Artificial Intelligence

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Outline

CHAPTER 7
QUANTIFYING UNCERTAINTY

7.1 Acting Under Uncertainty

7.1 Uncertainty

Let action A_t = leave for airport t minutes before flight

Will A_t get me there on time?

Problems:

1) partial observability (road state, other drivers' plans, etc.)
2) noisy sensors (traffic reports)
3) uncertainty in action outcomes (flat tire, etc.)
4) immense complexity of modelling and predicting traffic

7.1 Uncertainty

Let action A_t = leave for airport t minutes before flight
Will A_t get me there on time?

<u>Conclusion:</u>
1) hard to conclude: "A_{25} will get me there on time"
2) weak conclusions: "A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc etc."

7.1 Uncertainty

Let action A_t = leave for airport t minutes before flight
Will A_t get me there on time?

<u>Methods for handling uncertainty:</u>

Given the available evidence, A_{25} will get me there on time with probability 0.04

7.1 Probability

- Probability provides a way of summarizing the uncertainty that comes from our laziness and ignorance
  - Laziness: it is too much work to list the complete set of antecedents or consequents, too hard to use such rules.
  - Ignorance: we have no complete theory for the domain, lack of relevant facts, initial conditions, etc.
- Probabilities relate propositions to one's own state of knowledge

  $P(A_{25} | \text{no reported accidents}) = 0.06$
- Probabilities of propositions change with new evidence:

  $P(A_{25} | \text{no reported accidents, 5 a.m.}) = 0.15$

7.1 Making decisions under uncertainty

- Suppose I believe the following:
  $P(A_{25}$ gets me there on time$|\ldots) = 0.04$
  $P(A_{90}$ gets me there on time$|\ldots) = 0.70$
  $P(A_{120}$ gets me there on time$|\ldots) = 0.95$
  $P(A_{1440}$ gets me there on time$|\ldots) = 0.9999$
  Which action to choose?
  Depends on my preferences for missing flight vs. airport cuisine, etc.

- Utility theory is used to represent and reason with preferences;
  "Utility" ~ the quality of being useful
- Decision theory = utility theory + probability theory
  - an agent is rational if and only if it chooses the action that yields the highest expected utility, averaged over all the possible outcomes of the action; ~ maximum expected utility (MEU).

7.2 Basic Probability Notation

7.2 Probability basics

- Begin with Ω - a sample space, a set of all possible worlds
  e.g., 6 possible rolls of a die.
- $\omega \in \Omega$ is a sample point / possible world / atomic event.

- A probability space or probability model is a sample space with an assignment $P(\omega)$ for every $\omega \in \Omega$, such that
  $0 \leq P(\omega) \leq 1$
  $\Sigma_\omega P(\omega) = 1$
  e.g., $P(1)=P(2)=P(3)=P(4)=P(5)=P(6)=1/6$

- An event A is any subset of Ω:
  $P(A) = \Sigma_{\{\omega \in A\}} P(\omega)$
  e.g., $P(\text{die roll} < 4) = P(1) + P(2) + P(3) = 1/6 + 1/6 + 1/6 = 1/2$

7.2 Random variables

- Variables in probability theory are called **random variables**, their names begin with an uppercase letter,
  e.g., Total, Die
- Every random variable has a domain - the set of possible values it can take on.
  - domain of Total for two dice: {2,...,12}
  - domain of Die: {1,...,6}
  - domain of a boolean random variable: {true, false}

7.2 Syntax for propositions

- **Propositional or Boolean random variables**
  e.g., Cavity (do I have a cavity?)
  Cavity = true is a proposition, also written cavity

- **Discrete random variables** (finite or infinite)
  e.g., Weather is one of ⟨sunny, rain, cloudy, snow⟩
  Weather = rain is a proposition
  Values must be exhaustive and mutually exclusive

- **Continuous random variables** (bounded or unbounded)
  e.g., $Temp = 21.6$, $Temp < 22.0$.

- Arbitrary Boolean combinations of basic propositions

7.2 Prior probability

- **Prior or unconditional probabilities** of propositions
  e.g., P(Cavity = true) = 0.1 and P(Weather = sunny) = 0.72
  correspond to belief prior to arrival of any (new) evidence

- **Probability distribution** gives values for all possible assignments:
  P(Weather) = $\langle 0.72, 0.1, 0.08, 0.1 \rangle$ (normalized, i.e., sums to 1)

- **Joint probability distribution** for a set of r.v.s gives the probability of every atomic event on those r.v.s (i.e., every sample point)
  P(Weather, Cavity) = a 4 × 2 matrix of values:

$Weather =$	$sunny$	$rain$	$cloudy$	$snow$
$Cavity = true$	0.144	0.02	0.016	0.02
$Cavity = false$	0.576	0.08	0.064	0.08

7.2 Conditional probability

Conditional or posterior probabilities

Example:

P (cavity|toothache) = 0.8 i.e., given that toothache is all I know

P (Cavity=true | Toothache=true) = 0.8

If we know more, e.g., cavity is also given, then we have

P (cavity|toothache, cavity) = 1

New evidence may be irrelevant, allowing simplification, e.g.,

P (cavity|toothache, 49ersWin) = P (cavity|toothache) = 0.8

7.2 Conditional probability

- Definition of **conditional probability**:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

- **Product rule** gives an alternative formulation:

$P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

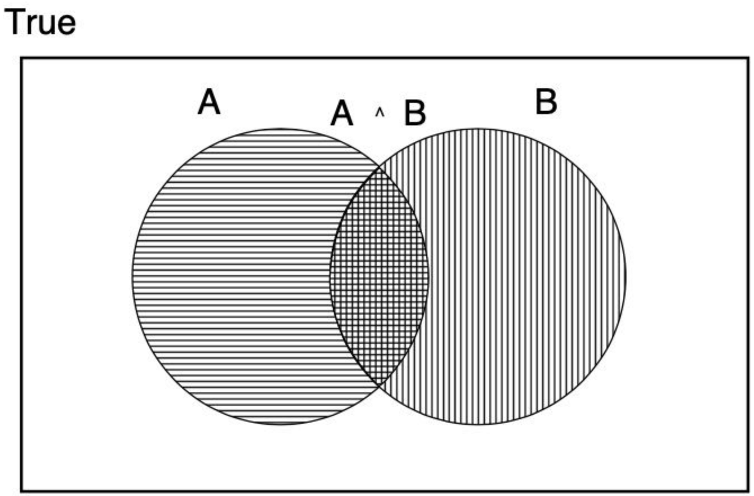
- **Chain rule** is derived by successive application of product rule:

$$\begin{aligned} P(X_1,\ldots,X_n) &= P(X_1,\ldots,X_{n-1})\, P(X_n|X_1,\ldots,X_{n-1}) \\ &= P(X_1,\ldots,X_{n-2})\, P(X_{n1}|X_1,\ldots,X_{n-2})\, P(X_n|X_1,\ldots,X_{n-1}) \\ &= \ldots \\ &= \Pi^n_{i=1} P(X_i|X_1,\ldots,X_{i-1}) \end{aligned}$$

7.2 Axioms of Probability

- Certain logically related events must have related probabilities

$$P(a \vee b) = P(a) + P(b) - P(a \wedge b)$$



7.3 Inference Using Full Joint Distributions

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Start with the joint distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

P (toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2

7.3 Inference Using Full Joint Distributions

Start with the joint distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

P (cavity $\vee$ toothache) = 0.108+0.012+0.072+0.008+0.016+0.064 = 0.28

7.3 Inference Using Full Joint Distributions

Start with the joint distribution:

	$toothache$		$\neg\, toothache$	
	$catch$	$\neg\, catch$	$catch$	$\neg\, catch$
$cavity$	**.108**	**.012**	**.072**	**.008**
$\neg\, cavity$	**.016**	**.064**	**.144**	**.576**

$$
\begin{aligned}
P(cavity \mid toothache) &= \frac{P(cavity \wedge toothache)}{P(toothache)} \\
&= \frac{0.108 + 0.012}{0.108 + 0.012 + 0.016 + 0.064} = 0.6 \,.
\end{aligned}
$$

7.3 Inference Using Full Joint Distributions

Start with the joint distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬ cavity	**.016**	**.064**	**.144**	**.576**

$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

$$= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$$

7.3 Normalization

- X: a single variable (e.g., Cavity)
- E: the list of evidence variables (e.g., Toothache), e: the list of their observed values
- Y: be the remaining unobserved variables (e.g., Catch)

- The query is $\mathbf{P}(X|\mathbf{e})$ and can be evaluated as

  $$\mathbf{P}(X|\mathbf{e}) = \alpha\mathbf{P}(X,\mathbf{e}) = \alpha \sum_{y}^{ys} \mathbf{P}(X,\mathbf{e},y)$$

  ys: all possible combinations of values of the unobserved variables Y

- General idea: compute distribution on query variable by **fixing evidence variables** and **summing over unobserved variables**

7.3 Normalization

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	$\boxed{.108}$	$\boxed{.012}$	.072	.008
¬ cavity	$\boxed{.016}$	$\boxed{.064}$	.144	.576

Denominator can be viewed as a normalization constant α

$$\mathbf{P}(Cavity|toothache) = \alpha\, \mathbf{P}(Cavity, toothache)$$

$$= \alpha[\mathbf{P}(Cavity, toothache, catch) + \mathbf{P}(Cavity, toothache, \neg catch)]$$

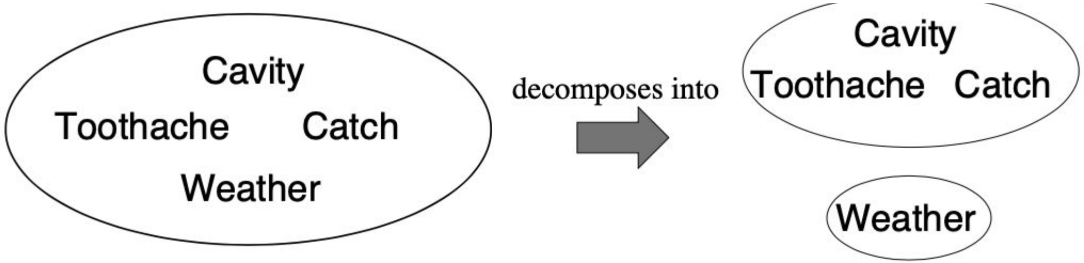
$$= \alpha[\langle 0.108, 0.016\rangle + \langle 0.012, 0.064\rangle]$$

$$= \alpha\langle 0.12, 0.08\rangle = \langle 0.6, 0.4\rangle$$

Cavity as a single variable, Toothache as evidence variable, Catch as unobserved variables

7.3 Independence

A and B are independent iff $P(A|B) = P(A)$ or $P(B|A) = P(B)$ or $P(A, B) = P(A)P(B)$



- It seems safe to say that the weather does not influence the dental variables
  => Dental variables are independent of weather.
  $P(Toothache, Catch, Cavity, Weather) = P(Toothache, Catch, Cavity)\ P(Weather)$

- Absolute independence is powerful but rare
- Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

- **P**(Toothache, Cavity, Catch) has $2^3 - 1 = 7$ independent entries

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  (1) P(catch|toothache, cavity) = P (catch|cavity)

- The same independence holds if I haven't got a cavity:
  (2) P(catch|toothache, ¬cavity) = P (catch|¬cavity)

- Catch is conditionally independent of Toothache given Cavity:
  P(Catch|Toothache, Cavity) = **P**(Catch|Cavity)

- Equivalent statements:
  P(Toothache|Catch, Cavity) = **P**(Toothache|Cavity)

Conditional independence contd.

Write out full joint distribution using chain rule:

$\mathbf{P}(Toothache, Catch, Cavity)$

$= \mathbf{P}(Toothache|Catch, Cavity)P(Catch, Cavity)$

$= \mathbf{P}(Toothache|Catch, Cavity)P(Catch|Cavity)P(Cavity)$

$= \mathbf{P}(Toothache|Cavity)P(Catch|Cavity)P(Cavity)$

In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n.

7.4 Bayes' Rule And Its Use

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Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

=> Bayes' rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

Or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha \mathbf{P}(X|Y)\mathbf{P}(Y)$$

7.4 Bayes' Rule And Its Use

P(effect | cause) quantifies the relationship in the **causal** direction,

P(cause | effect) describes the **diagnostic** direction

Useful for assessing **diagnostic probability** from **causal probability**

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

7.4 Using Bayes' rule: Combining evidence

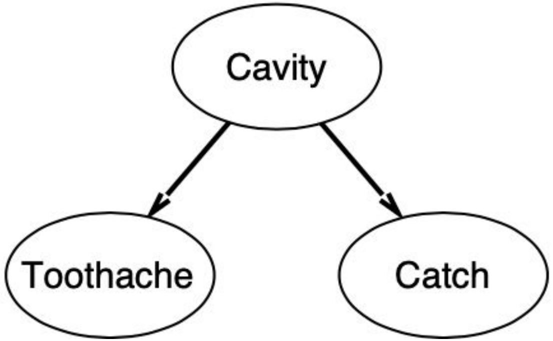
$\mathbf{P}(Cavity \mid toothache \wedge catch) = \alpha\, \mathbf{P}(toothache \wedge catch \mid Cavity)\, P(Cavity)$

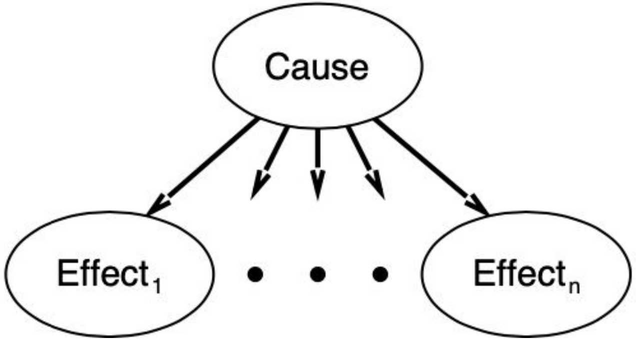
// toothache and catch are independent given the presence or the absence of a cavity

// conditional independence of toothache and catch given Cavity

$\mathbf{P}(toothache \wedge catch \mid Cavity) = \mathbf{P}(toothache \mid Cavity) P(catch \mid Cavity)$

$\Leftrightarrow \mathbf{P}(Cavity \mid toothache \wedge catch) = \alpha\, \mathbf{P}(toothache \mid Cavity)\, \mathbf{P}(catch \mid Cavity)\, \mathbf{P}(Cavity)$





Examples

Given the full joint distribution shown in the slides, calculate the following:

a. $P(toothache)$.

b. $\mathbf{P}(Cavity)$.

c. $\mathbf{P}(Toothache|cavity)$.

d. $\mathbf{P}(Cavity|toothache \vee catch)$.

Examples

Given the full joint distribution shown in the slides, calculate the following:

a. P(toothache): the probability that Toothache is true

b. **P**(Cavity): the vector of probability values for the random variable Cavity

c. **P**(Toothache|cavity): the vector of probability values for Toothache, given that Cavity is true

d. **P**(Cavity|toothache $\vee$ catch): the vector of probability values for Cavity, given that either Toothache or Catch is true

Examples

After your yearly check up, the doctor has bad news and good news. The bad news is that you tested positive for a serious disease and that the test is 99% accurate (i.e., the probability of testing positive when you do have the disease is 0.99, as is the probability of testing negative when you don't have the disease). The good news is that this is a rare disease, striking only 1 in 10,000 people of your age. Why is it good news that the disease is rare? What are the chances that you actually have the disease?

Examples

$P(test|disease) = 0.99 => P(test|-disease) = 0.01$
$P(-test|-disease) = 0.99$
$P(disease) = 0.0001 => P(-disease) = 1-0.0001 = 0.9999$

$P(disease|test) = P(test|disease) * P(disease) / P(test)$

$P(test|disease) * P(disease) = 0.99 * 0.0001$
$P(test) = P(test/disease)\, P(disease) + P(test/-disease)\, P(-disease)$
$= 0.99 * 0.0001 + 0.01 * 0.9999$