Multivariate Volatility with BEKK and DCC

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This assignment involves applying multivariate GARCH models to analyze the volatility dynamics of Intel stock and the Dow Jones Index, focusing on capturing autocorrelations and volatility clustering. By estimating both DCC and BEKK models, we aim to understand which model better captures the data's characteristics, using statistical tests and out-of-sample evaluations to compare their performance. Theoretically, the BEKK model is slightly more flexible since BEKK jointly models both the conditional conditional variances and covariances, meanwhile DCC separates the modeling of variances from modeling of means, preventing overfitting.

1 Data

In this part we analyze the daily log-returns of Intel and Dow Jones, with a focus on autocorrelations, volatility clustering, and distributional assumptions.

The 2 time series considered in this paper is the daily log-return of the Intel stock (INTC) and Dow Jones Industrial Average Index (DJI) , from 1992-02-25 to 2019-01-01. Denote it by $r_t = (\Delta \log \mathrm{INTC}_t, \Delta \log \mathrm{DJI}_t\ )$, and its plot is given in Figure 1.

General Trends in Returns (1992–2019)

Over the long period from 1992 to 2019, the Dow Jones Industrial Average showed a positive upward trend, though with significant dips during major financial crises (such as the dot-com crash in the early 2000s and the global financial crisis in 2007–2009). There were also periods of higher volatility, which were typically associated with major economic and geopolitical events.

Intel's stock exhibited more fluctuation compared to the Dow, given its tech-heavy nature and dependence on industry cycles. Intel saw strong growth during the 1990s and early 2000s as a leader in semiconductors but experienced more volatility afterward due to industry shifts and increased competition.

Both Intel and the Dow Jones exhibited **volatility clustering** over the long run, with periods of high volatility followed by lower volatility, particularly after major market corrections. These patterns were especially evident during the crisis periods and after major economic shocks, where investor sentiment and uncertainty caused large price swings.

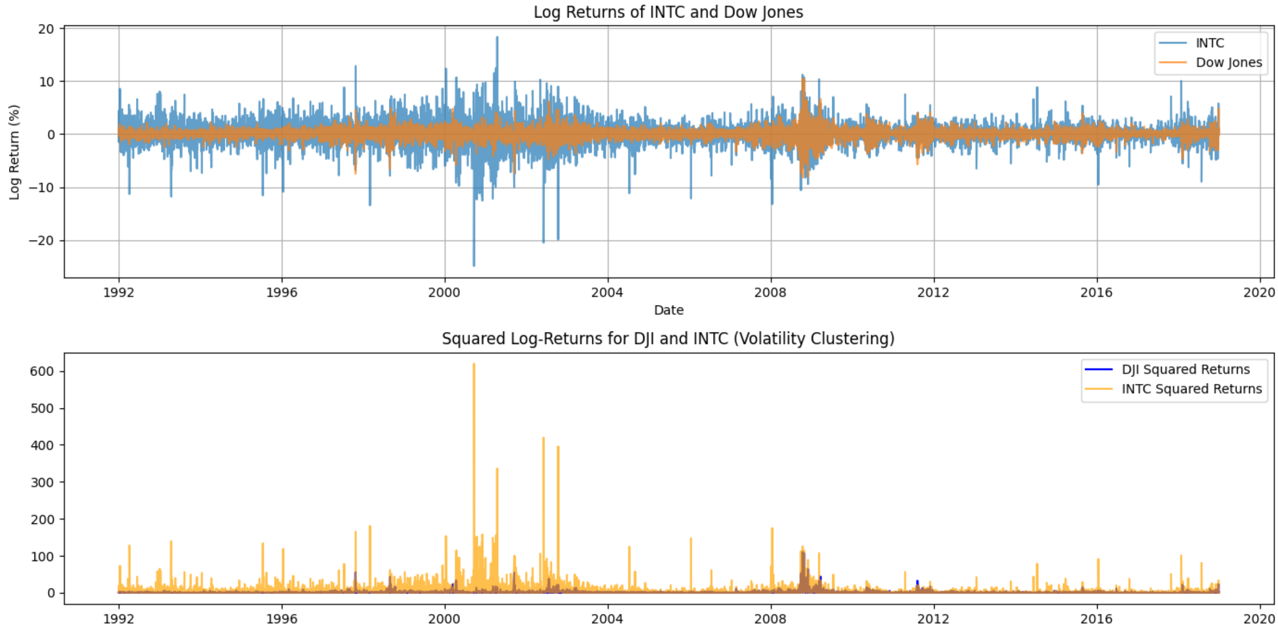

Figure 1: Log-return of INTC and DJI

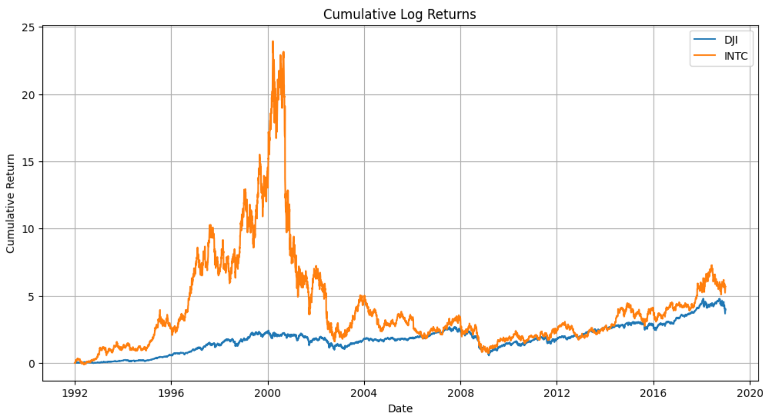

Figure 2: Cumulative Log-Returns

Distribution and Autocorrelation

Now, our aim is to model the covariance matrix of the return of the two stock. Firstly, we analyze the correlation structure. The sameple covariance matrix of INTC and DJI is

$$\begin{bmatrix} 5.63361183 & 1.42294899 \\ 1.42294899 & 1.1343994 \end{bmatrix}$$

To analyze the autocorrelation structure we look at ACF and PACF of individual stocks, which is diplayed in Figure 3.

It can be observed that all ACFs and PACFs do not cut off or exponentially decay, thus it is hard to conclude which order of ARCH/GARCH from the ACF and PACF. Instead, we will select the parameters based on BIC for each DCC and BEKK models. We have only looked at the individual ACF and PACF of each stock. To learn more about the joint autocorrelation of two stocks, we perform Ljung-Box tests.

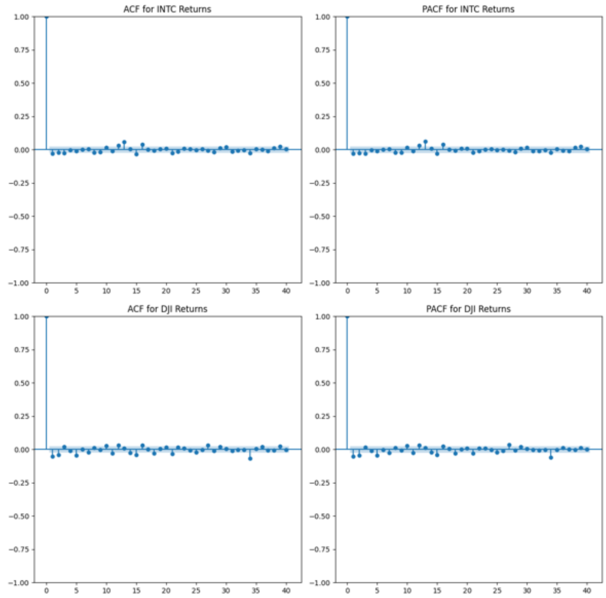

Figure 3: ACF and PACF of INTC and DJI.

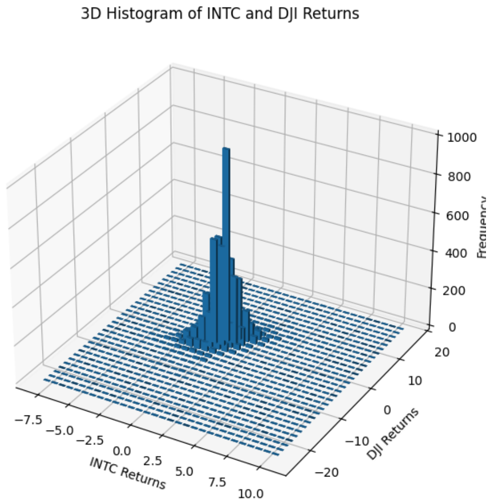

Figure 4: Histogram of joint density of INTC and DJI.

Regarding the distributional assumptions, we have plotted the histogram of joint INTC and DJI stocks in Figure 4. From the histogram, it is possible to see that the sample distibution is not really bivariate normal. To confirm this, we performed Shapiro-Wilk test on the log-return of both stocks, which yield p-values $1.046 \cdot 10^{-45}$ for INTC and $3.177 \cdot 10^{-51}$ for DJI. This strongly rejects the normality of both stocks. Thus, to model the reiduals we will assume a more fat tail Generalized Error distribution (GED).

2 DCC

The first model for r_t is the Dynamic Conditional Correlation (DCC) model (Engle, 2002):

$$\Sigma_t = D_t \rho_t D_t,$$
$$\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t} q_{jj,t}}},$$
$$Q_t = (1 - \theta_1 - \theta_2)\bar{Q} + \theta_1 \epsilon_{t-1}\epsilon'_{t-1} + \theta_2 Q_{t-1},$$

with $\epsilon_t = D_t^{-1} a_t$, where $Q_t = \mathrm{Var}(\epsilon_t | F_{t-1})$ and $\bar{Q} = \mathrm{Var}(\epsilon_t)$.

For our case, we model the disturbance with Generalized Error distribution $\epsilon_t \sim$ GED.

Model Selection: DCC Order Determination

The DCC model specification requires selecting appropriate GARCH orders p and q, where p (we don't model the mean). The GARCH orders are determined based on BIC, which yields the optimal $p = 1$, $q = 1$ order.

DCC Model estimation

The result of estimation of DCC model is given in the following table.

Parameter	Estimate	Std. Error	t value	p-value
α_1^{DJI}	0.067381	0.006616	10.1845	0.000000
ϕ_1^{DJI}	0.881029	0.008398	104.9119	0.000000
θ_1^{DJI}	-0.904669	0.007178	-126.0315	0.000000
ω^{DJI}	0.011952	0.002635	4.5354	0.000006
α_1^{DJI}	0.095663	0.011005	8.6926	0.000000
β_1^{DJI}	0.895128	0.011368	78.7392	0.000000
λ^{DJI}	1.351029	0.036586	36.9275	0.000000
α_1^{INTC}	0.074178	0.019040	3.8959	0.000098
ϕ_1^{INTC}	0.771680	0.005173	149.1866	0.000000
θ_1^{INTC}	-0.785642	0.005079	-154.6935	0.000000
ω^{INTC}	0.012551	0.004281	2.9315	0.003373
α_1^{INTC}	0.033893	0.001531	22.1396	0.000000
β_1^{INTC}	0.964103	0.000171	5628.0560	0.000000
λ^{INTC}	1.333462	0.041449	32.1713	0.000000

Table 1: DCC Model Estimation Result

The DCC model estimation provides insights into the volatility dynamics and correlations between Intel (INTC) and the Dow Jones Index (DJI). The results highlight several important aspects of the model fit and parameter estimates:

Parameter Estimates

DJI Parameters

- α_1^{DJI}: The estimate of 0.0674 indicates the magnitude of the ARCH effect, showing how past shocks influence current volatility.
- ϕ_1^{DJI} and θ_1^{DJI}: These parameters, with estimates of 0.8810 and -0.9047 respectively, suggest a strong persistence in volatility, indicating that past volatility significantly influences current volatility levels.
- ω^{DJI}: The estimate of 0.01195 represents the constant term in the GARCH model, indicating the baseline level of volatility.
- β_1^{DJI}: An estimate of 0.8951 suggests a high degree of volatility persistence, meaning that volatility tends to remain elevated over time.
- λ^{DJI}: The shape parameter of 1.351 indicates the distribution's tail behavior, suggesting heavier tails than a normal distribution.

INTC Parameters

- α_1^{INTC}: The estimate of 0.0742 indicates the ARCH effect for INTC, showing how past shocks influence current volatility.
- ϕ_1^{INTC} and θ_1^{INTC}: With estimates of 0.7717 and -0.7856, these parameters suggest a strong persistence in volatility for INTC.
- ω^{INTC}: The estimate of 0.01255 represents the baseline volatility level for INTC.
- β_1^{INTC}: An estimate of 0.9641 indicates a very high degree of volatility persistence.
- λ^{INTC}: The shape parameter of 1.333 suggests heavier tails in the distribution of returns.

Model Fit

The log-likelihood value of -22069.35 and average log-likelihood of -3.25 indicate a reasonable fit of the model to the data. Information criteria (Akaike, Bayes, Shibata, Hannan-Quinn) provide measures of model fit, with values around 6.5, suggesting a balance between model complexity and goodness of fit.

Model Diagnostics

Ljung-Box Test Results

DJI Residuals

- **X-squared = 56.9, df = 10, p-value = 1.392e-08**: The Ljung-Box test indicates significant autocorrelation in the residuals up to lag 10. This suggests that the DCC model may not have fully captured the serial correlation in the DJI residuals.

INTC Residuals

- **X-squared = 23.027, df = 10, p-value = 0.01065**: The test indicates some autocorrelation in the INTC residuals, though it is less significant than for DJI. This suggests that while the DCC model captures most of the dynamics, there may still be some residual autocorrelation.

ARCH Test Results

DJI Residuals

- **GARCH Model Fit**: The ARCH test results show that the residuals exhibit conditional heteroskedasticity, as evidenced by significant alpha and beta values. The model fit indicates that past shocks influence current volatility, justifying the use of a GARCH model.
- **Weighted Ljung-Box Test on Standardized Residuals**: The p-values are relatively high, suggesting no significant autocorrelation in the standardized residuals.
- **Weighted ARCH LM Tests**: The p-values are not significant, indicating that the GARCH model adequately captures the ARCH effects.

INTC Residuals

- **GARCH Model Fit**: Similar to DJI, the INTC residuals show significant conditional heteroskedasticity, with high alpha and beta values. This indicates that the GARCH model is appropriate for capturing the volatility dynamics.
- **Weighted Ljung-Box Test on Standardized Residuals**: The p-values are high, suggesting no significant autocorrelation in the standardized residuals.
- **Weighted ARCH LM Tests**: The p-values are not significant, indicating that the GARCH model adequately captures the ARCH effects.

Why DCC(1,1)?: The diagnostic results indicate that while the DCC(1,1) model captures much of the volatility dynamics and correlations between Intel (INTC) and the Dow Jones Index (DJI), there is still some residual autocorrelation, particularly in the DJI residuals. The ARCH tests confirm the presence of conditional heteroskedasticity, validating the use of GARCH models. The parameter estimates reveal strong persistence in volatility for both assets, with significant ARCH effects indicating the influence of past shocks on current volatility.

The model's ability to capture these dynamics is crucial for risk management and investment strategies, as it provides insights into how volatility and correlations evolve over time. The diagnostic results further validate the model's performance, confirming its suitability for analyzing the joint behavior of these financial time series.

3 BEKK

The second model is BEKK (Baba-Engle-Kraft-Kroner) model, which is given by:

$$\Sigma_t = AA' + \sum_{i=1}^{m} A_i a_{t-1} a'_{t-1} A'_i + \sum_{j=1}^{m} B_j \Sigma_{t-1} B'_j,$$

Model Selection: BEKK Order Determination

The BEKK model specification requires selecting appropriate orders p and q, where p denotes the order of the autoregressive component and q represents the order of the moving average component. To find the optimal order, multiple BEKK models were estimated for different values of p and q, evaluating them based on the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC).

A loop was implemented over p values (ranging from 1 to 3) and q values (ranging from 1 to 4) and fitted each BEKK specification using the `mgarchBEKKpackage` in R. The model was estimated using the Nelder-Mead optimization method, ensuring robust convergence.

Issues with Singular Hessian Matrix

During estimation, the error message indicating a singular Hessian matrix was encountered. A singular Hessian means that the estimated parameters do not have a positive definite Covariance Matrix.

To address this, modifications were made:

- **Scaling the Data** – Standardization was applied using `StandardScaler()` to mitigate numerical instability.
- **Regularization** – A small perturbation ϵ was added to the returns to ensure positive definiteness of covariance matrices.
- **Alternative Optimization** – Switching to Nelder-Mead methods in `mgarchBEKK` improved convergence.
- **Reducing Model Complexity** – Restricting q minimized the number of parameters, reducing estimation difficulties.

Therefore we obtained hugely inflated AIC and BIC values compared to DCC, making a direct comparison using this criteria impossible.

Simulation

The best-fitting model, according to AIC and BIC, is **BEKK(2,4)**. This model outperforms all others, indicating that it provides the best balance of fit and complexity.

Model	p	q	AIC	BIC
BEKK	1	1	37520.136	37535.172
BEKK	1	2	38212.658	38235.212
BEKK	1	3	37977.099	38007.171
BEKK	1	4	38343.714	38381.303
BEKK	2	1	37974.775	37997.329
BEKK	2	2	37314.681	37344.753
BEKK	2	3	37153.625	37191.214
BEKK	2	4	37093.105	37138.212
BEKK	3	1	37501.204	37531.275
BEKK	3	2	37234.829	37272.418
BEKK	3	3	37373.025	37418.132
BEKK	3	4	37378.299	37430.924
Best Model	**2**	**4**	**37093.105**	**37138.212**

Table 2: Fitted BEKK Models with AIC and BIC Values

Model Diagnostics

After estimating the best BEKK model (with the lowest AIC and BIC), diagnostics were carried out.

Correlation Matrix of Log-Returns

DJI and INTC have a moderate positive correlation of 0.563, indicating that their returns move together but not perfectly. This is relevant because BEKK models capture not just the volatility of each asset but also the correlation between their volatilities.

Given this moderate correlation, modeling the joint volatility dynamics using BEKK(p=2, q=4) is beneficial for understanding how volatility and correlations evolve over time.

Ljung-Box Test

DJI residuals: The Ljung-Box test shows no significant autocorrelation up to lag 10, as most p-values are relatively large (e.g., 0.654, 0.878, etc.). This suggests that the residuals are approximately white noise.

INTC residuals: The p-values are slightly lower, but there is no strong indication of autocorrelation, suggesting that the residuals are largely uncorrelated.

Squared returns: The presence of autocorrelation in squared returns suggests volatility clustering, where high-volatility periods tend to be followed by more high-volatility periods. The BEKK model effectively captures this through its conditional variance structure.

ARCH Test Results

DJI Residuals: The ARCH test confirms the presence of conditional heteroskedasticity (time-varying volatility). The significant alpha[1] and beta[1] values suggest that past shocks influence current volatility.

INTC Residuals: Similar to DJI, INTC exhibits strong ARCH effects, justifying the use of a BEKK model to capture dynamic volatility interactions between the two assets.

GARCH Model Results for DJI and INTC

DJI:

- Beta[1] = 0.9913, indicating very persistent volatility, meaning past volatility has a strong influence on current levels.
- Alpha[1] = 0.00599, suggesting that past shocks contribute to volatility, though to a lesser extent than persistence (beta).

INTC:

- Beta[1] = 0.9759, showing even greater volatility persistence than DJI.
- Alpha[1] = 0.0164, indicating that past shocks still have a measurable effect on future volatility.

Why BEKK(p=2, q=4)? These tests support the hypothesis that BEKK(2,4) model is best at balancing goodness-of-fit with model complexity better than alternatives. Given the persistence in volatility (high beta values) and the presence of conditional heteroskedasticity, this structure is well-suited to capture the dynamics between DJI and INTC.

4 F-test Comparision

In line with the measurement discussed by Engle (2002) p. 343 in form of an F-test, we have extracted the residuals of both the DCC and BEKK. Then we have calculated the F-test measurement specified by Engel (2002) p. 343, which yields $\mathbf{1.491 \cdot 10^{36}}$ for DCC and $\mathbf{1.146 \cdot 10^{36}}$ for BEKK. As we can see, BEKK model yields a lower F-statistic measurement, which by Engel (2002) p. 343, it means that DCC has a stronger time-varying volatility structre than BEKK. However, both models have very large F-statistics, implying both of them does not fully capture the GARCH structure. However, the BEKK model performs slightly better than DCC in this regard.

5 Value at Risk forecast comparison

The next measurement considered by Engel (2002) that we will consider is the hitting rate of Value at Risk. First, we have performed one-day ahead forecasting of Value at Risk defined in Engel(2002) p. 343 for each model, which is displayed in figures 5 and 6. It can be seen from the figure that the DCC VaR forecast is much more volatile than BEKK.

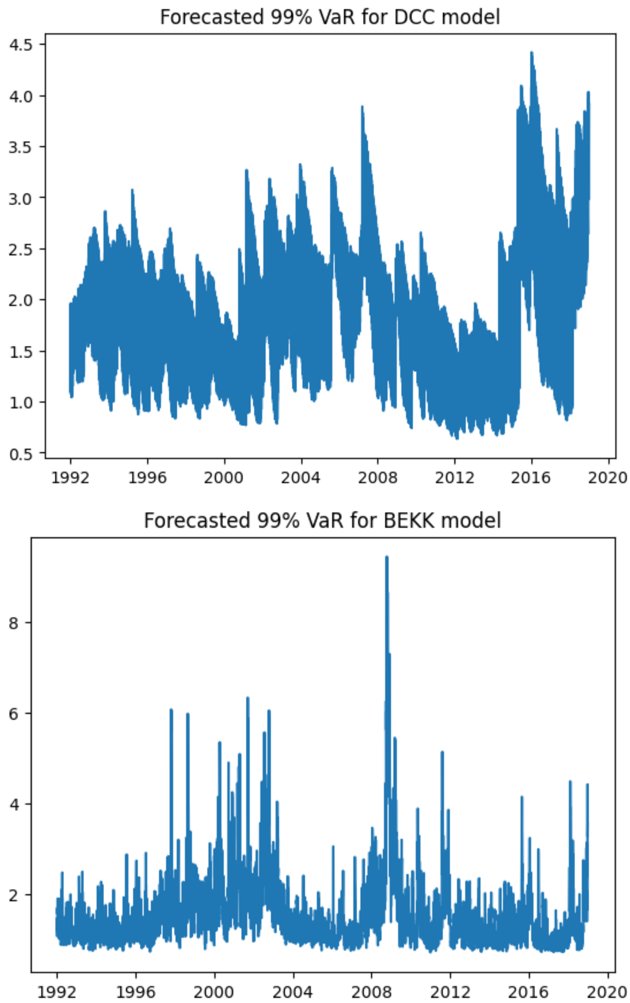

Figure 5: VaR forecast of DCC model

For the DCC model, the number of hits is based on forecasted VaR is 885, and the hitting rate is 0.1301. Meanwhile, for the BEKK model, the number of VaR exceedance is 878 and the exceedance ratio is 0.1291. This implies that the BEKK model again has a better performance than the DCC, since BEKK has a fewer number of empirical defaults.

6 Conclusion

The historical returns and market behavior of Intel and the Dow Jones from 1992 to 2019 align well with the characteristics of the BEKK and DCC models: volatility clustering, auto-correlation in returns, and conditional heteroskedasticity were key features throughout various market events such as the dot-com bubble, the global financial crisis, and the post-crisis recovery. However, the result of estimation shows that both DCC and BEKK do not fully capture the conditional variance and autocorrelation structure. Although, the BEKK model has consistenly outperform the DCC model in this empirical research, based on both the measurements of Engel (2002). The BEKK model has lower F-statistic, which imply lower rejection frequencies, and the VaR hitting rate of BEKK is also slightly lower than DCC. This enables us to conclude that for our particular problem, the BEKK model is preferred, which is in line with the theory that BEKK is generally more flexible than DCC.

Bibliography

Engle, R. F. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. Journal of Business Economic Statistics, 20(3).