Chapter 1a
Propositional Logic Review I

Mathematical Modeling (CO2011)

(Materials drawn from **Chapter 1** in:
"Michael Huth and Mark Ryan. *Logic in Computer Science: Modelling and Reasoning about Systems*, 2nd Ed., Cambridge University Press, 2006.")

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5. Conjunctive Normal Form

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Propositional Calculus

Study of atomic propositions

Propositions are built from sentences whose internal structure is not of concern.

Building propositions

Boolean operators are used to construct propositions out of simpler propositions.

Example for Propositional Calculus

- **Atomic proposition:** One plus one equals two.
- **Atomic proposition:** The earth revolves around the sun.
- **Combined proposition:** One plus one equals two *and* the earth revolves around the sun.

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Goals and Main Result of Propositional Calculus

Meaning of formula

Associate meaning to a set of formulas by assigning a value *true* or *false* to every formula in the set.

Proofs

Symbol sequence that formally establishes whether a formula is always true.

Soundness and completeness

The set of provable formulas is the same as the set of formulas which are always true.

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Contents

Uses of Propositional Calculus

Hardware design

The production of logic circuits uses propositional calculus at all phases; specification, design, testing.

Verification

Verification of hardware and software makes extensive use of propositional calculus.

Problem solving

Decision problems (scheduling, timetabling, etc) can be expressed as satisfiability problems in propositional calculus.

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Predicate Calculus: Central ideas

Richer language

Instead of dealing with atomic propositions, predicate calculus provides the formulation of statements involving sets, functions and relations on these sets.

Quantifiers

Predicate calculus provides statements that all or some elements of a set have specified properties.

Compositionality

Similar to propositional calculus, formulas can be built from composites using logical connectives.

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The uses of Predicate Calculus

Progamming Language Semantics

The meaning of programs such as

$$\texttt{if}\texttt{x} >= \texttt{0}\texttt{then}\texttt{y} := \texttt{sqrt}(\texttt{x})\texttt{else}\texttt{y} := \texttt{abs}(\texttt{x})$$

can be captured with formulas of predicate calculus:

$$\forall x \forall y (x' = x \wedge (x \geq 0 \rightarrow y' = \sqrt{x}) \wedge (\neg(x \geq 0) \rightarrow y' = |x|))$$

Other Uses of Predicate Calculus

- **Specification:** Formally specify the purpose of a program in order to serve as input for software design,
- **Verification:** Prove the correctness of a program with respect to its specification.

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An Example for Specification

Let P be a program of the form

```
while a <> b do
   if a > b then a := a - b else a:= b - a;
```

The specification of the program is given by the formula

$$\{a \geq 0 \wedge b \geq 0\}\ P\ \{a = \mathit{gcd}(a, b)\}$$

Logic in Theorem Proving, Logic Programming, and Other Systems of Logic

Theorem proving

Formal logic has been used to design programs that can automatically prove mathematical theorems.

Logic programming

Research in theorem proving has led to an efficient way of proving formulas in predicate calculus, called *resolution*, which forms the basis for *logic programming*.

Some Other Systems of Logic

- **Three-valued logic:** A third truth value (denoting "don't know" or "undetermined") is often useful.
- **Intuitionistic logic:** A mathematical object is accepted only if a finite construction can be given for it.
- **Temporal logic:** Integrates time-dependent constructs such as ("always" and "eventually") explicitly into a logic framework; useful for reasoning about real-time systems.

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Declarative Sentences

The language of propositional logic is based on *propositions* or *declarative sentences*.

Declarative Sentences

Sentences which one can—in principle—argue as being true or false.

Examples

1. The sum of the numbers 3 and 5 equals 8.
2. Jane reacted violently to Jack's accusations.
3. Every natural number > 2 is the sum of two prime numbers.
4. All Martians like pepperoni on their pizza.

Not Examples

- Could you please pass me the salt?
- Ready, steady, go!
- May fortune come your way.

Putting Propositions Together

Example 1.1

If *the train arrives late* and
there are no taxis at the station then
John is late for his meeting.

John is not late for his meeting.

The train did arrive late.

Therefore, *there were taxis at the station.*

Example 1.2

If *it is raining* and
Jane does not have her umbrella with her then
she will get wet.

Jane is not wet.

It is raining.

Therefore, *Jane has her umbrella with her.*

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Focus on Structure

We are primarily concerned about the structure of arguments in this class, not the validity of statements in a particular domain.

We therefore simply abbreviate sentences by letters such as p, q, r, p_1, p_2 etc.

From Concrete Propositions to Letters - Example 1.1

If *the train arrives late* and
there are no taxis at the station then
John is late for his meeting.

John is not late for his meeting.

The train did arrive late.

Therefore, *there were taxis at the station.*

becomes

Letter version

If p and not q, then r. Not r. p. Therefore, q.

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Focus on Structure

From Concrete Propositions to Letters - Example 1.2

If *it is raining* and
Jane does not have her umbrella with her then
she will get wet.

Jane is not wet.

It is raining.

Therefore, *Jane has her umbrella with her.*

has

the same letter version

If p and not q, then r. Not r. p. Therefore, q.

Logical Connectives

Notations/Symbols

Sentences like "**If** p **and not** q, **then** r." occur frequently. Instead of English words such as "**if...then**", "**and**", "**not**", it is more convenient to use symbols such as $\rightarrow$, $\wedge$, $\neg$.

- $\neg$: negation of p is denoted by $\neg p$.
- $\vee$: disjunction of p and r is denoted by $p \vee r$, meaning at least one of the two statements is true.
- $\wedge$: conjunction of p and r is denoted by $p \wedge r$, meaning both are true.
- $\rightarrow$: implication between p and r is denoted by $p \rightarrow r$, meaning that r is a logical consequence of p. p is called the *antecedent*, and r the *consequent*.

Example 1.1 Revisited

From Example 1.1

If *the train arrives late* and
there are no taxis at the station then
John is late for his meeting.

Symbolic Propositions

We replaced "*the train arrives late*" by p, etc.

The statement becomes: If p and not q, then r.

Symbolic Connectives

With symbolic connectives, the statement becomes:

$p \wedge \neg q \to r$

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- Sequents
- Rules for natural deduction
- Basic and Derived Rules
- Excursion: Intuitionistic Logic

3 Propositional Logic as a Formal Language

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5 Conjunctive Normal Form

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Introduction

Objective

We would like to develop a *calculus* for reasoning about propositions, so that we can establish the validity of statements such as Example 1.1.

Idea

We introduce *proof rules* that allow us to derive a formula ψ from a number of other formulas $\phi_1, \phi_2, \ldots \phi_n$.

Notation

We write a *sequent* $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$
to denote that we can derive ψ from $\phi_1, \phi_2, \ldots, \phi_n$.

Example 1.1 Revisited

English

If *the train arrives late* and
there are no taxis at the station then
John is late for his meeting.

John is not late for his meeting.

The train did arrive late.

Therefore, *there were taxis at the station.*

Sequent

$$p \wedge \neg q \to r, \neg r, p \vdash q$$

Remaining task

Develop a set of proof rules that allows us to establish such sequents.

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Rules for Conjunction

Introduction of Conjunction

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}[\wedge i]$$

Elimination of Conjunction

$$\frac{\phi \wedge \psi}{\phi}[\wedge e_1] \qquad \frac{\phi \wedge \psi}{\psi}[\wedge e_2]$$

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Example of Proof

To show

$$p \wedge q, r \vdash q \wedge r.$$

How to start?

$$p \wedge q \qquad\qquad r,$$

$$q \wedge r.$$

Proof Step-by-Step

1. $p \wedge q$ (premise)
2. r (premise)
3. q (by using Rule $\wedge e_2$ and Item 1)
4. $q \wedge r$ (by using Rule $\wedge i$ and Items 3 and 2)

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Graphical Representation of Proof

$$\dfrac{\dfrac{p \wedge q}{q}[\wedge e_2] \qquad r}{q \wedge r}[\wedge i]$$

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Where are we heading with this?

- We would like to prove sequents of the form $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$
- We introduce rules that allow us to form "legal" proofs
- Then any proof of any formula ψ using the premises $\phi_1, \phi_2, \ldots, \phi_n$ is considered "correct".
- Can we say that sequents with a correct proof are somehow "valid", or "meaningful"?
- What does it mean to be meaningful?
- Can we say that any meaningful sequent has a valid proof?
- ...but first back to the proof rules...

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Rules of Double Negation and Eliminating Implication

Double Negation

$$\frac{\neg\neg\phi}{\phi}[\neg\neg e] \qquad \frac{\phi}{\neg\neg\phi}[\neg\neg i]$$

Eliminating Implication

$$\frac{\phi \qquad \phi \to \psi}{\psi}[\to e]$$

Example

p:= "It rained," and $p \to q$:= "If it rained, then the street is wet."
We can conclude from these two that the street is indeed wet.

Modus Ponens

The rule

$$\frac{\phi \qquad \phi \to \psi}{\psi} [\to e]$$

is often called "Modus Ponens" (or MP)

Origin of term

"Modus ponens" is an abbreviation of the Latin "modus ponendo ponens" which means in English "mode that affirms by affirming". More precisely, we could say "mode that affirms the antecedent of an implication".

Modus Tollens

A similar rule of "Modus Ponens",

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi}[MT]$$

is called "Modus Tollens" (or MT).

Origin of term

"Modus tollens" is an abbreviation of the Latin "modus tollendo tollens" which means in English "mode that denies by denying". More precisely, we could say "mode that denies the consequent of an implication".

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Example

$$p \to (q \to r), p, \neg r \vdash \neg q$$

1	$p \to (q \to r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \to r$	$\to_e$ 1,2
5	$\neg q$	MT 4,3

How to introduce implication?

Compare the sequent (MT)

$$p \to q, \neg q \vdash \neg p$$

with the sequent

$$p \to q \vdash \neg q \to \neg p$$

The second sequent should be provable, but we don't have a rule to introduce implication yet!

A Proof We Would Like To Have

$$p \to q \vdash \neg q \to \neg p$$

1	$p \to q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \to \neg p$	$\to_i$ 2–3

We can start a box with an *assumption*, and use previously proven propositions (including premises) from the outside in the box.

We cannot use assumptions from inside the box in rules outside the box.

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Rule for Introduction of Implication

Introduction of Implication

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi}\,[\to i]$$

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Rule for Disjunction

Introduction of Disjunction

$$\frac{\phi}{\phi \vee \psi}[\vee i_1] \qquad \frac{\psi}{\phi \vee \psi}[\vee i_2]$$

Elimination of Disjunction

$$\frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}[\vee e]$$

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Example

1	$p \land (q \lor r)$	premise
2	p	$\land e_1$ 1
3	$q \lor r$	$\land e_2$ 1
4	q	assumption
5	$p \land q$	$\land i$ 2,4
6	$(p \land q) \lor (p \land r)$	$\lor i_1$ 5
7	r	assumption
8	$p \land r$	$\land i$ 2,7
9	$(p \land q) \lor (p \land r)$	$\lor i_2$ 8
10	$(p \land q) \lor (p \land r)$	$\lor e$ 3, 4–6, 7–9

Special Propositions

- Recall: We are only interested in the truth value of propositions, not the subject matter that they refer to.
- Therefore, all propositions that we all agree must be true are the same!
- Example: $p \to p$, $p \vee \neg p$
- We denote the proposition that is always true (**tautology**) using the symbol $\top$.

Another Special Proposition

- Similarly, we denote the proposition that is always false (**contradiction**) using the symbol $\perp$.
- Example: $p \wedge \neg p$

Rule for Negation

Elimination of Negation

$$\frac{\phi \qquad \neg\phi}{\bot}\,[\neg e]$$

Introduction of Negation

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}\,[\neg i]$$

Elimination of $\perp$

Elimination of $\perp$

$$\frac{\perp}{\phi}[\perp e]$$

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Basic Rules (conjunction and disjunction)

$$\frac{\phi \qquad \psi}{\phi \land \psi}[\land i] \qquad \frac{\phi \land \psi}{\phi}[\land e_1] \qquad \frac{\phi \land \psi}{\psi}[\land e_2]$$

$$\frac{\phi}{\phi \lor \psi}[\lor i_i] \qquad \frac{\psi}{\phi \lor \psi}[\lor i_2] \qquad \frac{\phi \lor \psi \qquad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi}[\lor e]$$

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Basic Rules (implication)

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi}\,[\to i] \qquad \frac{\phi \qquad \phi \to \psi}{\psi}\,[\to e]$$

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Basic Rules (negation)

$$\frac{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}{\neg\phi}[\neg i] \qquad \frac{\phi \qquad \neg\phi}{\bot}[\neg e]$$

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Basic Rules ($\perp$ and double negation)

$$\frac{\perp}{\phi}\,[\perp e]$$

$$\frac{\neg\neg\phi}{\phi}\,[\neg\neg e]$$

Some Derived Rules: Introduction of Double Negation

$$\frac{\phi}{\neg\neg\phi}\,[\neg\neg i]$$

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Example: Deriving [¬¬ i] from [¬ i] and [¬ e]

1	ϕ	premise
2	$\neg\phi$	assumption
3	$\bot$	¬e 1,2
4	$\neg\neg\phi$	¬i 2–3

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Some Derived Rules: Modus Tollens

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi}\,[MT]$$

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Some Derived Rules: Proof By Contradiction

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\,[\mathrm{PBC}]$$

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Some Derived Rules: Law of Excluded Middle

$$\frac{}{\phi \vee \neg\phi}\,[\text{LEM}]$$

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Motivation

Consider the following theorem.

Theorem

There exist irrational numbers a and b such that a^b is rational.

Let us call this theorem χ. We give a Proof Outline for χ.
Let p be the following proposition.

Proposition p

$\sqrt{2}^{\sqrt{2}}$ is rational.

Then the proof of χ goes like this:

$$\dfrac{\dfrac{}{p \lor \neg p}[LEM] \qquad \boxed{\begin{array}{c} p \\ \vdots \\ \chi \end{array}} \qquad \boxed{\begin{array}{c} \neg p \\ \vdots \\ \chi \end{array}}}{\chi}[\lor e]$$

In detail (1)

$$p$$
$$\vdots$$
$$\chi$$

Assume $\sqrt{2}^{\sqrt{2}}$ is rational. Choose a and b to be $\sqrt{2}$, and we have found irrational a and b such that a^b is rational. Thus Theorem χ holds under the assumption p.

In detail (2)

$$\begin{array}{c} \neg p \\ \vdots \\ \chi \end{array}$$

Assume $\sqrt{2}^{\sqrt{2}}$ is irrational. Choose a to be $\sqrt{2}^{\sqrt{2}}$ and b to be $\sqrt{2}$. Then we have

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = (\sqrt{2})^2 = 2.$$

As 2 is rational, Theorem χ holds under the assumption $\neg p$.

Summary of Proof for χ

Proposition p

$\sqrt{2}^{\sqrt{2}}$ is rational.

$$\dfrac{\dfrac{}{p \vee \neg p}[LEM] \qquad \boxed{\begin{array}{c} p \\ \vdots \\ \chi \end{array}} \qquad \boxed{\begin{array}{c} \neg p \\ \vdots \\ \chi \end{array}}}{\chi}[\vee e]$$

There exist irrational numbers a and b such that a^b is rational...

The Magic of LEM

- There exist irrational numbers a and b such that a^b is rational.
- But: If they exist, do you have an example?
- Probably $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$..., but we haven't proven that $\sqrt{2}^{\sqrt{2}}$ is irrational!
- Note: $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot}}}} = 2$
- Using LEM, we can make use of the "probable irrationality" of $\sqrt{2}^{\sqrt{2}}$ without having to prove it!

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Intuitionistic Logic

Intuitionistic logic does not accept the derived rule LEM. The underlying argument for LEM is elimination of double negation.

$$\frac{\neg\neg\phi}{\phi}[\neg\neg e]$$

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Deriving LEM using Basic Rules

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee i_1$ 2
4	$\bot$	$\neg$ e 3,1
5	$\neg\phi$	$\neg$ i 2–4
6	$\phi \vee \neg\phi$	$\vee$ i_2 5
7	$\bot$	$\neg$ e 6,1
8	$\neg\neg(\phi \vee \neg\phi)$	$\neg$i 1–7
9	$\phi \vee \neg\phi$	$\neg\neg$e

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Intuitionistic Logic

Intuitionistic logic is obtained from natural deduction by removing the rule $\neg\neg e$.

History of Intuitionistic Logic

- Late 19th century: Gottlob Frege proposes to reduce mathematics to set theory.
- Russell destroys this programme via paradox.
- In response, L.E.J. Brouwer proposes *intuitionistic* mathematics, with *intuitionistic logic* as its formal foundation.
- An alternative response is Hilbert's *formalistic* position.

Applications of Intuitionistic Logic

- Intuitionistic logic has a strong connection to *computability*
- For example, if we have an intuitionistic proof of

Theorem

There exist irrational numbers a and b such that a^b is rational.

then we would know irrational a and b such that a^b is rational.

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Recap: Logical Connectives

- $\neg$: negation of p is denoted by $\neg p$.
- $\vee$: disjunction of p and r is denoted by $p \vee r$, meaning at least one of the two statements is true.
- $\wedge$: conjunction of p and r is denoted by $p \wedge r$, meaning both are true.
- $\rightarrow$: implication between p and r is denoted by $p \rightarrow r$, meaning that r is a logical consequence of p.

Formal itemize Required

Use of Meta-Language

When we describe rules such as $\dfrac{}{\phi \vee \neg\phi}[LEM]$

we mean that letters such as ϕ can be replaced by *any* formula.

But what exactly is the set of formulas that can be used for ϕ?

Allowed

$(p \wedge (\neg q))$

Not allowed

$) \wedge p \quad q\neg($

Definition of Well-formed Formulas

Definition

- Every propositional atom $p, q, r, \ldots$ and $p_1, p_2, p_3, \ldots$ is a well-formed formula.
- If ϕ is a well-formed formula, then so is $(\neg\phi)$.
- If ϕ and ψ are well-formed formulas, then so is $(\phi \wedge \psi)$.
- If ϕ and ψ are well-formed formulas, then so is $(\phi \vee \psi)$.
- If ϕ and ψ are well-formed formulas, then so is $(\phi \rightarrow \psi)$.

Definition very restrictive

How about this formula?

$$p \land \neg q \lor r$$

Usually, this is understood to mean

$$((p \land (\neg q)) \lor r)$$

...but for the formal treatment of this section and the first homework, we insist on the strict definition, and exclude such formulas.

Backus Naur Form: A more compact definition

Backus Naur Form for propositional formulas

$$\phi ::= p \mid (\neg\phi) \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \to \phi)$$

where p stands for any atomic proposition.

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Inversion principle

How can we show that a a formula such as

$$(((\neg p) \land q) \to (p \land (q \lor (\neg r))))$$

is well-formed?

Answer: We look for the only applicable rule in the definition (the last rule in this case), and proceed on the parts.

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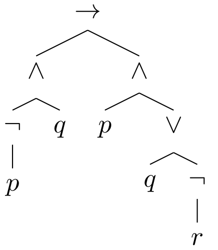
Parse trees

A formula

$$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$$

...and its parse tree:



1. Propositional Calculus: Declarative Sentences
2. Propositional Calculus: Natural Deduction
3. Propositional Logic as a Formal Language
4. **Semantics of Propositional Logic**
   - Meaning of Logical Connectives
   - Preview: Soundness and Completeness
5. Conjunctive Normal Form

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Meaning of propositional formula

Meaning as mathematical object

We define the meaning of formulas as a function that maps formulas and valuations to truth values.

Approach

We define this mapping based on the structure of the formula, using the meaning of their logical connectives.

Truth Values

The set of truth values contains two elements $\mathtt{T}$ and $\mathtt{F}$, where $\mathtt{T}$ represents "**true**" and $\mathtt{F}$ represents "**false**".

Valuations

A *valuation* or *model* of a formula ϕ is an assignment of each propositional atom in ϕ to a truth value.

Meaning of logical connectives

The meaning of a connective is defined as a truth table that gives the truth value of a formula, whose root symbol is the connective, based on the truth values of its components.

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

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Truth tables of formulas

Truth tables use placeholders of formulas such as ϕ:

ϕ	ψ	$\phi \land \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Build the truth table for given formula:

p	q	r	$(p \land q)$	$((p \land q) \land r)$
T	T	T	T	T
T	T	F	T	F
$\vdots$				

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Truth tables of other connectives

ϕ	ψ	$\phi \lor \psi$
T	T	T
T	F	T
F	T	T
F	F	F

ϕ	ψ	$\phi \to \psi$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	$\neg\phi$
T	F
F	T

$\top$
T

$\bot$
F

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Constructing the truth table of a formula

p	q	$(\neg p)$	$\neg q$	$p \to \neg q$	$q \vee \neg p$	$(p \to \neg q) \to (q \vee \neg p)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

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Validity and Satisfiability

Validity

A formula is *valid* if it computes `T` for all its valuations.

Satisfiability

A formula is *satisfiable* if it computes `T` for at least one of its valuations.

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Semantic Entailment, Soundness and Completeness of Propositional Logic

Semantic Entailment

If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to $\mathtt{T}$, the formula ψ evaluates to $\mathtt{T}$ as well, we say that $\phi_1, \phi_2, \ldots, \phi_n$ **semantically entail** ψ, written:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

Soundness

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional formulas. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ valid (has a proof), then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$.

Completeness

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional formulas. If $\phi_1, \phi_2, \ldots, \phi_n \models \psi$, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ valid (has a proof).

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1 Propositional Calculus: Declarative Sentences

2 Propositional Calculus: Natural Deduction

3 Propositional Logic as a Formal Language

4 Semantics of Propositional Logic

5 Conjunctive Normal Form

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Conjunctive Normal Form

Definition

A literal L is either an atom p or the negation of an atom $\neg p$. A formula C is in *conjunctive normal form* (CNF) if it is a conjunction of clauses, where each clause is a disjunction of literals:

$$
\begin{array}{rcl}
L & ::= & p|\neg p, \\
D & ::= & L|L \vee D, \\
C & ::= & D|D \wedge C.
\end{array}
$$

Examples

- $(\neg p \vee q \vee r) \wedge (\neg q \vee r) \wedge (\neg r)$ is in CNF.
- $(\neg p \vee q \vee r) \wedge ((p \wedge \neg q) \vee r) \wedge (\neg r)$ is not in CNF.
- $(\neg p \vee q \vee r) \wedge \neg(\neg q \vee r) \wedge (\neg r)$ is not in CNF.

Usefulness of CNF

Lemma

A disjunction of literals $L_1 \vee L_2 \vee \cdots \vee L_m$ is valid iff there are $1 \leq i, j \leq m$ such that L_i is $\neg L_j$.

How to disprove

$$\models (\neg q \vee p \vee r) \wedge (\neg p \vee r) \wedge q?$$

Disprove any of:

$$\models (\neg q \vee p \vee r) \qquad \models (\neg p \vee r) \qquad \models q.$$

How to prove

$$\models (\neg q \vee p \vee q) \wedge (p \vee r \neg p) \wedge (r \vee \neg r)?$$

Prove all of:

$$\models (\neg q \vee p \vee q) \qquad \models (p \vee r \neg p) \qquad \models (r \vee \neg r).$$

Usefulness of CNF (cont.) and Transformation to CNF

Proposition

Let ϕ be a formula of propositional logic. Then ϕ is satisfiable iff $\neg\phi$ is not valid.

Satisfiability test

We can test satisfiability of ϕ by transforming $\neg\phi$ into CNF, and show that some clause is not valid.

Theorem-Transformation to CNF

Every formula in the propositional calculus can be transformed into an equivalent formula in CNF.

Algorithm for CNF Transformation

1. Eliminate implication using:
   $A \to B \equiv \neg A \lor B$.
2. Push all negations inward using De Morgan's laws:

$$\neg(A \land B) \equiv (\neg A \lor \neg B),$$

$$\neg(A \lor B) \equiv (\neg A \land \neg B).$$

3. Eliminate double negations using the equivalence $\neg\neg A \equiv A$.
4. The formula now consists of disjunctions and conjunctions of literals. Use the distributive laws to eliminate conjunctions within disjunctions:

$$A \lor (B \land C) \equiv (A \lor B) \land (A \lor C),$$

$$(A \land B) \lor C \equiv (A \lor C) \land (B \lor C).$$

Example

$$
\begin{aligned}
(\neg p \rightarrow \neg q) \rightarrow (p \rightarrow q) & \equiv \neg(\neg\neg p \vee \neg q) \vee (\neg p \vee q) \\
& \equiv (\neg\neg\neg p \wedge q) \vee (\neg p \vee q) \\
& \equiv (\neg p \wedge q) \vee (\neg p \vee q) \\
& \equiv (\neg p \vee \neg p \vee q) \wedge (q \vee \neg p \vee q) \\
& \equiv \top .
\end{aligned}
$$

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Homeworks

I. Write down the explanations (in Vietnamese, or in English if possible) of the following terms, find examples for each term, what are the differences between them:
   1) fallacy, contradiction, paradox, counterexample;
   2) premise, assumption, axiom, hypothesis, conjecture;
   3) tautology, valid, contradiction, satisfiable;
   4) soundness, completeness;
   5) sequent, consequence, implication, (semantic) entailment;
   6) argument, variable, arity;

II. What are the differences between the following notations: '$\longrightarrow$', '$\Longrightarrow$', '$\vdash$', '$\models$'? And what are the differences between the following notations: '$\longleftrightarrow$', '$\Longleftrightarrow$', '$\dashv\vdash$', '$\equiv$', '$=$'? Find examples to illustrate these differences.

III. It is recommended that you should do as much as you can ALL marked exercises in [2] (notice that sample solutions for these exercises are available in [3]). For this lecture, the following are recommended exercises [2]:
1.1: 2d), 2g);
1.2: 1d), 1g), 1m), 1q), 1u), 1w), 3a), 3b), 3c), 3f), 3g), 3l), 3o);
1.4: 12d);
1.5: 3b), 3c), 7c).

Next Week?

- Exercises Session;
- [2, Section 1.6]: SAT Solvers;
- Application of SAT Solving.

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