

# COMP 4102: Assignment 1

Minh Thong Mai - ID: 101128263

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## 1 Theory questions

1. (5 points) Are three dimensional rotations expressed as  $R_x$ , followed by  $R_y$ , and then  $R_z$  (rotations around the x, y and z axis) commutative? That is, does the order in which they are applied matter.

No, the three dimensional rotations expressed as  $R_x$ , followed by  $R_y$ , and then  $R_z$  are not commutative. And the order in which they are applied matter.

That is because they are applied in different planes. The different order of rotations will lead to different rotations of the final object. Mathematically, it is because the matrices' multiplication are not commutative ( $R_x R_y R_z \neq R_z R_y R_x$ ).

Prove that matrices' multiplication are not commutative:

Given 2 matrices A and B. Such that  $AB^{-1} = B^{-1}A^{-1}$

We have:  $AB(AB)^{-1} = ABB^{-1}A^{-1} = AA^{-1} = I$

But:  $BA(AB)^{-1} = BAB^{-1}A^{-1} \neq I$

$\therefore AB \neq BA$

2. (8 points) Find the SVD of A,  $U\Sigma V^T$ , where  $A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

First find  $A^T A$ :  $A^T A = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\det(A^T A - \lambda I) = \begin{vmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 10\lambda^2 - 16\lambda = -\lambda(\lambda^2 - 10\lambda + 16) = -\lambda(\lambda - 2)(\lambda - 8) = 0$$

$\therefore \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 8$

$\lambda_1 = 0$  :

$$\begin{bmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

row-reduces:  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , so the eigenvector is  $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  and the unit-length vector in the kernel of

that matrix is  $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 2 :$

$$\begin{bmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

row-reduces:  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ , so the eigenvector is  $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$  and the unit-length vector in the kernel of

that matrix is  $v_2 = \begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$

$\lambda_3 = 8 :$

$$\begin{bmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

row-reduces:  $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ , so the eigenvector is  $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  and the unit-length vector in the kernel of

that matrix is  $v_3 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$

So the columns of the matrix V are the unit-length vectors:  $V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The square root of the nonzero eigenvalues is :  $\sigma_1 = 2\sqrt{2}, \sigma_2 = \sqrt{2}$

The  $\Sigma$  matrix is a zero matrix with  $\sigma_i$  in its diagonal:  $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$

Compute U by the formula  $u_i = \frac{1}{\sigma_i} Av_i$

$$v_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$v_2 = \frac{1}{\sigma_2} Av_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So the SVD is:

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. (4 points) Scale a vector  $\begin{bmatrix} x & y \end{bmatrix}^T$  in the plane can be achieved by  $x' = sx$  and  $y' = sy$  where  $s$  is a scalar.

(a) Write out the matrix form of this transformation.

(b) Write out the transformation matrix for homogeneous coordinates.

(c) If the transformation also includes a translation

$$x' = sx + t_x \text{ and } y' = sy + t_y$$

Write out the transformation matrix of the homogeneous coordinates.

(d) What is the equivalent of the above matrix for three-dimensional vectors?

(a)  $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(b)  $\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c)  $\begin{bmatrix} s & 0 & t_x \\ 0 & s & t_y \\ 0 & 0 & 1 \end{bmatrix}$

(d)  $\begin{bmatrix} s & 0 & 0 & t_x \\ 0 & s & 0 & t_y \\ 0 & 0 & s & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. (5 points) Find the least square solution  $\hat{x}$  for  $Ax = b$  if

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Verify that the error vector  $b - A\hat{x}$  is orthogonal to the columns of  $A$ .

$$\hat{x} = (A^T A)^{-1} A^T b = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{x} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{5}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{5}{24} \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{5}{12} & -\frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

We have:  $b - A\hat{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$

$$b - A\hat{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

We have  $\text{col}(A) = \text{span}\{(2, -1, 0), (0, 1, 2)\}$

and

$$\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0$$

so the error vector  $b - A\hat{x}$  is orthogonal to the columns of  $A$

5. Matrix  $K$  is a discrete, separable 2D filter kernel of size  $k \times k$ . Assume  $k$  is an odd number. After applying filter  $K$  on an image  $I$ , we get a resulting image  $I_K$ .

(a) (3 points) Given an image point  $(x, y)$ , find its value in the resulting image,  $I_K(x, y)$ . Express your answer in terms of  $I$ ,  $k$ ,  $K$ ,  $x$  and  $y$ . You don't need to consider the case when  $(x, y)$  is near the image boundary.

(b) (5 points) One property of this separable kernel matrix  $K$  is that it can be expressed as the product of two vectors  $g \in R^{k \times 1}$  and  $h \in R^{1 \times k}$ , which can also be regarded as two 1D filter kernels. In other words,  $K = gh$ . The resulting image we get by first applying  $g$  and then applying  $h$  to the image  $I$  is  $I_{gh}$ . Show that  $I_K = I_{gh}$ .

(a)  $I_K(x, y) = \sum_{i=1}^k \sum_{j=1}^k K_{ij} I(x - i + \frac{k}{2}, y - j + \frac{k}{2})$

(b) We have:  $I_K = K \circ I$ , and  $K = gh$  (by definition). So  $I_K = gh \circ I = I_{gh}$   
Mathematically:

$$\begin{aligned} I_K(x, y) &= \sum_{i=1}^k \sum_{j=1}^k K_{ij} I(x - (i - \frac{k}{2}), y - (j - \frac{k}{2})) \\ &= \sum_{i=1}^k \sum_{j=1}^k g_i h_j I(x - i + \frac{k}{2}, y - j + \frac{k}{2}) \\ &= \sum_{j=1}^k h_j \sum_{i=1}^k g_i I(x - i + \frac{k}{2}, y - j + \frac{k}{2}) \\ &= \sum_{j=1}^k h_j I(x, y - j + \frac{k}{2}) \\ &= I_{gh}(x, y) \end{aligned} \tag{1}$$