b$ is on the axis of $x$ at a distance $c$ from the origin.

The area which can be considered as composed of two parts is that of the larger circle, the two parts being, first, the smaller circle and, second, the irregular ring whose center of gravity we wish to find Now the center of gravity of a circle is known to be at its center. Therefore, in the formula of the theorem, we know $(\bar{x}, \bar{y})$, which is on the left of the equation, to be $(0, 0)$, and $(\bar{x}_1, \bar{y}_1)$ to be $(c, 0)$, and wish to find $(\bar{x}_2, \bar{y}_2)$.

Since we are dealing with areas, we take the masses to be equal to the areas, and have, accordingly, $M = \pi a^2$ (the mass of the larger circle), $M_1 = \pi b^2$ (the mass of the smaller circle), and $M_2 = \pi(a^2 - b^2)$ (the mass of the ring). Substituting in the formula, we have

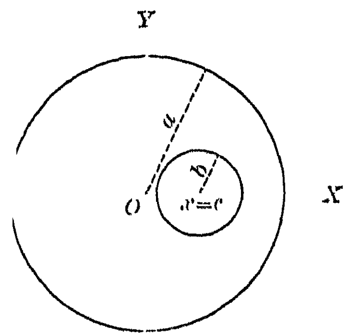



FIG. 105

$$\pi a^2 \cdot 0 = \pi b^2 c + \pi(a^2 - b^2)\bar{x}_2;$$

whence, by solving for $\bar{x}_2$, $\quad \bar{x}_2 = -\dfrac{b^2 c}{a^2 - b^2}$.

It is unnecessary to find $\bar{y}_2$, since, by symmetry, the center of gravity lies on $OX$

**EXERCISES**

1. Show that if there are only two component masses $M_1$ and $M_2$ in formulas (1) of the theorem, the center of gravity of the composite mass lies on the line connecting the centers of gravity of the component masses at such a point as to divide that line into segments inversely proportional to the masses.

2. Prove that if a mass $M_1$ with center of gravity $(\bar{x}_1, \bar{y}_1)$ has cut out of it a mass $M_2$ with center of gravity $(\bar{x}_2, \bar{y}_2)$, the center of gravity of the remaining mass is

$$\bar{x} = \frac{M_1\bar{x}_1 - M_2\bar{x}_2}{M_1 - M_2}, \qquad \bar{y} = \frac{M_1\bar{y}_1 - M_2\bar{y}_2}{M_1 - M_2}.$$

3. Two circles of radii $r_1$ and $r_2$ are tangent externally. Find their center of gravity

4. Find the center of gravity of a hemispherical shell bounded by two concentric hemispheres of radii $r_1$ and $r_2$.

5. Place $r_2 = r_1 + \Delta r$ in Ex. 4, let $\Delta r$ approach zero, and thus find the center of gravity of a hemispherical surface.

6. Find the center of gravity of a hollow right circular cone bounded by two parallel conical surfaces of altitudes $h_1$ and $h_2$ respectively and with their bases in the same plane.

7. Place $h_2 = h_1 + \Delta h$ in Ex 6, let $\Delta h$ approach zero, and thus find the center of gravity of a conical surface

8. Find the center of gravity of a carpenter's square each arm of which is 15 in on its outer edge and 2 in wide

9. From a square of edge 8 in a quadrant of a circle is cut out, the center of the quadrant being at a corner of the square and the radius of the quadrant being 4 in. Find the center of gravity of the figure remaining

10. Two iron balls of radius 4 in and 6 in respectively are connected by an iron rod of length 1 in. Assuming that the rod is a cylinder of radius 1 in., find the center of gravity of the system.

11. A cubical pedestal of side 4 ft. is surmounted by a sphere of radius 2 ft Find the center of gravity of the system, assuming that the sphere rests on the middle point of the top of the pedestal.

**87. Theorems.** The following theorems involving the center of gravity may often be used to advantage in finding pressures, volumes of solids of revolution, or areas of surfaces of revolution.

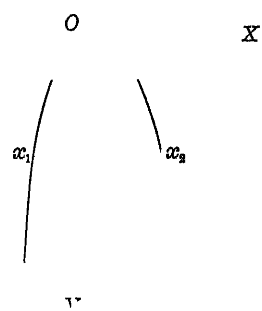


FIG. 106

I. *The total pressure on a plane surface immersed in liquid in a vertical position is equal to the area of the surface multiplied by the pressure at its center of gravity.*

Let the area be placed as in Fig 106, where the axis of $x$ is in the surface of the liquid and where the axis of $y$ is measured downward. Then, by § 25,

$$P = \int wy(x_2 - x_1)dy, \tag{1}$$

which may be written as a double integral in the form

$$P = \iint wy\,dy\,dx = w\iint y\,dx\,dy. \tag{2}$$

In fact, this may be written down directly, since the pressure on a small rectangle $dx\,dy$ is its area, $dx\,dy$, times its depth, $y$, times $w$.

Moreover, from § 85, we have

$$A\bar{y} = \iint y\,dx\,dy. \tag{3}$$

By comparison of (2) and (3) we have

$$P = w\bar{y}A.$$

But $w\bar{y}$ is the pressure at the center of gravity, and the theorem is proved for areas of the above general shape. If the area is not of this shape, it may be divided into such areas, and the theorem may be proved with the aid of the theorem of § 86.

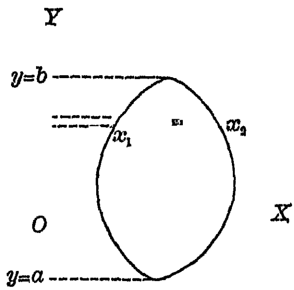


Fig 107

**Ex. 1.** A circular bulkhead which closes the outlet of a reservoir has a radius 3 ft, and its center is 12 ft below the surface of the water. Find the total pressure on it

Here $A = 9\pi$ and the depth of the center of gravity is 12. Therefore

$$P = 108\,\pi w = \tfrac{27}{8}\pi \text{ tons} = 10.6 \text{ tons}.$$

II. *The volume generated by revolving a plane area about an axis in its plane not intersecting the area is equal to the area of the figure multiplied by the circumference of the circle described by its center of gravity.*

To prove this take an area as in Fig. 107. Then, by § 26, if $V$ is the volume generated by the revolution about $OY$,

$$V = \pi\int_a^b (x_2^2 - x_1^2)\,dy, \tag{4}$$

which can be written as a double integral in the form

$$V = 2\pi\int_a^b\int_{x_1}^{x_2} x\,dy\,dx. \tag{5}$$

By § 85,

$$A\bar{x} = \int_a^b\int_{x_1}^{x_2} x\,dy\,dx;$$

and, by comparison of (4) and (5), we have

$$V = 2\pi\bar{x}A,$$

which was to be proved.

**Ex. 2.** Find the volume of the ring surface formed by revolving about an axis in its plane a circle of radius $a$ whose center is at a distance $c$ from the axis, where $c > a$.

We know that $A = \pi a^2$ and that the center of gravity of the circle is at the center of the circle and therefore describes a circumference of length $2\pi c$. Therefore

$$V = 2\pi^2 a^2 c.$$

III. *The area generated by revolving a plane curve about an axis in its plane not intersecting the curve is equal to the length of the curve multiplied by the circumference of the circle described by its center of gravity.*

To prove this we need a formula for the area of a surface of revolution which has not been given. It may be shown that if $S$ is this area, then

$$S = 2\pi \int x\,ds. \tag{6}$$

A rigorous proof of this will not be given here. However, the student may make the formula seem plausible by noticing that an element $ds$ of the curve will generate on the surface a belt of width $ds$ and length $2\pi x$. The product of length by breadth may be taken as the area of the belt.

Moreover, by § 85, we have

$$s\bar{x} = \int x\,ds; \tag{7}$$

and comparing the two equations (6) and (7), we have

$$S = 2\pi s\bar{x},$$

which was to be proved.

**Ex. 3.** Find the area of the ring surface described in Ex. 2.

We know that $s = 2\pi a$ and that the center of gravity of a circumference is at its center and therefore describes a circumference of length $2\pi c$ Therefore

$$S = 4\pi^2 ac.$$

Theorems II and III are known as the *theorems of Pappus.*

**EXERCISES**

1. Find by the theorems of Pappus the volume and the surface of a sphere.

2. Find by the theorems of Pappus the volume and the lateral surface of a right circular cone.

3. Find by the theorems of Pappus the volume generated by revolving a parabolic segment about its altitude.

4. Find by the theorems of Pappus the volume generated by revolving a parabolic segment about its base.

5. Find by the theorems of Pappus the volume generated by revolving a parabolic segment about the tangent at its vertex

6. Find the volume and the surface generated by revolving a square of side $a$ about an axis in its plane perpendicular to one of its diagonals and at a distance $b\left(b > \frac{a}{\sqrt{2}}\right)$ from its center.

7. Find the volume and the area generated by revolving a right triangle with legs $a$ and $b$ about an axis in its plane parallel to the leg of length $a$ on the opposite side from the hypotenuse and at a distance $c$ from the vertex of the right angle.

8. A circular water main has a diameter of 4 ft. One end is closed by a bulkhead, and the other is connected with a reservoir in which the surface of the water is 18 ft. above the center of the bulkhead Find the pressure on the bulkhead.

9. Find the pressure on an ellipse of semiaxes $a$ and $b$ completely submerged, if the center of the ellipse is $c$ units below the surface of the liquid.

10. Find the pressure on a semiellipse of semiaxes $a$ and $b$ $(a > b)$ submerged with the major axis in the surface of the liquid and the minor axis vertical.

11. Find the pressure on a parabolic segment submerged with the base horizontal, the axis vertical, the vertex above the base, and the vertex $c$ units below the surface of the liquid.

12. What is the effect on the pressure of a submerged vertical area in a reservoir if the level of the water in the reservoir is raised by $c$ feet?

**88. Moment of inertia.** The moment of inertia of a particle about an axis is the product of its mass and the square of its distance from the axis. The moment of inertia of a number of particles about the same axis is the sum of the moments of inertia of the separate particles about that axis. From these definitions we may derive the moment of inertia of a thin plate.

Let the surface of the plate be divided into elements of area $dA$. Then the mass of each element is $\rho\, dA$, where $\rho$ is the product of the thickness of the plate and its density. Let $R$ be the distance of any point in the element from the axis about which we wish the moment of inertia. Then the moment of inertia of element is approximately

$$R^2\rho\, dA.$$

We say "approximately" because not all points of the element are exactly a distance $R$ from the axis, as $R$ is the distance of some one point in the element. However, the smaller the element the more nearly can it be regarded as concentrated at one point and the limit of the sum of all the elements, as their size approaches zero and their number increases without limit, is the moment of inertia of the plate. Hence, if $I$ represents the moment of inertia of the plate, we have

$$I = \int R^2\rho\, dA. \qquad (1)$$

If in (1) we let $\rho = 1$, the resulting equation is

$$I = \int R^2 dA, \qquad (2)$$

where $I$ is called the *moment of inertia of the plane area.* When $dA$ in (1) or (2) is replaced by $dx\,dy$ or $r\,dr\,d\theta$, the double sign of integration must be used.

**Ex. 1.** Find the moment of inertia of a rectangle of dimensions $a$ and $b$ about the side of length $b$

Let the rectangle be placed as in Fig. 108. Let it be divided up into elements $dA = dx\,dy$. Then $x$ is the distance of some point in an element from $OY$. Hence, in (2), we have $R = x$ and $dA = dx\,dy$. Therefore

$$I = \int\int x^2\, dx\, dy.$$

We first sum the rectangles in a vertical strip, as $y$ ranges from 0 to $b$. We have

$$\int_0^b x^2\, dx\, dy = x^2 b\, dx.$$

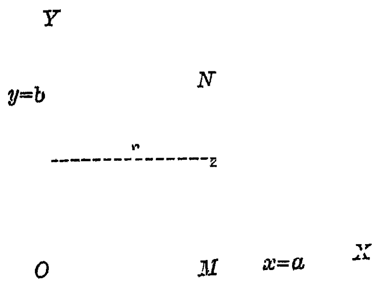


FIG. 108

This is the moment of inertia of the strip $MN$, and might have been written down at once, since all points on the left-hand boundary of the strip are at a distance $x$ from $OY$ and since the area of the strip is $b\,dx$

The second integration gives now

$$I = \int_0^a x^2 b\,dx = \tfrac{1}{3} a^3 b.$$

If, instead of asking for the moment of inertia of the area, we had asked for that of a plate of metal of thickness $t$ and density $D$, the above result would be multiplied by $\rho = Dt$. But in that case the total mass $M$ of the plate is $\rho ab$, so that we have

$$I = \tfrac{1}{3} Ma^2$$

**Ex. 2.** Find the moment of inertia of the quadrant of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ about its major axis

If we take any element of area as $dx\,dy$, we find the distance of its lower edge from the axis about which we wish the moment of inertia to be $y$ (Fig 109). Hence $R = y$ and

$$I = \iint y^2\,dx\,dy.$$

It will now be convenient to sum first with respect to $x$, since each point of a horizontal strip is at the same distance from $OX$ We therefore write

$$I = \iint y^2\,dy\,dx.$$

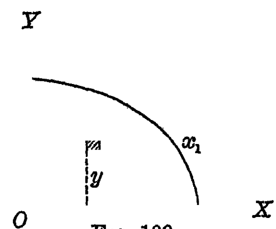


FIG. 109

Now, indicating by $x_1$ the abscissa of a point on the ellipse to distinguish it from the general $x$ which is that of a point inside the ellipse, we have for the first integration

$$\int_0^{x_1} y^2\,dy\,dx = y^2 x_1 dy = \frac{a}{b} y^2 \sqrt{b^2 - y^2}\,dy.$$

For the second integration

$$I = \frac{a}{b}\int_0^b y^2 \sqrt{b^2 - y^2}\,dy$$

To integrate, place $y = b \sin\phi$ Then

$$I = ab^3 \int_0^{\frac{\pi}{2}} \sin^2\phi \cos^2\phi\,d\phi = \frac{\pi a b^3}{16}.$$

If, instead of the area, we consider a thin plate of mass $M$, the above result must be multiplied by $\rho$, where $M = \tfrac{1}{4}\pi ab\rho$, whence

$$I = \tfrac{1}{4} Mb^2.$$

The *polar moment of inertia* of a plane area is defined as the moment of inertia of the area about an axis perpendicular to its plane. This may also be called conveniently the moment of inertia with respect to the point in which the axis cuts the plane of the area, for the distance of an element from the axis is simply its distance from that point. Thus we may speak, for example, of the polar moment of inertia with respect to an axis through the origin perpendicular to the plane of an area, or, more concisely, of the polar moment with respect to the origin.

If the area is divided into elements $dx\,dy$, and one point in the element has the coördinates $(x, y)$, the distance of that point from the origin is $\sqrt{x^2+y^2}$. That is, in (2), if we place $dA = dx\,dy$ and $R^2 = x^2 + y^2$, we shall have the formula for the polar moment of inertia with respect to the origin. Denoting this by $I_0$, we have

$$I_0 = \iint (x^2 + y^2)\,dx\,dy. \tag{3}$$

This integral may be split up into two integrals, giving

$$I_0 = \iint x^2 dx\,dy + \iint y^2 dy\,dx, \tag{4}$$

where the change in the order of the differentials in the two integrals indicates the order in which the integration may be most conveniently carried out.

The first integral in (4) is the moment of inertia about $OY$ and may be denoted by $I_y$; the second integral is the moment of inertia about $OX$ and may be denoted by $I_x$. Therefore formula (4) may be written as

$$I_0 = I_x + I_y, \tag{5}$$

so that the problem of finding the moment of inertia may be reduced to the solving of two problems of the type of the first part of this section.

**Ex. 3.** Find the polar moment of inertia of an ellipse with respect to the origin.

In Ex. 2 we found $I_x$ for a quadrant of the ellipse. For the entire ellipse it is four times as great, since moments of inertia are added by definition. Hence

$$I_x = \tfrac{1}{4}\,\pi a b^3.$$

By a similar calculation $I_y = \frac{1}{4}\pi a^3 b$

Therefore $I_0 = \frac{1}{4}\pi ab\,(a^2 + b^2).$

If the area is replaced by a plate of mass $M$, this result gives

$$I_0 = \frac{1}{4}M(a^2 + b^2)$$

If polar coordinates are used, the element of area is $r\,d\theta\,d$ and the distance of a point in an element from the origin is Hence, in (2), $dA = r\,d\theta\,dr$ and $R = r$. Therefore

$$I_0 = \iint r^3\,d\theta\,dr. \tag{6}$$

In practice it is usually convenient to integrate first wit respect to $r$, holding $\theta$ constant. This is, in fact, to find th polar moment of inertia of a sector with vertex at $O$.

**Ex. 4.** Find the polar moment of inertia of a circle with respect to point on its circumference

Let the circle be placed as in Fig 110 Its equation is then (Ex 1, § 51 $r = 2a\cos\theta$, where $a$ is the radius If we take any element $r\,d\theta\,dr$ an find $I_0$ for all elements which lie in the same sector with it, we have to add the elements $r^3\,d\theta\,dr$, with $r$ ranging from 0 to $r_1$, where $r_1$ is the value of $r$ on the circle, and therefore $r_1 = 2a\cos\theta$ We have

$$\int_0^{r_1} r^3\,d\theta\,dr = \frac{1}{4}r_1^4\,d\theta = 4a^4\cos^4\theta\,d\theta.$$

We have finally to sum these quantities, with $\theta$ ranging from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ We have

$$I_0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4a^4\cos^4\theta\,d\theta = \frac{3}{2}\pi a^4.$$

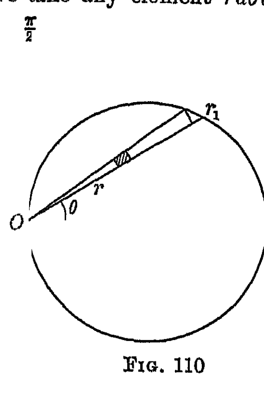



FIG. 110

If $M$ is the mass of a circular plate, this result, multiplied by $\rho$, give

$$I_0 = \frac{3}{2}Ma^2.$$

**Ex. 5.** Find the polar moment of inertia of a circle with respect to i center.

Here it will be convenient first to find the polar moment of inertia of ring (Fig 111). We integrate first with respect to $\theta$, keeping $r$ constan We have

$$\int_0^{2\pi} r^3\,dr\,d\theta = 2\pi r^3\,dr,$$

which is the approximate area of the ring $2\pi r dr$ multiplied by the square of the distance of its inner circumference from the center We then have, by the second integration,

$$I_0 = \int_0^a 2\pi r^3 dr = \tfrac{1}{2}\pi a^4.$$

If $M$ is the mass of a circular plate, this result, multiplied by $\rho$, gives

$$I_0 = \tfrac{1}{2} Ma^2.$$

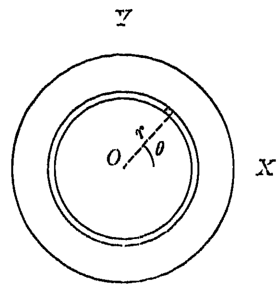



FIG. 111

The moment of inertia of a solid of revolution about the axis of revolution is the sum of the moments of inertia of the circular sections about the same axis; that is, of the polar moments of inertia of the circular sections about their centers. If the axis of revolution is $OY$, the radius of any circular section perpendicular to $OY$ is $x$ and its thickness is $dy$. Its mass is therefore $\rho\pi x^2 dy$; and therefore, by Ex. 5, its moment of inertia about $OY$ is $\frac{1}{2}\rho\pi x^4 dy$. The total moment of inertia of the solid is therefore

$$I = \tfrac{1}{2}\rho\pi \int x^4 dy.$$

**Ex. 6.** Find the moment of inertia of a circular cone about its axis.

Take the cone as in Ex. 4, § 85. Then we have

$$I = \frac{1}{2}\rho\pi \int_0^b \frac{a^4}{b^4} y^4 dy = \frac{1}{10}\rho\pi a^4 b.$$

But, if $M$ is the mass of the cone, we have $M = \frac{1}{3}\rho\pi a^2 b$.

Therefore $$I = \tfrac{3}{10} Ma^2.$$

## EXERCISES

**1.** Find the moment of inertia of a rectangle of base $b$ and altitude $a$ about a line through its center and parallel to its base.

**2.** Find the moment of inertia of a triangle of base $b$ and altitude $a$ about a line through its vertex and parallel to its base.

**3.** Find the moment of inertia of a triangle of base $b$ and altitude $a$ about its base.

**4.** Find the moment of inertia of an ellipse about its minor axis and also about its major axis.

5. Find the moment of inertia of a trapezoid about its lower base, taking the lower base as $b$, the upper base as $a$, and the altitude as $h$.

6. Find the moment of inertia about its base of a parabolic segment of base $b$ and altitude $a$.

7. Find the polar moment of inertia of a rectangle of base $b$ and altitude $a$ about its center.

8. Find the polar moment of inertia about its center of a circular ring, the outer radius being $r_2$ and the inner radius $r_1$.

9. Find the polar moment of inertia of a right triangle of sides $a$ and $b$ about the vertex of the right angle.

10. Find the polar moment of inertia about the origin of the area bounded by the hyperbola $xy=6$ and the straight line $x+y-7=0$.

11. Find the polar moment of inertia about the origin of the area bounded by the curves $y=x^2$ and $y=2-x^2$.

12. Find the polar moment of inertia about the origin of the area of one loop of the lemniscate $r^2=2a^2\cos 2\theta$.

13. Find the moment of inertia of a right circular cylinder of height $h$, radius $r$, and mass $M$, about its axis.

14. Find the moment of inertia about its axis of a hollow right circular cylinder of mass $M$, its inner radius being $r_1$, its outer radius $r_2$, and its height $h$.

15. Find the moment of inertia of a solid sphere about a diameter.

16. A ring is cut from a spherical shell whose inner and outer radii are respectively 5 ft. and 6 ft, by two parallel planes on the same side of the center and distant 1 ft and 3 ft respectively from the center Find the moment of inertia of this ring about its axis

17. The radius of the upper base and the radius of the lower base of the frustum of a right circular cone are respectively $r_1$ and $r_2$, and its mass is $M$. Find its moment of inertia about its axis.

**89. Moments of inertia about parallel axes.** The finding of a moment of inertia is often simplified by use of the following theorem:

*The moment of inertia of a body about an axis is equal to its moment of inertia about a parallel axis through its center of gravity plus the product of the mass of the body by the square of the distance between the axes.*

We shall prove this theorem only for a plane area, in the two cases in which the axes lie in the plane of the figure or are perpendicular to that plane. We shall also consider the mass of the area as equal to the area, as in § 88.

*Case I. When the axes lie in the plane of the figure.*

Let the area be placed as in Fig. 112, where the center of gravity $(\bar{x}, \bar{y})$ is taken as the origin (0, 0) and where the axis of $y$ is taken parallel to the axis $LK$, about which we wish to find the moment of inertia.

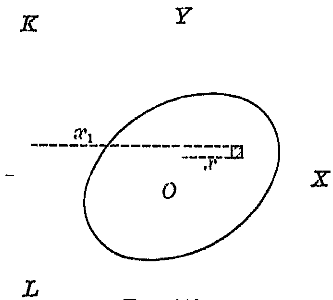



FIG. 112

Let $x$ be the distance of an element $dxdy$ from $OY$, and $x_1$ its distance from $LK$. Then, if $I_g$ is the moment of inertia about $OY$, and $I_l$ the moment of inertia about $LK$, we have

$$I_g = \iint x^2 dxdy, \qquad I_l = \iint x_1^2 dxdy. \tag{1}$$

Moreover, if $a$ is the distance between $OY$ and $LK$, we have

$$x_1 = x + a; \tag{2}$$

so that, by substituting from (2) in the second equation of (1), we have

$$I_l = \iint x^2 dxdy + 2a \iint xdxdy + a^2 \iint dxdy. \tag{3}$$

Now, by § 84, $\iint dxdy = A$; by § 85, $\iint xdxdy = A\bar{x} = 0$, since by hypothesis $\bar{x} = 0$; and, by (1), the first integral on the right hand of (3) is $I_g$. Therefore (3) can be written

$$I_l = I_g + a^2 A, \tag{4}$$

which proves the theorem for this case.

*Case II. When the axes are perpendicular to the plane of the figure.*

We have to do now with polar moments of inertia. Let the area be placed as in Fig. 113, where the center of gravity is taken as the origin, and $P$ is any point about which we wish the polar moment of inertia. Let $I_g$ be the polar moment of inertia about $O$, and $I_p$ the polar moment of inertia about $P$. Draw through $P$ axes $PX'$ and $PY'$ parallel to the axes of coördinates $OX$ and $OY$. Let $I_x$ and $I_y$ be the moments of inertia about $OX$ and $OY$ respectively, and let $I_{x'}$ and $I_{y'}$ be the moments of inertia about $PX'$ and $PY'$. Then, by (5), § 88,

$$I_g = I_x + I_y, \qquad I_p = I_{x'} + I_{y'}. \tag{5}$$

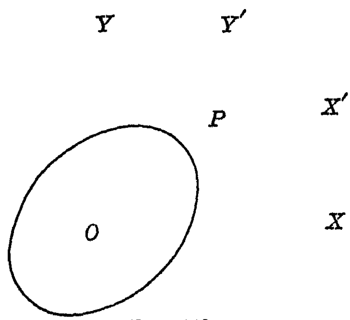



FIG. 113

Moreover, if $(a, b)$ are the coördinates of $P$, we have, by Case I,

$$I_{x'} = I_x + a^2A, \qquad I_{y'} = I_y + b^2A. \tag{6}$$

Therefore, from (5), we have

$$I_p = I_g + (a^2 + b^2)A, \tag{7}$$

which proves the theorem for this case also.

The student may easily prove that the theorem is true also for the moment of inertia of any solid of revolution about an axis parallel to the axis of revolution of the solid.

**Ex.** Find the polar moment of inertia of a circle with respect to a point on the circumference.

The center of gravity of a circle is at its center, and the distance of any point on its circumference from its center is $a$. By Ex 5, § 88, the polar moment of a circle about its center is $\frac{1}{2}\pi a^4$. Therefore, by the above theorem,

$$I_p = \tfrac{1}{2}\pi a^4 + a^2(\pi a^2) = \tfrac{3}{2}\pi a^4.$$

This result agrees with Ex 4, § 88, where the required moment of inertia was found directly

## EXERCISES

1. Find the moment of inertia of a circle about a tangent.

2. Find the polar moment of inertia about an outer corner of a picture frame bounded by two rectangles, the outer one being of dimensions 8 ft. by 12 ft., and the inner one of dimensions 5 ft. by 9 ft.

3. Find the moment of inertia about one of its outer edges of a carpenter's square of which the outer edges are 15 in. and the inner edges 13 in

4. Find the polar moment of inertia about the outer corner of the carpenter's square in Ex. 3.

5. From a square of side 20 a circular hole of radius 5 is cut, the center of the circle being at the center of the square. Find the moment of inertia of the resulting figure about a side of the square.

6. Find the polar moment of inertia about a corner of the square of the figure in Ex. 5.

7. Find the moment of inertia of a hollow cylindrical column of outer radius $r_2$ and inner radius $r_1$ about an element of the inner cylinder.

8. Find the moment of inertia of the hollow column of Ex. 7 about an element of the outer cylinder.

9. Find the moment of inertia of a circular ring of inner radius $r_1$ and outer radius $r_2$ about a tangent to the outer circle.

10. A circle of radius $a$ has cut from it a circle of radius $\frac{a}{2}$ tangent to the larger circle Find the moment of inertia of the remaining figure about the line through the centers of the two circles

11. Find the moment of inertia of the figure in Ex 10 about a line through the center of the larger circle perpendicular to the line of centers of the two circles and in the plane of the circles.

**90. Space coördinates.** In the preceding pages we have become familiar with two methods of fixing the position of a point in a plane; namely, by Cartesian coördinates $(x, y)$, and by polar coördinates $(r, \theta)$. If, now, any plane has been thus supplied with a coördinate system, and, starting from a point in that plane, we measure another distance, called $z$, at right angles to the plane, we can reach any point in space. The quantity $z$ will be considered positive if measured in one direction, and negative if measured in the other. We have, accordingly, two systems of space coördinates.

1. *Cartesian coördinates.* We take any plane, as $XOY$, in which are already drawn a pair of coördinate axes, $OX$ and $OY$, at right angles with each other. Perpendicular to this plane at

the origin we draw a third axis $OZ$ (Fig. 114). If $P$ is any point of space, we draw $PM$ parallel to $OZ$, meeting the plane $XOY$ at $M$, and from $M$ draw a line parallel to $OY$, meeting $OX$ at $L$. Then for the point $P\,(x,\ y,\ z)$, $OL = x$, $LM = y$, and $MP = z$. It is to be noticed that the three axes determine three planes, $XOY$, $YOZ$, and $ZOX$, called the coördinate planes, and that we may just as readily draw the line from $P$ perpendicular to either the plane $YOZ$ or $ZOX$ and then complete the construction as above.

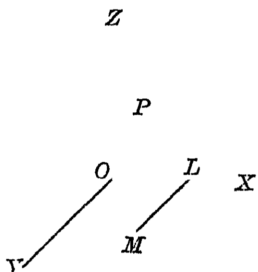


FIG 114

These possibilities are shown in Fig. 115, where it is seen that $x = OL = NM = SR = TP$, with similar sets of values for $y$ and $z$.

2. *Cylindrical coördinates.* Let $XOY$ be any plane in which a fixed point $O$ is the origin of a system of polar coördinates, and $OX$ is the initial line of that system (Fig. 116). Let $OZ$ be an axis perpendicular to the plane $XOY$ at $O$. If $P$ is any point in space, we draw from $P$ a straight line parallel to $OZ$ until it meets the plane $XOY$ at $M$. Then, if the polar coordinates of $M$ in the plane $XOY$ are $r = OM$, $\theta = XOM$, and we denote the distance $MP$ by $z$, the cylindrical coordinates of $P$ are $(r,\ \theta,\ z)$.

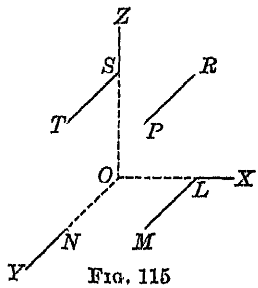


FIG. 115

It is evident that the axes $OX$ and $OZ$ determine a fixed plane, and that the angle $\theta$ is the plane angle of the dihedral angle between that fixed plane and the plane through $OZ$ and the point $P$. If $SP$ is drawn in the latter plane perpendicular to $OZ$, it is evident that $OM = SP = r$ and $OS = MP = z$. The coördinate $r$, therefore, measures the distance of the point $P$ from the axis $OZ$, and the coördinate $z$ measures the distance of $P$ from the plane $XOY$.

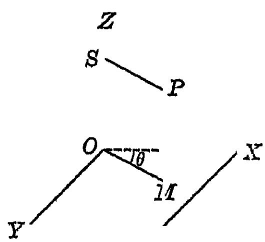


FIG. 116

If the line $OX$ of the cylindrical cöordinates is the same as the axis $OX$ of the Cartesian coördinates, and the axis $OZ$ is the same in both systems, it is evident, from § 51, that

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z. \tag{1}$$

These are formulas by which we may pass from one system to the other. It is convenient to notice especially that

$$r^2 = x^2 + y^2. \tag{2}$$

**91. Certain surfaces.** A single equation between the coördinates of a point in space represents a surface. We shall give examples of the equations of certain surfaces which are important in applications. In this connection it should be noticed that when we speak of the equation of a sphere we mean the equation of a spherical surface, and when we speak of the volume of a sphere we mean the volume of the solid bounded by a spherical surface. The word *sphere*, then, indicates a surface or a solid, according to the context. Similarly, the word *cone* is used to denote either a conical surface indefinite in extent or a solid bounded by a conical surface and a plane base. It is in the former sense that we speak of the equation of a cone, and in the latter sense that we speak of the volume of a cone. In the same way the word *cylinder* may denote either a cylindrical surface or a solid bounded by a cylindrical surface and two plane bases. This double use of these words makes no confusion in practice, as the context always indicates the proper meaning in any particular case.

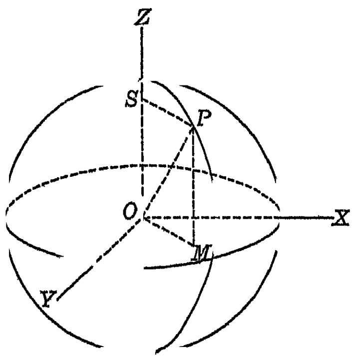


FIG. 117

1. *Sphere with center at origin.* Consider any sphere (Fig. 117) with its center at the origin of coördinates and its radius equal to $a$. Let $P$ be any point on the surface of the sphere. Pass a plane through $P$ and $OZ$, draw $PS$ perpendicular to $OZ$, and connect $O$ and $P$. Then, using cylindrical coördinates, in the right triangle $OPS$, $OS = z$, $SP = r$, and $OP = a$. Therefore

$$r^2 + z^2 = a^2. \tag{1}$$

This equation is satisfied by the cylindrical coordinates of any point on the surface of the sphere and by those of no other point It is therefore the equation of the sphere in cylindrical coordinates.

By means of (2), § 90, equation (1) becomes

$$x^2 + y^2 + z^2 = a^2, \tag{2}$$

which is the equation of the sphere in Cartesian coordinates.

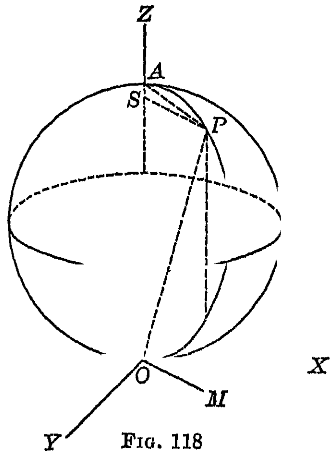


FIG. 118

2. *Sphere tangent at origin to a coordinate plane.* Consider a sphere tangent to the plane $XOY$ at $O$ (Fig 118). Let $P$ be any point on the surface of the sphere Let $A$ be the point in which the axis $OZ$ again meets the sphere Pass a plane through $P$ and $OZ$, connect $A$ and $P$, $O$ and $P$, and draw $PS$ perpendicular to $OZ$ Then, using cylindrical coordinates, $OS = z$, $SP = r$, and $OA = 2a$, where $a$ is the radius of the sphere Now $OAP$ is a right triangle, since it is inscribed in a semicircle, and $PS$ is the perpendicular from the vertex of the right angle to the hypotenuse. Therefore, by elementary plane geometry,

$$\overline{SP}^2 = OS \cdot SA = OS(OA - OS)$$

Substituting the proper values, we have

$$r^2 = 2az - z^2, \tag{3}$$

which is the equation of the sphere in cylindrical coordinates.

By (2), § 90, equation (3) becomes

$$x^2 + y^2 + z^2 - 2az = 0, \tag{4}$$

which is the equation of the sphere in Cartesian coordinates.

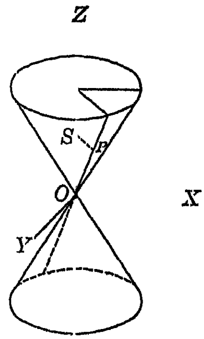


FIG. 119

3 *Right circular cone.* Consider any right circular cone with its vertex at the origin and its axis along $OZ$ (Fig 119). Let $\alpha$ be the angle which each element of the cone makes with $OZ$. Take $P$ any point on the surface of the cone, pass a plane through $P$ and $OZ$, and draw $PS$ perpendicular to $OZ$. Then $SP = r$ and $OS = z$. But $\frac{SP}{OS} = \tan SOP = \tan\alpha$ Therefore we have

$$r = z\tan\alpha \tag{5}$$

as the equation of the cone in cylindrical coördinates.

By 2, § 90, equation (5) becomes

$$x^2 + y^2 - z^2 \tan^2 \alpha = 0, \tag{6}$$

as the equation of the cone in Cartesian coordinates.

As explained above, we have here used the word *cone* in the sense of a conical surface If the cone is a solid with its altitude $h$ and the radius of its base $a$, then $\tan \alpha = \frac{a}{h}$. In this case equation (5) or (6) is that of the curved surface of the cone only.

4. *Surface of revolution.* Consider any surface of revolution with $OZ$ the axis of revolution (Fig 120). Take $P$ any point on the surface and pass a plane through $P$ and $OZ$ In the plane $POZ$ draw $OR$ perpendicular to $OZ$ and, from $P$, a straight line perpendicular to $OZ$ meeting $OZ$ in $S$ If we regard $OR$ and $OZ$ as a pair of rectangular axes for the plane $POZ$, the equation of the curve $CD$ in which the plane $POZ$ cuts the surface is

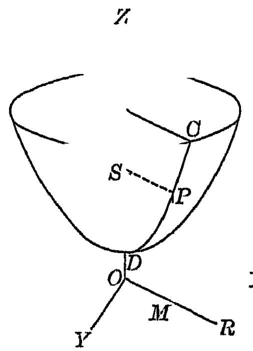


FIG 120

$$z = f(r), \tag{7}$$

exactly as $y = f(x)$ is the equation of a curve in § 12

But $CD$ is the same curve in all sections of the surface through $OZ$ Therefore equation (7) is true for all points $P$ and is the equation of the surface in cylindrical coordinates. When the plane $POZ$ coincides with the plane $XOZ$, $r$ is equal to $x$, and equation (7) becomes, for that section,

$$z = f(x). \tag{8}$$

Hence we have the following theorem:

*The equation of a surface of revolution formed by revolving about $OZ$ any curve in the plane $XOZ$ may be found in cylindrical coordinates by writing $r$ for $x$ in the equation of the curve.*

The equation of the surface in Cartesian coordinates may then be found by placing $r = \sqrt{x^2 + y^2}$. For example, the equation of the surface formed by revolving the parabola $z^2 = 4x$ about $OZ$ as an axis is $z^2 = 4r$ in cylindrical coördinates, or $z^4 = 16(x^2 + y^2)$ in Cartesian coördinates.

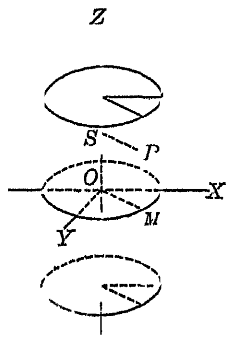


FIG. 121

5. *Cylinder.* Consider first a right circular cylinder with its axis along $OZ$ (Fig. 121). From any point $P$ of the surface of the cylinder draw $PS$ perpendicular to $OZ$. Then $SP$ is always equal to $a$, the radius of the cylinder. Therefore, for all points on the surface,

$$r = a, \tag{9}$$

which is the equation of the cylinder in cylindrical coördinates. Reduced to Cartesian coordinates equation (9) becomes

$$x^2 + y^2 = a^2, \tag{10}$$

the equation of the cylinder in Cartesian coordinates.

More generally, any equation in $x$ and $y$ only, or in $r$ and $\theta$ only, represents a cylinder. In fact, either of these equations, if interpreted in the plane $XOY$, represents a curve, but if a line is drawn from any point in this curve perpendicular to the plane $XOY$, and $P$ is any point on this line, the coordinates of $P$ also satisfy the equation, since $z$ is not involved in the equation. As examples, the equation $y^2 = 4x$ represents a parabolic cylinder, and the equation $r = a \sin 3\theta$ represents a cylinder whose base is a rose of three leaves (Fig. 65, p. 144).

6 *Ellipsoid* Consider the surface defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{11}$$

If we place $z = 0$, we get the points on the surface which lie in the $XOY$ plane These points satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{12}$$

and therefore form an ellipse.

Similarly, the points in the $ZOX$ plane lie on the ellipse

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \tag{13}$$

and those in the $YOZ$ plane lie on the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{14}$$

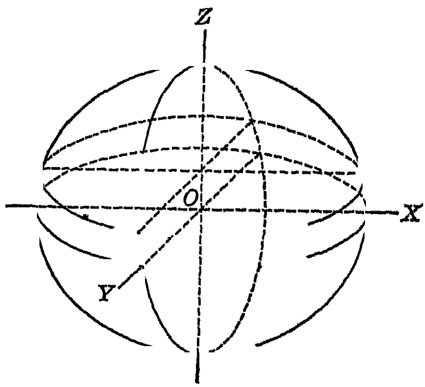



FIG. 122

The construction of these ellipses gives a general idea of the shape of the surface (Fig 122). To make this more precise, let us place $z = z_1$ in (11), where $z_1$ is a fixed value. We have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z_1^2}{c^2}, \tag{15}$$

which can be written

$$\frac{x^2}{a^2\left(1 - \frac{z_1^2}{c^2}\right)} + \frac{y^2}{b^2\left(1 - \frac{z_1^2}{c^2}\right)} = 1, \tag{16}$$

which is satisfied by all points which lie in the plane at the distance $z_1$ from the $XOY$ plane.

As long as $z_1^2 < c^2$, equation (16) represents an ellipse with semiaxes $a\sqrt{1-\frac{z_1^2}{c^2}}$ and $b\sqrt{1-\frac{z_1^2}{c^2}}$. By taking a sufficient number of these sections we may construct the ellipsoid with as much exactness as desired.

If $z_1^2 = c^2$ in (16), the axes of the ellipse reduce to zero, and we have a point If $z_1^2 > c^2$, the axes are imaginary, and there is no section.

7 *Elliptic paraboloid.* Consider the surface

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \tag{17}$$

where we shall assume, for definiteness, that $c$ is positive.

If we place $z = 0$, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0, \tag{18}$$

which is satisfied in real quantities only by $x = 0$ and $y = 0$. Therefore the $XOY$ plane simply touches the surface at the origin

If we place $z = c$, we get the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{19}$$

which lies in the plane $c$ units distant from the $XOY$ plane

If we place $y = 0$, we get the parabola

$$z = \frac{c}{a^2}x^2; \tag{20}$$

and if we place $x = 0$, we get the parabola

$$z = \frac{c}{b^2}y^2. \tag{21}$$

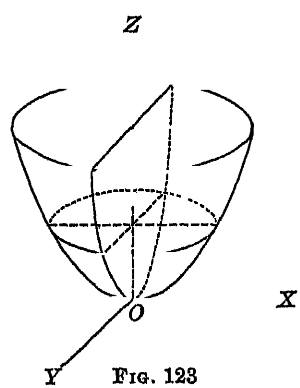


FIG. 123

The sections (19), (20), and (21) determine the general outline of surface. For more detail we place $z = z_1$ and find the ellipse

$$\frac{x^2}{\frac{a^2 z_1}{c}} + \frac{y^2}{\frac{b^2 z_1}{c}} = 1, \tag{22}$$

so that all sections parallel to the $XOY$ plane and above it are ellipses (Fig. 123).

8. *Elliptic cone.* Consider the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0. \tag{23}$$

Proceeding as in 7, we find that the section $z = 0$ is simply the origin and that the section $z = c$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{24}$$

If we place $x = 0$, we get the two straight lines

$$y = \pm \frac{b}{c} z, \tag{25}$$

and if we place $y = 0$, we get the two straight lines

$$x = \pm \frac{a}{c} z \tag{26}$$

The sections we have found suggest a cone with an elliptic base. To prove that the surface really is a cone, we change equation (23) to cylindrical coördinates, obtaining

$$\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}\right) r^2 = \frac{z^2}{c^2}. \tag{27}$$

Now if $\theta$ is held constant in (27), the coefficient of $r^2$ is constant, and the equation may be written

$$r = \pm kz, \tag{28}$$

which is the equation of two straight lines in the plane through $OZ$

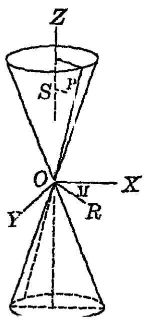



Fig 124

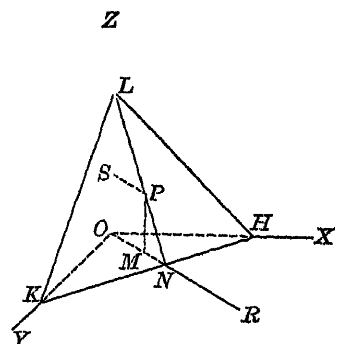



Fig 125

determined by $\theta = \text{const}$ Hence any plane through $OZ$ cuts the surface in two straight lines, and the surface is a cone (Fig. 124).

9. *Plane* Consider the surface

$$Ax + By + Cz + D = 0. \tag{29}$$

The section $z = 0$ is the straight line $KH$ (Fig 125) with the equation

$$Ax + By + D = 0, \tag{30}$$

the section $y = 0$ is the straight line $LH$ with the equation

$$Ax + Cz + D = 0, \tag{31}$$

and the section $x = 0$ is the straight line $LK$ with the equation

$$By + Cz + D = 0. \tag{32}$$

The two lines (31) and (32) intersect $OZ$ in the point $L\left(0, 0, -\frac{D}{C}\right)$, unless $C = 0$. Assuming for the present that $C$ is not zero, we change equation (29) to cylindrical coordinates, obtaining

$$(A \cos \theta + B \sin \theta)r + Cz + D = 0. \tag{33}$$

This is the equation of a straight line $LN$ in the plane $\theta = \text{const}$ It passes through the point $L$, which has the cylindrical coordinates $r = 0$, $z = -\frac{D}{C}$; and it meets the line $KH$, since when $z = 0$, equation (33) is the same as equation (30). Hence the surface is covered by straight lines which pass through $L$ and meet $KH$. The locus of such lines is clearly a plane

We have assumed that $C$ in (29) is not zero. If $C = 0$, equation (29) is

$$Ax + By + D = 0. \tag{34}$$

The point $L$ does not exist, since the lines corresponding to $HL$ and $KL$ are now parallel. But, by 5, equation (34) represents a plane parallel to $OZ$ intersecting $XOY$ in the line whose equation is (34) Therefore we have the following theorem

*Any equation of the first degree represents a plane.*

**92. Volume.** Starting from any point $(x, y, z)$ in space, we may draw lines of length $dx$, $dy$, and $dz$ in directions parallel to $OX$, $OY$, and $OZ$ respectively, and on these lines as edges construct a rectangular parallelepiped. The volume of this figure we call the element of volume $dV$ and have

$$dV = dx\,dy\,dz. \tag{1}$$

For cylindrical coordinates we construct an element of volume whose base is $r\,d\theta\,dr$ (§ 84), the element of plane area in polar coördinates, and whose altitude is $dz$. This figure has for its volume $dV$ the product of its base by its altitude, and we have

$$dV = r\,d\theta\,dr\,dz. \tag{2}$$

The two elements of volume $dV$ given in (1) and (2) are not equal to each other, since they refer to differently shaped figures. Each is to be used in its appropriate place. To find the volume of any solid we divide it into elements of one of these types.

To do this in Cartesian coördinates, note that the $x$-coördinate of any point will determine a plane parallel to the plane $YOZ$

and $x$ units from it, and that similar planes correspond to the values of $y$ and $z$. We may, accordingly, divide any required volume into elements of volume as follows:

Pass planes through the volume parallel to $YOZ$ and $dx$ units apart. The result is to divide the required volume into slices of thickness $dx$, one of which is shown in Fig. 126. Secondly, pass planes through the volume parallel to $XOZ$ and $dy$ units apart, with the result that each slice is divided into columns of cross section $dxdy$. One such column is shown in Fig. 126.

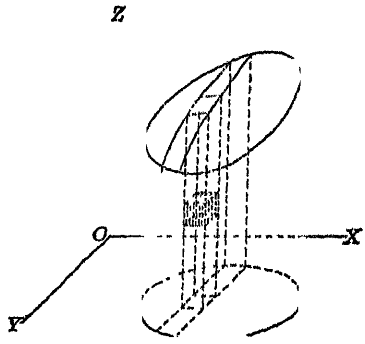



FIG. 126

Finally, pass planes through the required volume parallel to $XOY$ and $dz$ units apart, with the result that each column is divided into rectangular parallelepipeds of dimensions $dx$, $dy$, and $dz$. One of these is shown in Fig. 126.

It is to be noted that the order followed in the above explanation is not fixed and that, in fact, the choice of beginning with either $x$ or $y$ or $z$, and the subsequent order depend upon the particular volume considered.

A similar construction may be made for cylindrical coördinates. In this case the coördinate $\theta$ determines a plane through $OZ$. We accordingly divide the volume by means of planes through $OZ$ making the angle $d\theta$ with each other. The result is a set of slices one of which is shown in Fig. 127.

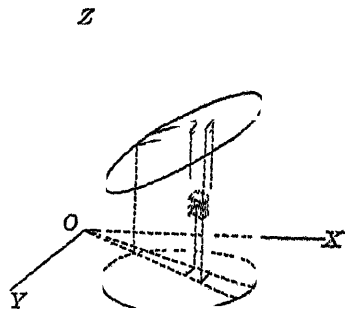



FIG. 127

The coördinate $r$ determines a cylinder with $OZ$ as its axis. We accordingly divide each slice into columns by means of cylinders with radii differing by $dr$. One such column is shown in Fig. 127.

Finally, these columns are divided into elements of volume by planes parallel to $XOY$ at a distance $dz$ apart. One such element is shown in Fig. 127.

When the volume has been divided in either of these ways, it is evident that some of the elements will extend outside the boundary surfaces of the solid. The sum of all the elements that are either completely or partially in the volume will be approximately the volume of the solid, and this approximation becomes better as the size of each element becomes smaller. In fact, the volume is the limit of the sum of the elements. The determination of this limit involves in principle three integrations, and we write

$$V=\iiint dx\,dy\,dz \tag{3}$$

or

$$V=\iiint r\,d\theta\,dr\,dz. \tag{4}$$

In carrying out the integrations we may, in some cases, find it convenient first to hold $z$ and $dz$ constant. We shall then be taking the limit of the sum of the elements which lie in a plane parallel to the $XOY$ plane. We may indicate this by writing (3) or (4) in the form

$$V=\int dz\iint dx\,dy \quad \text{or} \quad V=\int dz\iint r\,d\theta\,dr. \tag{5}$$

But, by § 84, $\iint dx\,dy=A$ and $\iint r\,d\theta\,dr=A$, where $A$ is the area of the plane section at a distance $z$ from $XOY$. Hence (5) is

$$V=\int A\,dz, \tag{6}$$

in agreement with § 26.

Hence, whenever it is possible to find $A$ by elementary means without integration, the use of (6) is preferable. This is illustrated in Ex. 1.

In some cases, however, this method of evaluation is not convenient, and it is necessary to carry out three integrations. This is illustrated in Ex. 2.

**Ex. 1.** Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

By 6, § 91, the section made by a plane parallel to $XOY$ is an ellipse with semiaxes $a\sqrt{1 - \frac{z^2}{c^2}}$ and $b\sqrt{1 - \frac{z^2}{c^2}}$. Therefore, by Ex. 1, § 77, its area is $\pi ab\left(1 - \frac{z^2}{c^2}\right)$ Hence we use formula (6) and have

$$V = \pi ab\int_{-c}^{c}\left(1 - \frac{z^2}{c^2}\right)dz = \frac{4}{3}\pi abc.$$

**Ex. 2.** Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = 5$ and below by the paraboloid $x^2 + y^2 = 4z$ (Fig 128).

As these are surfaces of revolution, this example may be solved by the method of Ex. 1, but in so doing we need two integrations — one for the sphere and the other for the paraboloid We shall solve the example, however, by the other method in order to illustrate that method

We first reduce our equations to cylindrical coordinates, obtaining respectively

$$r^2 + z^2 = 5 \quad (1)$$

and

$$r^2 = 4z \quad (2)$$

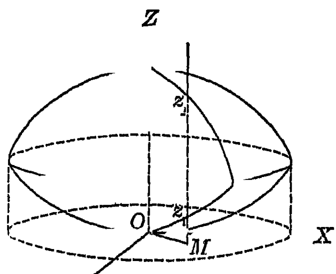


Fig. 128

The surfaces intersect when $r$ has the same value in both equations; that is, when

$$z^2 + 4z = 5, \quad (3)$$

which gives $z = 1$ or $z = -5$ The latter value is impossible, but when $z = 1$, we have $r = 2$ in both equations Therefore the surfaces intersect in a circle of radius 2 in the plane $z = 1$ This circle lies directly above the circle $r = 2$ in the $XOY$ plane.

We now imagine the element $r\,d\theta\,dr\,dz$ inside the surface and, holding $r, \theta, d\theta, dr$ constant, we take the sum of all the elements obtained by varying $z$ inside the volume These elements obviously extend from $z = z_1$ in the lower boundary to $z = z_2$ in the upper boundary. From (2), $z_1 = \frac{r^2}{4}$ and, from (1), $z_2 = \sqrt{5 - r^2}$. The first integration is therefore

$$r\,d\theta\,dr\int_{\frac{r^2}{4}}^{\sqrt{5-r^2}}dz = \left[r\sqrt{5 - r^2} - \frac{r^3}{4}\right]d\theta\,dr.$$

We must now allow $\theta$ and $r$ so to vary as to cover the entire circle $r = 2$ above which the required volume stands.

If we hold $\theta$ constant, $r$ varies from 0 to 2. The second integration is therefore

$$d\theta\int_0^2\left[r\sqrt{5 - r^2} - \frac{r^3}{4}\right]dr = \left(\frac{5\sqrt{5}}{3} - \frac{4}{3}\right)d\theta.$$

Finally, $\theta$ must vary from 0 to $2\pi$, and the third integration is

$$\left(\frac{5\sqrt{5}}{3}-\frac{4}{3}\right)\int_0^{2\pi}d\theta=\frac{2\pi}{3}(5\sqrt{5}-4)$$

If we put together what we have done, we have

$$V=\int_0^{2\pi}\int_0^{2}\int_{\frac{r^2}{4}}^{\sqrt{5-r^2}}r\,d\theta\,dr\,dz=\frac{2\pi}{3}(5\sqrt{5}-4).$$

**EXERCISES**

1. Find the volume bounded by the paraboloid $z=x^2+y^2$ and the planes $x=0$, $y=0$, and $z=4$.

2. Prove that the volume bounded by the surface $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ and the plane $z=c$ is one half the product of the area of the base by the altitude.

3. Find the volume bounded by the plane $z=0$ and the cylinders $x^2+y^2=a^2$ and $y^2=a^2-az$.

4. Find the volume cut from the sphere $r^2+z^2=a^2$ by the cylinder $r=a\cos\theta$.

5. Find the volume bounded below by the paraboloid $r^2=az$ and above by the sphere $r^2+z^2-2az=0$

6. Find the volume bounded by the plane $XOY$, the cylinder $x^2+y^2-2ax=0$, and the right circular cone having its vertex at $O$, its axis coincident with $OZ$, and its vertical angle equal to 90°

7. Find the volume bounded by the surfaces $r^2=bz$, $z=0$, and $r=a\cos\theta$.

8. Find the volume bounded by a sphere of radius $a$ and a right circular cone, the axis of the cone coinciding with a diameter of the sphere, the vertex being at an end of the diameter, and the vertical angle of the cone being 90°.

9. Find the volume of the sphere of radius $a$ and with its center at the origin of coördinates, included in the cylinder having for its base one loop of the curve $r^2=a^2\cos 2\theta$.

10. Find the volume of the paraboloid $x^2+y^2=2z$ cut off by the plane $z=x+1$.

11. Find the volume of the solid bounded by the paraboloid $z=x^2+y^2$ and the plane $z=x$.

**93. Center of gravity of a solid.** The center of gravity of a solid has three coördinates, $\bar{x}$, $\bar{y}$, $\bar{z}$, which are defined by the equations

$$\bar{x}=\frac{\int x\,dm}{\int dm},\quad \bar{y}=\frac{\int y\,dm}{\int dm},\quad \bar{z}=\frac{\int z\,dm}{\int dm},\tag{1}$$

where $dm$ is the mass of one of the elements into which the solid may be divided, and $x$, $y$, and $z$ are the coordinates of the point at which the element $dm$ may be regarded as concentrated. The derivation of these formulas is the same as that in § 85 and is left to the student.

When $dm$ is expressed in terms of space coördinates, the integrals become triple integrals, and the limits of integration are to be substituted so as to include the whole solid.

We place $dm=\rho\,dV$, where $\rho$ is the density. If $\rho$ is constant, it may be placed outside the integral signs and canceled from numerators and denominators. Formulas (1) then become

$$V\bar{x}=\int x\,dV,\quad V\bar{y}=\int y\,dV,\quad V\bar{z}=\int z\,dV.\tag{2}$$

**Ex.** Find the center of gravity of a body bounded by one nappe of a right circular cone of vertical angle $2\alpha$ and a sphere of radius $a$, the center of the sphere being at the vertex of the cone.

If the center of the sphere is taken as the origin of coordinates and the axis of the cone as the axis of $z$, it is evident from the symmetry of the solid that $\bar{x}=\bar{y}=0$. To find $\bar{z}$ we shall use cylindrical coördinates, the equations of the sphere and the cone being respectively

$$r^2+z^2=a^2 \quad\text{and}\quad r=z\tan\alpha.$$

As in Ex. 2, § 92, the surfaces intersect in the circle $r=a\sin\alpha$ in the plane $z=a\cos\alpha$. Therefore

$$V=\int_0^{2\pi}\int_0^{a\sin\alpha}\int_{r\,\text{ctn}\,\alpha}^{\sqrt{a^2-r^2}} r\,d\theta\,dr\,dz=\tfrac{2}{3}\pi a^3(1-\cos\alpha)$$

and
$$\int z\,dV=\int_0^{2\pi}\int_0^{a\sin\alpha}\int_{r\,\text{ctn}\,\alpha}^{\sqrt{a^2-r^2}} rz\,d\theta\,dr\,dz=\tfrac{1}{4}\pi a^4\sin^2\alpha.$$

Therefore, from (2), $\bar{z}=\tfrac{3}{8}a(1+\cos\alpha)$.

**EXERCISES**

1. Find the center of gravity of a solid bounded by the paraboloid $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ and the plane $z=c$

2. A ring is cut from a spherical shell, the inner radius and the outer radius of which are respectively 4 ft. and 5 ft., by two parallel planes on the same side of the center of the shell and distant 1 ft and 3 ft. respectively from the center. Find the center of gravity of this ring.

3. Find the center of gravity of a solid in the form of the frustum of a right circular cone the height of which is $h$, and the radius of the upper base and the radius of the lower base of which are respectively $r_1$ and $r_2$.

4. Find the center of gravity of that portion of the solid of Ex. 2, p. 73, which is above the plane determined by $OA$ and $OB$ (Fig. 31).

5. Find the center of gravity of a body in the form of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

6. Find the center of gravity of a solid bounded below by the paraboloid $az=r^2$ and above by the right circular cone $z+r=2a$.

7. Find the center of gravity of a solid bounded below by the cone $z=r$ and above by the sphere $r^2+z^2=1$.

8. Find the center of gravity of a solid bounded by the surfaces $z=0$, $r^2+z^2=b^2$, and $r=a\,(a<b)$.

**94. Moment of inertia of a solid.** If a solid body is divided into elements of volume $dV$, then, as in § 88, the moment of inertia of the solid about any axis is

$$I=\int R^2\rho\, dV=\rho\int R^2\, dV, \tag{1}$$

where $R$ is the distance of any point of the element from the axis, and $\rho$ is the density of the solid, which we have assumed to be constant and therefore have been able to take out of the integral sign. If $M$ is the total mass of the solid, $\rho$ may be determined from the formula $M=\rho V$.

If the moment of inertia about $OZ$, which we shall call $I_z$, is required, then in cylindrical coördinates $R = r$ and $dV = r\,d\theta\,dr\,dz$, so that (1) becomes

$$I_z = \rho\iiint r^3\,d\theta\,dr\,dz. \tag{2}$$

If we use Cartesian coördinates to determine $I_z$, we have $R^2 = x^2 + y^2$ and $dV = dx\,dy\,dz$, so that

$$I_z = \rho\iiint (x^2 + y^2)\,dx\,dy\,dz. \tag{3}$$

Similarly, if $I_y$ and $I_x$ are the moments of inertia about $OY$ and $OX$ respectively, we have

$$I_y = \rho\iiint (x^2 + z^2)\,dx\,dy\,dz, \quad I_x = \rho\iiint (y^2 + z^2)\,dx\,dy\,dz. \tag{4}$$

In evaluating (2) it is sometimes convenient to integrate with respect to $z$ last. We indicate this by the formula

$$I_z = \rho\int dz\iint r^3\,d\theta\,dr. \tag{5}$$

But $\iint r^3 d\theta\,dr$ is, by § 88, the polar moment of inertia of a plane section perpendicular to $OZ$ about the point in which $OZ$ intersects the plane section. Consequently, if this polar moment is known, the evaluation of (5) reduces to a single integration. This has already been illustrated in the case of solids of revolution.

A similar result is obtained by considering (3). In fact, the ease with which a moment of inertia is found depends upon a proper choice of Cartesian or cylindrical coördinates and, after that choice has been made, upon the order in which the integrations are carried out.

Equation (3) may be written in the form

$$I_z = \rho\iiint x^2 dx\,dy\,dz + \rho\iiint y^2 dx\,dy\,dz, \tag{6}$$

and the order of integration in the two integrals need not be the same. Similar forms are derived from (4).

The theorem of § 89 holds for solids. This is easily proved by the same methods used in that section.

**Ex.** Find the moment of inertia about $OZ$ of a cylindrical solid of altitude $h$ whose base is one loop of the curve $r = a\sin 3\,\theta$.

The base of this cylinder is shown in Fig 65, p. 144. We have, from formula (2),

$$I_z = \rho\int_0^{\frac{\pi}{3}}\int_0^{a\sin 3\theta}\int_0^h r^3 d\theta\, dr\, dz,$$

where the limits are obtained as follows:

First, holding $r$, $\theta$, $d\theta$, $dr$ constant, we allow $z$ to vary from the lower base $z = 0$ to the upper base $z = h$, and integrate. The result $\rho h r^3 d\theta dr$ is the moment of inertia of a column such as is shown in Fig 127. We next hold $\theta$ and $d\theta$ constant and allow $r$ to vary from its value at the origin to its value on the curve $r = a\sin 3\,\theta$, and integrate. The result $\frac{1}{4}\rho h a^4 \sin^4 3\,\theta d\theta$ is the moment of inertia of a slice as shown in Fig. 127. Finally, we sum all these slices while allowing $\theta$ to vary from its smallest value 0 to its largest value $\frac{\pi}{3}$ The result is $\frac{1}{32}\rho h a^4\pi$.

The volume of the cylinder may be computed from the formula

$$V = \int_0^{\frac{\pi}{3}}\int_0^{a\sin 3\theta}\int_0^h r\, d\theta\, dr\, dz = \tfrac{1}{12}ha^2\pi.$$

Therefore $\qquad M = \frac{1}{12}\rho h a^2\pi \quad$ and $\quad I_z = \frac{3}{8}Ma^2.$

### EXERCISES

**1.** Find the moment of inertia of a rectangular parallelepiped about an axis through its center parallel to one of its edges.

**2.** Find the moment of inertia about $OZ$ of a solid bounded by the surface $z = 2$ and $z^2 = r$.

**3.** Find the moment of inertia of a right circular cone of radius $a$ and height $h$ about any diameter of its base as an axis.

**4.** Find the moment of inertia about $OZ$ of a solid bounded by the paraboloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ and the plane $z = c$.

**5.** Find the moment of inertia of a right circular cone of height $h$ and radius $a$ about an axis perpendicular to the axis of the cone at its vertex.

**6.** Find the moment of inertia of a right circular cylinder of height $h$ and radius $a$ about a diameter of its base.

**7.** Find the moment of inertia about $OZ$ of the portion of the sphere $r^2 + z^2 = a^2$ cut out by the plane $z = 0$ and the cylinder $r = a\cos\theta$.

8. Find the moment of inertia about $OX$ of a solid bounded by the paraboloid $z = r^2$ and the plane $z = 2$.

9. Find the moment of inertia about its axis of a right elliptic cylinder of height $h$, the major and the minor axis of its base being respectively $2a$ and $2b$.

10. Find the moment of inertia about $OZ$ of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

## GENERAL EXERCISES

1. Find the center of gravity of the arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, which is above the axis of $x$.

2. A wire is bent into a curve of the form $9y^2 = x^3$. Find the center of gravity of the portion of the wire between the points for which $x = 0$ and $x = 5$ respectively.

3. Find the center of gravity of the area bounded by the curve $ay^2 = x^3$ and any double ordinate.

4. Find the center of gravity of the area bounded by the axis of $x$, the axis of $y$, and the curve $y^2 = 8 - 2x$.

5. Find the center of gravity of the area bounded by the curves $y = x^3$ and $y = \frac{1}{x^2}$, the axis of $x$, and the line $x = 2$

6. Find the center of gravity of the area bounded by the axes of $x$ and $y$ and the curve $x = a\cos^3\phi$, $y = a\sin^3\phi$

7. Find the center of gravity of the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$, the circle $x^2 + y^2 = a^2$, and the axis of $y$.

8. Find the center of gravity of the area bounded by the parabola $x^2 = 8y$ and the circle $x^2 + y^2 = 128$

9. Find the center of gravity of the area bounded by the curves $x^2 - a(y - b) = 0$, $x^2 - ay = 0$, the axis of $y$, and the line $x = c$.

10. Find the center of gravity of an area in the form of a semicircle of radius $a$ surmounted by an equilateral triangle having one of its sides coinciding with the diameter of the semicircle.

11. Find the center of gravity of an area in the form of a rectangle of dimensions $a$ and $b$ surmounted by an equilateral triangle one side of which coincides with one side of the rectangle which is $b$ units long.

12. Find the center of gravity of the segment of a circle of radius $a$ cut off by a straight line $b$ units from the center.

13. From a rectangle $b$ units long and $a$ units broad a semicircle of diameter $a$ units long is cut, the diameter of the semicircle coinciding with a side of the rectangle. Find the center of gravity of the portion of the rectangle left.

14. Find the center of gravity of a plate in the form of one half of a circular ring the inner and the outer radii of which are respectively $r_1$ and $r_2$

15. In the result of Ex. 14, place $r_2 = r_1 + \Delta r$ and find the limit as $\Delta r \rightarrow 0$, thus obtaining the center of gravity of a semicircumference.

16. Find the center of gravity of a plate in the form of a T-square 10 in across the top and 12 in. tall, the width of the upright and that of the top being each 2 in.

17. From a plate in the form of a regular hexagon 5 in. on a side, one of the six equilateral triangles into which it may be divided is removed. Find the center of gravity of the portion left.

18. Find the center of gravity of a plate, in the form of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$, in which there is a circular hole of radius $c$, the center of the hole being on the major axis of the ellipse at a distance $d$ from its center.

19. Find the center of gravity of the solid formed by revolving about $OY$ the surface bounded by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the lines $y = 0$ and $y = b$.

20. Find the center of gravity of the solid generated by revolving about the line $x = a$ the area bounded by that line, the axis of $x$, and the parabola $y^2 = kx$.

21. Find the center of gravity of the segment cut from a sphere of radius $a$ by two parallel planes distant respectively $h_1$ and $h_2$ $(h_2 > h_1)$ from the center of the sphere.

22. Find the moment of inertia of a plane triangle of altitude $a$ and base $b$ about an axis passing through its center of gravity parallel to the base.

23. Find the moment of inertia of a parallelogram of altitude $a$ and base $b$ about its base as an axis.

24. Find the moment of inertia of a plane circular ring, the inner radius and the outer radius of which are respectively 3 in. and 5 in., about a diameter of the ring as an axis.

25. A square plate 10 in. on a side has a square hole 5 in. on a side cut in it, the center of the hole being at the center of the plate and its sides parallel to the sides of the plate. Find the moment of inertia of the plate about a line through its center parallel to one side as an axis.

26. Find the moment of inertia of the plate of Ex. 25 about one of the outer sides as an axis.

27. Find the moment of inertia of the plate of Ex. 25 about one side of the hole as an axis.

28. Find the moment of inertia of the plate of Ex. 25 about one of its diagonals as an axis.

29. A square plate 8 in. on a side has a circular hole 4 in. in diameter cut in it, the center of the hole coinciding with the center of the square. Find the moment of inertia of the plate about a line passing through its center parallel to one side as an axis.

30. Find the moment of inertia of the plate of Ex. 29 about a diagonal of the square as an axis.

31. Find the moment of inertia of a semicircle about a tangent parallel to its diameter as an axis.

32. Find the polar moment of inertia of the plate of Ex. 25 about its center.

33. Find the polar moment of inertia of the entire area bounded by the curve $r^2 = a^2 \sin 3\theta$ about the pole.

34. Find the polar moment of inertia of the area bounded by the cardioid $r = a(1 + \cos\theta)$ about the pole.

35. Find the polar moment of inertia of that area of the circle $r = a$ which is not included in the curve $r = a \sin 2\theta$ about the pole.

36. Find the moment of inertia about $OY$ of a solid bounded by the surface generated by revolving about $OY$ the area bounded by the curve $y^2 = x$, the axis of $y$, and the line $y = 2$.

37. A solid is in the form of a hemispherical shell the inner radius and the outer radius of which are respectively $r_1$ and $r_2$. Find its moment of inertia about any diameter of the base of the shell as an axis.

38. A solid is in the form of a spherical cone cut from a sphere of radius $a$, the vertical angle of the cone being 90°. Find its moment of inertia about its axis.

39. A solid is cut from a hemisphere of radius 5 in by a right circular cylinder of radius 3 in, the axis of the cylinder being perpendicular to the base of the hemisphere at its center Find its moment of inertia about the axis of the cylinder as an axis.

40. An anchor ring of mass $M$ is bounded by the surface generated by revolving a circle of radius $a$ about an axis in its plane distant $b(b > a)$ from its center. Find the moment of inertia of this anchor ring about its axis.

41. Find the moment of inertia of the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$, its height being $h$, about the major axis of its base.

42. Find the center of gravity of the solid bounded by the cylinder $r = 2a \cos \theta$, the cone $z = r$, and the plane $z = 0$

43. Find the moment of inertia about $OZ$ of the solid of Ex. 42.

44. Find the volume of the cylinder having for its base one loop of the curve $r = 2a \cos 2\theta$, between the cone $z = 2r$ and the plane $z = 0$

45. Find the center of gravity of the solid of Ex. 44.

46. Find the moment of inertia about $OZ$ of the solid of Ex. 44.

47. Find the volume of the cylinder having for its base one loop of the curve $r = a \cos 2\theta$ and bounded by the planes $z = 0$ and $z = x + 2a$.

48. Find the moment of inertia about $OZ$ of the solid of Ex. 47

49. Find the volume of the cylinder $r = 2a \cos \theta$ included between the planes $z = 0$ and $z = 2x + a$.

50. Find the moment of inertia about $OZ$ of the solid of Ex. 49.

51. Through a spherical shell of which the inner radius and the outer radius are respectively $r_1$ and $r_2$, a circular hole of radius $a(a < r_1)$ is bored, the axis of the hole coinciding with a diameter of the shell. Find the moment of inertia of the ring thus formed about the axis of the hole.

# ANSWERS

[The answers to some problems are intentionally omitted.]

## CHAPTER I

### Page 4 (§ 2)

1. $21\frac{10}{41}$.
2. $1\frac{13}{78}$.
3. $40\frac{10}{11}$.
4. 106 ft. per second
5. 33.97 mi. per hour for entire trip.
6. 1.26.
7. $2\frac{1}{2}$
8. $1\frac{1}{2}$ mi. per hour.
9. 98.4.

### Page 5 (§ 2)

10. 196.8.

### Page 7 (§ 3)

1. 96 ft. per second.
2. 128 ft. per second.

### Page 8 (§ 3)

3. 128 ft. per second.
4. 74 ft. per second.
5. 68 ft. per second.
6. 52 ft. per second.

### Page 11 (§ 5)

1. $12\,t_1^2$; $24\,t_1$.
2. 16; 14.
3. 85; 32; 6.
4. 42; 57.
5. 5, 4, when $t = 2$; 10, 6, when $t = 3$
8. $3\,at^2 + 2\,bt + c$; $6\,at + 2\,b$.

### Page 13 (§ 6)

1. $\frac{15}{\pi}$ sq. ft. per second.
2. $\frac{1}{2\,\pi}\,C$.
3. $8\,\pi r^2$ cu. in. per second.
4. $4\,\pi r^2$.
5. $16\,\pi r$ sq. in. per second.

### Page 14 (§ 6)

6. $8\,\pi r$.
7. $3\,(\text{edge})^2$.
8. $6\,\pi r^2$.
9. 18.
10. $2\,\pi$.

## CHAPTER II

### Page 18 (§ 7)

1. $3\,x$.
2. $3\,x^2 + 4\,x$.
3. $4\,x^3 - 2\,x$.
4. $-\frac{3}{x^4}$.
5. $2\,x - \frac{2}{x^3}$.
6. $-\frac{2}{(2+x)^2}$.
7. $x^2 + x + 1$.
8. $3 - \frac{1}{x^2}$.

## Page 21 (§ 9)

1. Increasing if $x > 2$; decreasing if $x < 2$.
2. Increasing if $x > -\frac{5}{3}$, decreasing if $x < -\frac{5}{3}$.
3. Increasing if $x < \frac{5}{2}$; decreasing if $x > \frac{5}{2}$.
4. Increasing if $x < -\frac{1}{2}$, decreasing if $x > -\frac{1}{2}$.
5. Increasing if $x < -2$ or $x > 1$, decreasing if $-2 < x < 1$.
6. Increasing if $x < -5$ or $x > 3$, decreasing if $-5 < x < 3$.
7. Increasing if $x < -1$ or $x > \frac{5}{3}$; decreasing if $-1 < x < \frac{5}{3}$.
8. Always increasing.
9. Increasing if $x < -1$ or $-\frac{1}{2} < x < 1$; decreasing if $-1 < x < -\frac{1}{2}$ or $x > 1$.
10. Increasing if $x > 1$, decreasing if $x < 1$.

## Page 24 (§ 10)

1. When $t < -1$ or $t > 1$, when $-1 < t < 1$.
2. When $t < 5$; when $t > 5$.
3. When $t < 2$ or $t > 4$; when $2 < t < 4$.
4. Always moves in direction in which $s$ is measured.
5. When $t > \frac{3}{4}$, when $t < \frac{3}{4}$.
6. Always increasing.
7. Always decreasing
8. Increasing when $t > 2$, decreasing when $t < 2$.
9. Increasing when $t > \frac{5}{3}$, decreasing when $t < \frac{5}{3}$
10. Increasing when $t < \frac{2}{3}$, decreasing when $t > \frac{2}{3}$.

## Page 26 (§ 11)

1. $\frac{1}{3}\pi h^2$.
2. $6\pi h$ sq. ft. per second.

## Page 27 (§ 11)

3. 0.2 cm. per second.
4. 20.9 sq in. per second.
5. 0.26
6. 64 cu. ft per second.
7. $3\frac{\pi}{8}$ (total height)$^2$.
8. $4\pi(t^2 + 12t + 36)$, $t$ is thickness.

## Page 31 (§ 13)

1. 1.46.
2. $-2.07$.
3. 0.46; 2.05.
4. 1.12; 3.93.
5. 2.41.
6. $-2.52$.

## Page 35 (§ 14)

1. $3x - y - 9 = 0$.
2. $2x + 3y + 3 = 0$
3. $21x - 2y - 13 = 0$.
4. $y + 3 = 0$
5. $\sqrt{3}x - y - 2\sqrt{3} - 2 = 0$.
6. $x + y + 1 = 0$
7. $x + 2y + 8 = 0$.
8. $4x - 3y - 1 = 0$.
9. $12x - 4y - 5 = 0$.
10. $5x - 6y - 4 = 0$.

*12. $\tan^{-1}\frac{1}{8}$.

13. $\tan^{-1}12$.
14. $\frac{\pi}{2}$.
15. $2x - 3y - 16 = 0$.
16. $(-1\frac{7}{13}, 2\frac{4}{13})$.

* The symbol $\tan^{-1}\frac{1}{8}$ represents the angle whose tangent is $\frac{1}{8}$ (cf. § 46).

**Page 39 (§ 15)**

1. $(-\frac{2}{3}, 2\frac{2}{3})$.
2. $(\frac{3}{4}, 4\frac{1}{8})$.
3. $(0, 4), (2, 0)$.
4. $(1, 7), (3, 3)$.
5. $(-2, 0), (1, -9)$.
6. $(-1, -3), (3, 29)$.
7. $2x - y - 1 = 0$
8. $4x + y + 4 = 0$
9. $(-3, 10), (1, 2)$
10. $27x + 27y - 86 = 0,\ x + y - 2 = 0$
11. $18x - 27y + 80 = 0,\ 18x - 27y - 28 = 0$.
12. $\tan^{-1}\frac{2}{3}$

**Page 43 (§ 17)**

1. 25 sq. in.
2. Length is twice breadth.
3. 5 ft.
4. 50.
5. $\dfrac{4\pi a^3\sqrt{3}}{9}$.

**Page 44 (§ 17)**

6. Depth is one-half side of base.
7. 2 portions 4 ft. long; 4 portions 1 ft. long
8. Breadth $= \dfrac{2a\sqrt{3}}{3}$; depth $= \dfrac{2a\sqrt{6}}{3}$.
9. Altitude $= \dfrac{p\sqrt{2}}{4}$; base $= \dfrac{p}{4}$. ($p$ = perimeter)
10. 2000 cu. in.; 2547 cu. in
11. Height of rectangle = radius of semicircle, semicircle of radius, $\dfrac{a}{\pi}$
12. $\dfrac{5}{\sqrt{3}}$ in.

**Page 46 (§ 18)**

1. 426 ft.

**Page 47 (§ 18)**

2. 46 ft.
3. 95 ft.
4. $38\frac{2}{3}$ ft.
5. 576 ft.
6. $y = x^2 + 3x - 5$.
7. $y = 2x^3 + x^2 - 4x + 6$.
8. $y = 7 + 4x - \frac{3}{2}x^2 - \frac{1}{3}x^3$.
9. $y = \frac{1}{4}(x^2 - 33)$.
10. $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + x + \frac{8}{3}$.

**Page 49 (§ 19)**

1. $7\frac{1}{3}$. 2. $13\frac{1}{3}$. 3. $62\frac{1}{4}$. 4. 36. 5. $1\frac{1}{3}$ 6. $1\frac{5}{12}$. 7. $2\frac{1}{4}$. 8. $21\frac{1}{3}$

**Page 53 (§ 20)**

8. 0.0001; 0.000001; 0.00000001
9. 0.000009001; 0.000000090001.
10. 0 000003 sq. in.
11. 456 58 cu. in.

**Page 54 (§ 21)**

1. 72 sq. in.
2. $\dfrac{5\pi}{16}$ cu. in.; $\dfrac{\pi}{2}$ sq. in.
3. $\dfrac{2\pi}{5}$ cu. in.; $\dfrac{3\pi}{25}$ cu. in.
4. 27.0054 cu. in.
5. 28.2749 cu in.
6. 606 0912.
7. 0 0012.
8. 5.09934

**Page 55 (General Exercises)**

1. $\dfrac{5}{(1-x)^2}$.
2. $\dfrac{2a}{(a-x)^2}$.
3. $-\dfrac{2x}{(x^2+1)^2}$.
4. $\dfrac{4x}{(x^2+1)^2}$.
5. $\dfrac{1}{2\sqrt{x}}$.
6. $-\dfrac{1}{2\sqrt{x^3}}$.
7. $\dfrac{x}{\sqrt{x^2+1}}$.

13. Increasing if $x > -2$, decreasing if $x < -2$.

14. Increasing if $-\frac{2}{3} < x < \frac{2}{3}$ or $x > 2$, decreasing if $x < -\frac{2}{3}$ or $\frac{2}{3} < x < 2$.

15. Increasing if $x > -\dfrac{b}{2a}$, decreasing if $x < -\dfrac{b}{2a}$.

16. Increasing if $x < -\dfrac{a}{\sqrt{3}}$ or $x > \dfrac{a}{\sqrt{3}}$, decreasing if $-\dfrac{a}{\sqrt{3}} < x < \dfrac{a}{\sqrt{3}}$.

17. Increase if $x < \dfrac{4a}{3}$, decrease if $x > \dfrac{4a}{3}$.

**Page 56 (General Exercises)**

18. $1 < t < 5$
19. $2 < t < 5$, $4\frac{1}{2}$.
20. Up when $0 < t < 6\frac{1}{4}$, down when $6\frac{1}{4} < t < 12\frac{1}{2}$.
21. Increasing when $t > 4$, decreasing when $t < 4$.
22. $v$ increasing when $t < 3$, $v$ decreasing when $t > 3$, speed increasing when $2 < t < 3$ or $t > 4$, speed decreasing when $t < 2$ or $3 < t < 4$
23. Increasing when $1 < t < 2$ or $t > 3$, decreasing when $t < 1$ or $2 < t < 3$
24. 0 0055 in. per minute
25. 8 6 in. per second
26. 1 sq in per minute.
27. $x + 2y + 6 = 0$.
28. $7x + 5y + 1 = 0$.

**Page 57 (General Exercises)**

29. $x - 2 = 0$
30. $x - 2y - 7 = 0$.
31. $2x - y + 3 = 0$.
32. $\tan^{-1}\frac{7}{9}$
33. $(-\frac{1}{2}, 3\frac{1}{4})$.
34. $(1\frac{1}{4}, 0)$
35. $(2, -2)$.
36. $(-1, 13)$, $(5, -95)$.
37. $(-3, 13)$, $(1, -19)$.
38. $(-4, 20)$.
41. $10\frac{2}{3}$.
42. $\tan^{-1}\frac{8}{19}$.
43. $x - y - 11 = 0$
44. $(1, -1)$, $(-\frac{1}{9}, -\frac{13}{243})$.

**Page 58 (General Exercises)**

46. $6\frac{2}{3}$ ft. long
47. Altitude of cone is $\frac{4}{3}$ radius of sphere.
48. Altitude $= \sqrt[4]{\dfrac{4k^2}{243}}$; side of base $= \sqrt[4]{\dfrac{4k^2}{27}}$
49. 2 pieces 3 in long, 3 pieces 1 in. long
50 600 ft.
51. 56 ft
52. $y = x^2 + 3x - 13$
53. $y = \frac{1}{3}x^3 - x^2 + 7x$
54. $85\frac{1}{3}$.
55. $28\frac{1}{2}$
56. $20\frac{5}{6}$
57. 72.
59. 0.0003.

**Page 59 (General Exercises)**

60. 0.00629.
62. $288\pi$ cu in.
63. 161.16 cu. in.
64. 0.09 cu in
65. 0.0003.
66. 354.1028; 353.8972.
67. 24.0024 sq. in.
68. 0.4698.

## CHAPTER III

**Page 66 (§ 23)**

1. $3\frac{1}{3}$
2. $23\frac{1}{3}$
3. $52\frac{1}{12}$.
4. $166\frac{2}{3}$.
5. $5\frac{1}{3}$
6. $42\frac{2}{3}$
7. $36\frac{13}{48}$.
8. $2\frac{1}{4}$.
9. 96.
10. $10\frac{2}{3}$.
11. $42\frac{2}{3}$
12. $10\frac{2}{3}$.

**Page 67 (§ 24)**

1. 150 ft
2. 140 ft.
3. $57\frac{1}{6}$ ft.

**Page 68 (§ 24)**

4. When $\frac{1}{2} < t < \frac{11}{2}$; $83\frac{1}{3}$ ft.
5. $8000\,\pi$ ft.-lb.

**Page 70 (§ 25)**

1. $8\frac{1}{2}$ T.
2. $2\frac{1}{4}$ T.
3. 3 T.
4. $1\frac{1}{2}$ T

**Page 71 (§ 25)**

5. Approx. 2413 lb.
6. $4\frac{1}{12}$ T., $4\frac{5}{12}$ T.
7. $585\frac{15}{16}$ T.
8. $117\frac{3}{16}$ T
9. $234\frac{3}{8}$ T
10. $2\frac{7}{18}$ T.
11. 2 1 ft. from upper side

**Page 75 (§ 26)**

1. $21\frac{1}{3}\pi$.
2. $\dfrac{625\sqrt{3}}{4}$.
3. $34\frac{2}{15}\pi$.
4. $1\frac{23}{105}\pi$.
5. $557\frac{2}{5}\pi$.
6. $333\frac{1}{3}$ cu in.
7. $8\frac{8}{15}\pi$.
8. $2\frac{1}{4}$
9. $25\frac{3}{5}\pi$
10. $213\frac{1}{3}\pi$.
11. $38\frac{2}{5}$

**Page 76 (General Exercises)**

1. $6\frac{1}{3}$ ft.
2. 81 ft.
3. $10\frac{2}{3}$ ft
4. $8\frac{13}{24}$ mi.
5. 20.
6. $\dfrac{4a^2\sqrt{2}}{3}$.
7. 8.
8. $21\frac{1}{12}$.
9. $\frac{35}{48}$ T
12. Reduced to $\frac{1}{8}$ original pressure

**Page 77 (General Exercises)**

13. Twice as great.
14. $\frac{13}{20}$ T.
15. $16\,w$.
16. $68\frac{4}{15}\pi$
17. $6\frac{14}{15}\pi$.
18. $\dfrac{144\pi\sqrt{3}}{35}$.
19. $96\,\pi$.
20. $341\frac{1}{3}$ cu. in.
21. $\dfrac{32\sqrt{3}}{3}$.
23. $(ah^2 - \frac{1}{3}h^3)\pi$

**Page 78 (General Exercises)**

24. $8\,\pi$.
25. $115\frac{1}{5}$.
26. $34\frac{2}{15}\pi$.
27. 9
28. $204\frac{4}{5}$
29. 728,949 ft.-lb.
30. 5301 ft -lb.

## CHAPTER IV

**Page 81 (§ 28)**

1. $x^2 + y^2 - 8x + 4y + 11 = 0$.
2. $x^2 + y^2 + 2y - 24 = 0$.
3. $(-3, 5)$; 5.
4. $(-\frac{4}{5}, \frac{3}{5})$; 2.
5. $3x - 2y + 4 = 0$.

**Page 84 (§ 30)**

1. $(-2, 0)$.
2. $(0, 1)$.
3. $(1\frac{1}{2}, 0)$.
4. $(0, -1\frac{3}{4})$
5. $3\frac{29}{32}$ ft.
6. $10\sqrt{10}$ ft.
7. $28\frac{7}{16}$ ft
8. $\frac{10\pi\sqrt{6}}{3}$ in

**Page 85 (§ 30)**

9. $y^2 + 6x - 9 = 0$.
10. $x^2 - 4x - 12y + 16 = 0$.

**Page 87 (§ 31)**

1. $(\pm 4, 0)$, $(\pm\sqrt{7}, 0)$, $\frac{\sqrt{7}}{4}$.
2. $(0, \pm 3)$, $(0, \pm\sqrt{5})$, $\frac{\sqrt{5}}{3}$.
3. $\left(\pm\frac{\sqrt{6}}{3}, 0\right)$, $\left(\pm\frac{\sqrt{6}}{6}, 0\right)$, $\frac{1}{2}$.
4. $\left(\pm\frac{\sqrt{2}}{2}, 0\right)$; $\left(\pm\frac{\sqrt{6}}{6}, 0\right)$, $\frac{\sqrt{3}}{3}$.
5. $9x^2 + 25y^2 - 36x - 189 = 0$
6. $49x^2 + 24y^2 - 120y - 144 = 0$.

**Page 91 (§ 32)**

1. $(\pm 3, 0)$, $(\pm\sqrt{13}, 0)$, $2x \pm 3y = 0$; $\frac{\sqrt{13}}{3}$.
2. $(\pm 2, 0)$, $(\pm\sqrt{13}, 0)$, $3x \pm 2y = 0$; $\frac{\sqrt{13}}{2}$.
3. $(0, \pm\sqrt{2})$, $(0, \pm\sqrt{5})$, $\sqrt{2}x \pm \sqrt{3}y = 0$, $\frac{\sqrt{10}}{2}$.
4. $(\pm 2\sqrt{2}, 0)$; $(\pm 4, 0)$, $x \pm y = 0$, $\sqrt{2}$
5. $\left(\pm\frac{\sqrt{2}}{2}, 0\right)$, $\left(\pm\frac{\sqrt{30}}{6}, 0\right)$, $\sqrt{2}x \pm \sqrt{3}y = 0$, $\frac{\sqrt{15}}{3}$.
6. $\left(0, \pm\frac{1}{2}\right)$, $\left(0, \pm\frac{\sqrt{5}}{2}\right)$, $x \pm 2y = 0$; $\sqrt{5}$.
7. $3x^2 - y^2 - 12x + 9 = 0$.
8. $3x^2 - y^2 + 4y - 16 = 0$.

**Page 102 (§ 36)**

1. $18x^2 + 22x - 3$
2. $4x^2(x + 3)$.
3. $6x(x^4 - 1)$.
4. $-\frac{36x}{(x^2 - 9)^2}$.
5. $-\frac{x^2 + 4x + 1}{(x^2 + x + 1)^2}$.
6. $\frac{2}{3}\left(\frac{1}{\sqrt[3]{x}} + \frac{1}{\sqrt[3]{x^5}}\right)$.
7. $2x - 1 + \frac{4}{x^2} - \frac{2}{x^3}$.
8. $2(8x + 3)(4x^2 + 3x + 1)$.
9. $\frac{3x^2 + 8x}{2\sqrt{x^3 + 4x^2 + 1}}$.
10. $3x\sqrt{x^2 + 4}$.
11. $-3x\sqrt{a^2 - x^2}$.
12. $\frac{x}{\sqrt{(9 - x^2)^3}}$.
13. $\frac{3x(x^3 + 4)}{\sqrt[5]{(x^5 + 10x^2 + 3)^2}}$.
14. $-\frac{1}{(x - 1)\sqrt{x^2 - 1}}$.

15. $\dfrac{a-b}{2(x-b)\sqrt{(x-a)(x-b)}}$.
16. $\dfrac{9-2x^2}{\sqrt{9-x^2}}$.
17. $\dfrac{2x^2+x-1}{\sqrt{x^2-1}}$.
18. $\dfrac{a^2}{\sqrt{(a^2+x^2)^3}}$.
19. $\dfrac{1}{(x+1)\sqrt{x^2-1}}$.
20. $\dfrac{3x^2+6}{\sqrt[3]{(x^3+3)^4}}$.
21. $\dfrac{(x+1)(3x^2+x+2)}{\sqrt{x^2+1}}$.
22. $\dfrac{x^3}{\sqrt{(x^2+9)^3}}$.
23. $\dfrac{1}{\sqrt[3]{(1+x^3)^4}}$.

**Page 104 (§ 37)**

1. $\dfrac{ay-x^2}{y^2-ax}$.
2. $-\dfrac{2xy}{x^2+4a^2}$.
3. $\dfrac{y}{x+y(x+y)^2}$.
4. $\dfrac{\sqrt{y+x}-\sqrt{y-x}}{a}$.
5. $-\dfrac{2x}{3y}$, $-\dfrac{4}{3y^3}$.
6. $\dfrac{4x}{9y}$; $-\dfrac{16}{9y^3}$.
7. $-\dfrac{x^2}{y^2}$, $-\dfrac{2a^3x}{y^5}$.
8. $-\dfrac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$, $\dfrac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}$.
9. $-\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$; $\dfrac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}$.
10. $-\dfrac{y+2}{x+3}$; $\dfrac{2y+4}{(x+3)^2}$.
11. $-\dfrac{2x+y}{x+2y}$; $-\dfrac{6a^2}{(x+2y)^3}$.

**Page 105 (§ 38)**

1. $3x-4y+2=0$. 2. $x-7y+5=0$. 3. $(-2, -1)$. 4. $\tan^{-1}\frac{12}{5}$.

**Page 106 (§ 38)**

8. $\dfrac{\pi}{4}$.
9. $\dfrac{\pi}{2}$; $\tan^{-1}\frac{9}{13}$.
10. $\tan^{-1}3$.
11. $\tan^{-1}\frac{4}{3}$.
12. $\tan^{-1}\frac{6}{7}$; $\tan^{-1}\frac{3}{5}$.
13. $\tan^{-1}\frac{11}{2}$.

**Page 110 (§ 40)**

1. $x^3=8y$; $\sqrt{4+9t^4}$.
2. $x=(y-1)^2$, $\sqrt{4t^2+1}$.
3. $x^2-6x+9y=0$; $\frac{2}{3}\sqrt{117-48t+16t^2}$.
4. $(3, 1)$.
5. $(y-2)^2=(x+3)^3$; $t\sqrt{4+9t^2}$.
6. $x^{\frac{1}{2}}+y^{\frac{1}{2}}=2$; $8\sqrt{1-2t+2t^2}$.
7. $\left(\dfrac{v_0^2\sin 2\alpha}{2g}, \dfrac{v_0^2\sin^2\alpha}{2g}\right)$.
8. $\left(\dfrac{v_0^2\sin 2\alpha}{2g}, \dfrac{v_0^2\sin^2\alpha}{2g}\right)$.
9. $\dfrac{v_0^2\sin 2\alpha}{g}$, $v_0$, $\alpha$.
10. $\dfrac{\pi}{4}$.
11. $y=x\tan\alpha-\dfrac{gx^2}{2v_0^2\cos^2\alpha}$.

**Page 112 (§ 41)**

1. $3\sqrt{2}$ ft. per second. 2. 12.5 ft. per second. 3. $\frac{1}{40}$ in. per minute.

4. Circle; $\dfrac{15}{x}$ ft. per second ($x=$ distance of point from wall).

5. 2.64 ft. per second.

**Page 113 (§ 41)**

6. 0.13 cm. per second. 7. 0.21 in. per minute. 8. 6.6 ft. per second.

9. $\dfrac{2x+4y}{2x+y}$ ft. per second, where $x$ is the distance of top of ladder, and $y$ is distance of foot of ladder, from base of pyramid.

**Page 114 (General Exercises)**

20. $31x + 8y + 9a = 0$.
21. $x_1^{-\frac{1}{3}}x + y_1^{-\frac{1}{3}}y = a^{\frac{2}{3}}$
26. $\frac{\pi}{4}$; $\tan^{-1}\frac{1}{3}$.
27. $\tan^{-1}\frac{3}{4}$.
28. $\frac{\pi}{2}$; $\tan^{-1}7$.
29. $\frac{\pi}{2}$
30. $\tan^{-1}\frac{12}{5}$.
31. $\frac{\pi}{2}$, $\tan^{-1}\frac{1}{2}$.
32. $\frac{\pi}{2}$, $\tan^{-1}\frac{3}{2}$
33. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, $3t$.
34. $y = \frac{8}{x^2+4}$; $\frac{2\sqrt{(t^2+1)^4+4t^2}}{(t^2+1)^2}$.
35. $2a^2y = 2abx - gx^2$; $\sqrt{a^2+b^2-2bgt+g^2t^2}$

**Page 115 (General Exercises)**

36. $(x+2)^2 = (y-1)^3$, $\frac{1}{2}\sqrt{9t+4}$
37. 20 ft per second, $10\sqrt{5}$ ft per second, (100, 20).
38. $v_x = \pm\frac{yv_0}{a}$, $v_y = \mp\frac{xv_0}{a}$.
39. Velocity in path $= \frac{2ct}{y}\sqrt{ax+a^2}$
40. $\frac{1}{\sqrt{t-t^2}}$.
41. 5.8 mi. per hour, 28.8 mi.
42. $9x^2 + 36y^2 = 4096$; $\frac{4}{3}\sqrt{\frac{256-3t^2}{64-t^2}}$.
43. $\frac{3s}{\sqrt{s^2-400}}$ ft per second, where $s$ is length of rope from man to boat.
44. 0.06 ft. per minute
45. $\frac{1}{500}$ ft. per minute.

**Page 116 (General Exercises)**

46. 0.08 ft. per minute
47. 0.01 in. per minute.
48. $\frac{40}{81}$ sq. in. per minute
49. 0.04 in. per second.
50. Length is twice breadth
51. Other sides equal.
52. Breadth, 9 in., depth, $9\sqrt{3}$ in.
53. Length $= \frac{3}{2}$ breadth
54. Side of base, 10 ft., depth, 5 ft
55. Depth $= \frac{1}{2}$ side of base.
56. Radius, 3 in., height, 3 in.

**Page 117 (General Exercises)**

57. 2.64 in.
58. $\frac{1}{\sqrt{2}}$.
59. $\frac{a\sqrt{6}}{3}$.
60. 8 mi. from point on bank nearest to $A$.
61. $4\frac{1}{16}\frac{3}{}$ mi. travel on land
62. $a - \frac{bm}{\sqrt{n^2-m^2}}$ mi. on land, $\frac{bn}{\sqrt{n^2-m^2}}$ mi. in water
63. $1\frac{8}{37}$ hr.
64. $\sqrt[3]{100}$ mi. per hour.

**Page 118 (General Exercises)**

65. Velocity in still water $= \frac{3a}{2}$ mi. per hour
66. Base $= a\sqrt{3}$, altitude $= \frac{3}{2}b$

## CHAPTER V

**Page 126 (§ 44)**

1. $15\cos 5x$
2. $\sec^2\frac{x}{2}$.
3. $2\sin^2 2x\cos 2x$.
4. $-5\sin 10x$.
5. $\sin^2 x$
6. $5\sin^2 5x\cos^3 5x$.
7. $5\sec^2\frac{5x}{2}\tan\frac{5x}{2}$.
8. $-3\csc^3 3x\operatorname{ctn} 3x$.

**Page 127 (§ 44)**

9. $\sin^3 \frac{x}{2}$.

10. $-\csc^4 \frac{x}{2}$.

11. $x \sin \frac{x}{2}$

12. $2 \sec x (\sec x + \tan x)^2$.

13. $2 \cos 4x$.

14. $9 \tan^4 3x$.

15. $2 \sec 2x (\sec^2 2x + \tan^2 2x)$

16 $\sin^3 2x \cos^2 2x$

17. $-\frac{2}{3} \cos 2x \cos^2 3y$.

18. $-\frac{y}{x}$.

**Page 129 (§ 45)**

2. $s = 3 \sin \frac{\pi t}{2}$.

3. $\pi$, 5.

4. At mean point of motion; at extreme points of motion.

5. At extreme points of motion, at mean point of motion.

6. $2\sqrt{(s-3)(5-s)}$; $4(4-s)$

7. $\pi$.

8. 10, $2\pi$.

**Page 134 (§ 47)**

1. $\frac{3}{\sqrt{1-9x^2}}$.

2 $-\frac{1}{x\sqrt{x^2-1}}$.

3. $\frac{1}{\sqrt{6x-x^2}}$.

4. $-\frac{3}{\sqrt{12x-9x^2}}$.

5. $\frac{1}{2+2x+x^2}$.

6. $\frac{1}{(x-1)\sqrt{x^2-2x}}$.

7. $\frac{2x}{x^4+1}$.

8. $\frac{1}{x\sqrt{25x^2-1}}$.

9. $-\frac{1}{x\sqrt{4x^2-1}}$.

10. $-\frac{2}{x^2+4}$.

11. $-\frac{1}{(x+1)\sqrt{x^2+2x}}$.

12. $-\frac{1}{\sqrt{1-x^2}}$.

13. $\frac{2}{x\sqrt{x^2-1}}$.

14. $2\sqrt{1-x^2}$.

15. $\frac{16}{(x^2+4)^2}$.

16. $\frac{\sqrt{x^2-4}}{x}$.

17. $\frac{2a}{x^2+a^2}$.

18. $\cos^{-1}\sqrt{1-x^2}$.

**Page 136 (§ 48)**

1. $v_x = \pm 9.42$ ft. per second; $v_y = \mp 36.46$ ft. per second.

2. 5 3

**Page 137 (§ 48)**

3. 3 radians per unit of time.

**Page 138 (§ 49)**

7. $\frac{b \sin \phi}{a - b \cos \phi}$; $\phi = \cos^{-1} \frac{b}{a}$.

**Page 141 (§ 50)**

1. $\frac{2}{3}(x+3)\sqrt{x^2+3x}$.

2. $3(axy)^{\frac{1}{3}}$.

3. $\frac{5\sqrt{5}}{4}$.

5. $\frac{17\sqrt{17}}{4}$.

6. $a\phi$

**Page 145 (§ 51)**

17. Origin, $\left(\sqrt{3}, \frac{\pi}{3}\right)$. 18. Origin, $\left(\pm a\sqrt[4]{\frac{3}{4}}, \frac{\pi}{6}\right)$. 19. Origin, $\left(2, \frac{\pi}{2}\right)$.

20 Origin, $\left(\pm a, \frac{\pi}{2}\right)$, $\left(\pm \frac{a}{\sqrt[4]{2}}, \frac{\pi}{4}\right)$, $\left(\pm \frac{a}{\sqrt[4]{2}}, \frac{3\pi}{4}\right)$.

21. $r^2 \sin 2\theta = 8$
22. $r = 4a(\cos\theta + \sin\theta)$.
23. $r = 2a\sin\theta$.
24. $r^2 = a^2 \cos 2\theta$.
25. $x - a = 0$
26. $x^2 + y^2 - 2ax = 0$
27. $x^4 + x^2y^2 = a^2y^2$.
28. $(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$.

**Page 148 (§ 52)**

1. $\tan^{-1}\frac{1}{3}$. 2. $\pi - \tan^{-1}\frac{2}{3}$. 3. 0.

**Page 149 (General Exercises)**

9. $2(1 + \sec^2 2x)$.
10. $\sec^4(3x + 2)$.
11. $\cos^2(2 - 3x)$.
12. $\dfrac{\sec^2(x-y) + \sec^2(x+y)}{\sec^2(x-y) - \sec^2(x+y)}$
13. $-3\operatorname{ctn}^2\frac{x}{5}\csc^4\frac{x}{5}$
14 $-8\csc^2 4x(\operatorname{ctn} 4x + 1)$
15. $a\tan ax\sec^2 ax$.
16. $8\cos^3 2x\sin 4x\cos 6x$.
17. $\frac{1}{2}\sin^3\frac{x}{4}$.
18. $\tan^4 2x$.
19. $\dfrac{1}{(x+1)\sqrt{x}}$
20. $-\dfrac{1}{\sqrt{2 + x - x^2}}$.
21. $\dfrac{1}{2(x+1)\sqrt{x}}$.
22. $\dfrac{1}{x\sqrt{49x^2 - 1}}$
23. $-\dfrac{4}{x^2 + 4}$.
24. $\dfrac{1}{(1-x)\sqrt{x^2 - 2x}}$
25. $\dfrac{2}{\sqrt{3 - 4x - 4x^2}}$.
26. $\dfrac{2 - 3x}{\sqrt{9 - x^2}}$.
27. $\dfrac{2}{(x^2 - 1)\sqrt{x^2 - 2}}$.
28. $s = 3$; 2

**Page 150 (General Exercises)**

29. $k\sqrt{a^2\sin^2 kt + b^2\cos^2 kt}$.
30. $2\frac{1}{2}$
31. $\sqrt{41}$, $\pi + 2\tan^{-1}\frac{4}{5}$.
32. $\dfrac{16\sqrt{2}}{\pi^3}$.
33. $\dfrac{(\pi^2 + 1)^{\frac{3}{2}}}{2\pi}$.
34. $2a\sqrt{3}$.
35. $\dfrac{(a^2 + b^2)^{\frac{3}{2}}}{2ab\sqrt{2}}$.
46. Origin, $\left(\pm\frac{\sqrt{6}}{2}, \frac{\pi}{6}\right)$; $\left(\pm\frac{\sqrt{6}}{2}, \frac{5\pi}{6}\right)$.
47. Origin; $\left(\frac{2a}{\sqrt{5}}, \tan^{-1}\frac{1}{2}\right)$.
48. Origin; $\left(\frac{4}{\sqrt{5}}, \tan^{-1} 2\right)$.
49. Origin; $\left(2a, \frac{\pi}{4}\right)$, $\left(2a, \frac{5\pi}{4}\right)$.

**Page 151 (General Exercises)**

50. $r = \dfrac{2a\sin^2\theta}{\cos\theta}$.
51. $r = a\operatorname{ctn}\theta$.
52. $(x^2 + y^2)^2 - 4a^2xy = 0$.
53. $(x^2 + y^2)^2 + 2ax(x^2 + y^2) - a^2y^2 = 0$.
54. $\tan^{-1}\frac{2}{3}$.

55. 0; $\tan^{-1}2$

56. $0, \frac{\pi}{2}, \tan^{-1}3\sqrt{3}$.

57. $\frac{\pi}{4}$.

58. $\frac{\pi}{4}$

59 $16\sqrt{3}$ ft

60. 72°.

61. $\sqrt{2}$ ft

62. At an angle $\tan^{-1}k$ with ground.

## Page 152 (General Exercises)

63. 12 m.

64. $5\sqrt{5}$ ft.

65. $a$.

67. 6.1 ft per second.

68. 15 sq ft, 10.04 sq. ft. per second.

69. 26 7 mi. per minute.

70. $\left(b\sin\theta + \frac{b^2\sin\theta\cos\theta}{\sqrt{a^2 - b^2\sin^2\theta}}\right)$ times angular velocity of $AB$, where $\theta =$ angle $CAB$.

71. $\frac{(x-2)^2}{9} + \frac{(y-3)^2}{4} = 1$; $\sqrt{9\sin^2 t + 4\cos^2 t}$, where $t = (2k+1)\frac{\pi}{2}$.

72. $\frac{x^2}{4} - \frac{y^2}{16} = 1$; $6\sec 3t\sqrt{\tan^2 3t + 4\sec^2 3t}$.

## Page 153 (General Exercises)

73. $6\sin 2\phi$.

74. $a\sqrt{1+\cos^2 x}$, fastest when $x = k\pi$; most slowly when $x = (2k+1)\frac{\pi}{2}$.

75. $\tan^{-1}2\sqrt{2}$

76. $\tan^{-1}\frac{4}{3}$

77. $0, \tan^{-1}3\sqrt{3}$.

78. $\tan^{-1}\frac{1}{2}, \tan^{-1}4\sqrt{2}$

79. $\tan^{-1}3$; $\tan^{-1}\frac{1}{3}$.

# CHAPTER VI

## Page 162 (§ 55)

(The student is not expected to obtain exactly these answers, they are given merely to indicate approximately the solution.)

1. $y = 0.62x - 0.76$.

2. $I = 0.0017D$

## Page 163 (§ 55)

3. $y = 0.30(2.7)^x$. 4. $c = 0.010(0.84)^t$. 5. $a = 0.0000000048\, l^{3.06}$. 6 $pv^{1.35} = 10$.

## Page 165 (§ 56)

1. $\frac{1}{x^2}e^{-\frac{1}{x}}$.

2. $\frac{1}{2}(e^x - e^{-x})$.

3. $2x\, a^{x^2-1}\ln a$.

4. $a^{\sin^{-1}x}\frac{\ln a}{\sqrt{1-x^2}}$.

5. $\frac{2x+4}{x^2+4x-1}$.

6. $\frac{2x+3}{2x^2+6x+9}$.

7. $\frac{1}{x^2-9}$.

8. $\frac{1}{\sqrt{x^2+4}}$.

9. $\frac{3}{\sqrt{9x^2+1}}$.

10. $-4\sec 2x$.

11. $\frac{2(e^{2x} - e^{-2x})}{e^{2x} + e^{-2x}}$.

12. $e^{-2x}(3\cos 3x - 2\sin 3x)$.

13. $\operatorname{ctn}^{-1}x$.

14. $27x^2e^{3x}$.

15. $5e^{2x}\sin x$.

16. $\frac{2}{e^x + e^{-x}}$.

17. $2\sec^3 x$.

18. $\frac{1}{x\sqrt{x+1}}$.

**Page 167 (§ 57)**

1. $y = 5e^{\frac{x}{2}}$ 2. $y = 45.22\,e^{0.01x}$. 3. $y = 7\,e^{0.847x}$ 4. \$739.

**Page 168 (§ 57)**

5. $P = 10000\,e^{0.0229t}$ 6. $c = 0.01\,e^{-0.0446t}$. 7. 2 min.

**Page 168 (General Exercises)**

*10. $p = 0.018t + 24$. *11. Load $= 192 - 6.6$ length.

**Page 169 (General Exercises)**

*12. $s = 25(0.40)^t$
*13. $c = 0.010(0.83)^t$
*14. $t = 0.1\sqrt{l}$.
*15. $I = 0.023\sqrt{\theta}$
*16. $y = 0.40x^{1.54}$.

**Page 170 (General Exercises)**

*17. $pv = 1620$
18. $\dfrac{1}{9x^2 - 4}$.
19. ctn $x$
20. $\dfrac{2}{e^x + e^{-x}}$
21. $2\csc^{-1}2x$.
22. $2(x+1)e^{-\frac{2}{x}}$
23. $\dfrac{2}{e^{2x} - e^{6x}}$.
24. $a\tan^3 ax$.
25. $\tan^{-1}x$.
26. 0.898
27. 15.8 hr
28. 1000 sec
29. $p = 14.7\,e^{-0.00004h}$.

**Page 171 (General Exercises)**

31. $2\sqrt{2}e^{-2t}$, $2e^{-2t}$ 33. $\dfrac{(e^{2\pi} + 4)^{\frac{3}{2}}}{8e^{2\pi}}$. 35. $\sqrt{2}\,e^t$.

37. $\dfrac{(x^2+1)^{\frac{3}{2}}}{x}$, $\dfrac{3\sqrt{3}}{2}$. 38. $\dfrac{(1 + e^{\pi})^{\frac{3}{2}}}{2e^{\frac{\pi}{2}}}$.

## CHAPTER VII

**Page 176 (§ 59)**

1. $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$
2. $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$.
3. $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + \cdots$.
4. $x + \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{x^5}{5} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{x^7}{7} + \cdots$.
5. $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$.
6. $\dfrac{1}{\sqrt{2}}\left(1 + x - \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots\right)$
7. $\ln 2 + \dfrac{x}{2} - \dfrac{1}{2}\cdot\dfrac{x^2}{2^2} + \dfrac{1}{3}\cdot\dfrac{x^3}{2^3} + \cdots$.
9. 0.0872
10. 0.4695.

* Statement in regard to answers to exercises in § 55 is true of this answer.

**Page 178 (§ 60)**

1. $\ln 5 + \frac{x-5}{5} - \frac{1}{2}\frac{(x-5)^2}{5^2} + \frac{1}{3}\frac{(x-5)^3}{5^3} + \cdots$.

2. $\frac{1}{3}\left(1 - \frac{x-2}{3} + \frac{(x-2)^2}{3^2} - \frac{(x-2)^3}{3^3} + \cdots\right)$.

3. $\frac{1}{\sqrt{2}}\left(1 + \left(x - \frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} - \frac{\left(x-\frac{\pi}{4}\right)^3}{3!} + \cdots\right)$

4. $\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}\cdot\frac{\left(x-\frac{\pi}{6}\right)^2}{2!} + \frac{1}{2}\frac{\left(x-\frac{\pi}{6}\right)^3}{3!} + \cdots$.

5. $e^3\left(1 + (x-3) + \frac{(x-3)^2}{2!} + \frac{(x-3)^3}{3!} + \cdots\right)$

6. $\frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{2}\cdot\frac{(x-1)^2}{2!} + \frac{1}{2}\frac{(x-1)^3}{3!} + \cdots$

7. $\sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{2\sqrt{2}}\frac{(x-1)^2}{2!} - \frac{3}{4\sqrt{2}}\frac{(x-1)^3}{3!} + \cdots$.

8. 0.7193　　9. 0.8480　　10. 3.0042

**Page 179 (General Exercises)**

1. $-\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\cdots\right)$.

2. $1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots$.

3. $1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$

4. $2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7}\cdots\right)$.

5. $\frac{1}{2} - \frac{\sqrt{3}}{2}x - \frac{1}{2}\cdot\frac{x^2}{2!} + \frac{\sqrt{3}}{2}\cdot\frac{x^3}{3!} + \cdots$.

6. $\frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{1}{2}\cdot\frac{x^2}{2!} - \frac{\sqrt{3}}{2}\cdot\frac{x^3}{3!} - \cdots$.

7. $1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$

10. $1 - 2x + 2x^2 - 2x^3 + \cdots$.

11. $1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!} + \cdots$

12. $x + \frac{2x^2}{2!} + \frac{2x^3}{3!} - \frac{4x^5}{5!} + \cdots$,

13. $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots$,

14. 0.9659.
15. 0.5150.
16. 1.6487.
17. 0.40546.
18. 0.69315; 1.0986.
19. 0.22314; 1.6094.
20. 0.8473; 1.046.

**Page 180 (General Exercises)**

23. 2.0305.
24. 2.9625.

25. $x - \frac{x^3}{3(3!)} + \frac{x^5}{5(5!)} - \frac{x^7}{7(7!)} + \cdots$.

26. $x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \cdots$.

27. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$.

28. $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$.

## CHAPTER VIII

### Page 183 (§ 61)

1. $3x^2 - 8xy + 8y^2$, $-4x^2 + 16xy + 15y^2$.
2. $\dfrac{y^3 - x^2y}{(x^2+y^2)^2}$, $\dfrac{x^3 - y^2x}{(x^2+y^2)^2}$
3. $\dfrac{y}{x^2+y^2}$, $-\dfrac{x}{x^2+y^2}$.
4. $\dfrac{y}{\sqrt{1-x^2y^2}}$, $\dfrac{x}{\sqrt{1-x^2y^2}}$
5. $\dfrac{y}{xy - x^2}$; $\dfrac{x}{xy - y^2}$.
6. $\dfrac{2y^2}{(x+y)^2}\cos\dfrac{2xy}{x+y}$, $\dfrac{2x^2}{(x+y)^2}\cos\dfrac{2xy}{x+y}$.
7. $\dfrac{1}{y}e^{\frac{x}{y}}$, $-\dfrac{x}{y^2}e^{\frac{x}{y}}$.
8. $\dfrac{1}{\sqrt{x^2+y^2}}$; $\dfrac{y}{\sqrt{x^2+y^2}\,(x+\sqrt{x^2+y^2})}$.

### Page 185 (§ 62)

1. $\dfrac{x^2-y^2}{x^2+y^2}$.
2. $-e^y\sin(x-y)$
3. $\dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$.

### Page 187 (§ 63)

1. 0 000061
2. 0 0012.
3. 2%.

### Page 188 (§ 63)

4. 0.018 in
5. 0 0105
6. 0 015 in.
7. 6320 ft.
8. 0.0064.

### Page 191 (§ 64)

1. $-z$.
2. $-\frac{2}{5}$
4. $-\frac{1}{2}$
5. $-\dfrac{1}{\sqrt{x^2+y^2}}$; 0.
6. 0; 0

### Page 192 (General Exercises)

8. $-14.33$ cu ft.
9. 1735
10. 0.5655 sq. in.; 0 5756 sq. in
11. 3.0 in
12. 0 35 in.
13. 2.206 sq. in. per second.
15. 4.4 sq. in. per second.

### Page 193 (General Exercises)

16. $-\dfrac{3\sqrt{2}}{5}$.
17. $\alpha = \tan^{-1}\frac{3}{4}$, $5k$.
18. $-\dfrac{1}{2}\cos\alpha - \dfrac{\sqrt{3}}{2}\sin\alpha$, 1.

## CHAPTER IX

### Page 198 (§ 66)

1. $2x^3 + 2x^2 + 6\ln x$.
2. $\frac{4}{21}(3x+7)x^{\frac{3}{4}}$
3. $\dfrac{2(x^3+5)}{5\sqrt{x}}$.
4. $\ln x - \dfrac{2}{x} - \dfrac{1}{2x^2}$.
5. $\frac{1}{2}[x^2 + \ln(x^2-1)]$.
6. $\frac{1}{8}(x^2+1)^4$
7. $\frac{1}{6}(x^4+4)^{\frac{3}{2}}$.
8. $\frac{1}{3}\ln(e^{3x}+6)$.
9. $\frac{1}{2}\ln(2x+\sin 2x)$
10. $-\dfrac{1}{3(x-\sin x)^3}$.
11. $\dfrac{1}{a}\ln(e^{ax} + \tan ax)$
12. $\frac{1}{3}\ln(x^3+3x^2+1)$.
13. $-\dfrac{1}{a}\ln(1+\cos ax)$.
14. $-\frac{1}{8}\cos^4 2x$.
15. $\frac{1}{9}\sin^3 3x$.
16. $\frac{1}{2}\sin^2(x+2)$.
17. $-\frac{1}{8}\cos^{\frac{8}{3}} 3x$.
18. $\frac{1}{9}(3\tan 3x + \tan^3 3x)$.
19. $-\frac{1}{6}\operatorname{ctn}^3(2x+1)$.
20. $\frac{1}{8}[\cos^3(2x-3) - 3\cos(2x-3)]$.

**Page 202 (§ 67)**

1. $\frac{1}{3}\sin^{-1}\frac{3x}{4}$.
2. $\frac{1}{\sqrt{3}}\sin^{-1}\frac{x\sqrt{21}}{7}$
3. $\frac{1}{3}\sec^{-1}\frac{2x}{3}$.
4. $\sec^{-1}x\sqrt{3}$.
5. $\frac{1}{\sqrt{3}}\sin^{-1}\frac{3x-2}{\sqrt{10}}$.
6. $\frac{1}{\sqrt{7}}\tan^{-1}\frac{x}{\sqrt{7}}$.
7. $\frac{1}{\sqrt{21}}\tan^{-1}\frac{x\sqrt{21}}{7}$.
8. $\frac{1}{2}\tan^{-1}\frac{x+3}{2}$
9. $\sin^{-1}\frac{x-3}{3}$.
10. $\frac{1}{2}\sin^{-1}\frac{4x-3}{3}$.
11. $\frac{1}{\sqrt{3}}\sin^{-1}\frac{6x-5}{5}$.
12. $\sin^{-1}\frac{x+2}{\sqrt{5}}$.
13. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{3x-2}{\sqrt{2}}$
14. $\frac{1}{2}\left[3\ln(x^2+4)+11\tan^{-1}\frac{x}{2}\right]$.
15. $5\sin^{-1}\frac{x}{2}-2\sqrt{4-x^2}$.
16. $\frac{\pi}{6}$
17. $\frac{\pi}{4}$.
18. $\pi$.
19. $\frac{5\pi}{36}$.
20. $\frac{\pi}{18}$

**Page 204 (§ 68)**

1. $\ln(x+\sqrt{x^2+2})$.
2. $\frac{1}{3}\ln(3x+\sqrt{9x^2-1})$.
3. $\frac{1}{\sqrt{3}}\ln(3x+\sqrt{9x^2-12})$.
4. $\ln(x+1+\sqrt{x^2+2x})$
5. $\frac{1}{\sqrt{3}}\ln(3x+1+\sqrt{9x^2+6x+9})$.
6. $\frac{1}{20}\ln\frac{2x-5}{2x+5}$.
7. $\frac{1}{2\sqrt{2}}\ln\frac{x\sqrt{2}-1}{x\sqrt{2}+1}$.
8. $\frac{1}{2\sqrt{15}}\ln\frac{3x-\sqrt{15}}{3x+\sqrt{15}}$.
9. $\frac{1}{2}\ln(x^2-5)+\frac{3}{2\sqrt{5}}\ln\frac{x-\sqrt{5}}{x+\sqrt{5}}$.
10. $\frac{1}{4}\ln\frac{x}{x+4}$.
11. $\frac{1}{5}\ln\frac{x}{3x+5}$.
12. $\frac{1}{\sqrt{5}}\ln\frac{2x-3-\sqrt{5}}{2x-3+\sqrt{5}}$.
13. $\frac{1}{\sqrt{33}}\ln\frac{2x+5-\sqrt{33}}{2x+5+\sqrt{33}}$.
14. $\frac{1}{2\sqrt{13}}\ln\frac{4x-1-\sqrt{13}}{4x-1+\sqrt{13}}$.
15. $\ln\frac{3+\sqrt{5}}{4}$.
16. $\frac{1}{3}\ln(3+\sqrt{10})$.
17. $\frac{1}{3}\ln\frac{4+2\sqrt{3}}{3+\sqrt{5}}$.
18. $\frac{1}{\sqrt{2}}\ln(2+\sqrt{5})$.
19. $\frac{1}{2}\ln\frac{2}{3}$.
20. $\frac{1}{5}\ln\frac{7}{2}$.

**Page 207 (§ 69)**

1. $-\frac{1}{3}\cos(3x-2)$.
2. $-\frac{1}{2}\sin(4-2x)$.
3. $\frac{1}{3}\sec(3x-1)$.
4. $4\tan\frac{x}{4}$.
5. $\frac{2}{3}\ln\sec\frac{3x}{2}$.
6. $\frac{1}{5}\ln\sin 5x$.
7. $\frac{1}{2}\ln[\csc(2x+3)-\operatorname{ctn}(2x+3)]$.
8. $-2\csc\frac{x}{2}$.
9. $\frac{1}{4}\ln[\sec(4x+2)+\tan(4x+2)]$.
10. $\frac{1}{2}\operatorname{ctn}(3-2x)$.
11. $\ln(\csc x-\operatorname{ctn}x)+2\cos x$.
12. $\frac{1}{2}(x-\sin x)$.
13. $\frac{1}{2}\left(x+3\sin\frac{x}{3}\right)$.
14. $x-\cos x$.

15. $3\left(\tan\frac{2x}{3} - \sec\frac{2x}{3}\right) - x.$

16. $\frac{1}{3}(x - \sin x)$

17. $\frac{4\sqrt{2}}{3}\sin\frac{3x}{4}.$

18. $-\frac{1}{\sqrt{2}}\cos 2x.$

19. $\frac{2}{3}.$

20. $\ln\frac{4}{3}$

21. $\sqrt{3} - 1 - \frac{\pi}{12}.$

22. $\frac{1}{2}\ln\frac{2+\sqrt{3}}{1+\sqrt{2}}$

23. $\frac{\pi}{4}.$

24. 1

## Page 208 (§ 70)

1. $\frac{1}{2}e^{2x+5}$
2. $\frac{1}{2}e^{x^2}$
3. $e^x + \frac{x^{e+1}}{e+1}$
4. $\frac{e^{a+bx}c^{a+bx}}{b(1+\ln c)}$
5. $\frac{1}{2}(e^{2x} - e^{-2x}) + 2x.$
6. $e^x + e^{-x}.$
7. $\ln(e^{2x} - 1) - x.$
8. $2(e^{\frac{1}{2}} - e^{-\frac{1}{2}}).$
9. $\frac{99}{\ln 10}$
10. $\frac{3}{2\ln 2}.$
11. $\frac{1}{2}(e - 1)$
12. $\ln\frac{e^2+1}{2e}.$

## Page 212 (§ 72)

1. $\frac{1}{2}x^2 - 4x + 12\ln(x+2) + \frac{8}{x+2}.$
2. $\frac{2}{15}(3x^2 + 4x + 8)\sqrt{x-1}$
3. $\frac{3}{112}(16x^2 - 6x - 27)\sqrt[3]{2x-3}$
4. $\sqrt{x^2-1} - \tan^{-1}\sqrt{x^2-1}$
5. $\frac{x^2+8}{\sqrt{x^2+4}}.$
6. $\frac{x}{\sqrt{4-x^2}} - \sin^{-1}\frac{x}{2}.$
7. $\frac{x^5}{15(3-x^2)^{\frac{5}{2}}}.$
8. $-\frac{\sqrt{4x^2+1}}{x}.$
9. $-\frac{x}{4\sqrt{x^2-4}}.$
10. $\frac{3}{5}.$
11. $\frac{\pi}{12}.$
12. $\frac{\pi}{4}$
13. $\frac{1}{24}(9\sqrt{3} - 10\sqrt{2}).$
14. $\frac{\sqrt{2}}{360}.$
15. $2\frac{2}{3}$

## Page 216 (§ 74)

1. $\frac{1}{9}(3x - 1)e^{3x}$
2. $\frac{1}{4}(2x^2 - 2x + 1)e^{2x}$
3. $x\cos^{-1}x - \sqrt{1-x^2}$
4. $x\tan^{-1}3x - \frac{1}{6}\ln(1 + 9x^2).$
5. $\frac{1}{2}x^2\sec^{-1}2x - \frac{1}{8}\sqrt{4x^2-1}.$
6. $\sin x(\ln\sin x - 1)$
7. $\frac{1}{5}(2\cos x + \sin x)e^{2x}.$
8. $\frac{1}{4}(x^2 + 2x\sin x + 2\cos x).$
9. $4 - 2\sqrt{e}$
10. $\frac{1}{9}(81\ln 3 - 26).$
11. $\frac{1}{4}(\pi - 2).$
12. $\frac{1}{8}(\pi - 2).$

## Page 217 (§ 75)

1. $\ln\frac{(x-4)^7}{(x-2)^6}$
2. $\ln(x+3)^2\sqrt{2x-1}$
3. $\ln(x-2)\sqrt{\frac{x-1}{x-3}}$
4. $\ln\frac{x^2-1}{(x+2)^2}.$
5. $\ln\frac{2x^2+x}{x-1}.$
6. $\frac{1}{6}\ln(x-1)(x+2)^2(x-3)^3.$

## Page 220 (General Exercises)

1. $x^3 + 2x^2 + \frac{1}{x} + \frac{1}{2x^2}.$
2. $\frac{3}{4}x^{\frac{8}{3}} + \frac{3}{4}x^{\frac{4}{3}} + 3x^{\frac{2}{3}}.$
3. $\frac{5}{6}x^{\frac{6}{5}} - \frac{5}{2}x^{\frac{2}{5}}.$
4. $\frac{1}{4}x^4 + \frac{4}{3}x^3 + 2x^2.$
5. $\frac{1}{4}(x^2 - 4)^2$
6. $\frac{1}{4}(x^3 + 3)^{\frac{4}{3}}.$

**Page 221 (General Exercises)**

7. $\frac{1}{6}(2 + e^{2x})^3$.

8. $\frac{3}{4}(1 + 2x + x^4)^{\frac{4}{3}}$.

9. $\sqrt[3]{3x + x^3}$

10. $\frac{1}{2}x^2 + x + \ln(x - 1)$.

11. $\frac{1}{30}[5\sin^3(2x - 1) - 3\sin^5(2x - 1)]$.

12. $\dfrac{1}{3}\sin\dfrac{x}{5}\left(3\cos^4\dfrac{x}{5} + 4\cos^2\dfrac{x}{5} + 8\right)$.

13. $-\frac{1}{60}\operatorname{ctn} 4x(15 + 10\operatorname{ctn}^2 4x + 3\operatorname{ctn}^4 4x)$.

14. $\frac{1}{15}[3\sec^5(x - 2) - 5\sec^3(x - 2)]$.

15. $\frac{1}{2}\tan^2(x - 1) + \ln\tan(x - 1)$

16. $\frac{1}{70}(7\csc^5 2x - 5\csc^7 2x)$.

17. $\frac{1}{7}(\sec^2 3x - 7)\sqrt[3]{\sec 3x}$.

18. $-\sqrt{\csc 2x}$

19. $-\frac{1}{48}(8\operatorname{ctn} 5x + 3\operatorname{ctn}^3 5x)\sqrt[5]{\operatorname{ctn} 5x}$

20. $\frac{1}{153}(153 - 34\sin^2 4x + 9\sin^4 4x)\sqrt[4]{\sin 4x}$.

21. $-\frac{1}{72}(9\operatorname{ctn} 5x + 4\operatorname{ctn}^3 5x)\sqrt[5]{\operatorname{ctn}^3 5x}$.

22. $\dfrac{1}{2}\sin^{-1}\dfrac{2x}{5}$.

23. $\sin^{-1}\dfrac{x}{\sqrt{5}}$.

24. $\dfrac{1}{3}\sin^{-1}\dfrac{3x}{2\sqrt{2}}$.

25. $\dfrac{1}{\sqrt{5}}\sec^{-1}\dfrac{3x}{\sqrt{5}}$.

26. $\dfrac{1}{4}\sec^{-1}\dfrac{x\sqrt{5}}{2}$.

27. $\dfrac{1}{2}\sec^{-1}\dfrac{x+1}{2}$.

28. $\dfrac{1}{3\sqrt{3}}\sec^{-1}\dfrac{3x-1}{\sqrt{3}}$.

29. $\dfrac{1}{5}\sec^{-1}\dfrac{2x+3}{5}$.

30. $\sin^{-1}\dfrac{2x-5}{5}$

31. $\sin^{-1}\dfrac{x-1}{2}$.

32. $\dfrac{1}{2}\sin^{-1}\dfrac{2x-1}{2\sqrt{2}}$

33. $\dfrac{1}{\sqrt{2}}\sin^{-1}\dfrac{2x+3}{\sqrt{3}}$.

34. $\dfrac{1}{2\sqrt{5}}\tan^{-1}\dfrac{2x}{\sqrt{5}}$.

35. $\dfrac{1}{\sqrt{15}}\tan^{-1}\dfrac{x\sqrt{15}}{3}$

36. $\dfrac{1}{\sqrt{6}}\tan^{-1}\dfrac{x-2}{\sqrt{6}}$.

37. $\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{2x+2}{\sqrt{3}}$

38 $\dfrac{1}{2}\ln\dfrac{3x+1}{x+1}$.

39. $\dfrac{2}{\sqrt{11}}\tan^{-1}\dfrac{18x+5}{\sqrt{11}}$

40. $\dfrac{1}{6}\tan^{-1}\dfrac{x^2}{3}$.

**Page 222 (General Exercises)**

41. $\dfrac{1}{3\sqrt{21}}\tan^{-1}\dfrac{x^3\sqrt{21}}{7}$.

42. $\dfrac{1}{2}\sin^{-1}\dfrac{x^2-2}{2}$.

43. $\dfrac{1}{2\sqrt{6}}\sec^{-1}\dfrac{x^2}{\sqrt{6}}$.

44. $\dfrac{1}{6}\sec^{-1}\dfrac{x^3}{2}$.

45. $\dfrac{1}{4}\sin^{-1}\dfrac{2x^2+3}{\sqrt{10}}$.

46. $\dfrac{1}{3}\ln(3x^2 + 7) - \dfrac{3}{\sqrt{21}}\tan^{-1}\dfrac{x\sqrt{21}}{7}$.

47. $2\sqrt{2}\sin^{-1}\dfrac{x\sqrt{2}}{3} + \dfrac{3}{2}\sqrt{9 - 2x^2}$.

48. $\ln(x + \sqrt{x^2 - 7})$.

49. $\frac{1}{2}\ln(2x + \sqrt{4x^2 + 3})$.

50. $\dfrac{1}{\sqrt{2}}\ln(2x + \sqrt{4x^2 + 10})$.

51. $\frac{1}{4}\ln(2x^2 + \sqrt{4x^4 - 5})$.

52. $\frac{1}{3}\ln(x^3 + \sqrt{x^6 + 7})$.

53. $2\sqrt{x^2 + 4} + \ln(x + \sqrt{x^2 + 4})$.

54. $\sqrt{3x^2+1} - \dfrac{4}{\sqrt{3}}\ln(3x + \sqrt{9x^2+3})$.

55. $\dfrac{1}{\sqrt{2}}\ln(4x - 3 + 2\sqrt{4x^2 - 6x})$.

56. $\frac{1}{2}\ln\left(2x+1+\sqrt{4x^2+4x+7}\right)$.

57. $\dfrac{1}{\sqrt{5}}\ln\left(5x+2+\sqrt{25x^2+20x-5}\right)$

58. $\frac{1}{3}\ln\left(3x-4+\sqrt{9x^2-24x+14}\right)$

59. $\dfrac{1}{6\sqrt{7}}\ln\dfrac{3x-\sqrt{7}}{3x+\sqrt{7}}$.

60. $\dfrac{1}{4\sqrt{5}}\ln\dfrac{x\sqrt{5}-2}{x\sqrt{5}+2}$.

61. $\dfrac{1}{4\sqrt{6}}\ln\dfrac{x\sqrt{6}+2}{x\sqrt{6}-2}$.

62. $\frac{1}{3}\ln(3x^2-7)-\dfrac{11}{2\sqrt{21}}\ln\dfrac{3x-\sqrt{21}}{3x+\sqrt{21}}$.

63. $\dfrac{1}{6}\ln\dfrac{2x-3}{x}$.

64. $\dfrac{1}{5}\ln\dfrac{2x-5}{x}$

65. $\dfrac{1}{12}\ln\dfrac{x-1}{3x+1}$.

66. $\dfrac{1}{4\sqrt{6}}\ln\dfrac{2x-1-\sqrt{6}}{2x-1+\sqrt{6}}$.

67. $\dfrac{1}{15}\ln\dfrac{5x-2}{5x+1}$.

68. $\dfrac{1}{8}\ln\dfrac{x-2}{3x+2}$.

69. $\dfrac{1}{2\sqrt{6}}\ln\dfrac{5x+1-\sqrt{6}}{5x+1+\sqrt{6}}$

70. $\dfrac{1}{5}\ln\dfrac{2x-1}{x+2}$.

71. $\frac{1}{3}(\tan 3x-\operatorname{ctn} 3x)$.

72. $\ln\left[\sec(x-\frac{2}{3})+\tan(x-\frac{2}{3})\right]$.

73. $-\cos 2x$.

74. $\ln(\sec x+\tan x)$.

## Page 223 (General Exercises)

75 $\dfrac{3}{8}x-\dfrac{3}{4}\sin^3\dfrac{x}{3}\cos\dfrac{x}{3}-\dfrac{9}{16}\sin\dfrac{2x}{3}$.

76 $\frac{1}{2}\tan 2x-x$

77. $\frac{1}{2}(\sin 2x-\cos 2x)$.

78. $x+2\left(\operatorname{ctn}\dfrac{x}{2}-\csc\dfrac{x}{2}\right)$

79. $\tan 2x-x$

80 $\sin x-\cos x$

81. $2\sqrt{e^x}$

82. $\frac{3}{2}\sqrt[3]{e^{x^2}}$.

83. $x-\frac{1}{2}\ln(1+e^{2x})$.

84. $\frac{1}{24}(3x-1)(5x+1)^{\frac{3}{5}}$.

85. $\frac{1}{14}(2x^6+x^3-6)\sqrt[3]{x^3+2}$.

86. $\frac{2}{5}\sqrt{x^5+3}$

87. $\frac{1}{6}x^3-\frac{1}{12}\ln(2x^3+1)$.

88. $\dfrac{3x^2-2}{3(1-x^2)^{\frac{3}{2}}}$

89. $\dfrac{x^3}{27(4x^2+9)^{\frac{3}{2}}}$

90. $\dfrac{(2x^2+25)\sqrt{x^2-25}}{1875x^3}$.

91. $(x\ln 5-1)\dfrac{5^{x+2}}{(\ln 5)^2}$.

92. $\frac{1}{8}e^{2x}(4x^3-6x^2+6x-3)$

93. $\frac{1}{2}(x^2+4)\tan^{-1}\dfrac{x}{2}-x$.

94. $\frac{1}{27}(2-9x^2)\cos 3x+\frac{2}{9}x\sin 3x$

95. $x\left[(\ln 2x)^2-2\ln 2x+2\right]$

96. $x\ln\left(3x+\sqrt{9x^2-4}\right)-\frac{1}{3}\sqrt{9x^2-4}$.

97. $\frac{1}{16}\ln(2x-1)(2x+3)^3$.

98. $\dfrac{1}{2}\ln\dfrac{x^4(x-3)}{(x+3)^3}$.

99. $\ln\dfrac{(x-2)^2}{\sqrt{4x^2-9}}$.

100. $\dfrac{1}{8}\ln\dfrac{(2x+1)(2x+3)^2}{2x-1}$

101. $\dfrac{1}{2}\ln\dfrac{(x^2-4)(x+3)^2}{(x-3)^2}$.

102. $\dfrac{\pi}{12}$

103. $\dfrac{\pi}{12}$.

104 $\frac{1}{11}\ln\frac{25}{14}$.

105. $\frac{1}{6}\ln 4$

106. $\dfrac{1+\sqrt{3}}{4\sqrt{3}}$.

## Page 224 (General Exercises)

107. $\frac{\sqrt{3}-2}{\sqrt{3}}$.
108. 0
109. $e^2 - e$.
110. $\frac{e^6 - 1}{2e^9}$.
111. $\frac{1}{\ln 5}\left(5^{\frac{\pi}{3}} - 5^{\frac{\pi}{4}}\right)$.
112. $6\frac{10}{21}$
113. $7\frac{5}{9}$.
114. $\frac{1}{9}\ln 2$.
115. $\frac{1}{4\sqrt{2}}\ln\frac{9+4\sqrt{2}}{14}$.
116. $3\pi$
117. $\frac{13}{64}\sqrt{3}$.
118. $\frac{1}{27}(18\sqrt{2} - 8\sqrt{3})$.
119. $\frac{1}{48}(3\sqrt{3} - \pi)$
120. $\frac{2}{5}(e^{\frac{\pi}{2}} + 1)$
121. $2 - \ln 3$.
122. $\frac{1}{4}\pi^2$.
123. $\frac{1}{12}(\pi + 2\ln 2 - 2)$.

# CHAPTER X

## Page 228 (§ 77)

1. 2.
2. $a^2\left(e^{\frac{h}{a}} - e^{-\frac{h}{a}}\right)$
3. $4\pi a^2$
4. $8\frac{8}{15}$.
5. $\frac{1}{12}$
6. $3\pi a^2$
7. $\frac{32}{105}\pi a^3$
8. $\frac{2}{3}ba^2$
9. $\frac{\pi}{2}(\pi + 4a)$.

## Page 229 (§ 77)

10. $2\pi^2 a^3$.
11. $\frac{16}{35}\pi a^3$
12. $\frac{\pi a^3}{3}(2 - \sqrt{3})$.
13. $\frac{1}{12}$ T
14. $259\frac{1}{5}\,w$.
15. $3\frac{9}{20}$ T.
16. $\frac{1}{4}\pi w$.

## Page 232 (§ 79)

1. $2a^2$.
2. $\frac{\pi a^2}{4n}$.
3. $\frac{3}{2}\pi a^2$.
4. $\frac{59\pi}{2}$.
5. $\frac{a^2\sqrt{2}}{3}$.
6. $\frac{3}{4}\pi a^2$
7. $11\pi$.
8. $\frac{4}{3}$.
9. $40\pi$.
10. $\frac{a^2}{4}(\pi - 2)$.

## Page 234 (§ 80)

1. $\frac{\pi a}{4}$; $a$ is radius of semicircle.
2. $\frac{2a}{\pi}$; $a$ is radius of semicircle.
3. $\frac{2}{\pi}$.
4. $\frac{2v_0^2}{\pi g}$.
5. $\frac{1}{3}\pi a^2$.

## Page 235 (§ 80)

6. $\frac{2}{9}\pi a^3$.
7. 100 revolutions per minute.
8. 5.54 lb. per square inch.

## Page 236 (§ 81)

1. $\frac{1}{9}(13\sqrt{13} - 8)$.
2. $\frac{a}{2}\left(e^{\frac{h}{a}} - e^{-\frac{h}{a}}\right)$.
3. $6a$.
4. $6a$.
5. $8\pi^2 a$.
6. $\sqrt{2}\left(1 - e^{-\frac{\pi}{2}}\right)$.
7. $\frac{8a}{3}$.
8. $8a$.

**Page 238 (§ 82)**

1. $\frac{m}{a}$.
2. $22\frac{2}{9}$ ft.-lb
3. $\frac{\pi k a}{12}$
4. 196,350 ft.-lb
5. $586\frac{2}{3}$ ft.-lb
6. $2kca^2$, $k$ is the constant ratio
7. $\frac{Raw}{R+a}$ mi.-lb ; $R$ is radius of earth in miles
8. $2\pi C$

**Page 239 (§ 82)**

9. 1.76 ft.-lb., 1 56 ft.-lb.

**Page 239 (General Exercises)**

1. $3\sin^{-1}\frac{3}{5}$
2. $12\pi - \frac{9\sqrt{3}}{5}$, $24\pi + \frac{9\sqrt{3}}{5}$.
3. $16 - 12\ln 3$
4. $\frac{2a^2}{3}(3\pi - 2)$
5. $\frac{8a^2}{15}$.
6. $1\frac{23}{30}$, $15\frac{3}{10}$
7. $2\pi ab$
8. $4a^2$
9. $\frac{3\pi ab}{4}$.
10. $\frac{3}{8}\pi a^2$
11. $\frac{1}{2}\pi h^2\sqrt{k_1 k_2}$, $k_1$ and $k_2$ are the values for $k$ in the equation $y^2 = kx$
12. $\frac{272\pi}{15}$.
13. $\pi a^3 \tan\theta$.

**Page 240 (General Exercises)**

14. $\frac{32}{105}a^3$.
15. $4\pi a^3(\ln 4 - 1)$
16. $\frac{204\pi}{5}$
17. $\frac{64\pi\sqrt{2}}{5}$
18. $\frac{4\pi\sqrt{2}}{15}$.
19. $\pi a^3(6 - 8\ln 2)$, $\frac{2\pi a^3}{3}(12\ln 2 - 7)$
20. $\frac{208\pi}{15}$, $\frac{4\pi}{15}(25\sqrt{5} - 52)$
21. $\frac{\pi a^3}{6}(11 + 3e^2)$.
22. $\frac{2\pi\sqrt{3}}{3}$
23. $\pi^2 - 4$

**Page 241 (General Exercises)**

24. $400\pi$ lb.
25. $\frac{125}{6}(8\pi + 9\sqrt{3})w$.
26. 4 4 lb.
27. 307 lb.
28. 12 3 T
29. $\frac{a^2}{n}$
30. $\frac{a^2}{8}(4 - \pi)$
31. $21\frac{1}{3}$.
32. $\frac{a^2}{8}(\pi - 2)$
33. $\frac{a^2}{6}(3\sqrt{3} - \pi)$.
34. $\frac{a^2}{16}(8\pi + 9\sqrt{3})$
35. $\frac{a^2}{72}(3\sqrt{3} - \pi)$.

**Page 242 (General Exercises)**

36. $\frac{a^2\sqrt{3}}{4}$.
37. $\frac{a^2}{4}(8 + \pi)$, $\frac{a^2}{4}(5\pi - 8)$, $\frac{a^2}{4}(8 - \pi)$
38. $\frac{\pi a}{2}$.
40. $\frac{2}{3}$
41. 950
42. $\frac{a}{2}(\ln 9 - 1)$.

43. $\ln(e + e^{-1})$

44. $\dfrac{4(b^2 + ba + a^2)}{b + a}$.

45. $\dfrac{3\pi a}{2}$

46. $\dfrac{a}{2}[2\pi\sqrt{4\pi^2+1} + \ln(2\pi + \sqrt{4\pi^2+1})]$.

47. $\dfrac{9k}{8a^{\frac{2}{3}}}$.

48. $\dfrac{3ka^2}{2}$.

**Page 243 (General Exercises)**

49. $\dfrac{3k}{8a^2}$.

50. 50,000 ft.-lb.

51. 438.1 ft.-lb.

## CHAPTER XI

**Page 245 (§ 83)**

1. $30 - \ln 3$.

2. $\ln 3$.

3. $2\frac{22}{27}$.

4. $\pi - 1$

**Page 246 (§ 83)**

5. $\frac{1}{8}(\pi - 3\sqrt{3})$

6. 14.

7. $\dfrac{\pi a^2}{4}$.

8. $\dfrac{a}{4}(\pi - 2)$

9. $\dfrac{7a^2}{6}$.

10. $\dfrac{a^3}{9}(22 - \pi)$.

11. $\sqrt{a}(\pi\sqrt{2} - 4)$.

12. $\dfrac{7\sqrt{2}}{30}$.

**Page 254 (§ 85)**

3. $\left(\dfrac{2a}{e+1}, \dfrac{a(e^4 + 4e^2 - 1)}{4e(e^2 - 1)}\right)$.

4. $\left(\pi a, \dfrac{4a}{3}\right)$

5. On axis, $\dfrac{3a}{5}$ from vertex.

6. On axis of quadrant, $\dfrac{4a\sqrt{2}}{3\pi}$ from center of circle.

7. Intersection of medians.

8. $\left(\dfrac{\pi}{2}, \dfrac{\pi}{8}\right)$.

9. (9, 9).

10. On axis, distant $\dfrac{4a^2 + 3\pi ab + 6b^2}{3\pi a + 12b}$ from base.

11. $\left(\pi a, \dfrac{5a}{6}\right)$.

13. $\left(\dfrac{5a}{6}, 0\right)$.

14. $(\frac{20}{11}, 0)$.

15. On axis, distant $\frac{3}{8}$ (radius) from base.

16. On axis, distant $\dfrac{a}{3}$ from base.

17. On axis, distant $\dfrac{k}{6}$ from base.

18. Middle point of axis.

**Page 256 (§ 86)**

3. On line of centers, distant $\dfrac{(r_1 + r_2)r_2^2}{r_1^2 + r_2^2}$ from center of circle of radius $r_1$.

4. On axis of shell, distant $\dfrac{3(r_2^4 - r_1^4)}{8(r_2^3 - r_1^3)}$ from common base of spherical surfaces.

5. Middle point of axis.

**Page 257 (§ 86)**

6. On axis, $\dfrac{h_2^4 - h_1^4}{4(h_2^3 - h_1^3)}$ distant from base.

7. On axis, $\frac{3}{4}$ of distance from vertex to base.

8. $(4\frac{27}{56}, 4\frac{27}{56})$, the outer edges of the square being taken as $OX$ and $OY$.

9. On axis, distant 4 8 from corner of square
10. On axis, distant 3.98 in from center of cylinder in direction of larger ball.
11. On axis, distant 3.4 ft. from base of pedestal.

## Page 260 (§ 87)

3. $\frac{1}{2}$ base × altitude
4. $\frac{16\pi a^2 b}{15}$, $a =$ altitude and $2b =$ base of segment.
5. $\frac{8\pi a^2 b}{5}$; $a =$ altitude and $2b =$ base of segment.
6. $2\pi a^2 b$, $8\pi ab$.
7. $\frac{\pi ab}{3}(b + 3c)$, $\pi\left[2ca + 2bc + b^2 + (b + 2c)\sqrt{a^2 + b^2}\right]$
8. 7 07 T.
9. $\pi abcw$
10. $\frac{2ab^2 w}{3}$.
11. $\frac{4wab}{15}(5c + 3a)$, $a =$ altitude and $2b =$ base of segment
12. $cw$ (area).

## Page 265 (§ 88)

1. $\frac{1}{12}Ma^2$.
2. $\frac{1}{2}Ma^2$.
3. $\frac{1}{6}Ma^2$
4. $\frac{1}{4}Ma^2$, $\frac{1}{4}Mb^2$

## Page 266 (§ 88)

5. $M\frac{h^2(b + 3a)}{6(b + a)}$.
6. $\frac{8}{35}Ma^2$
7. $\frac{1}{12}M(a^2 + b^2)$
8. $\frac{1}{2}M(r_2^2 + r_1^2)$.
9. $\frac{1}{6}M(a^2 + b^2)$.
10. $145\frac{5}{6}$.
11 $3\frac{17}{21}$.
12. $\frac{\pi a^4}{4}$
13. $\frac{1}{2}Mr^2$.
14. $\frac{1}{2}M(r_2^2 + r_1^2)$.
15. $\frac{2}{5}Ma^2$, $a =$ radius.
16. $575\frac{2}{3}\pi$.
17. $\frac{3}{10}M\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}$

## Page 268 (§ 89)

1 $\frac{5}{4}Ma^2$, $a =$ radius.
2. $3918\frac{1}{2}$

## Page 269 (§ 89)

3. $2284\frac{2}{3}$.
4. $4569\frac{1}{3}$.
5. 44,990
6. 89,980
7 $\frac{1}{2}M(r_2^2 + 3r_1^2)$
8. $\frac{1}{2}M(r_1^2 + 3r_2^2)$.
9 $\frac{1}{4}M(r_1^2 + 5r_2^2)$
10. $\frac{5}{16}Ma^2$
11. $\frac{11}{48}Ma^2$.

## Page 281 (§ 92)

1. $2\pi$.
4. $\frac{2a^3}{9}(3\pi - 4)$.
6. $\frac{32a^3}{9}$.
3. $\frac{3\pi a^3}{4}$.
5. $\frac{7\pi a^3}{6}$
7. $\frac{3\pi a^4}{32b}$.
8. $\pi a^3$
9. $\frac{a^3}{9}(3\pi + 20 - 16\sqrt{2})$.
10. $\frac{9\pi}{4}$
11. $\frac{\pi}{32}$.

**Page 283 (§ 93)**

1. $\left(0, 0, \frac{2c}{3}\right)$
2. On axis of ring, distant 2 ft. from center of shell.
3. On axis, distant $\frac{h(r_1^2 + 2r_1r_2 + 3r_2^2)}{4(r_1^2 + r_1r_2 + r_2^2)}$ from upper base
4. On axis, $\frac{3a}{8}$ from base.
5. $\left(\frac{3a}{8}, \frac{3b}{8}, \frac{3c}{8}\right)$.
6. $\left(0, 0, \frac{9a}{10}\right)$.
7. $\left(0, 0, \frac{6 + 3\sqrt{2}}{16}\right)$.
8. $\left(0, 0, \frac{3a^2(2b^2 - a^2)}{8[b^3 - (b^2 - a^2)^{\frac{3}{2}}]}\right)$.

**Page 285 (§ 94)**

1. $\frac{1}{12}M(a^2 + b^2)$, where $M$ is mass, and $a$ and $b$ are the lengths of the sides perpendicular to the axis.
2. $\frac{40}{9}M$.
3. $\frac{1}{20}M(3a^2 + 2h^2)$
4. $\frac{1}{6}M(a^2 + b^2)$.
5. $\frac{3}{20}M(a^2 + 4h^2)$
6. $\frac{1}{12}M(3a^2 + 4h^2)$.
7. $M\frac{2a^2(15\pi - 26)}{25(3\pi - 4)}$.

**Page 286 (§ 94)**

8. $\frac{14\pi}{3}$.
9. $\frac{1}{4}M(a^2 + b^2)$.
10. $\frac{1}{6}M(a^2 + b^2)$.

**Page 286 (General Exercises)**

1. $\left(0, \frac{2a}{5}\right)$.
2. $\left(\frac{253}{95}, \frac{12\sqrt{5} + 2\ln\frac{7 + 3\sqrt{5}}{2}}{19}\right)$.
3. $\left(\frac{5b}{7}, 0\right)$, $x = b$ is the ordinate.
4. $\left(\frac{8}{5}, \frac{3\sqrt{2}}{4}\right)$.
5. $\left(\frac{4(1 + 5\ln 2)}{15}, \frac{73}{252}\right)$.
6. $\left(\frac{256a}{315\pi}, \frac{256a}{315\pi}\right)$.
7. $\left(\frac{4a}{3\pi}, \frac{4(a + b)}{3\pi}\right)$.
8. $\left(0, \frac{352}{5(2 + 3\pi)}\right)$.
9. $\left(\frac{c}{2}, \frac{b}{2} + \frac{c^2}{3a}\right)$.
10. On axis, distant $\frac{2a}{3(\pi + 2\sqrt{3})}$ from base of triangle and away from semicircle.
11. On axis, distant $\frac{4a^2 + 2ab\sqrt{3} + b^2}{2(4a + b\sqrt{3})}$ from base.

**Page 287 (General Exercises)**

12. On axis of segment, distant $\frac{4(a^2 - b^2)^{\frac{3}{2}}}{3\left(\pi a^2 - 2b\sqrt{a^2 - b^2} - 2a^2\sin^{-1}\frac{b}{a}\right)}$ from center of circle.

13. On axis, distant $\dfrac{2(6b^2 - a^2)}{3(8b - \pi a)}$ from center of semicircle.

14. On axis of plate, distant $\dfrac{4(r_2^2 + r_2 r_1 + r_1^2)}{3\pi(r_2 + r_1)}$ from center of circles.

15. On axis, distant $\dfrac{2r_1}{\pi}$ from center of circle

16. On axis of square, 8 in. from bottom.

17. On axis, distant $\dfrac{\sqrt{3}}{3}$ in. from center of hexagon

18. On axis, distant $\dfrac{c^2 d}{ab - c^2}$ from center of ellipse

19. On axis of solid, distant $\dfrac{9b}{16}$ from smaller base

20. On axis, distant from base $\frac{5}{16}$ of distance to top.

21. On axis of segment, distant $\dfrac{3[2a^2(h_2^2 - h_1^2) - (h_2^4 - h_1^4)]}{4[3a^2(h_2 - h_1) - (h_2^3 - h_1^3)]}$ from center of sphere

22. $\frac{1}{18} Ma^2$

23. $\frac{1}{3} Ma^2$

## Page 288 (General Exercises)

24. $\frac{17}{2} M.$
25. $\frac{125}{12} M.$
26. $\frac{425}{12} M.$
27. $\frac{50}{3} M$
28. $\frac{125}{12} M$
29. $\frac{1024}{3} - 4\pi.$
30. $\frac{1024}{3} - 4\pi.$
31. $\dfrac{a^4}{24}(15\pi - 32)$; $a$ is radius.
32. $\frac{125}{6} M.$
33. $\dfrac{\pi a^4}{4}.$
34. $\dfrac{35\pi a^4}{16}.$
35. $\dfrac{5\pi a^4}{16}.$
36. $\frac{40}{9} M.$
37. $\dfrac{4\pi}{15}(r_2^5 - r_1^5).$

## Page 289 (General Exercises)

38. $\dfrac{\pi a^5}{30}(8 - 5\sqrt{2}).$
39. $\frac{1322}{305} M$
40. $\frac{1}{4} M(4b^2 + 3a^2)$
41. $\frac{1}{12} M(3b^2 + 4h^2).$
42. $\left(\dfrac{6a}{5}, 0, \dfrac{27\pi a}{128}\right).$
43. $\frac{48}{25} Ma^2.$
44. $\dfrac{32a^3}{9}.$
45. $\left(\dfrac{32a\sqrt{2}}{35}, 0, \dfrac{27\pi a}{64}\right).$
46. $\frac{48}{25} Ma^2.$
47. $\dfrac{a^3}{420}(105\pi + 64\sqrt{2}).$
48. $\dfrac{3\pi a^5}{32} + \dfrac{256 a^5\sqrt{2}}{3465}.$
49. $3\pi a^3$
50. $\frac{11}{6} Ma^2.$
51. $\dfrac{4\pi}{15}\left[(2r_2^2 + 3a^2)(r_2^2 - a^2)^{\frac{3}{2}} - (2r_1^2 + 3a^2)(r_1^2 - a^2)^{\frac{3}{2}}\right].$

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