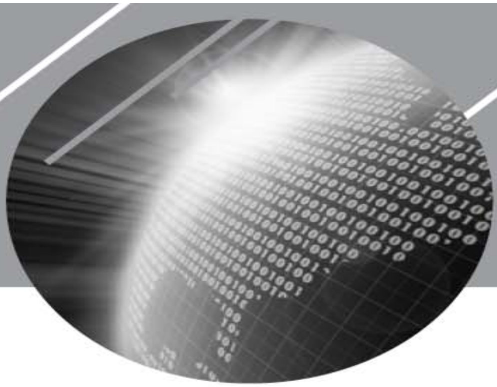
Chapter 7.

Basic Calculus

Part 1. Limit and Derivatives



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7.1 Limit – Basic Concept

The Limit Concept

- The notion of a limit is a *fundamental* concept of calculus.
  https://www.khanacademy.org/math/differential-calculus/dc-limits
- Suppose you drive 200 miles, and it takes you 4 hours.

Then your average speed is: $200 \text{ mi} \div 4 \text{ hr} = 50 \dfrac{\text{mi}}{\text{hr}}$

$$\text{average speed} = \frac{\text{distance}}{\text{elapsed time}} = \frac{\Delta x}{\Delta t}$$

- **Average vs. instantaneous speed**
  - If you look at your speedometer during this trip, it might read 65 mph. This is your instantaneous speed.

The Limit Concept (Cont.)

- *Consider a* rock falls from a high cliff
  - The position of the rock is given by: $y = 16t^2$
    - Then, after 2 seconds: $y = 16 \cdot 2^2 = 64$

- Average vs. instantaneous speed @ 2 seconds
  - <u>average</u> speed: $V_{\mathrm{av}} = \dfrac{64\ \mathrm{ft}}{2\ \mathrm{sec}} = 32\dfrac{\mathrm{ft}}{\mathrm{sec}}$
  - <u>instantaneous</u> speed: $V_{\mathrm{instantaneous}} \approx \dfrac{\Delta y}{\Delta t} = \dfrac{16(2+h)^2 - 16(2)^2}{h}$

for some very small change in t

where h = some very small change in t

Cont.

$$V_{\text{instantaneous}} \approx \frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

$$\left(16*(2+h)\wedge 2-64\right) \div h \,\Big|\, h=\{1,.1,.01,.001,.0001,.00001\}$$

We can see that the velocity approaches 64 ft/sec as h becomes very small.

We say that the velocity has a limiting value of 64 as h approaches zero.

(Note that h never actually becomes zero.)

h	$\frac{\Delta y}{\Delta t}$
1	80
0.1	65.6
.01	64.16
.001	64.016
.0001	64.0016
.00001	64.0002

Definition of Limit

- If *f(x)* becomes arbitrarily close to a unique number *L* as *x* approaches *c* from either side (i.e., left or right), the limit of *f(x)* as *x* approaches *c* is *L*.

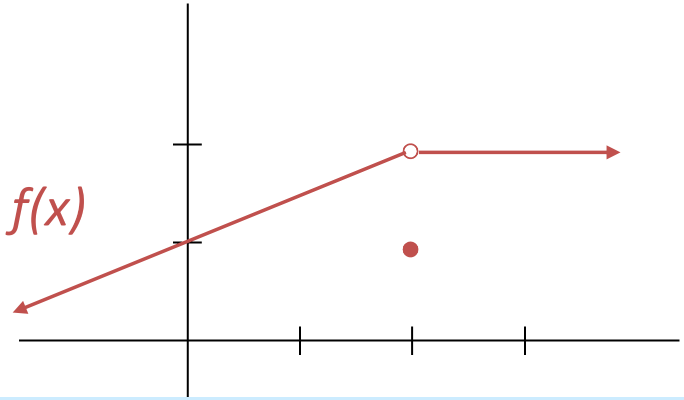
Limit notation: $\lim_{x \to c} f(x) = L$

https://www.khanacademy.org/math/differential-calculus/dc-limits/dc-epsilon-delta/v/epsilon-delta-definition-of-limits

"The limit of f of x as x approaches c is L."

Example: $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$

(Cont.)



$$\lim_{x \to 2} f(x) = \boxed{?}$$

<u>not 1</u>

The <u>limit</u> of a function refers to the value that the function <u>approaches</u>, <u>not</u> the actual value (if any).

- Continuity at a point

Related video: https://www.khanacademy.org/math/differential-calculus/dc-limits/dc-point-continuity/v/continuity-at-a-point

Properties of Limits:

- Limits can be added, subtracted, multiplied, multiplied by a constant, divided, and raised to a power.
  - Study via: https://www.khanacademy.org/math/differential-calculus/dc-limits/dc-limit-prop/v/limit-properties
- For a limit to exist, the function must approach the same value from both sides.

One-sided limits approach from either the left or right side only.

https://www.khanacademy.org/math/differential-calculus/dc-limits/dc-limits-from-graphs/v/one-sided-limits-from-graphs

Examples

- **Video 1. Limits of piecewise functions**
  - https://www.khanacademy.org/math/differential-calculus/dc-limits/dc-direct-substitution/v/limits-of-trigonometric-functions
- **Video 2. Limits of piecewise functions: absolute value**
  - https://www.khanacademy.org/math/differential-calculus/dc-limits/dc-direct-substitution/v/limit-at-a-point-of-discontinuity

- **HW:** take a quiz
  - https://www.khanacademy.org/math/differential-calculus/dc-limits/quiz/dc-direct-substitution-quiz

More: The Sandwich (Squeeze) Theorem

Show that: $$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

- **Squeeze Theorem :**

If $g(x) \le f(x) \le h(x)$ for all $x \neq c$ in some interval about c and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} f(x) = L$.

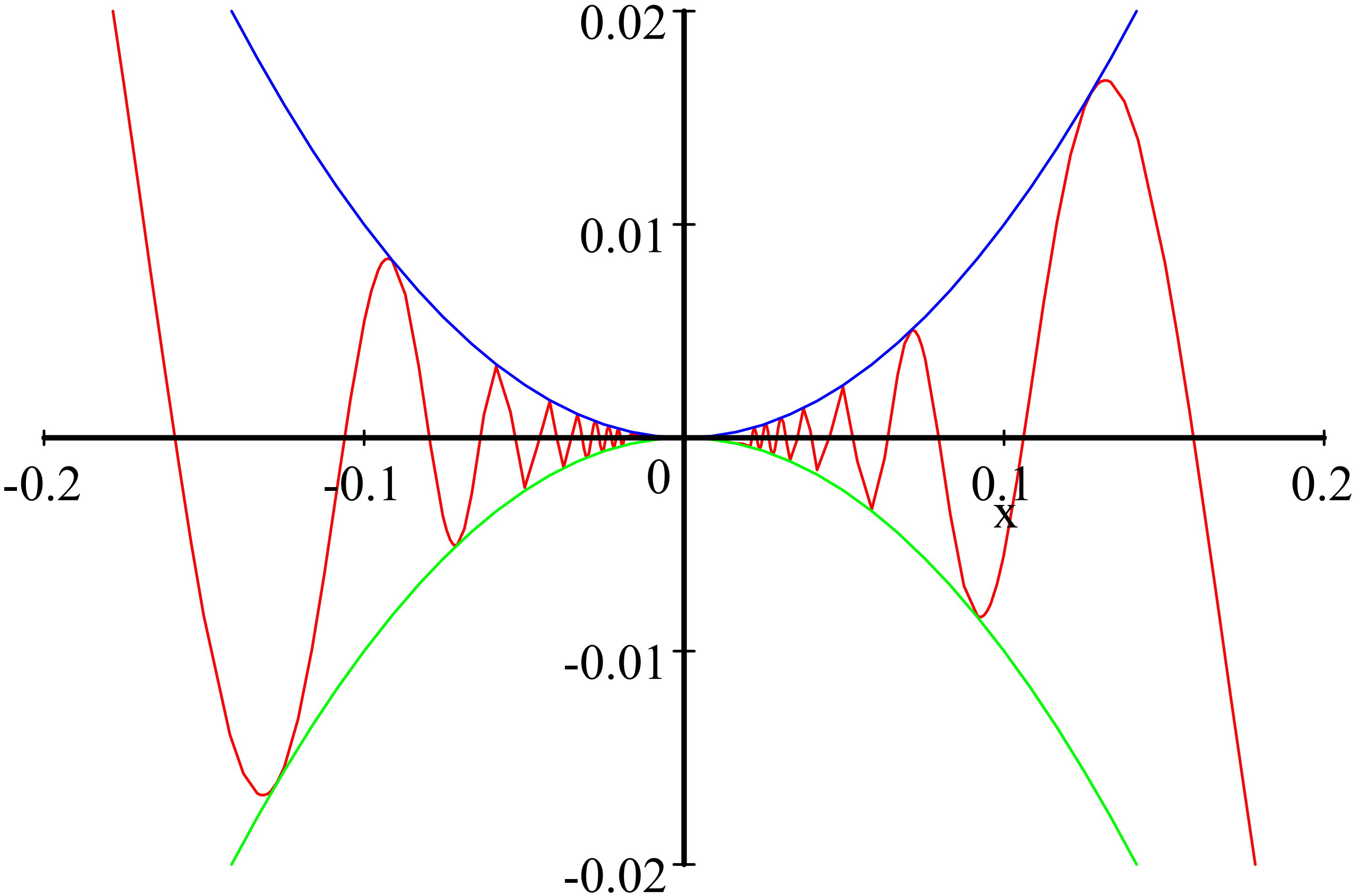
The maximum value of sine is 1, so $$x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

The minimum value of sine is -1, so $$x^2 \sin\left(\frac{1}{x}\right) \ge -x^2$$

So: $$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-new/ab-1-8/v/squeeze-sandwich-theorem

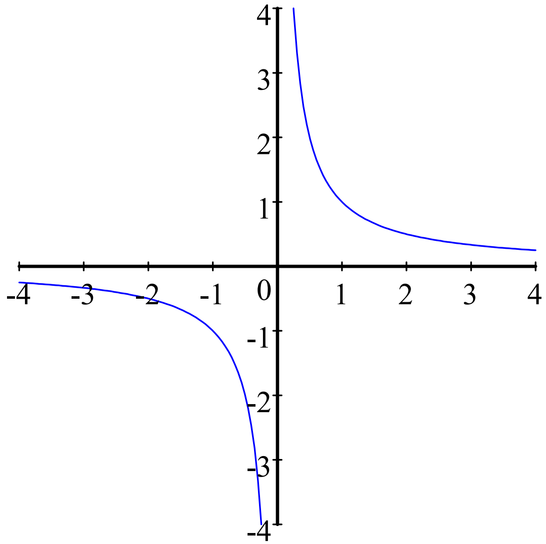
Show that: $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$



Infinite Limits

$$f(x) = \frac{1}{x}$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

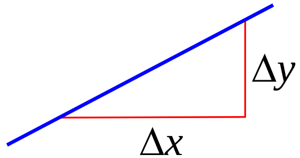


- As the denominator approaches zero, the value of the fraction gets very large. $\lim_{x\to\infty} \frac{1}{x} = 0$
- If the denominator is positive then the fraction is positive. $\lim_{x\to 0^+} \frac{1}{x} = \infty$
- If the denominator is negative then the fraction is negative. $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

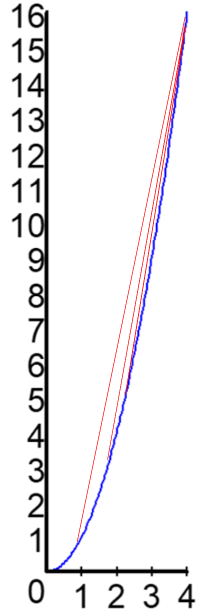
- Limits involving infinity
  - https://www.khanacademy.org/math/differential-calculus/dc-limits/dc-infinite-limits/v/introduction-to-infinite-limits

Rates of Change and *Tangent Lines**(접선)

- The slope of a line is given by: $m = \dfrac{\Delta y}{\Delta x}$



$f\left(x\right) = x^2$



- The slope at (1,1) can be approximated by the slope of the secant through (4,16).

$$\frac{\Delta y}{\Delta x} = \frac{16-1}{4-1} = \frac{15}{3} = 5$$

We could get a better approximation if we move the point closer to (1,1). ie: (3,9)

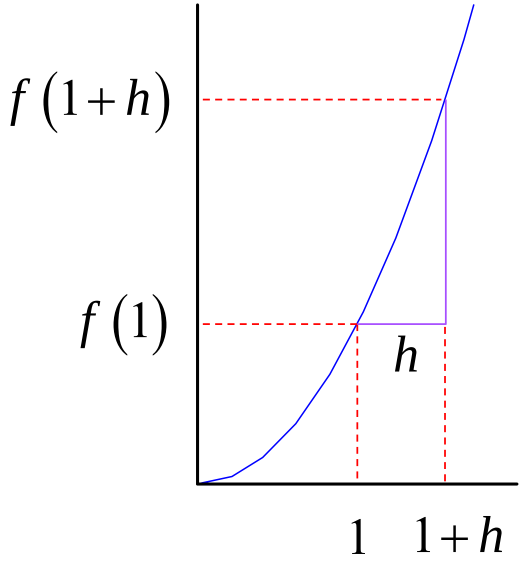
$$\frac{\Delta y}{\Delta x} = \frac{9-1}{3-1} = \frac{8}{2} = 4$$

Even better would be the point (2,4).

$$\frac{\Delta y}{\Delta x} = \frac{4-1}{2-1} = \frac{3}{1} = 3$$

How far can we go?

The slope of the curve



$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(1+h)-f(1)}{h}$$

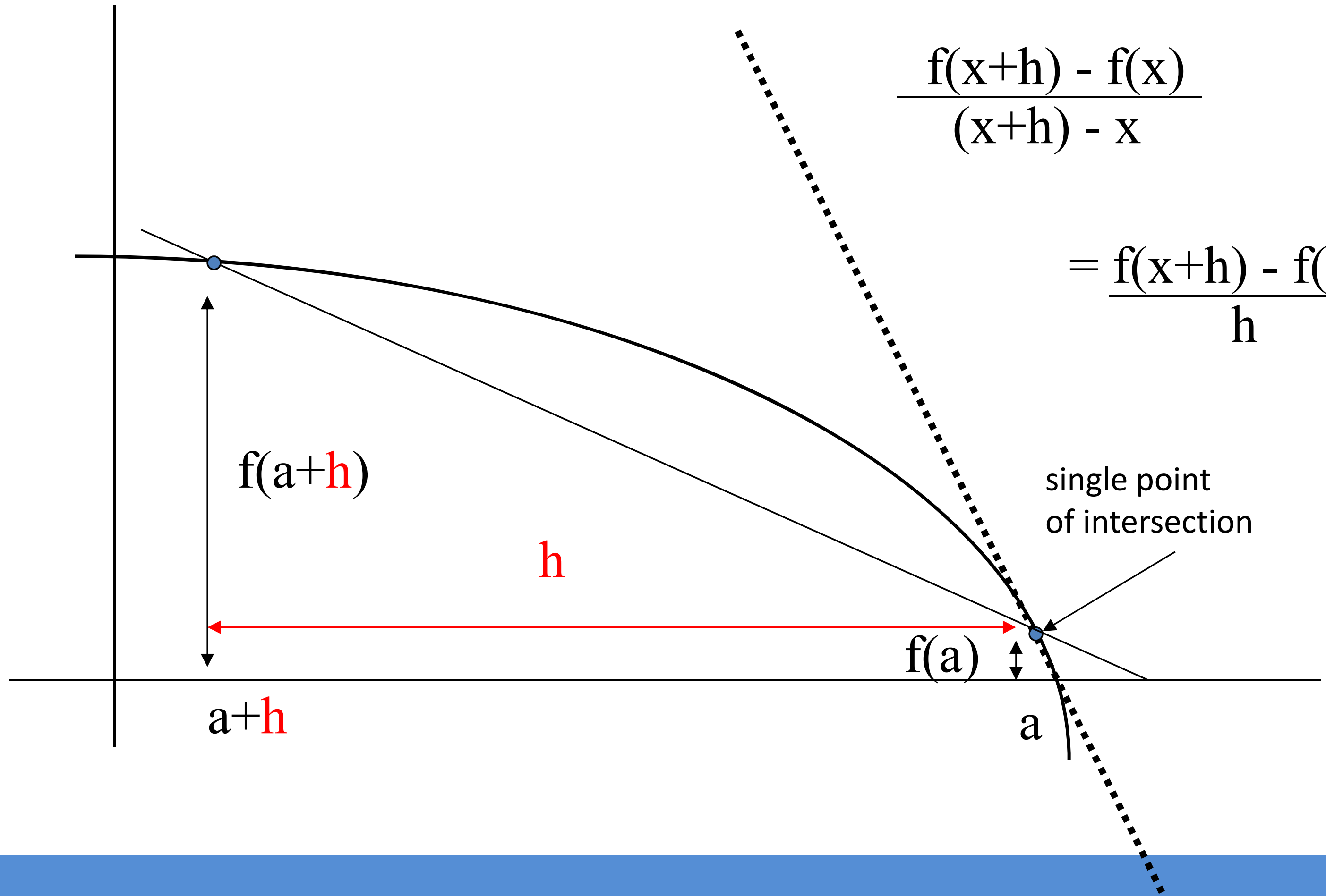
$$\text{slope at } (1,1) = \lim_{h\to 0}\frac{(1+h)^2-1}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{1}+2h+h^2-\cancel{1}}{h} = \lim_{h\to 0}\frac{\cancel{h}(2+h)}{\cancel{h}} = 2$$

The slope of the curve $y=f(x)$ at the point $P(a, f(a))$ is:

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

The tangent line: the slope at a point



$$\frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

The slope at a point

The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

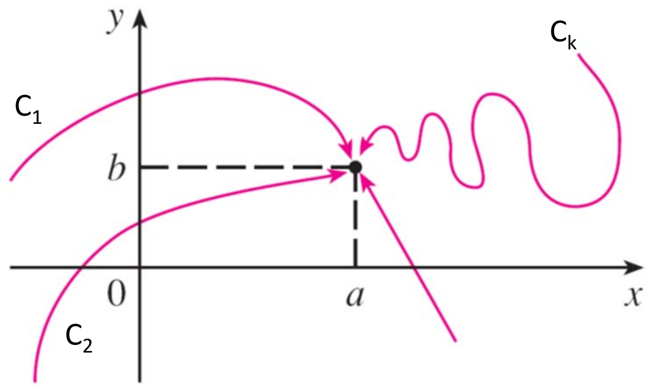
- The slope of a curve at a point is the same as the slope of the tangent line at that point.

$\frac{f(a+h) - f(a)}{h}$ is called the **difference quotient** of f at a.

Limits of Multiple Variables

- For functions of two or more variables, the situation is not as simple.

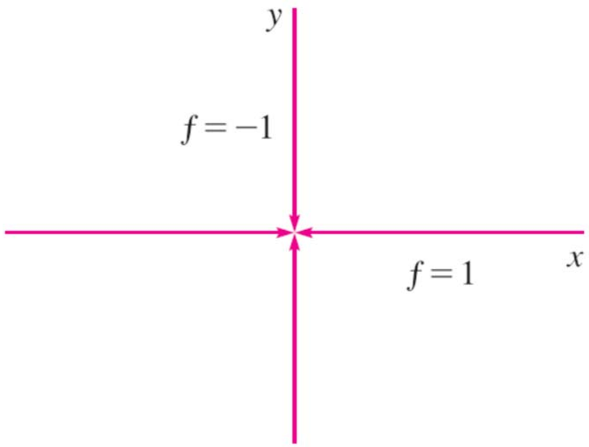
$$\lim_{(x,y)\to(a,b)} f(x,y) = ?$$



- For example, if we can find two different paths C_1 and C_2 of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Limits of Multiple Variables (Cont.)

- If
  $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and
  $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2,
  where $L_1 \neq L_2$, then
  $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.



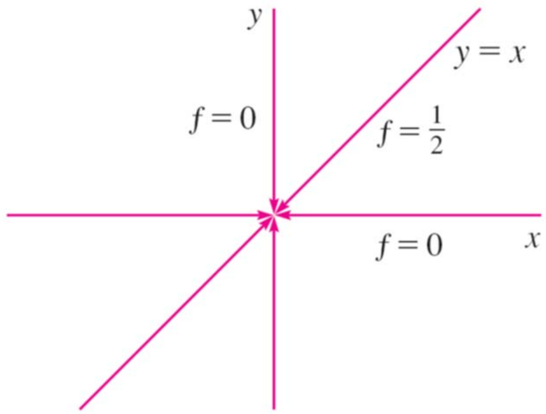
- Example: $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.
  - Path C_1: approach (0, 0) along the *x*-axis
    - Then, $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$. So, $f(x, y) \to \mathbf{1}$ as $(x, y) \to (0, 0)$ along the *x*-axis.
  - Path C_2: approach (0, 0) along the *y*-axis
    - Then, $f(0, y) = -y^2/y^2 = -1$ for all $y \neq 0$. So, $f(x, y) \to \mathbf{-1}$ as $(x, y) \to (0, 0)$ along the *y*-axis
  - Thus, the given limit does not exist.

Cont.

- Another Question: Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist for $f(x,y) = \frac{xy}{x^2+y^2}$?

  - Hint.
    If $y = 0$ and x → 0, then $f(x, 0) = 0/x^2 \to 0$. (x-axis)
    If $x = 0$ and y → 0, then $f(0, y) = 0/y^2 \to 0$. (y-axis)

- But!
  - If we approach (0, 0) along another line, say $y = x$.
  - For all x ≠ 0, $f(x,x) = x^2/(x^2 + x^2) \to 1/2$



To See More

- **Average vs. instantaneous rate of change**
  - https://www.khanacademy.org/math/differential-calculus/dc-diff-intro#dc-diff-calc-intro
  - Newton, Leibniz, and Usain Bolt
  - Derivative as a concept
  - Secant lines & average rate of change
  - Derivative notation review
  - Derivative as slope of curve
  - The derivative & tangent line equations

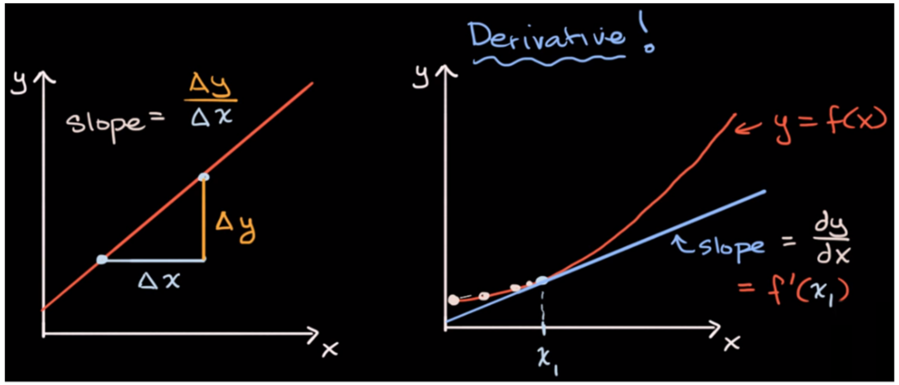
7.2 Derivatives / Differentiation

미분(differentiation), 도함수(derivative)

Derivatives

- What is a Derivative?
  - https://www.khanacademy.org/math/differential-calculus/dc-diff-intro

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ is called the <u>derivative of f at a.</u>



We write: $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

"The derivative of f with respect to x is …"

There are many ways to write the derivative of $y=f(x)$

There are many ways to write the derivative of $y = f(x)$

$f'(x)$ "f prime x" or "the derivative of *f* with respect to x"

y' "y prime"

$\frac{dy}{dx}$ "dee why dee ecks" or "the derivative of y with respect to x"

$\frac{df}{dx}$ "dee eff dee ecks" or "the derivative of f with respect to x"

$\frac{d}{dx} f(x)$ "dee dee ecks uv eff uv ecks" or "the derivative of f of x"

$(\; d \;\; dx \;\text{ of }\; f \;\text{ of }\; x \;)$

Differentiation

- Differentiation
  - Process of finding the derivative of a function

- A function is differentiable if it has a derivative everywhere in its domain. It must be continuous and smooth. Functions on closed intervals must have one-sided derivatives defined at the end points.

Rules for Differentiation

- **The derivative of a constant**

$$\boxed{\frac{d}{dx}(c) = 0}$$

If the derivative of a function is its slope, then for a constant function, the derivative must be zero.

- **The derivative of a power function**

If n is an integer, then:

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}}$$

examples:

$$f(x) = x^4 \qquad y = x^8$$

$$f'(x) = 4x^3 \qquad y' = 8x^7$$

https://www.khanacademy.org/math/differential-calculus/dc-diff-intro/dc-power-rule/v/power-rule

- **constant multiple rule:**

$$\boxed{\frac{d}{dx}(cu) = c\frac{du}{dx}}$$

examples:

$$\frac{d}{dx}cx^n = cnx^{n-1} \qquad \frac{d}{dx}7x^5 = 7 \cdot 5x^4 = 35x^4$$

- **sum and difference rules:**

https://www.khanacademy.org/math/differential-calculus/dc-diff-intro/dc-combine-power-rule-with-others/v/differentiating-polynomials-example?modal=1

$$\boxed{\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}} \qquad \boxed{\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}}$$

$$y = x^4 + 12x \qquad y' = 4x^3 + 12$$

$$y = x^4 - 2x^2 + 2 \qquad \frac{dy}{dx} = 4x^3 - 4x$$

(Each term is treated separately)

Differentiation: Product rule

- **product rule:**

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Notice that this is not just the product of two derivatives.

This is sometimes memorized as: $d(uv) = u\ dv + v\ du$

$$\frac{d}{dx}\left[\left(x^2+3\right)\left(2x^3+5x\right)\right] = \left(x^2+3\right)\left(6x^2+5\right) + \left(2x^3+5x\right)\left(2x\right)$$

$$\frac{d}{dx}\left(2x^5+5x^3+6x^3+15x\right) \qquad 6x^4+5x^2+18x^2+15+4x^4+10x^2$$

$$\frac{d}{dx}\left(2x^5+11x^3+15x\right)$$

$$10x^4+33x^2+15 \qquad 10x^4+33x^2+15$$

Differentiation: Quotient rule

- **quotient rule:**

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$$

or

$$d\left(\frac{u}{v}\right)=\frac{v\ du-u\ dv}{v^2}$$

$$\frac{d}{dx}\frac{2x^3+5x}{x^2+3}=\frac{\left(x^2+3\right)\left(6x^2+5\right)-\left(2x^3+5x\right)\left(2x\right)}{\left(x^2+3\right)^2}$$

Some important derivatives

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
e^x	e^x
e^{ax}	ae^{ax}
$\ln x$	$\frac{1}{x}$
$\ln ax$	$\frac{1}{ax}\frac{d}{dx}[ax] = \frac{1}{x}$

https://www.khanacademy.org/math/differential-calculus/dc-diff-intro/dc-more-diff-rules/v/derivatives-of-sinx-and-cosx

Higher Order Derivatives:

$$y' = \frac{dy}{dx}$$ is the first derivative of y with respect to x.

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$ is the <u>second</u> derivative.

(y double prime)

$$y''' = \frac{dy''}{dx}$$ is the third derivative.

$$y^{(4)} = \frac{d}{dx}y'''$$ is the fourth derivative.

Example 1.

- Differentiate the following function:

$$(a) \quad f(x) = x^3 - 2x$$
$$(b) \quad f(x) = -x^{-3} + 3x^2$$
$$(c) \quad f(x) = x^5 - 20$$

Example 2.

- Differentiate the following function:

$$(a)\quad y = (x^2 - 2)(3x^4 - x)$$
$$(b)\quad y = x^2 \sin 4\,x$$
$$(c)\quad y = e^x x^3$$
$$(d)\quad y = 2e^{3x} \ln 5\,x$$
$$(e)\quad y = \frac{x^2 + 1}{x^3 - 2x}$$

Example 3.

- Differentiate and simplify the following function:

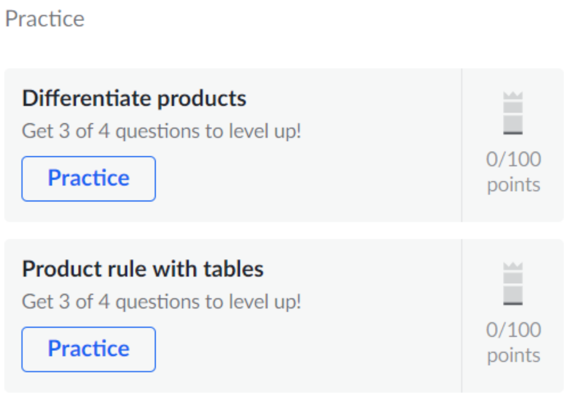
$(a)\quad y = e^{2x}\cos x$

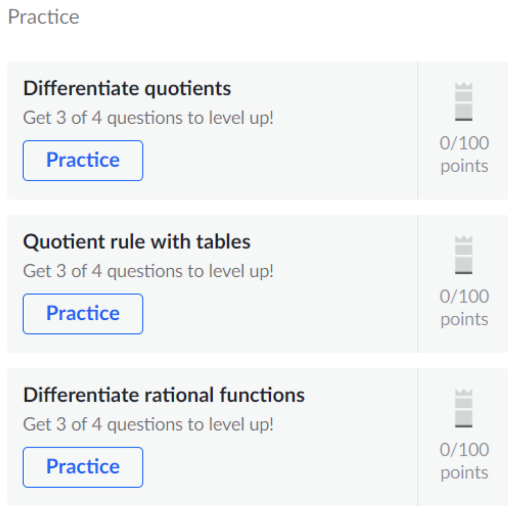
$(b)\quad y = (x^3 - 2)\sin 2\,x$

Practice More @ (Important)

- https://www.khanacademy.org/math/differential-calculus/dc-diff-intro/dc-quotient-rule/v/quotient-rule

- https://www.khanacademy.org/math/differential-calculus/dc-diff-intro





Derivative of Composite Functions

The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

– If *g* is differentiable at point *x* and f is differentiable at the point *g(x)*, then $f \cdot g$ is differentiable at *x*.

If $f \cdot g$ is the composite of $y = f(x)$ and $u = g(x)$, then:

$$(f \circ g)' = f'_{\text{at } u = g(x)} \cdot g'_{\text{at } x}$$

"Outside-Inside" Rule

– Alternative method for Chain Rule:

– If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)).g'(x)$

Examples

- Differentiate the following function:

$$(a) \quad y = \ln(\sin 3x)$$

$$(b) \quad y = e^{\sin 2x}$$

$$(c) \quad y = x^2 \ln(e^{2x})$$

Implicit Differentiation* (*음함수 P(x,y)=0 미분)

- When we cannot put an equation F(x,y)=0 in the form y = f(x), use implicit differentiation to find $\frac{dy}{dx}$
  for an **implicit** relation between variables *x* and *y*.
  - Example:

$$x^2 + y^2 - 25 = 0, \quad x^3 + y^3 - 9xy = 0$$

- Steps:
  - Differentiate both sides of the equation with respect to *x*, treating *y* as a differentiable function of *x*
  - Collect the terms with $\frac{dy}{dx}$ on one side of the equation
  - Solve for $\frac{dy}{dx}$

Example

- Find $\frac{dy}{dx}$ for $3x^2y + 2x\ln y = 2x^3$.

Solution

- Step 1: Differentiate w.r.t *x*

$$\frac{d}{dx}\left(3x^2 y\right)+\frac{d}{dx}\left(2x\ln y\right)=\frac{d}{dx}\left(3x^3\right)$$

$$\left(3x^2\frac{d}{dx}(y)+y\frac{d}{dx}\left(3x^2\right)\right)+\left(2x\frac{d}{dx}(\ln y)+\ln y\frac{d}{dx}(2x)\right)=9x^2$$

  – *Simplify* $3x^2\frac{dy}{dx}+6xy+2x\frac{1}{y}\frac{dy}{dx}+2\ln y=9x^2$

- Step 2: Rearrange & Factorize

$$3x^2\frac{dy}{dx}+\frac{2x}{y}\frac{dy}{dx}=9x^2-2\ln y-6xy \qquad \frac{dy}{dx}\left(3x^2+\frac{2x}{y}\right)=9x^2-2\ln y-6xy$$

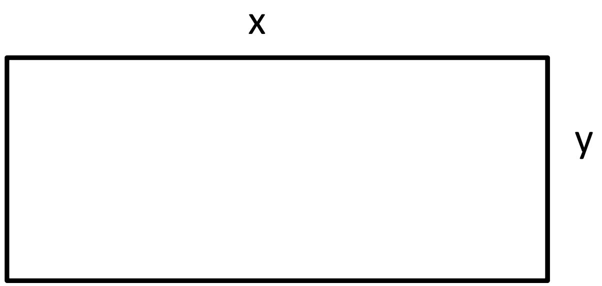
- Step 3: Obtained $\frac{dy}{dx}=\frac{9x^2-2\ln y-6xy}{3x^2+\frac{2x}{y}}$

Example – Related rates

- A process of finding a rate at which a quantity changes by relating that quantity to the other quantities.
  - The rate is usually with respect to time, *t*.

Example Question:

- **How fast is the area of a rectangle changing** from one side 10cm long and the side increase at a rate of 2cm/s and the other side is 8cm long and decrease at a rate of 3cm/s?



$$\text{At } x = 10, \quad \frac{dx}{dt} = 2cm/s. \quad \text{At } y = 8, \quad \frac{dy}{dt} = -3cm/s$$

$$\text{Area of rectangle}: \quad A = xy \cdots (1) \qquad \frac{dA}{dt} = ?$$

Solution

$$\text{At } x = 10, \quad \frac{dx}{dt} = 2cm/s. \quad \text{At } y = 8, \quad \frac{dy}{dt} = -3cm/s$$

$$\text{Area of rectangle:} \quad A = xy \cdots (1)$$

Differentiate (1) wrt *t*:

$$\frac{d}{dt}(A) = \frac{d}{dt}(xy)$$

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$= 10(-3) + 8(2)$$

$$= -14cm^2/s$$

Minimum and Maximum Values

Maximum & Minimum

- Use 1st derivative to locate and identify extreme values(stationary values) of a continuous function from its derivative

- **Definition: Absolute Maximum and Absolute Minimum**
- Let f be a function with domain D. Then f has an ABSOLUTE MAXIMUM value on D at a point *c* if:

$$f(x) \leq f(c), \ \forall x \in D$$

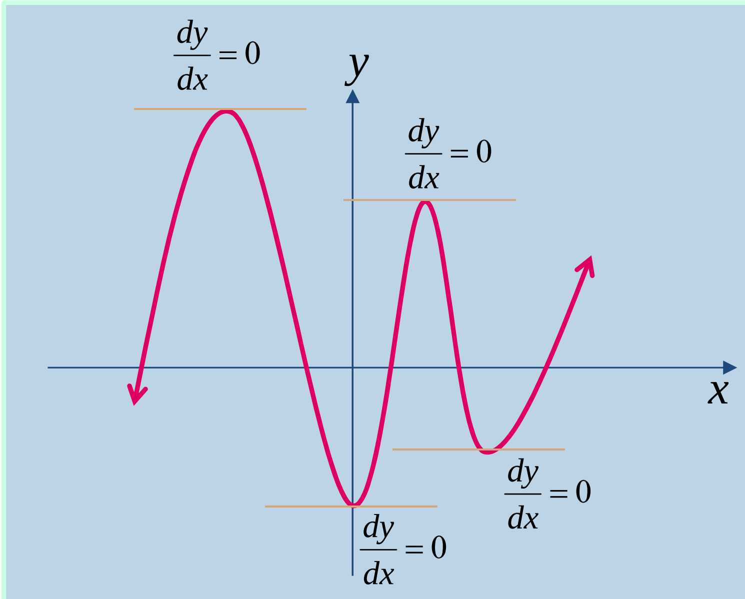
- ABSOLUTE MINIMUM

$$f(x) \geq f(c), \ \forall x \in D$$

Stationary Point (or Critical Point*) (*임계점, 정류점)

- A point on the graph of a function *y* = *f*(*x*) where the rate of change is zero.

$$\frac{dy}{dx} = 0$$



Example:

Find stationary points:

$$(1)\quad y = x^2 - 4x + 3$$

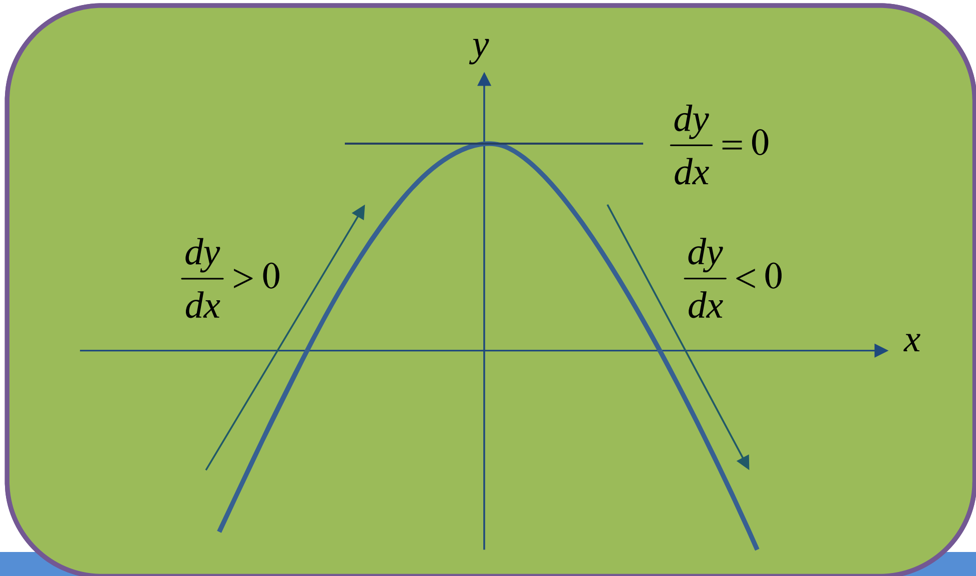
$$(2)\quad y = x^3 - 3x + 3$$

1^{st} Derivative Test

Suppose that *f* is continuous on [a,b] and differentiable on (a,b).

a) If $\dfrac{dy}{dx} > 0$ at each point $x \in [a,b]$, then y is said to be increasing on $[a,b]$

b) If $\dfrac{dy}{dx} < 0$ at each point $x \in [a,b]$, then y is said to be decreasing on $[a,b]$

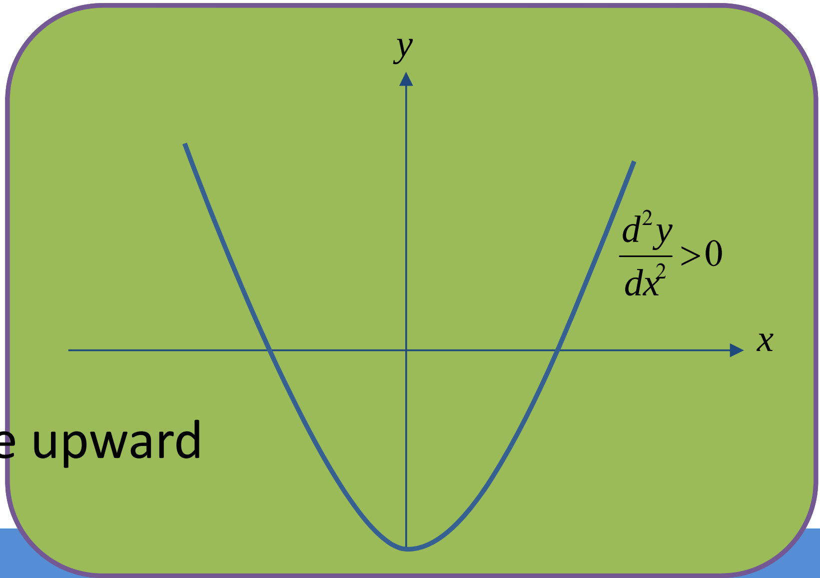


2nd Derivative Test for Concavity

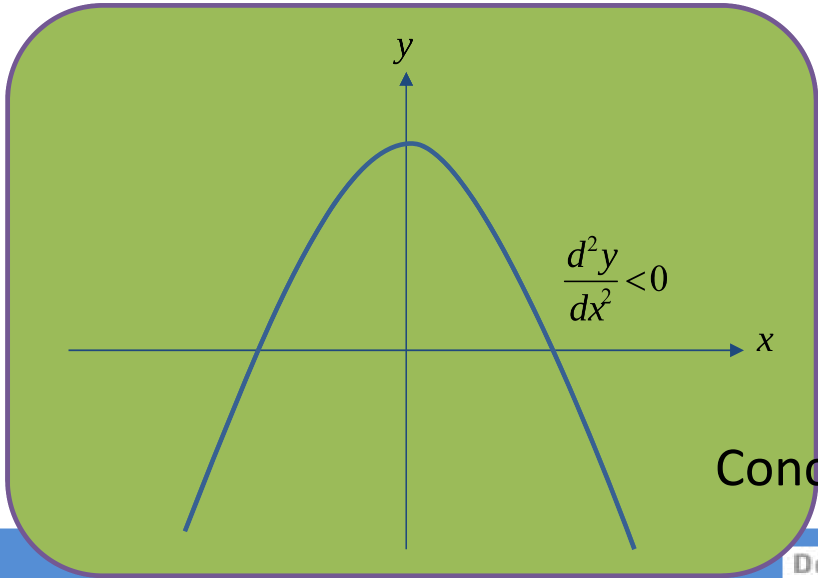
Let *y=f(x)* be twice-differentiable on an interval *I*

a) Concave up on an open interval if $\frac{d^2y}{dx^2} > 0$ on $[a,b]$

b) Concave down on an open interval if $\frac{d^2y}{dx^2} < 0$ on $[a,b]$



Concave upward



Concave downward

Maximum point & Minimum point

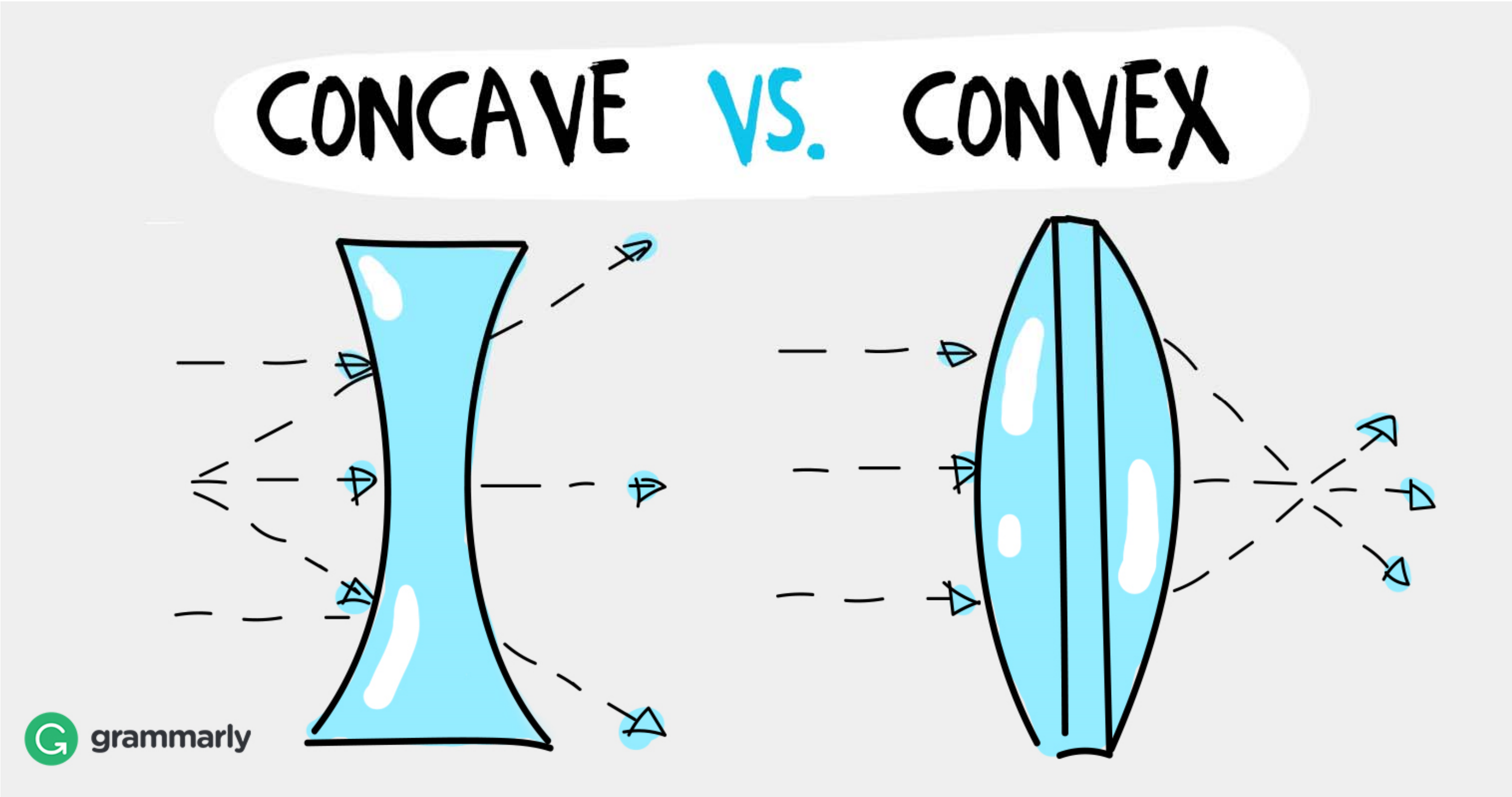
- If $\frac{d^2y}{dx^2} > 0 \rightarrow y$ is minimum

  Therefore (x,y) is a minimum point.

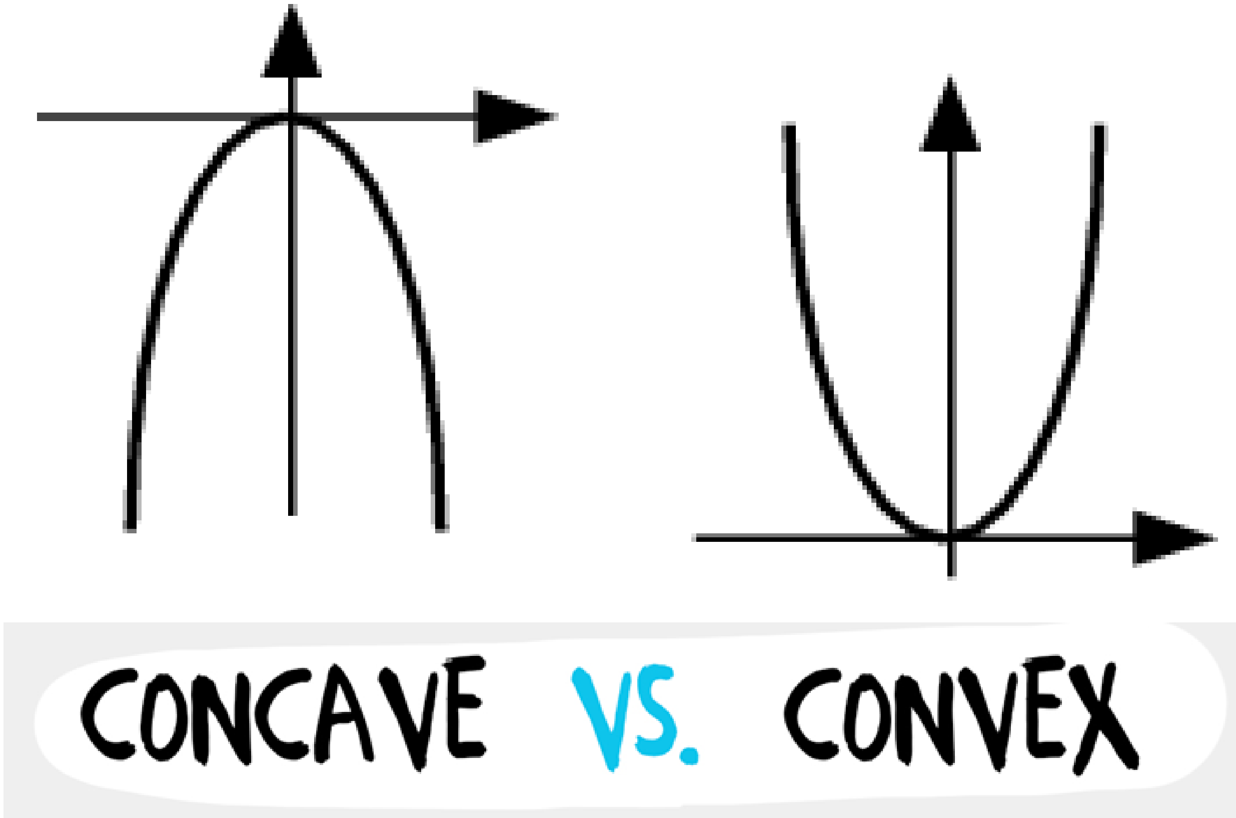
- If $\frac{d^2y}{dx^2} < 0 \rightarrow y$ is maximum

  Therefore (x,y) is a maximum point.

Concave vs. Convex



Concave vs. Convex Functions



Convex Function

- Convex function: A function is convex on set D if and only if for any two points $x^{(1)}$ and $x^{(2)}$ Є D and $0 \leq \lambda \leq 1$,
  - $f[\lambda x^{(1)} + (1-\lambda)x^{(2)}] \leq \lambda f(x^{(1)}) + (1-\lambda)\, x^{(2)}$

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

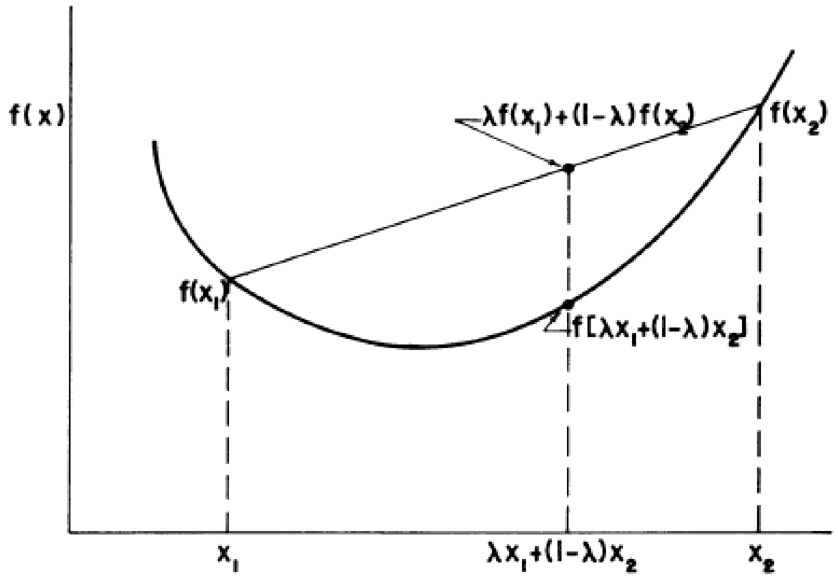
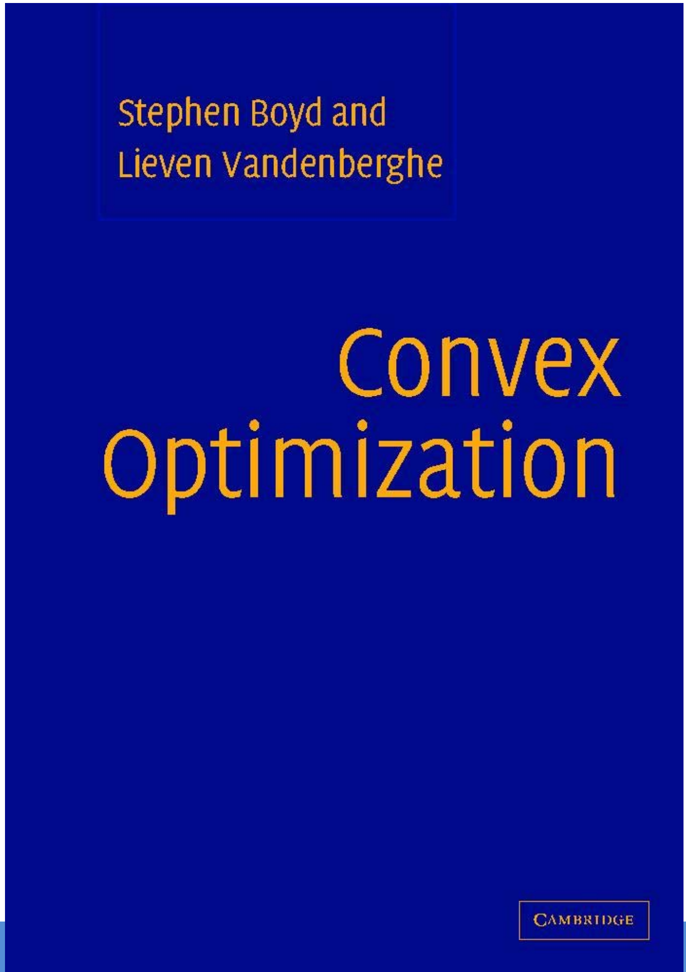


Figure B.1. Convex function.

(Appendix) Convex Optimization



(mathematical) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \ldots, m \end{array}$$

Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \ldots, m \end{array}$$

- objective and constraint functions are convex:

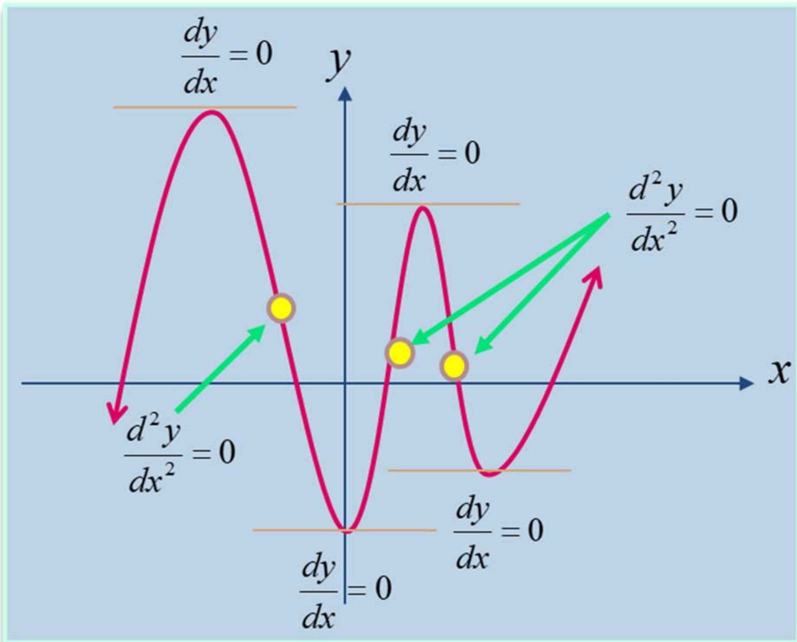
$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

if $\alpha + \beta = 1$, $\alpha \ge 0$, $\beta \ge 0$

A Point of Inflexion* & Concavity (*변곡점)

A point where the graph of a function has a tangent line and where the concavity changes is a **POINT OF INFLEXION.**

$$\frac{d^2y}{dx^2} = 0$$



For More @

- Concavity and inflection points intro
  - https://www.khanacademy.org/math/differential-calculus/dc-analytic-app#dc-concavity-intro

- Second Derivative Test
  - https://www.khanacademy.org/math/differential-calculus/dc-analytic-app/dc-second-derivative-test/v/second-derivative-test

- Extreme value theorem
  - https://www.khanacademy.org/math/differential-calculus/dc-analytic-app/dc-evt/v/extreme-value-theorem

References

- Lecture Notes from Greg Kelly's Lecture Notes, Richland, Washington
- Dr. Nazuhusna and Dr. Shahrir Rizal's Lecture Notes (EQT 101 Engineering Mathematics I) @ UniMAP: Universiti Malaysia Perlis
- and
- KhanAcademy