COMP 4102: Assignment 1

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1 Theory questions

1. (5 points) Are three dimensional rotations expressed as R_x, followed by R_y, and then R_z (rotations around the x, y and z axis) commutative? That is, does the order in which they are applied matter.

   No, the three dimensional rotations expressed as R_x, followed by R_y, and then R_z are not commutative. And the order in which they are applied matter.

   That is because they are applied in different planes. The different order of rotations will lead to different rotations of the final object. Mathematically, it is because the matrices' multiplication are not commutative ($R_xR_yR_z \neq R_zR_yR_x$).

   Prove that matrices' multiplication are not commutative:
   Given 2 matrics A and B. Such that $AB^{-1} = B^{-1}A^{-1}$
   We have: $AB(AB)^{-1} = ABB^{-1}A^{-1} = AA^{-1} =$ I
   But: $BA(AB)^{-1} = BAB^{-1}A^{-1} \neq I$
   $\therefore AB \neq BA$

2. (8 points) Find the SVD of A, UΣV^T, where A $= \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

   First find A^TA: $A^TA = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   $\det(A^TA - \lambda I) = \begin{bmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = -\lambda^3 + 10\lambda^2 - 16\lambda = -\lambda(\lambda^2 - 10\lambda + 16) = -\lambda(\lambda-2)(\lambda-8) = 0$
   $\therefore \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 8$

   $\lambda_1 = 0:$

   $$\begin{bmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   row-reduces: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the eigenvector is $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and the unit-length vector in the kernel of that matrix is $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 2:$

$$\begin{bmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

row-reduces: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so the eigenvector is $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and the unit-length vector in the kernel of that matrix is $v_2 = \begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$

$\lambda_3 = 8:$

$$\begin{bmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

row-reduces: $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so the eigenvector is $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and the unit-length vector in the kernel of that matrix is $v_3 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$

So the columns of the matrix V are the unit-length vectors: $V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The square root of the nonzero eigenvalues is : $\sigma_1 = 2\sqrt{2}$, $\sigma_2 = \sqrt{2}$
The Σ matrix is a zero matrix with σ_i in its diagonal: $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$

Compute U by the formula $u_i = \frac{1}{\sigma_i} A v_i$

$$v_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$v_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So the SVD is:

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. (4 points) Scale a vector $\begin{bmatrix} x & y \end{bmatrix}^T$ in the plane can be achieved b $x' = sx$ and $y' = sy$ where s is a scalar.
(a) Write out the matrix form of this transformation.
(b) Write out the transformation matrix for homogeneous coordinates.
(c) If the transformation also includes a translation
$x' = sx + t_x$ and $y' = sy + t_y$
Write out the transformation matrix of the homogeneous coordinates.
(d) What is the equivalent of the above matrix for three-dimensional vectors?

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(b) $\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} s & 0 & t_x \\ 0 & s & t_y \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} s & 0 & 0 & t_x \\ 0 & s & 0 & t_y \\ 0 & 0 & s & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. (5 points) Find the least square solution $\hat{x}$ for Ax = b if
$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
Verify that the error vector b - A $\hat{x}$ is orthogonal to the columns of A.

$$\hat{x} = (A^T A)^{-1} A^T b = \left(\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{x} = \left(\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{5}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{5}{24} \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{5}{12} & \frac{-1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

We have: $b - A\hat{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$

$b - A\hat{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$

We have col(A) = span{(2, -1, 0), (0, 1, 2)}

and

$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0$

$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0$

so the error vector b - A$\hat{x}$ is orthogonal to the columns of A

5. Matrix K is a discrete, separable 2D filter kernel of size k x k. Assume k is an odd number. After applying filter K on an image I, we get a resulting image I_k.

   (a) (3 points) Given an image point (x,y), find its value in the resulting image, $I_k(x, y)$. Express your anwser in terms of I, k, K, x and y. You don't need to consider the case when (x, y) is near the image boundary.

   (b) (5 points) One property of this separable kernel matrix K is that it can be expressed as the product of two vectors g $\in R^{kx1}$ and h $\in R^{1xk}$, which can also be regarded as two 1D filter kernels. In other words, K = gh. The resulting image we get by first applying g and then applying h to the image I is I_{gh}. Show that $I_K = I_{gh}$.

   (a) Applying the convolution to matrix K with a kernel k x k, we have:
   $I_K(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{k} K_{ij} I(x - (i - \frac{k}{2}), y - (j - \frac{k}{2}))$

   (b) We have: $I_K = K \circ I$, and K = gh (by definition). So $I_K = gh \circ I = I_{gh}$
   Mathematically:

$$\begin{aligned} I_K(x, y) &= \sum_{i=1}^{k} \sum_{j=1}^{k} K_{ij} I(x - (i - \frac{k}{2}), y - (j - \frac{k}{2})) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} g_i h_j I(x - i + \frac{k}{2}, y - j + \frac{k}{2}) \\ &= \sum_{j=1}^{k} h_j \sum_{i=1}^{k} g_i I(x - i + \frac{k}{2}, y - j + \frac{k}{2}) \\ &= \sum_{j=1}^{k} h_j I(x, y - j + \frac{k}{2}) \\ &= I_{gh}(x, y) \end{aligned} \tag{1}$$