Chapter 2-2.

Basic Structures : Functions

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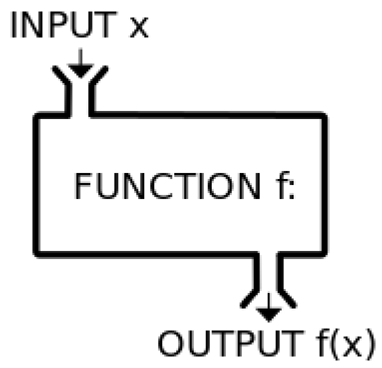
Functions

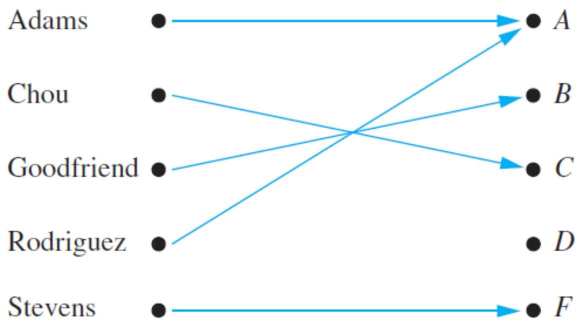
- Functions assign/produce a single output for each of their inputs.
  - "A function **relates** an input to an output."



- *A function **f** from a set A to a set B is an assignment of exactly one element of B to each element of A.*
  - let A and B be nonempty sets
- $f : A \rightarrow B$
  - f is a function from A to B
- f(a) = b
  - *b* is the unique element of B assigned by f to the element *a* of A

*Remark: Functions are sometimes also called **mappings** or **transformations**.

Example:



Functions (extra slide)

- A function $f: A \to B$ can also be defined as a subset of $A \times B$ (in terms of a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

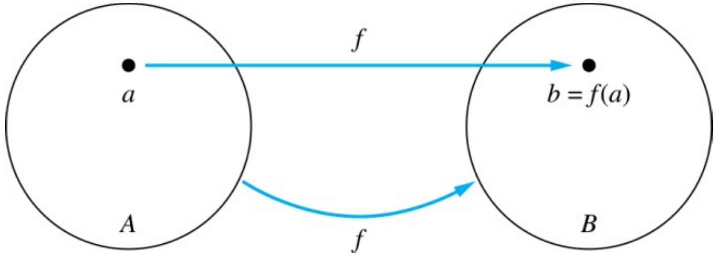
and

$$\forall x[x \in A \to \exists y[y \in B \land (x, y) \in f]]$$

$$\forall x, y_1, y_2[[(x, y_1) \in f \land (x, y_2)] \to y_1 = y_2]$$

Domain, Codomain and Range

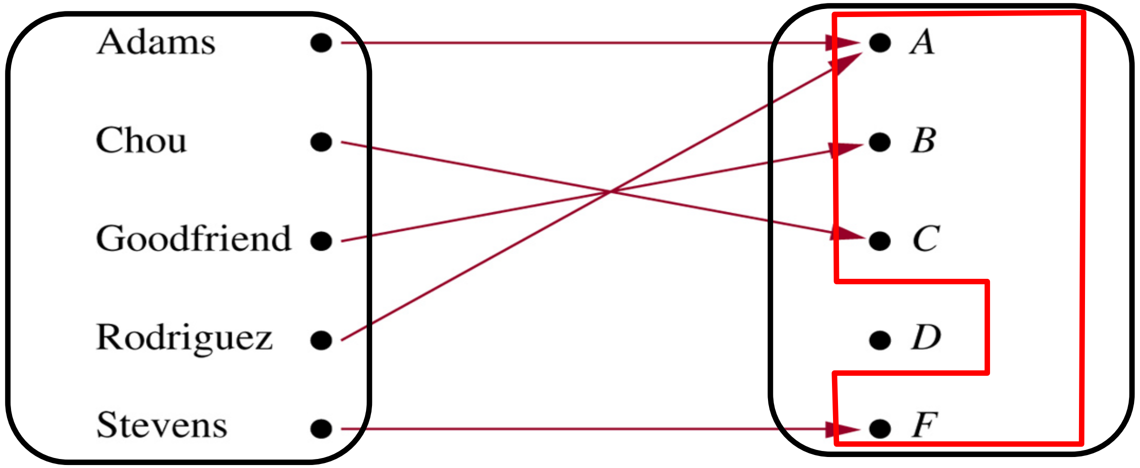
- f : A → B
  - A is the domain of f
  - B is the codomain of f
- f (a)=b
  - b is the image of a
  - a is a preimage of b



We say that f:A→B maps A to B or f is a *mapping* from *A* to *B*.

- The **range** or **image** of f : A → B is the set of all images
- Example

Domain



Codomain

Range

Representation of Functions

- There are several ways to represent a function

e.g., let us specify *f* as follows:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

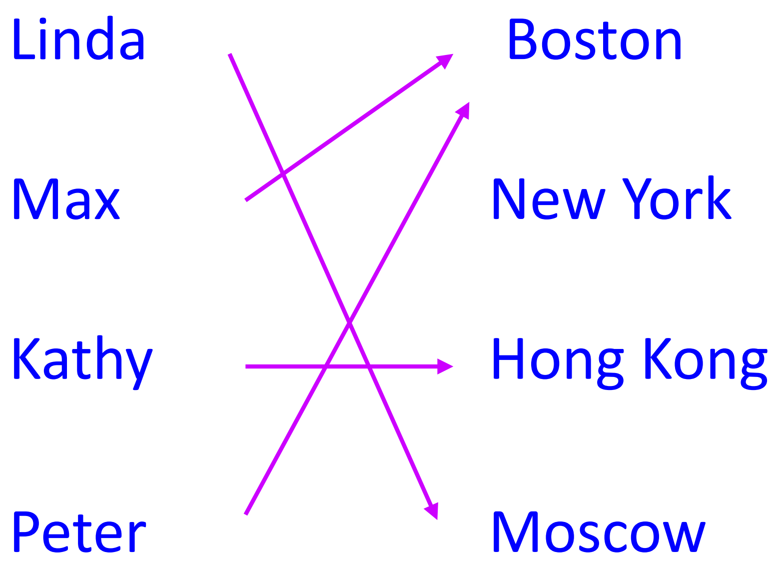
- Is *f* a function? yes

- What is its range? {Moscow, Boston, Hong Kong}

Cont.

- Other ways to represent f:

x	f(x)
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



- If the domain of our function f is large, it is convenient to specify f with a formula, e.g.:
  - $f: \mathbf{R} \to \mathbf{R}$
  - $f(x) = 2x$

This leads to:
$f(1) = 2$, $f(3) = 6$, $f(-3) = -6$, …

Cont.

- Functions may be specified in different ways:
  - A computer program.
    - A Java program that when given an integer n, produces the nth Fibonacci Number (covered in the next section and also in Ch 5).

Range or Image of Function

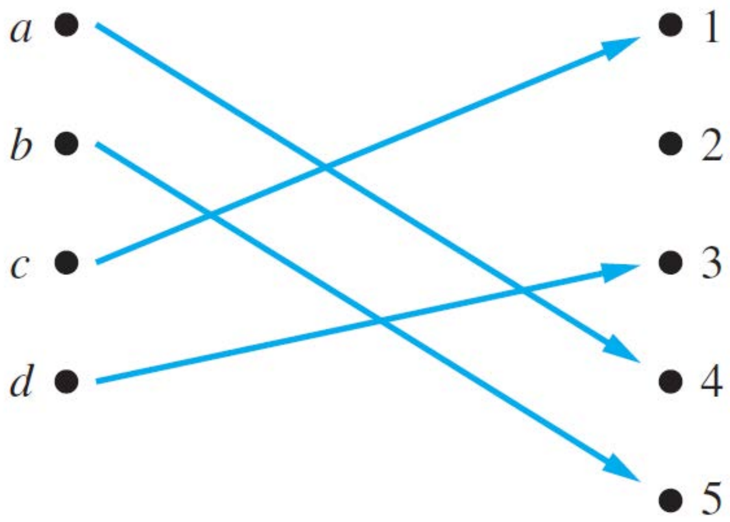
- If we only regard a subset $S \subseteq A$, the set of all images of elements $s \in S$ is called the image of S.
- We denote the image of S by **f(S)**:
  $f(S) = \{f(s) \mid s \in S\}$

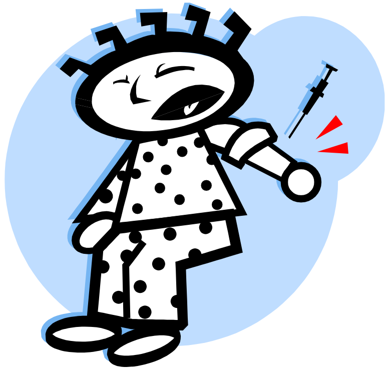
- f(Linda) = Moscow
- f(Max) = Boston
- f(Kathy) = Hong Kong
- f(Peter) = Boston

- What is the image of S = {Linda, Max} ?
  - f(S) = {Moscow, Boston}
- What is the image of S = {Max, Peter} ?
  - f(S) = {Boston}

Properties:
One-to-One (**Injective**) Function

- Definition
  - A function f is ***one-to-one*** (**injunctive** or **injunction**)
    if and only if f (a)=f (b) implies that a=b for all a and b in the domain of f
  - $\forall a \forall b\ (f(a)=f(b) \rightarrow a=b)$
- Example





Properties: Increasing/Decreasing Functions

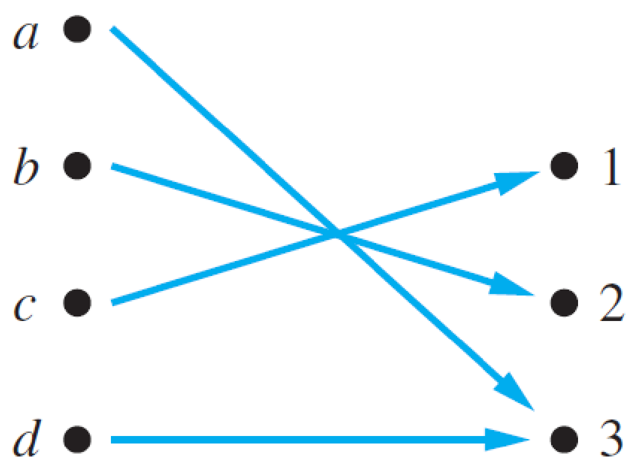
- A function f: $A \rightarrow B$ with $A, B \subseteq \mathbf{R}$ is called strictly increasing, if
  $\forall x,y \in A\ (x < y \rightarrow f(x) < f(y))$,

- and strictly decreasing, if
  $\forall x,y \in A\ (x < y \rightarrow f(x) > f(y))$.

- Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.

Properties:
Onto (Surjective) Function

- Definition
  - A function f from A to B is called **onto** (**surjection**) if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$ (f is called ***surjective***)
  - $\forall y \exists x (f(x) = y)$
- Example



Examples

- Example 10.
  - Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

Solution: The function $f(x) = x + 1$ is a one-to-one function. To demonstrate this, note that $x + 1 \neq y + 1$ when $x \neq y$. ◀

- Example 12.
  - Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto. ◀

- Example 13.
  - Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

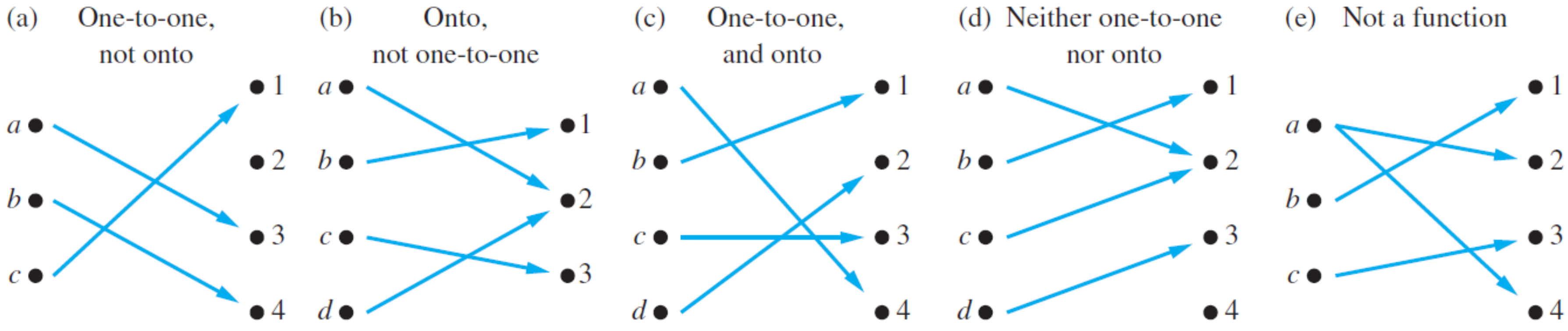
Solution: The function f is not onto because there is no integer x with $x^2 = -1$, for instance. ◀

Properties:
Bijection Function

- <u>Definition:</u>

  A function *f* is a ***bijection*** or ***one-to-one correspondence*** if it is both one-to-one and onto

Examples:



Showing that *f* is one-to-one or onto

Suppose that $f : A \to B$.

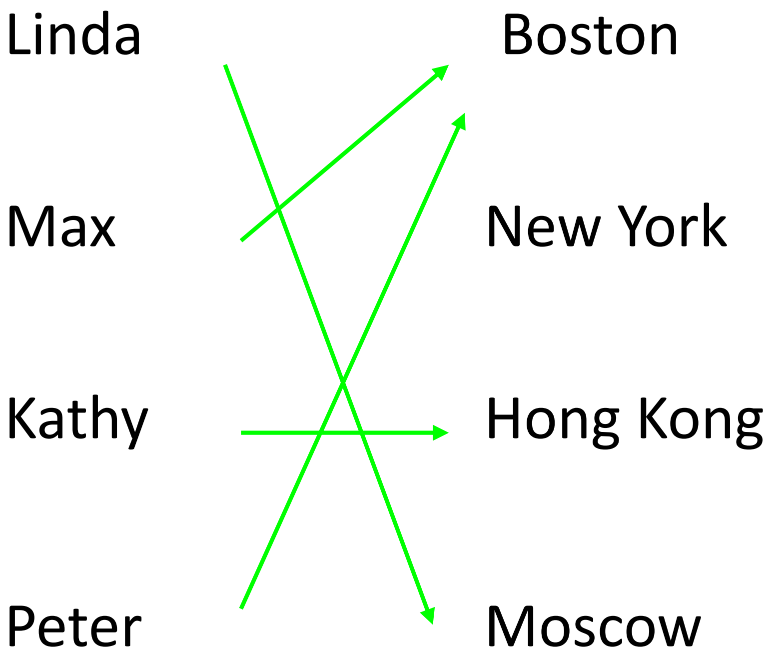
To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Examples: Properties of Functions



Is f injective?

- No.

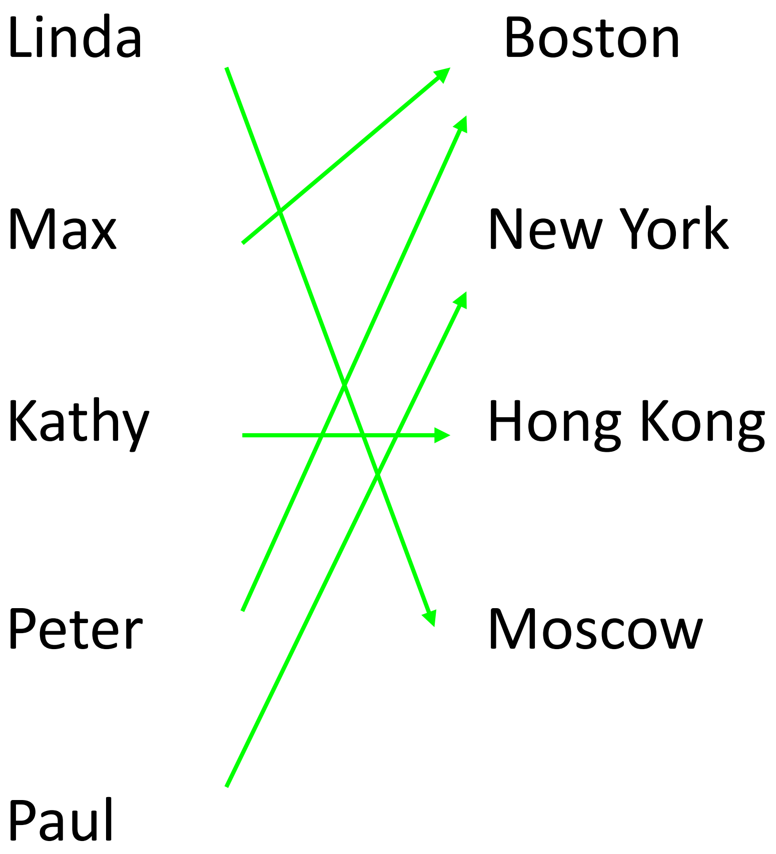
Is f surjective?

- No.

Is f bijective?

- No.

Properties of Functions



Is f injective?

- No.

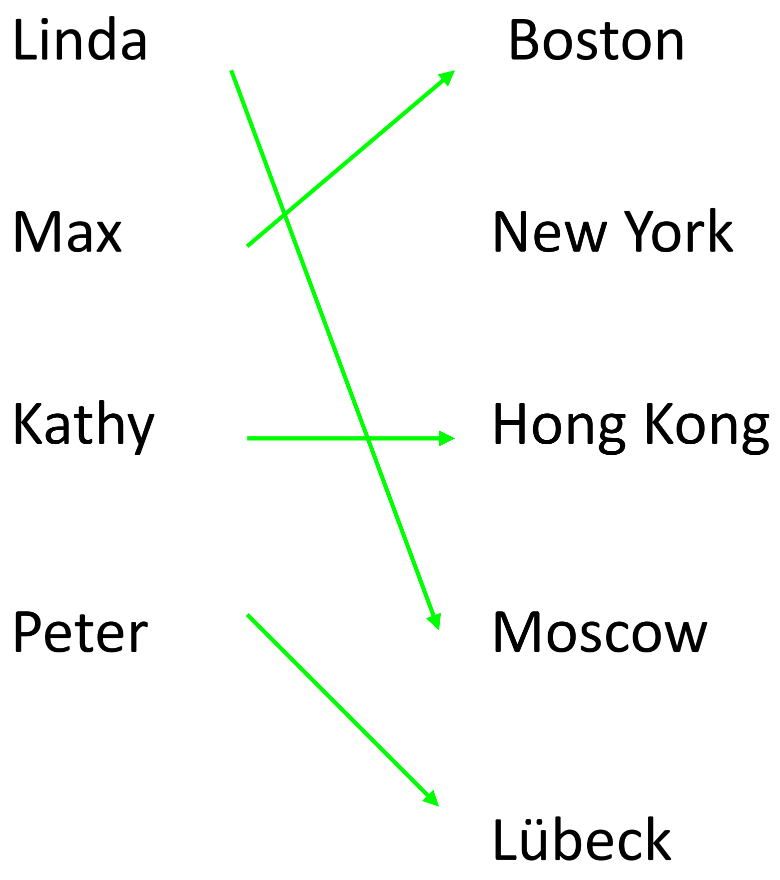
Is f surjective?

- Yes.

Is f bijective?

- No.

Properties of Functions



Is f injective?

- Yes.

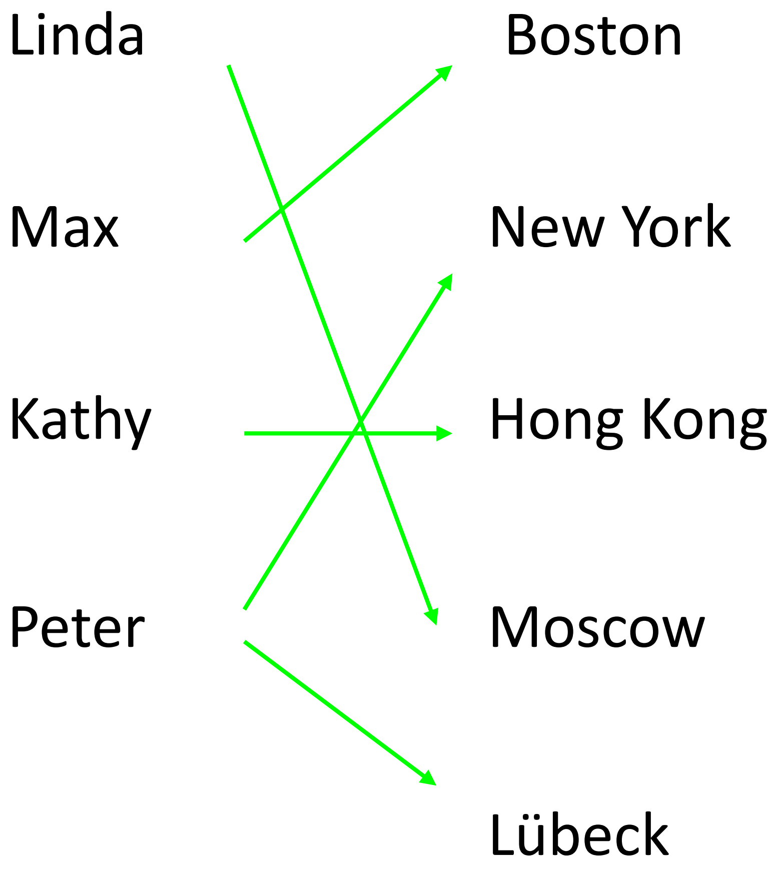
Is f surjective?

- No.

Is f bijective?

- No.

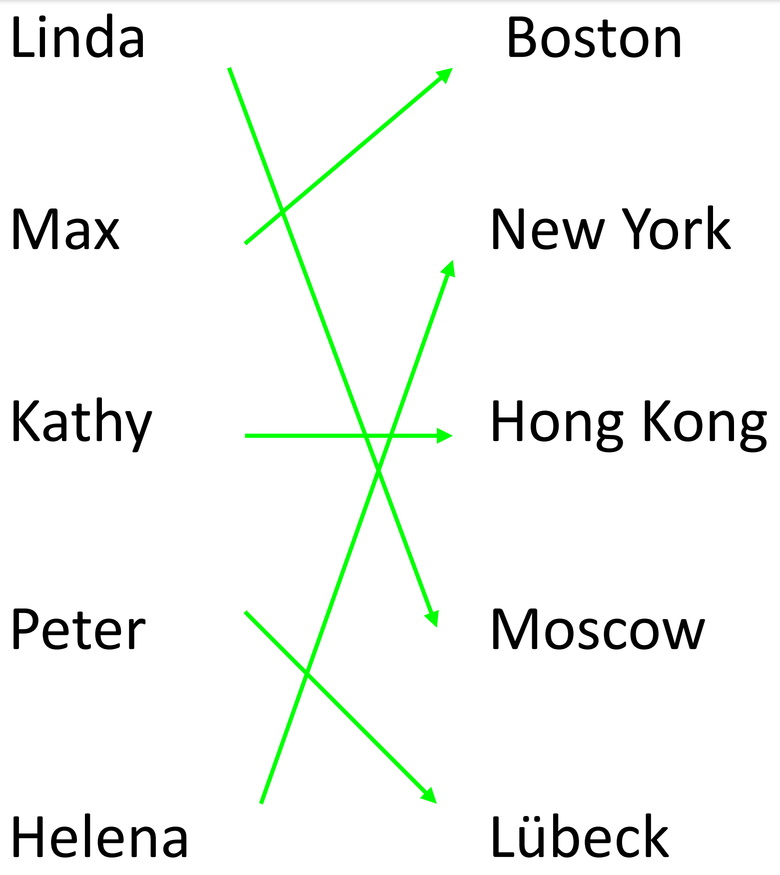
Properties of Functions



Is f injective?

- No! f is not even a function!

Properties of Functions



Is f injective?

- Yes.

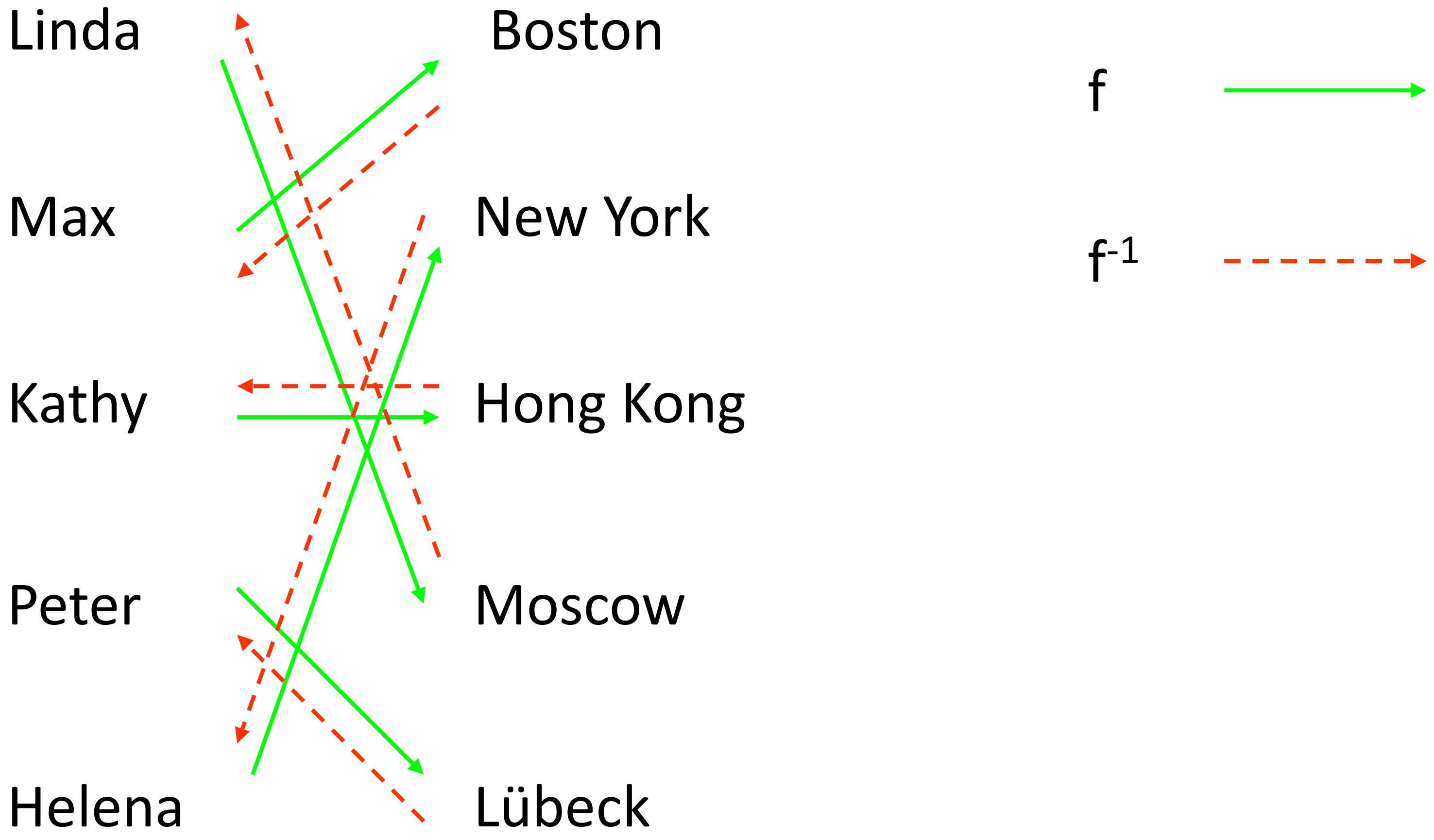
Is f surjective?

- Yes.

Is f bijective?

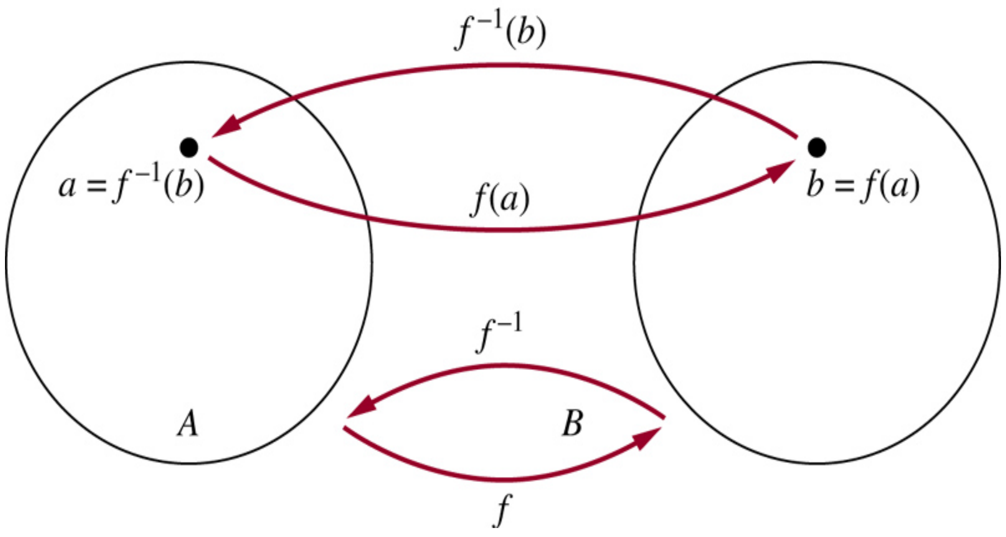
- Yes.

Inversion


Linda
Max
Kathy
Peter
Helena
Boston
New York
Hong Kong
Moscow
Lübeck
f
f-1

Inverse Functions

- An interesting property of *bijections* is that they have an **inverse function**.
- The inverse function of the *bijection* f:A→B is the function f^{-1}:B→A with $f^{-1}(b) = a$ whenever f(a) = b.

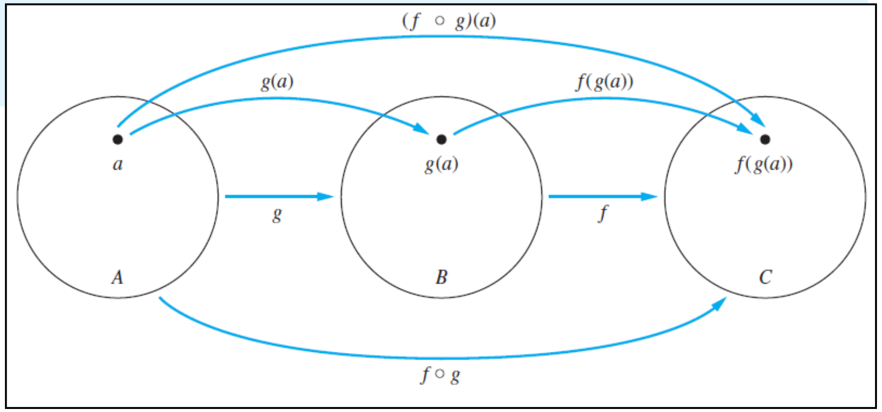


Example 19:
Let f :Z→Z be such that f (x)=x+1
$f^{-1}(y) = y-1$

Composition of Functions : $f \circ g$

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$

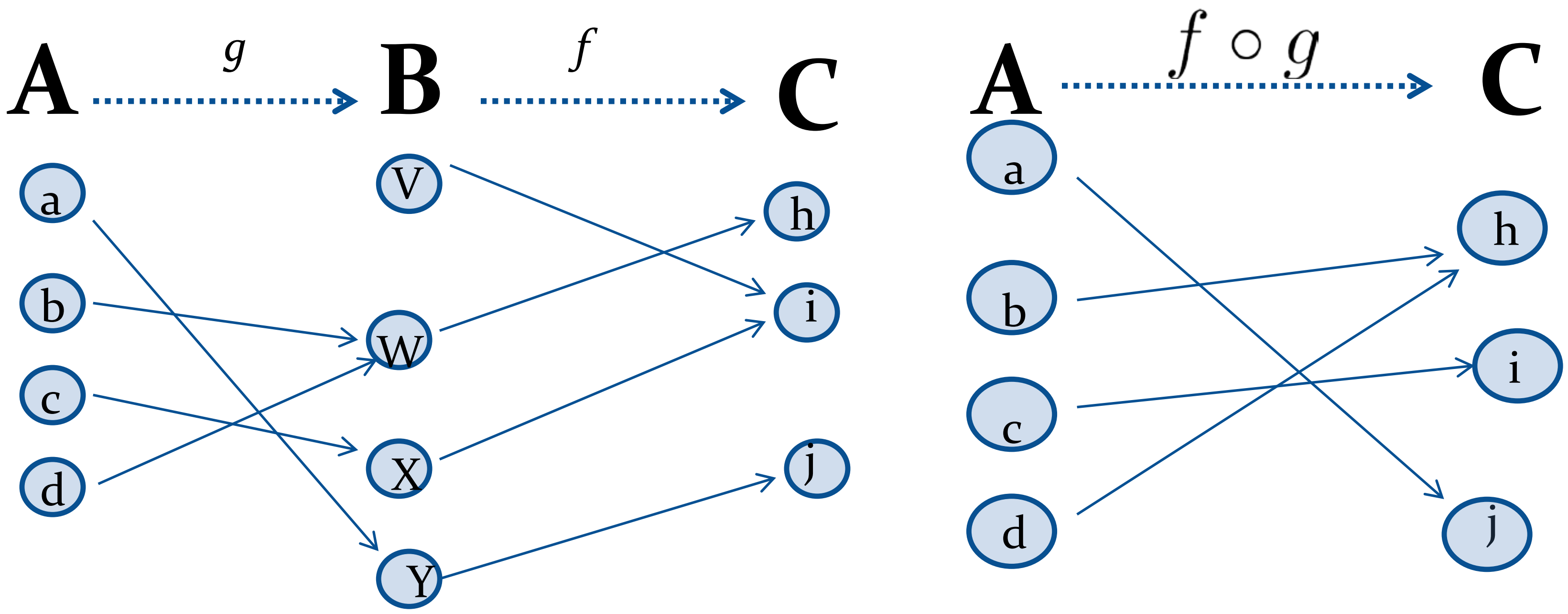


- For two functions $g{:}A \rightarrow B$, $f{:}B \rightarrow C$, composition $(f \circ g)(a) = f(g(a))$
  - This means that

first, function g is applied to element $a \in A$, mapping it onto an element of B,
then, function f is applied to this element of B, mapping it onto an element of C.

  - **Therefore**, the composite function maps from A to C.

Composition : $f \circ g$



Examples

- Example 23.
  - Let *f* and *g* be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of *f* and *g*? What is the composition of *g* and *f* ?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

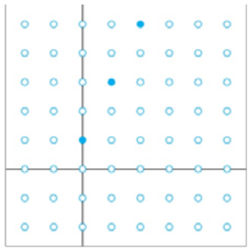
$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

- Problem:
  - Composition of a function *f(x)* and its inverse $f^{-1}(x)$.

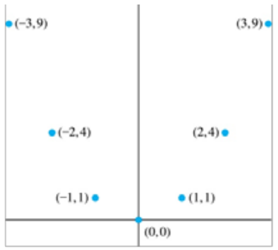
Sol: $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$

Graph

- The **graph** of a function f:A→B is the set of ordered pairs {(a, b) | a∈A and f(a) = b}.
- The graph is a subset of A×B that can be used to visualize f in a two-dimensional coordinate system.



Graph of $f(n) = 2n + 1$ from Z to Z

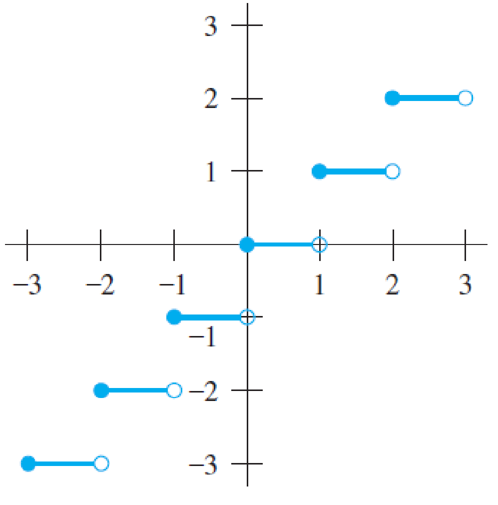


Graph of $f(x) = x^2$ from Z to Z

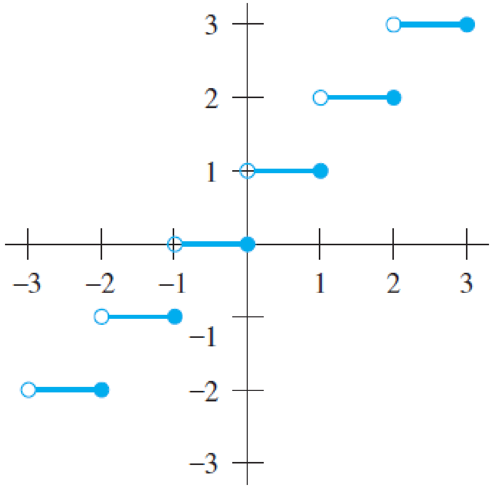
- Example:
  – Display the graph of the function f(x) = | x | from the set of integers to the set of integers.

Floor and Ceiling Functions

- Floor function
  - Assigns to the real number x the largest integer that is less than or equal to x
  - $\lfloor x \rfloor$
  - Examples
    - $\lfloor 2.3 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -3.5 \rfloor = -4$
- Ceiling function
  - Assigns to the real number x the smallest integer that is greater than or equal to x
  - $\lceil x \rceil$
  - Examples:
    - $\lceil 2.3 \rceil = 3$, $\lceil 2 \rceil = 2$, $\lceil 0.5 \rceil = 1$, $\lceil -3.5 \rceil = -3$



(a) $y = \lfloor x \rfloor$



(b) $y = \lceil x \rceil$

Exponential and Logarithmic Functions

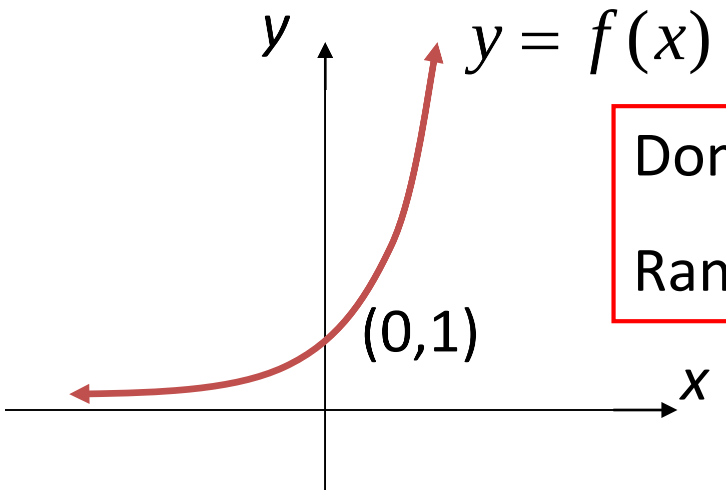
Exponential Function

- An ***exponential function*** with **base *b*** and **exponent *x*** is defined by

$$f(x) = b^x \qquad (b > 0, b \neq 1)$$

Ex. $f(x) = 3^x$

x	y
-1	$1/3$
0	1
1	3
2	9



Domain: All reals

Range: $y > 0$

Properties of the Exponential Function

$$y = f(x) = b^x \qquad (b > 0, b \neq 1)$$

1. The domain is $(-\infty, \infty)$.
2. The range is $(0, \infty)$.
3. It passes through (0, 1).
4. It is continuous everywhere.
5. If $b > 1$ it is increasing on $(-\infty, \infty)$.
   If $b < 1$ it is decreasing on $(-\infty, \infty)$.

Laws of Exponents

- Let *a* and *b* be positive numbers and let *x* and *y* be real numbers. Then,

1. $$b^x \cdot b^y = b^{x+y}$$
2. $$\frac{b^x}{b^y} = b^{x-y}$$
3. $$\left(b^x\right)^y = b^{xy}$$
4. $$(ab)^x = a^x b^x$$
5. $$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Examples

- Ex1. Simplify the expression

$$\frac{\left(3x^2y^{1/2}\right)^4}{x^3y^7}$$

$$=\frac{3^4x^8y^2}{x^3y^7}=\frac{81x^5}{y^5}$$

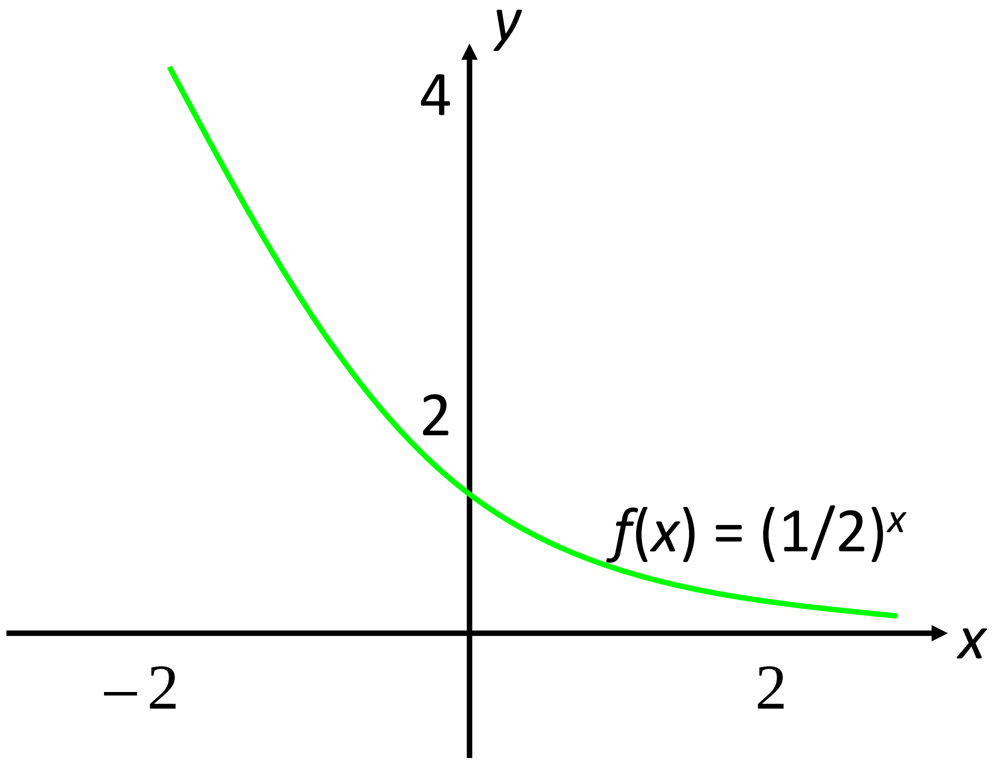
- Ex2. Solve the equation

$$4^{3x+1}=2^{4x-2}$$

$$2^{2(3x+1)}=2^{4x-2}$$
$$2^{6x+2}=2^{4x-2}$$
$$6x+2=4x-2$$
$$2x=-4$$
$$x=-2$$

Example

- Sketch the graph of the exponential function $f(x) = (1/2)^x$.



Logarithms

- The ***logarithm*** of *x* to the base *b* is defined by

$$y = \log_b x \quad \text{if and only if} \quad x = b^y \qquad (x > 0)$$

Ex.

$$\log_3 81 = 4; \qquad (3^4 = 81)$$

$$\log_7 1 = 0; \qquad (7^0 = 1)$$

$$\log_{1/3} 9 = -2; \qquad \left(\left(\frac{1}{3} \right)^{-2} = 9 \right)$$

$$\log_5 5 = 1; \qquad (5^1 = 5)$$

Examples

- a. Solve the equation

$$\log_2 x = 5$$

$$x = 2^5 = 32$$

- b. Solve the equation

$$\log_{27} 3 = x$$

$$3 = 27^x$$

$$3 = 3^{3x}$$

$$1 = 3x \qquad \left(a^m = a^n \Rightarrow m = n\right)$$

$$\frac{1}{3} = x$$

Laws of Logarithms

1. $\log_b mn = \log_b m + \log_b n$
2. $\log_b \left(\dfrac{m}{n}\right) = \log_b m - \log_b n$
3. $\log_b m^n = n \log_b m$
4. $\log_b 1 = 0$
5. $\log_b b = 1$

Notation:

Common Logarithm $\log x = \log_{10} x$

Natural Logarithm $\ln x = \log_e x$

Example

- Use the laws of logarithms to simplify the expression:

$$\log_5 \frac{25x^7 y}{\sqrt{z}}$$

$$= \log_5 25 + \log_5 x^7 + \log_5 y - \log_5 z^{1/2}$$

$$= 2 + 7\log_5 x + \log_5 y - \frac{1}{2}\log_5 z$$

Logarithmic Function

- The ***logarithmic function*** of x to the base b is defined by

$$f(x) = \log_b x \quad (b > 0, b \neq 1)$$

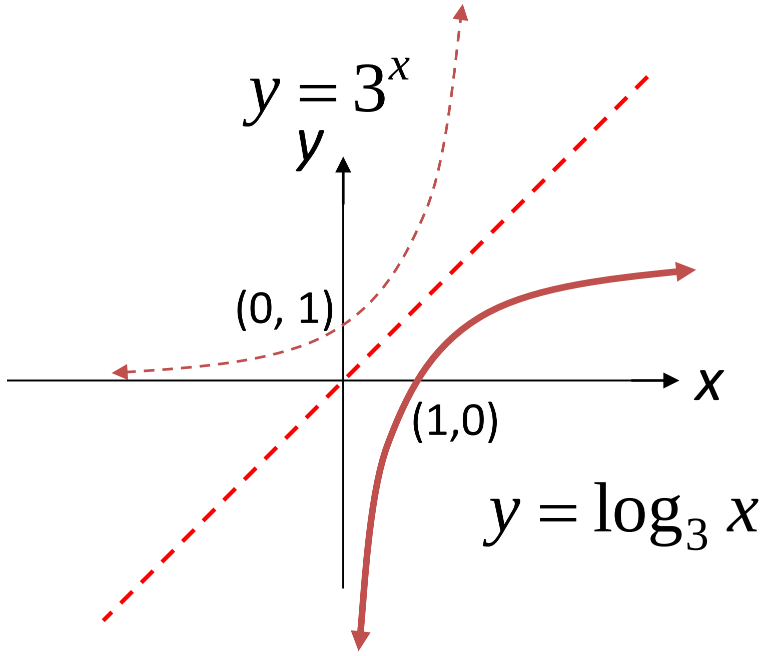
Properties:

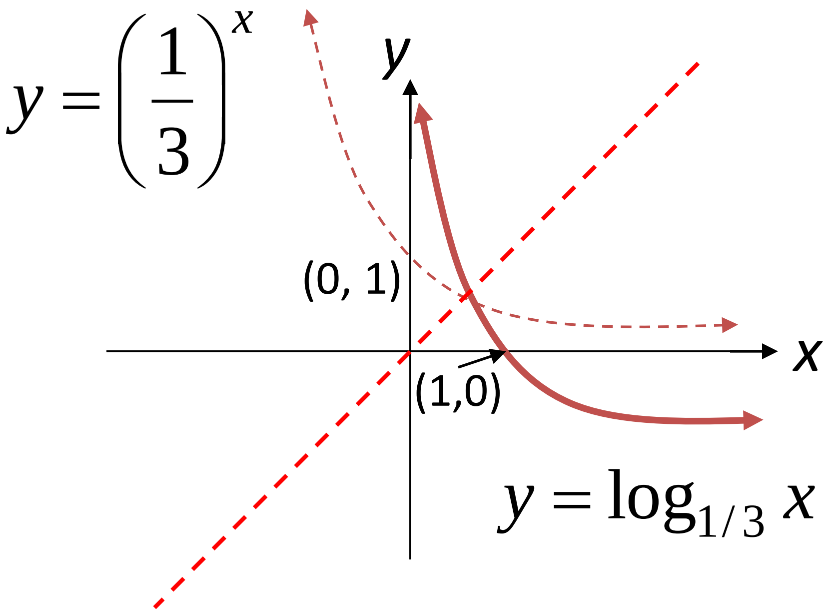
1. Domain: $(0, \infty)$
2. Range: $(-\infty, \infty)$
3. x-intercept: (1, 0)
4. Continuous on $(0, \infty)$
5. Increasing on $(0, \infty)$ if $b > 1$

   Decreasing on $(0, \infty)$ if $b < 1$

Graphs of Logarithmic Functions

Ex. $f(x) = \log_3 x$ $\qquad$ $f(x) = \log_{1/3} x$





The number "*e*" and the natural logarithm

- "***e***" is a number like π (pi).
  - $e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ or $e = \sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots$
  - It is a non repeating, never ending decimal.
  - An irrational number.
  - Sometimes called Euler's number
- e ≈ 2.71828…
- e is the base of the natural log (=ln)

The number "*e*"

- Question (from the study of compound interest/복리)
  - An account starts with \$1.00 and pays 100 percent interest per year. If the interest is credited once, at the end of the year, the value of the account at year-end will be \$2.00. What happens if the interest is computed and credited more frequently (e.g., 2, 4, or 12 … times) during the year?
    - (watch) https://www.khanacademy.org/math/algebra2/exponential-and-logarithmic-functions/e-and-the-natural-logarithm/v/e-through-compound-interest
- Ans:
  - If the interest is credited twice in the year, the interest rate for each 6 months will be 50%, so the initial \$1 is multiplied by 1.5 twice, yielding $\$1.00 \times 1.5^2 = \$2.25$ at the end of the year.
  - 4 times / year : $\$1.00 \times 1.25^4 = \$2.4414...$, and
  - Monthly (12 times) : $\$1.00 \times (1 + 1/12)^{12} = \$2.613035...$
  - If there are n compounding intervals, the interest for each interval will be $100\%/n$ and the value at the end of the year will be $\$1.00 \times (1 + 1/n)^n$.

Why we use "*e*" ?

- *(natural) Exponential function*
  - $f(x) = e^x$

- *The natural logarithm, or logarithm to base e*
  - *The inverse function to* e^x = $\log_e$ = *log* or *ln*

  - Because in the natural log (ln) the base is *e*. ALL PROPERTIES of logarithms apply to natural logarithms.

- Very very important in many fields of math (e.g., calculus/미적분)

$$\frac{d}{dx}e^x = e^x.$$

(We will study later)

Properties Relating Exponential and Logarithmic Functions

- Properties relating e^x and $\ln x$:

$$e^{\ln x} = x \qquad (x > 0)$$

$$\ln e^x = x \qquad (\text{for any real number } x)$$

Example

- Solve the equation $2e^{x+2} = 5$.

- sol:
  - Divide both sides of the equation by 2 to obtain:

$$e^{x+2} = \frac{5}{2} = 2.5$$

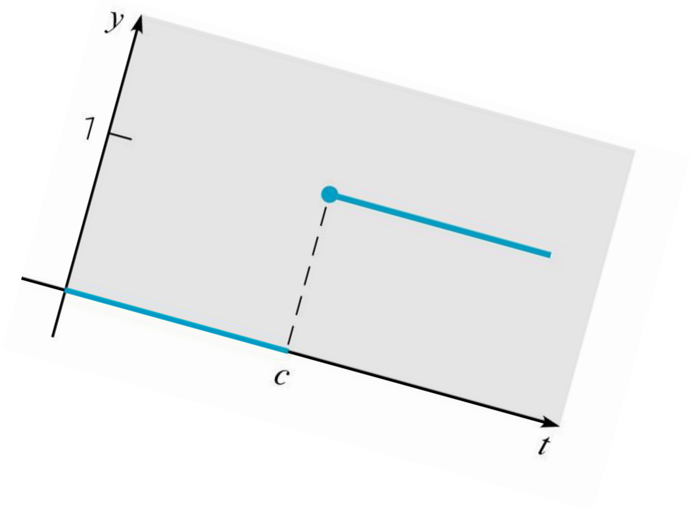
  - Take the natural logarithm of each side of the equation and solve:

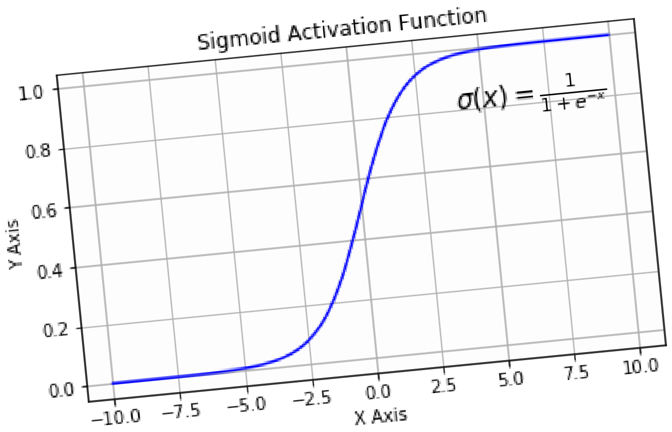
$$\ln e^{x+2} = \ln 2.5$$
$$(x+2)\ln e = \ln 2.5$$
$$x + 2 = \ln 2.5$$
$$x = -2 + \ln 2.5$$
$$x \approx -1.08$$

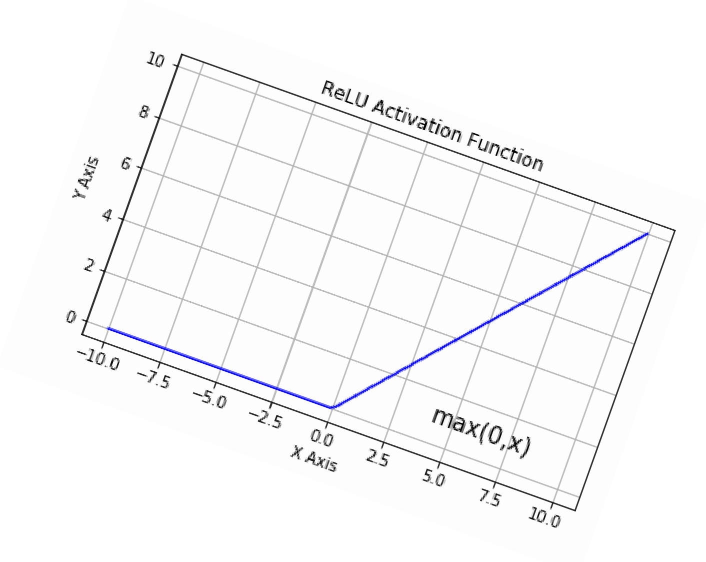
For more information

- Exponential & Logarithmic
  - https://www.khanacademy.org/math/algebra2/exponential-and-logarithmic-functions

Some Important Functions







Basic Component of Mathematical Model

- In mathematical modelling, a function is the key component.

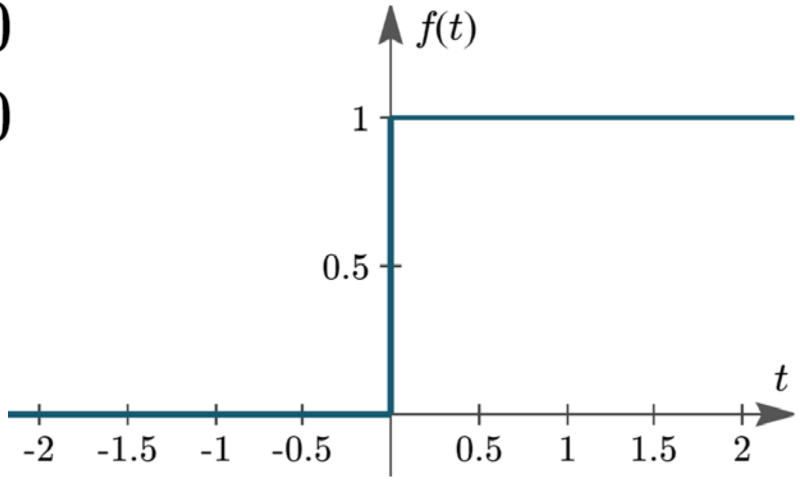
- A function is used as a mathematical description of the actual physical phenomenon, behavior, system or lots more.
  - For example,
    - The signal is the actual physical phenomenon that carries information, and the function is a mathematical description of the signal.

- There are many important functions that are used as basic components of mathematical modelling in lots of scientific and engineering fields.

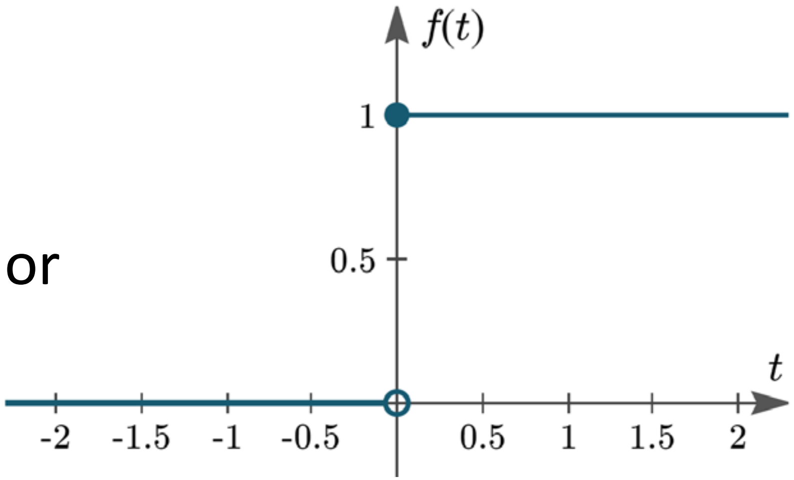
The Unit Step Function

- The **unit step function *u(t)***, (or Heaviside function), is defined by

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



or

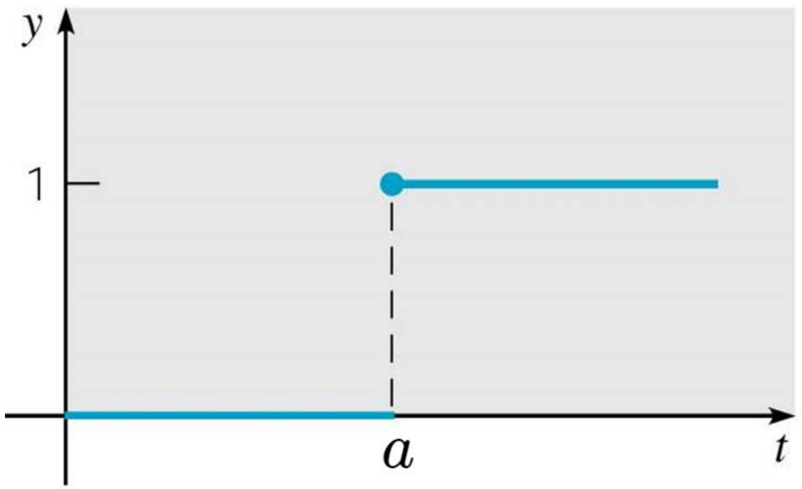


- For more:
  - https://www.intmath.com/laplace-transformation/1a-unit-step-functions-definition.php

Shifted Unit Step Function

- A function which has value 0 up to *t=a* and thereafter has value 1, is written:

$$u(t-a) = \begin{cases} 0 & \text{if} \quad t < a \\ 1 & \text{if} \quad t > a \end{cases}$$



- Example:
  – Sketch the graph of $f(t) = u(t-3)$

Example 1

- Sketch the graph of

$$h(t) = u(t - \pi) - u(t - 2\pi), \qquad t > 0$$

- Solution:

- For more exercise:
  - Visit: https://www.intmath.com/laplace-transformation/1a-unit-step-functions-definition.php

Understanding Step Function

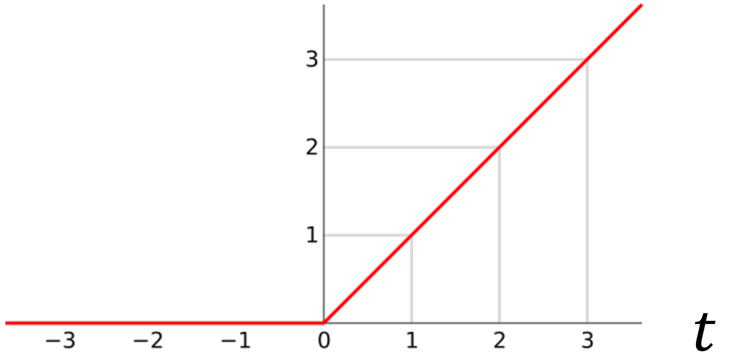
- The unit step function can mathematically describe the decision of whether or not to pass the signal.
  - a signal (or value) that is zero up to some point (i.e., a threshold* a) and non-zero after that.

*threshold - 임계값

The Unit Ramp Function

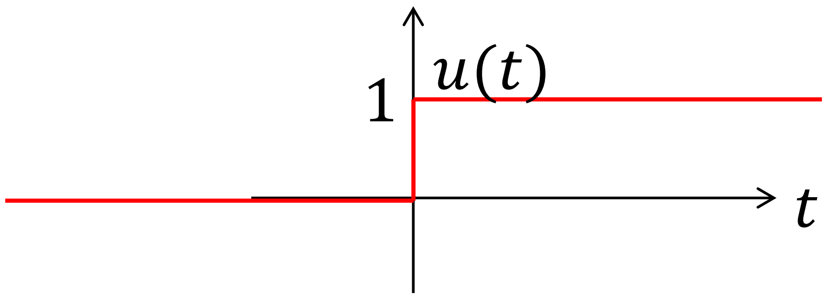
- The Ramp function is defined as follows:

$$ramp(t) = \begin{cases} t, t > 0 \\ 0, t \leq 0 \end{cases}$$



$$ramp(t) = tu(t)$$

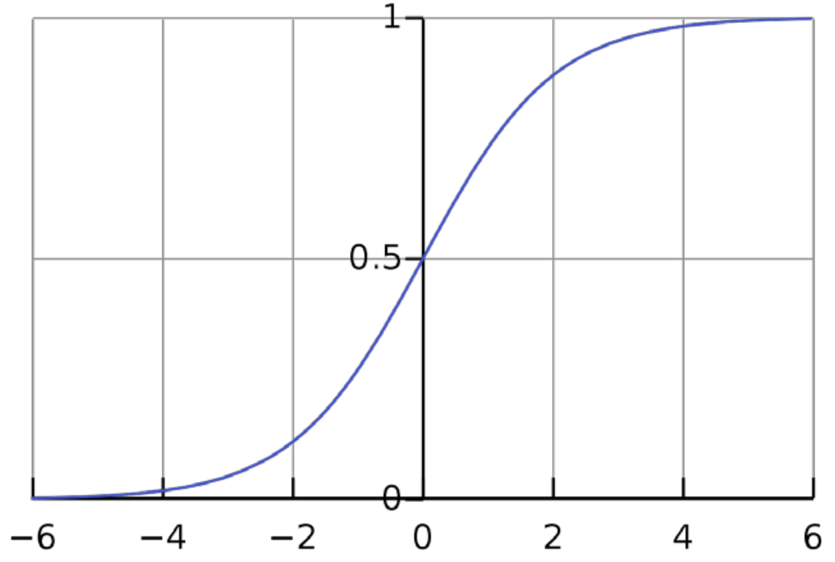
$$ramp(t) = \int_{-\infty}^{t} u(\tau)d\tau = \int_{0}^{t} 1d\tau = \tau|_0^t = t$$



- Some functions are used to map an input to something that is bounded (e.g., between 0 and 1)
  - Very important in Machine Learning
  - Visit & Read:
    - https://www.learnopencv.com/understanding-activation-functions-in-deep-learning/

Sigmoid (Logistic) Function

- Sigmoid Function
  - a mathematical function having a characteristic "S"-shaped curve or sigmoid curve
    - e.g., Logistic, tanh, Error function, so on
  - Often, sigmoid function refers to the special case of the **logistic function**



- Logistic Function

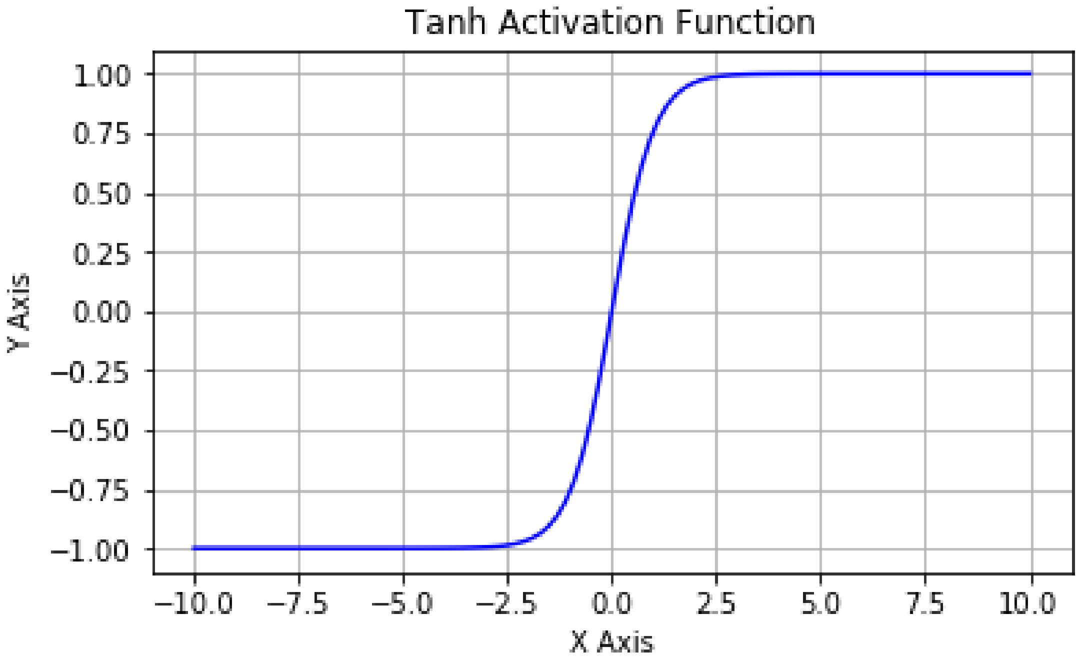
$$f(x) = \frac{1}{1 + e^{-x}}$$

  - converts large negative numbers to 0 and large positive numbers to 1.

Tanh (Hyperbolic tangent)

- Similar to sigmoid, tanh also takes a real-valued number but squashes it into a range between -1 and 1.

$$f(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

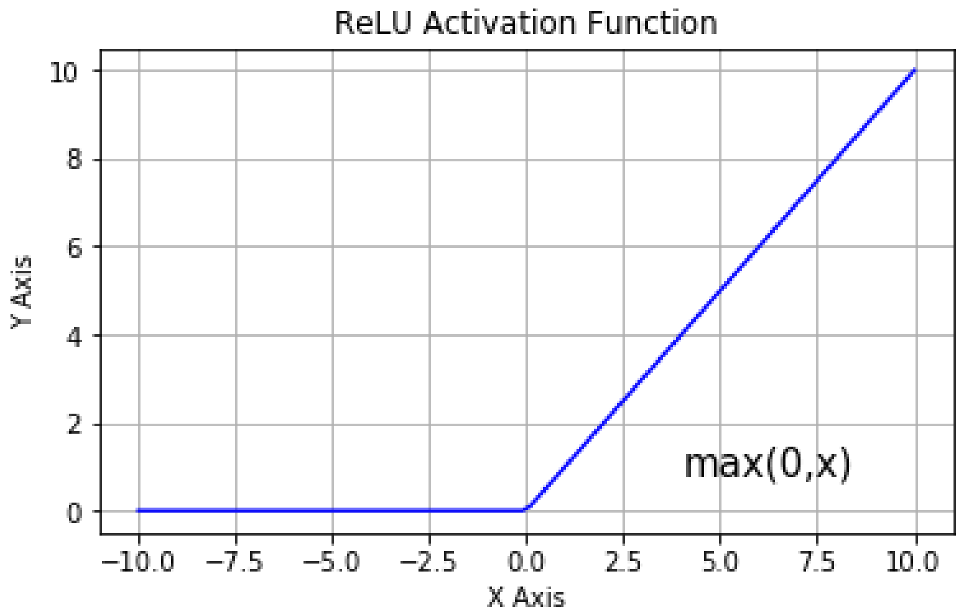


  - The negative inputs considered as strongly negative, zero input values mapped near zero, and the positive inputs regarded as positive.
  - In practice, tanh is preferable over sigmoid in the artificial neural networks.

Rectified Linear Unit (ReLU)

- In the context of artificial neural networks, the **rectifier** is an activation function defined as the positive part of its argument:

$$f(x) = x^{+} = \max(0, x)$$



- This is also known as a “ramp function”

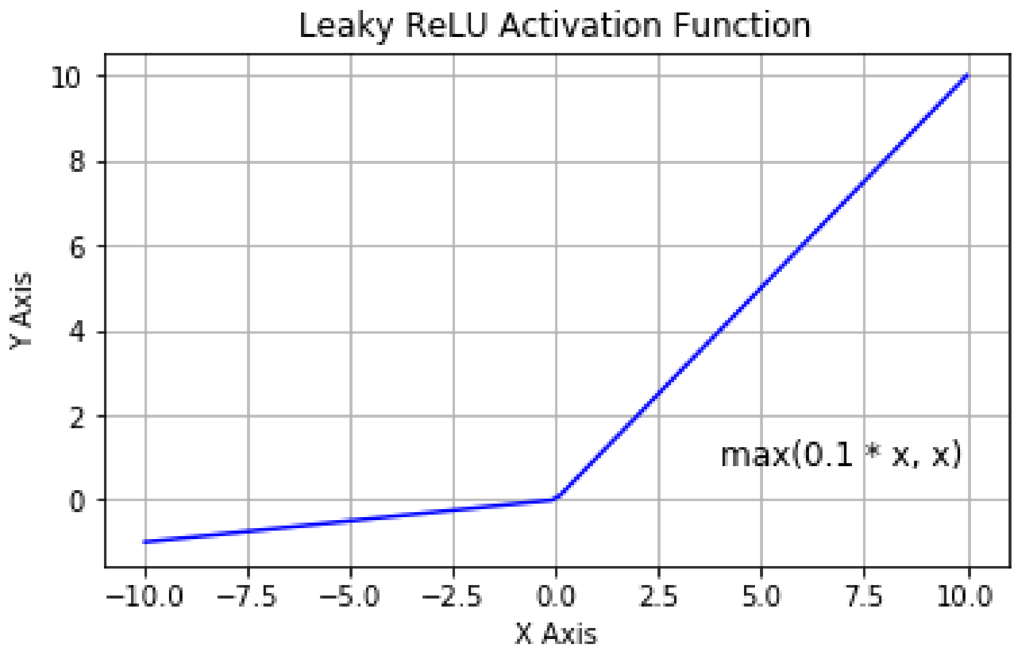
Leaky ReLU

- Leaky ReLUs
  - Variation of ReLU

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 0.01x & \text{otherwise} \end{cases}$$

$$f(x) = max(0.1x, x)$$

  - The concept of leaky ReLU is when x < 0, it will have a small positive slope of 0.1.



Section Summary

- Definition of a Function.
  - Domain, Codomain
  - Image, Pre-image
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Floor, Ceiling functions
- Exponential and Logarithmic Functions
- Some important functions