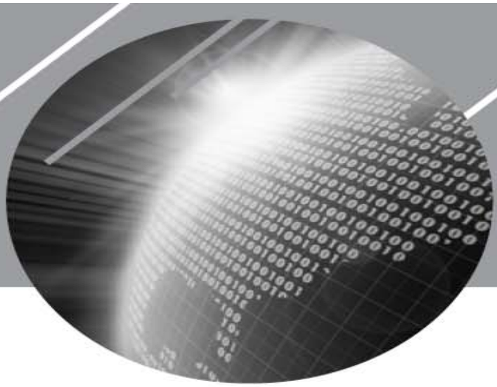
Chapter 2.

Basic Structures : Sets and Functions

Part I: Sets and Set Operations



Dept. of Software
Gachon University
Prof. Joohyung Lee
Spring 2019

Introduction to Sets

- **Definition:** A **set** is an ***unordered*** **collection** of **objects**
  - An **object** in a set is called **element** or **members** of the set. A set is said to contain its elements.

$a \in A$	"a is an element of the set A"
$a \notin A$	"a is not an element of the set A"
$A = \{a_1, a_2, \ldots, a_n\}$	"A contains…"

- **Two ways to define sets (notations for sets)**

i) to enumerate the elements (원소 나열법)

$A = \{a_1, a_2, \ldots, a_n\}$ finite

$A = \{a_1, a_2, \ldots\}$ infinite

ii) To specify condition with predicate (*set builder notation*, 조건 제시법)

$\mathbf{A = \{x \mid P(x)\}}$

$\mathbf{A = \{x \in U \mid P(x)\}}$ **U : universe(data type)**

Basic Properties of Sets

- Sets are inherently ***unordered***: (Order of elements is meaningless)
  - e.g., No matter what objects a, b, and c denote,
  - {a, b, c} = {a, c, b} = {b, a, c} = {b, c, a} = {c, a, b} = {c, b, a}.

- All elements are *distinct* (unequal); multiple listings make no difference! **(중복은 의미 없다!)**
  - If a = b, then {a, b, c} = {a, c} = {b, c} = {a, a, b, a, b, c, c, c, c}.
    - This set contains at most two elements!

- A set is either *finite* or *infinite* (*i.e.,* not *finite*, without end, unending).

Set Equality

- Definition
  - Two sets are equal *if and only if* they have the same elements

- In particular, it does not matter *how the set is defined or denoted.*

- Example : following sets are all the same
  - {1, 2, 3} = {x | x is an positive integer less than 4}
  - {1, 2, 3} = {3, 2, 1}
  - {1, 2, 3} = {1, 2, 2, 3, 3, 3}

Important Sets in Discrete Math

- Symbols for some "**standard**" infinite sets:
  - **N** = {**0,** 1, 2, 3, …} the set of **natural numbers***.
  - **Z** = {…, -2, -1, 0, 1, 2, …} the set of **integers**.
  - $\mathbf{Z}^+$ = {1, 2, 3, …} (=N^+) the set of **positive integers (natural numbers)**
  - **Q** = $\{p/q \mid p \in Z, q \in Z \text{ and } q \neq 0\}$ the set of **rational numbers**.
  - **R** the set of **real numbers.**

  Those sets are all *infinite* sets (*i.e.,* not *finite*, without end, unending sets).

* In mathematics, there are two conventions for the set of natural numbers: it is either the set of positive integers {1,2,3,…} according to the traditional definition; or the set of non-negative integers {0,1,2,…} according to a definition first appearing in the nineteenth century.

Examples for Sets

- $A = \varnothing$ "empty set/null set"
- $A = \{z\}$ Note: $z \in A$, but $z \neq \{z\}$
- $A = \{\{b, c\}, \{c, x, d\}\}$
- $A = \{\{x, y\}\}$ Note: $\{x, y\} \in A$, but $\{x, y\} \neq \{\{x, y\}\}$
- $A = \{x \mid P(x)\}$ "set of all x such that P(x)"
- $A = \{x \mid x \in \mathbf{N} \wedge x > 7\} = \{8, 9, 10, \ldots\}$

  "set builder notation"

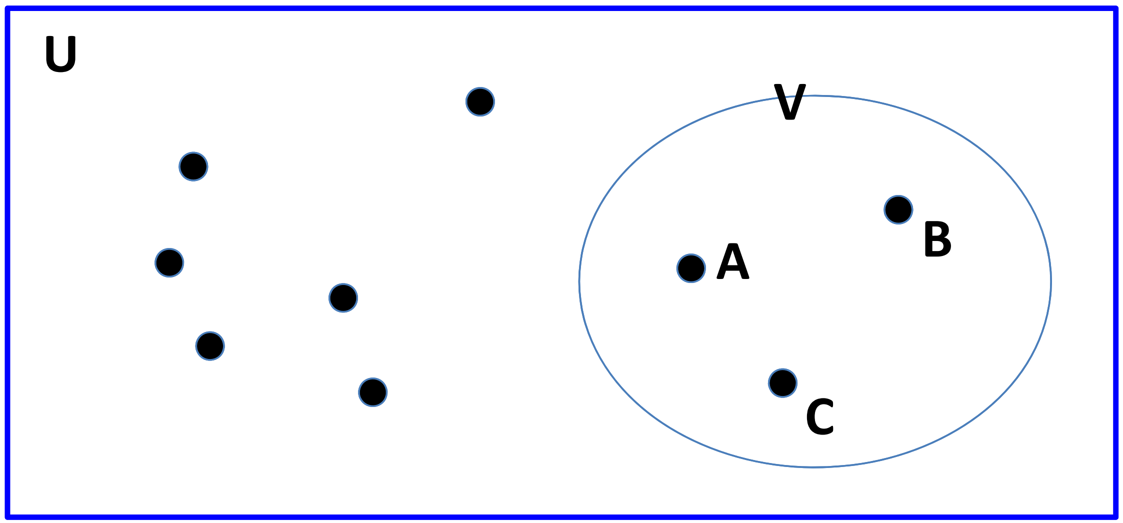
Special Sets

- **The universal set U**
  - the set of all objects under the consideration, i.e., the universe (domain) of discourse

- **The empty set:** {}, $\varnothing$ ("null", "the empty set")
  - is the unique set that contains no elements.
  - $\varnothing = \{\} = \{ x \mid \text{False}\}$
  - **Note:** $\varnothing \neq \{\varnothing\}$

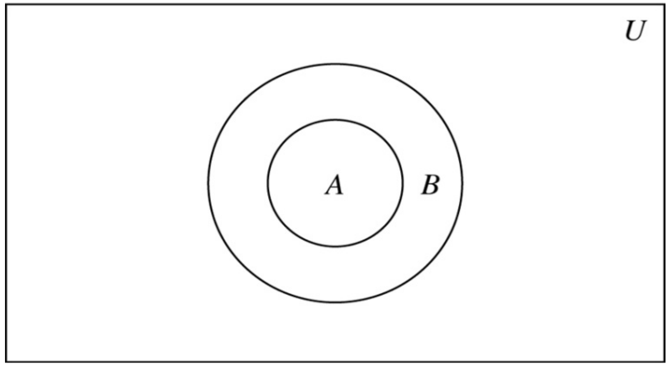
Venn Diagrams

- A set can be visualized using **Venn Diagrams**:
  - V={ A, B, C }



Subset

- **Definition : subset**
  - The set A is said to be a **subset** of B if and only if every element of A is also an element of B
  - $\forall x(x \in A \rightarrow x \in B)$



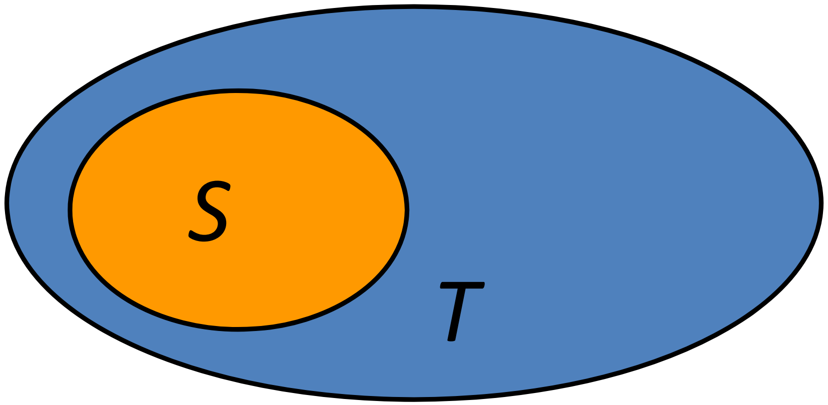
- Example
  - $\{1, 2\} \subseteq \{1, 2, 3\}$
  - $\{1, 2, 3\} \subseteq \{1, 2, 3\}$

- Theorem : the empty set is a subset of every set
  - **For every set S,**
    $\varnothing \subseteq$ S and S $\subseteq$ S

Proper Subset $\subset$ vs. $\subseteq$

- **Definition**: proper subset

  A **set A** is said to be a **proper subset** of B if and only if **A** $\subseteq$ **B** and **A** $\neq$ **B**.
  We denote that A is a proper subset of B with the notation A $\subset$ B.

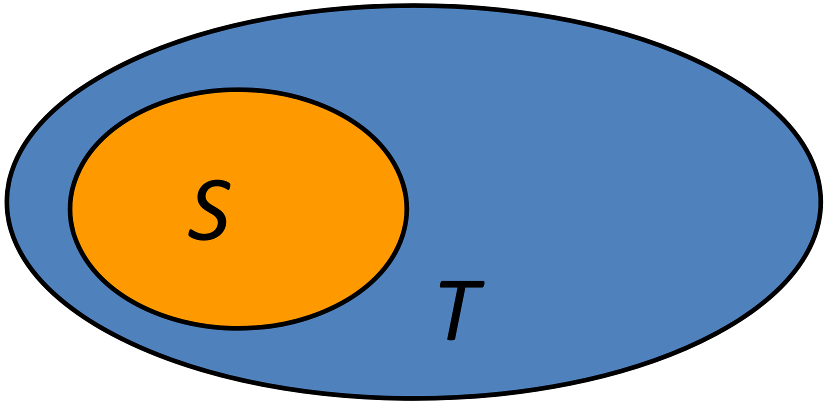


Example:
$\{1,2\} \subset \{1,2,3\}$

Any set is a subset of itself, but not a proper subset. S $\subseteq$ S

Superset

- "T is a superset of S" means $T \supseteq S$ (or $S \subseteq T$)



- $(S \subseteq T) \wedge (S \supseteq T) \Leftrightarrow S = T$
- $S \nsubseteq T$ means $\exists x(x \in S \wedge x \notin T)$

Sets are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.

- For example, let $S=\{x \mid x \subseteq \{1, 2, 3\}\}$
- then $S = \{ \varnothing,$
  $\{1\}, \{2\}, \{3\},$
  $\{1,2\}, \{1,3\}, \{2,3\},$
  $\{1,2,3\} \}$

- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Cardinality of Sets (Size of a Set)

- **Definition: |S|**
  Let S be a set. If there are exactly *n* distinct elements in S, where *n* is a nonnegative integer, we say S is a finite set and that *n* is the **cardinality of S**. The cardinality of S is denoted by **|S|**.
- **Examples:**
  - $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$,
  - $|\varnothing| =$?

$D = \{ x \in \mathbf{N} \mid x < 1000 \}$ $|D| = 1000$

$E = \{ x \in \mathbf{N} \mid x > 1000 \}$ E is infinite!

- If $|S| \in$ N, then we say *S* is *finite*.
- Otherwise, we say *S* is *infinite*.

Power Set

- Definition:

  The *power set* P(*S*) or 2^S of a set *S* is the set of all subsets of *S*.

  $P(S) = \{x \mid x \subseteq S\}$

- Examples:
  - $P(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.
  - $P(\emptyset) =$? $P(\{\emptyset\}) =$? .

Note: $|A| = 0$, $|P(A)|=|2^A| = 1$

- Sometimes P(*S*) is written 2^S.
- Note that for finite *S*, $|P(S)| = |2^S| = 2^{|S|}$.
- It turns out that $|P(N)| > |N|$.
  - *There are different sizes of infinite sets*!

The Power Set

- Cardinality of power sets:
- $|P(A)| = |2^A| = 2^{|A|}$
- Imagine each element in A has an "on/off" switch
- Each possible switch configuration in A corresponds to one element in 2^A

A	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

- For 3 elements in A, there are
  $2*2*2 = 8$ elements in 2^A

Ordered n-tuples

- The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_{\mathbf{n}})$ is an ordered collection of objects.
- Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain exactly the same elements in the same order, i.e. $a_i = b_i$ for $1 \leq i \leq n$.
- Example: Note $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.

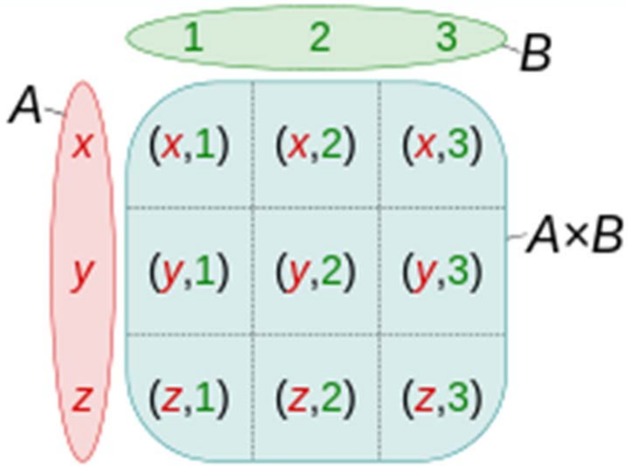
Ordered Pair

- Ordered Pair: Any two elements enclosed by parentheses
- Example: (x, y)

- $(x, y) \neq (y, x)$

Cartesian Products

- **Definition:**
  For sets *A*, *B*, their *Cartesian product*
  $$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$
  – ***set of all ordered pair (a,b), where*** $a \in A$ and $b \in B$.
- *E.g.* $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$



- Examples:
  – A deck of cards: {A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2} × {♠, ♥, ♦, ♣}
  – A two-dimensional coordinate system :
    the Cartesian product $\mathbb{R} \times \mathbb{R}$ with $\mathbb{R}$ denoting the real numbers

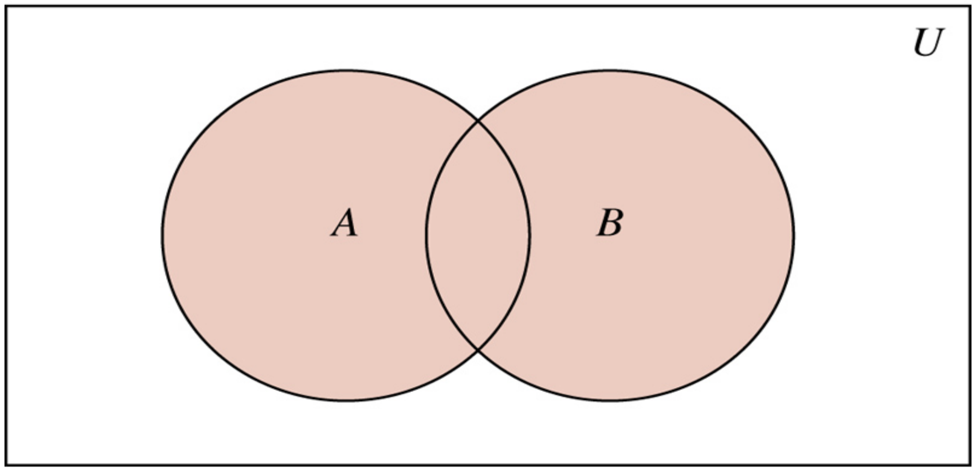
Cartesian Products

- Note that the Cartesian product is *not* commutative!!!:
  For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
  - *E.g.* $\{a,b\} \times \{1,2\} \neq \{1,2\} \times \{a,b\} = \{(1,a),(1,b),(2,a),(2,b)\}$
- Note that:
  - for finite sets *A* and *B*, $|A \times B| = |A||B|$.
  - $A \times \emptyset = \emptyset$
  - $\emptyset \times A = \emptyset$
  - $A^2 = A \times A$

- The Cartesian product of two or more sets is defined as:
- $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$
  - $A^n = \underbrace{A \times A \times \ldots \times A}_{n}$

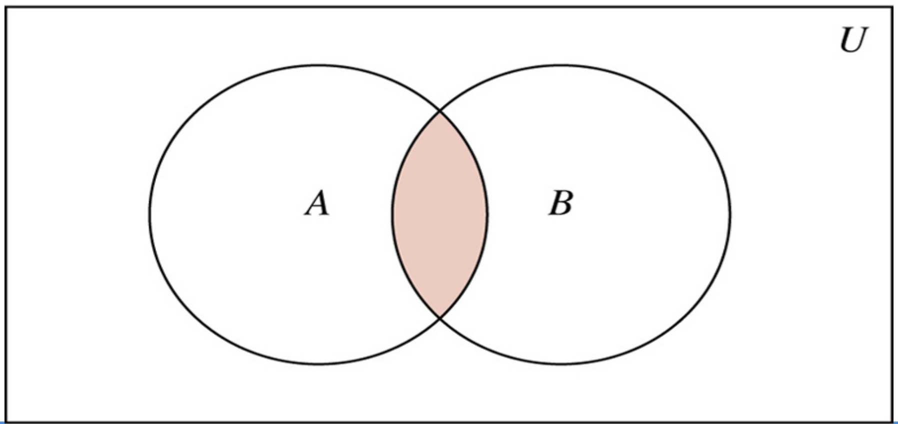
Union

- Definition
  - $A \cup B = \{ x \mid x \in A \lor x \in B \}$
  - The set that contains those elements that are either in A or in B, or in both
- Example
  - $\{1, 3, 5\} \cup \{1, 2, 3\} = \{ 1, 2, 3, 5 \}$



Intersection

- Definition
  - $A \cap B = \{ x \mid x \in A \wedge x \in B \}$
  - The set containing those elements in both A and B
- $|A \cup B| = |A| + |B| - |A \cap B|$
- Example
  - $\{1, 3, 5\} \cap \{1, 2, 3\} = \{ 1, 3 \}$

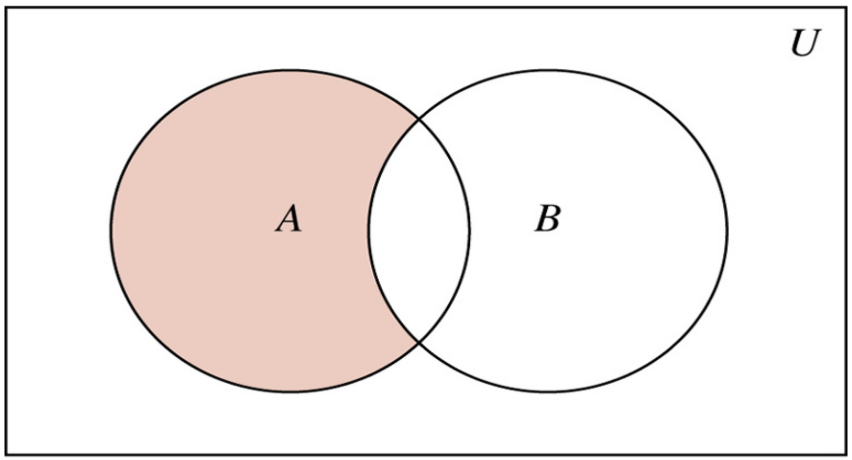


Disjoint

- Definition
  - Two sets are called disjoint if their intersection is the empty set
- Example
  - Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8}
  - Because $A \cap B = \varnothing$, A and B are disjoint

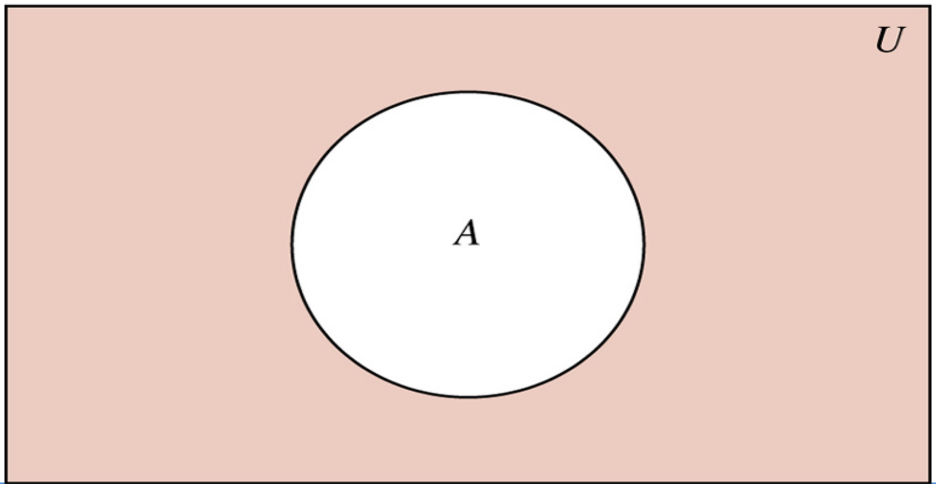
Difference

- Definition
  - $A - B = \{x \mid x \in A \wedge x \notin B\}$
  - The set containing those elements that are in A but not in B
- Example
  - $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$



Complement

- Definition
  - $\overline{A}$ = { x | x $\notin$ A } or U – A, where U is the universal set
- Example
  - Let A = {a, e, i, o, u} and U is the set of English alphabets
  - $\overline{A}$ = {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}



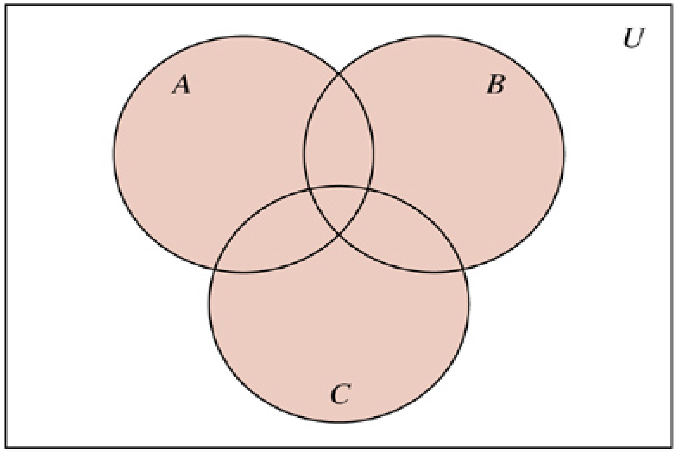
Set Identities

Equivalence	Name
$A \cup \varnothing = A$, $A \cap U = A$	Identity laws
$A \cup U = U$, $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$, $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$, $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$, $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$, $A \cap \overline{A} = \varnothing$	Negation laws

(Table 1, p. 132)

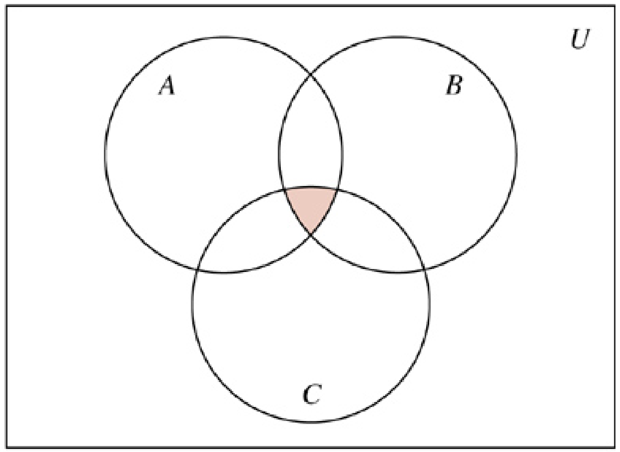
Generalized Union

- Union of a collection of sets
  - The set that contains those elements that are members of at least one set in the collection
  - $$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$
- Example
  - A={0, 2, 4, 6, 8}, B={0, 1, 2, 3, 4}, C={0, 3, 6, 9}
  - A∪B∪C={0, 1, 2, 3, 4, 6, 8, 9}



Generalized Intersection

- Intersection of a collection of sets
  - The set that contains those elements that are members of all the sets in the collection
  - $$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$
- Example
  - A={0, 2, 4, 6, 8}, B={0, 1, 2, 3, 4}, C={0, 3, 6, 9}
  - A∩B∩C={0}



Example:
Representation of Sets with Bit Strings

- There are various ways to represent sets using a computer.
- One method for storing elements is to express each of these sets with bit strings where the *i*-th bit in the string is 1 if *i* is in the set and 0 otherwise.
- Example:
  - Suppose the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
  - Express each of these sets:
    - {1, 3, 5, 7, 9} 10 1010 1010.
    - {2, 4, 6, 8, 10} 01 0101 0101.
    - {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, 11 1111 1111.

(Cont.) Bit String Representation

- Example 20:

  Use bit strings to find the union and intersection of the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9}.

  – Solution:

$$11\ 1110\ 0000 \vee 10\ 1010\ 1010 = 11\ 1110\ 1010,$$ {1, 2, 3, 4, 5, 7, 9}.

$$11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000,$$ {1, 3, 5}.

- Example 19:

  What is the complements of {1, 3, 5, 7, 9} ?

  – Solution:

10 1010 1010. → 01 0101 0101, {2, 4, 6, 8, 10}.

Acknowledgement

- Some slides are taken from lecture notes of Prof. Ok-ran Jeong at GCU and Marc Pomplun at UMB.