Algorithms

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1. Basics of Algorithm Design and Analysis (part II)

Contents

- Analyzing and Designing algorithms
  - Kinds of analyses
  - Machine-independent time
  - The running time of the algorithm
  - Asymptotic performance

Kinds of analyses

- Worst-case (usually)
  - $T(n)$ = maximum time of algorithm on any input of size n.
  - Analysis for the worst-case input(s)
- Average-case (sometimes)
  - $T(n)$ = expected time of algorithm over all inputs of size n.
    (Analysis for all inputs)
  - Need assumption of statistical distribution of inputs
  - More difficult to analyze
- Best-case (bogus)
  - Analysis for the best-case input(s)

Machine-independent time

- *What is insertion sort's worst-case time?*
  - It depends on the speed of our computer:
  - relative speed (on the same machine)
  - absolute speed (on different machines)
- BIG IDEA:
  - Ignore machine-dependent constants.
  - Look at ***growth*** of $T(n)$ as n .

  "Asymptotic Analysis"

Insertion sort

```
INSERTION-SORT (A, n)]
    for j ← 2 to n
        do key ← A[j]             ◁   ◁   ◁    n-1 times
           i ← j – 1              ◁   ◁   ◁    n-1 times
           while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
           A[i+1] = key
```

Θ-notation

Math:

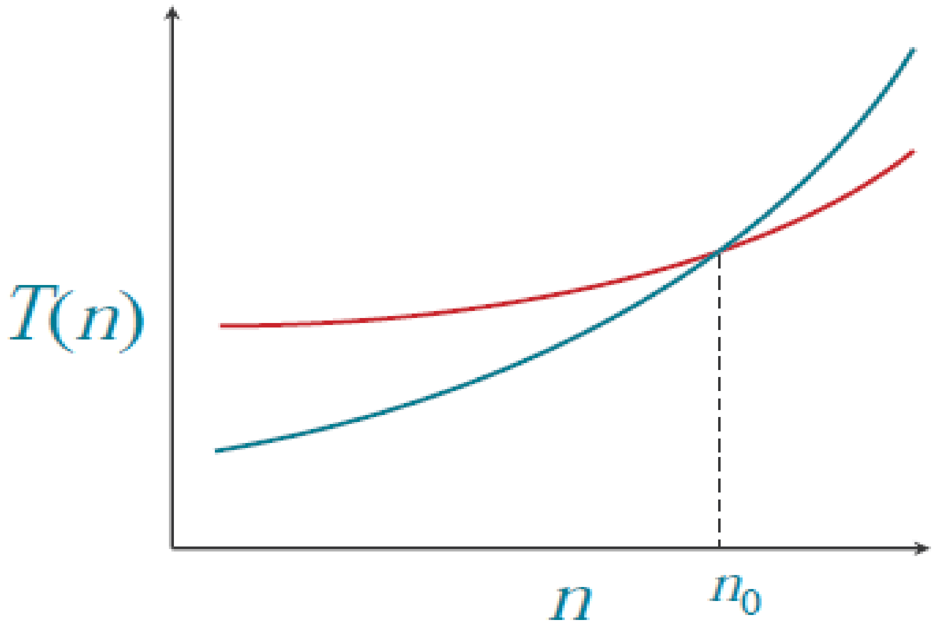
$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1\, g(n) \leq f(n) \leq c_2\, g(n)$ for all $n \geq n_0 \}$

Engineering:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.

The running time of the algorithm

$\sum$ *(cost of statement)•(number of times statement is executed).*

INSERTION-SORT (A, n)]

```
for j ← 2 to n
    do key ← A[j]           ◁   ◁   ◁   n-1 times
       i ← j – 1            ◁   ◁   ◁   n-1 times
       while i > 0 and A[i] > key
           do A[i+1] ← A[i]
              i ← i – 1
       A[i+1] = key
```

of Insertion sort.

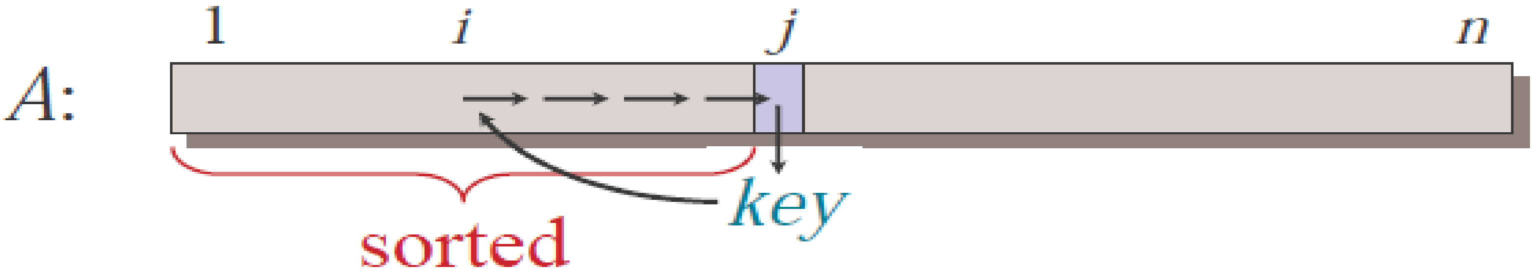
$- c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1)$

$+ c_7(n-1).$

Insertion sort

“pseudocode”

```
INSERTION-SORT (A, n)        ◁ A[1 . . n]
    for j ← 2 to n
          do key ← A[ j]
             i ← j – 1
             while i > 0 and A[i] > key
                  do A[i+1] ← A[i]
                     i ← i – 1
             A[i+1] = key
```



Insertion sort analysis

Worst case: Input reverse sorted.

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2) \qquad \text{[arithmetic series]}$$

Average case: All permutations equally likely.

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \Theta(n^2)$$

Is insertion sort a fast sorting algorithm?

- Moderately so, for small n.
- Not at all, for large n.

Designing algorithms

- There are many ways to design algorithms

  For example, insertion sort is incremental:

  having sorted A[1 .. j-1], place A[j] correctly,

  So that A[1 .. J] is sorted.

Another common approach

- Divide and Conquer
  - Divide: the problem into a number of subproblems that are smaller instances of the same problem.
  - Conquer: the subproblems by solving them recursively.
  - Combine: the subproblem solutions to give a solution to the original problem.

Merge sort

- MERGE-SORT $A[1\,.\,.\,n]$

  1. If $n = 1$, done.

  2. Recursively sort $A[\,1\,.\,.\,n/2\,]$

  and $A[\,n/2+1\,.\,.\,n\,]$.

  3. "***Merge***" the 2 sorted lists.

  Key subroutine: **MERGE**

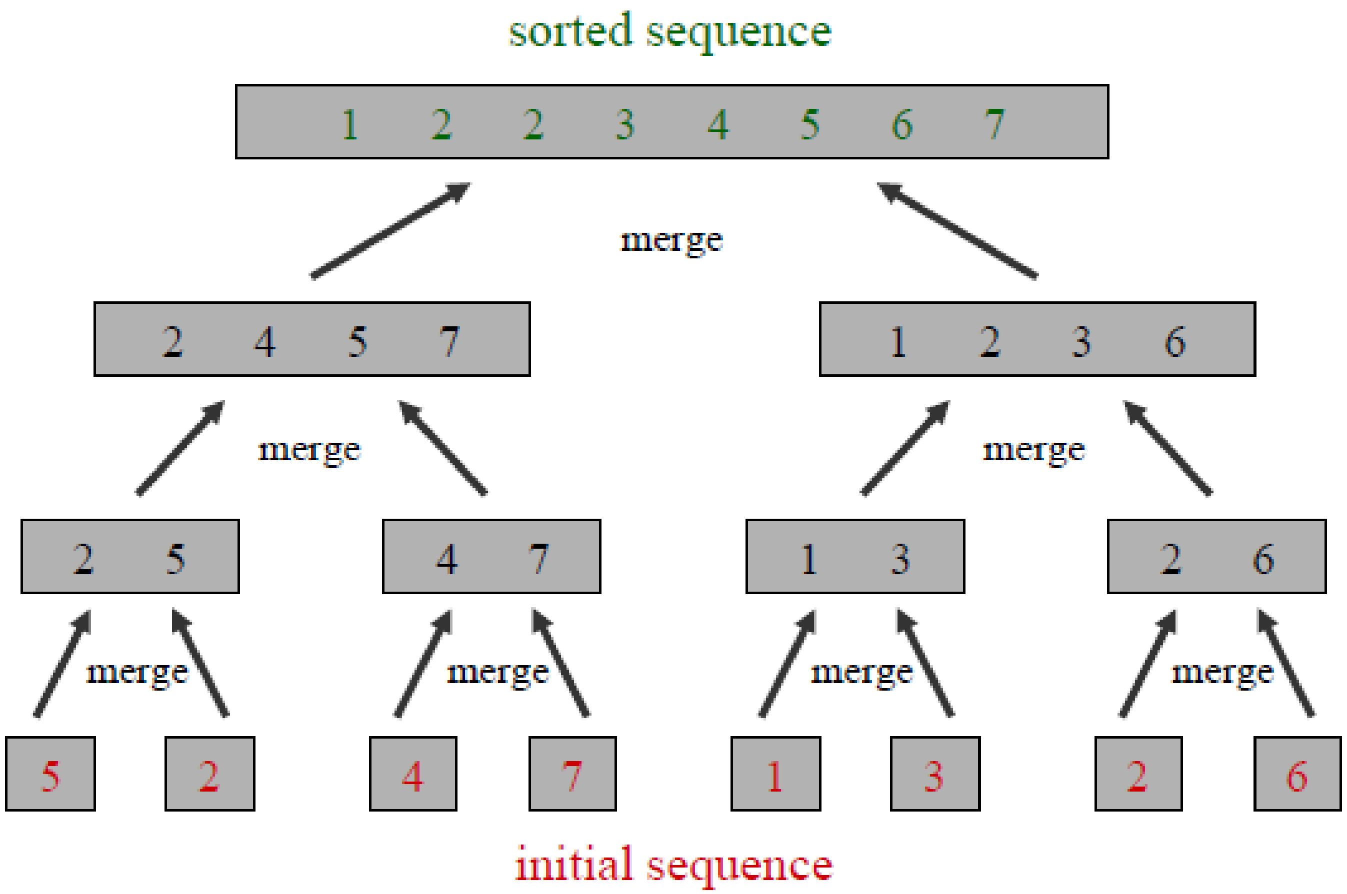
Merge sort

MERGE-SORT $A[1 \,.\,.\, n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
3. "*Merge*" the 2 sorted lists.

Key subroutine: MERGE

Operation of merge sort

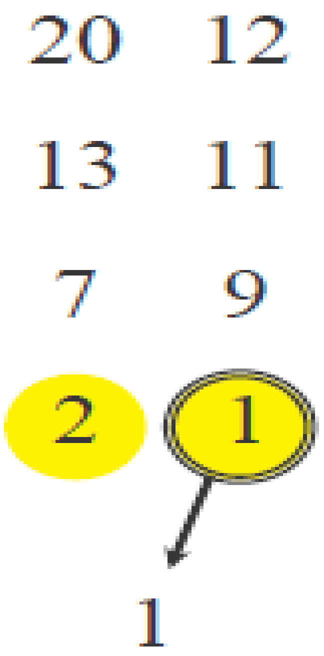


https://www.youtube.com/watch?v=JSceec-wEyw

Merging two sorted arrays

20	12
13	11
7	9
2	1

Merging two sorted arrays



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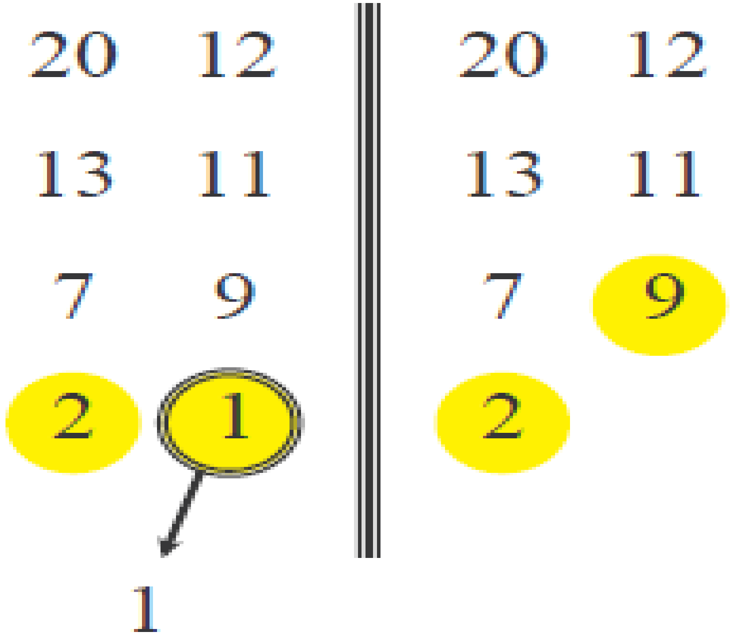

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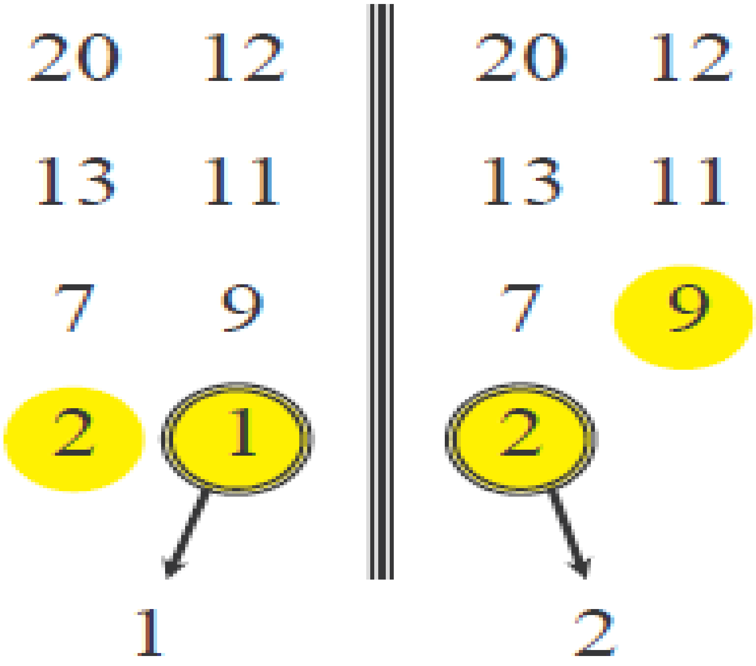

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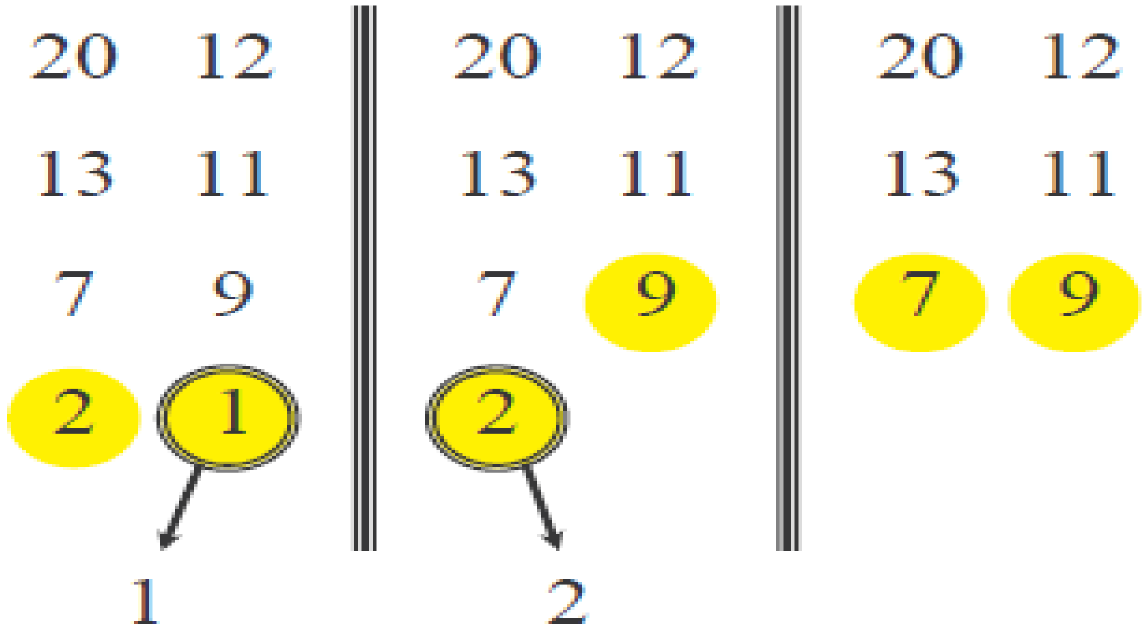

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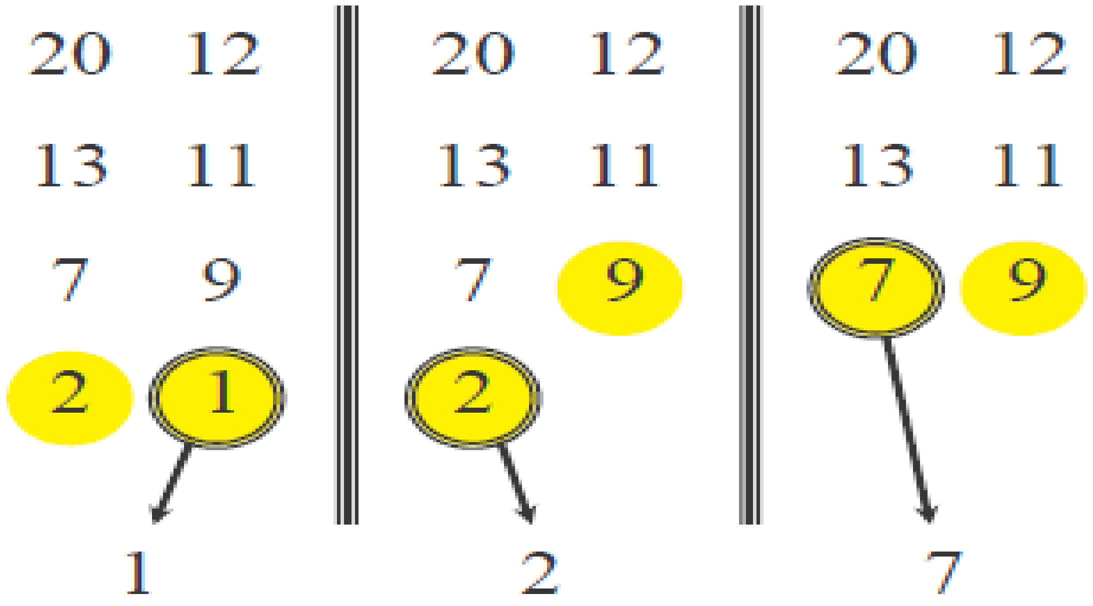

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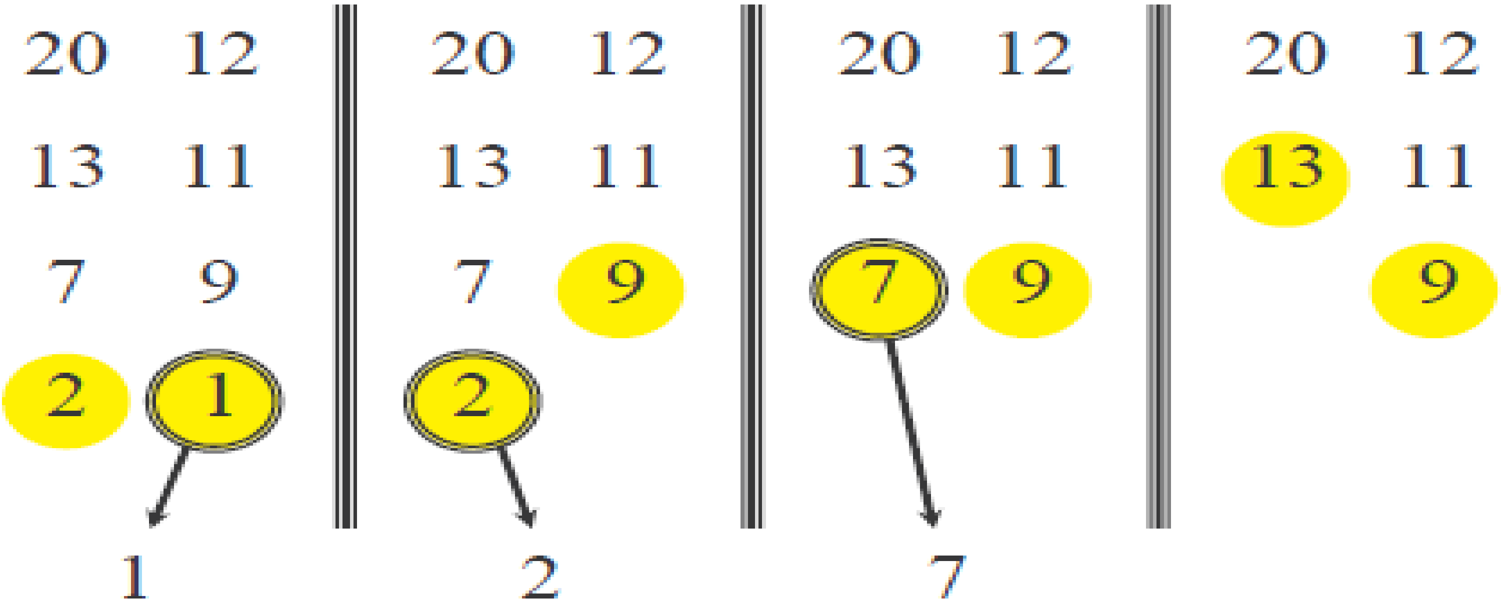

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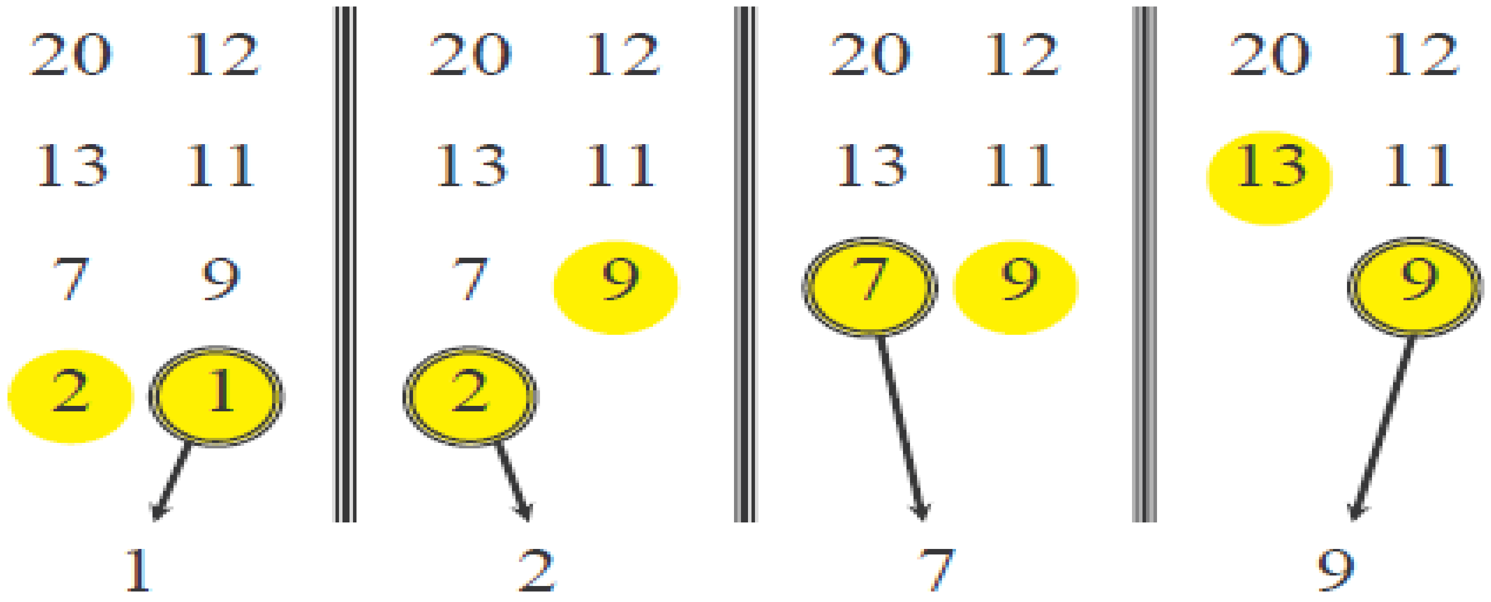

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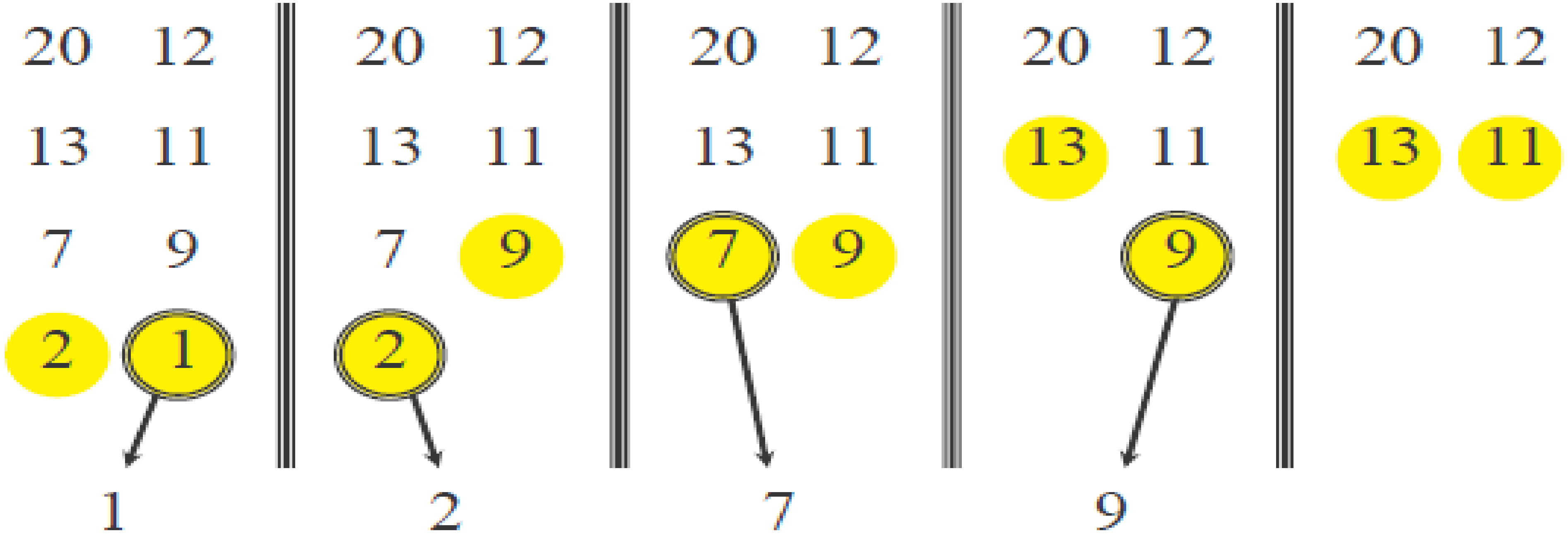

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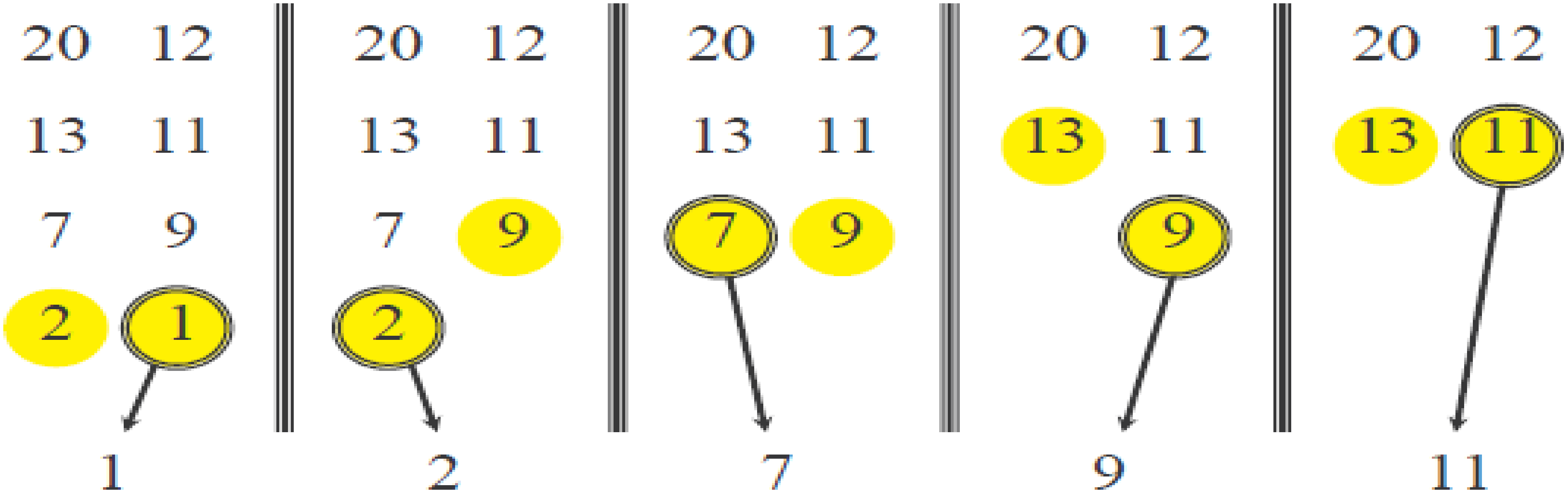

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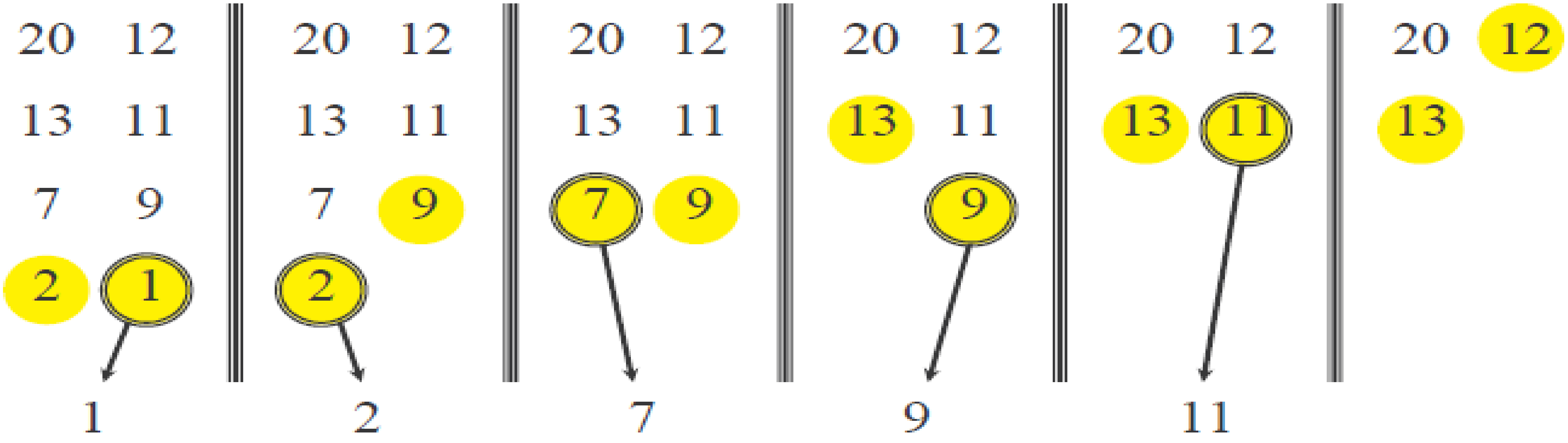

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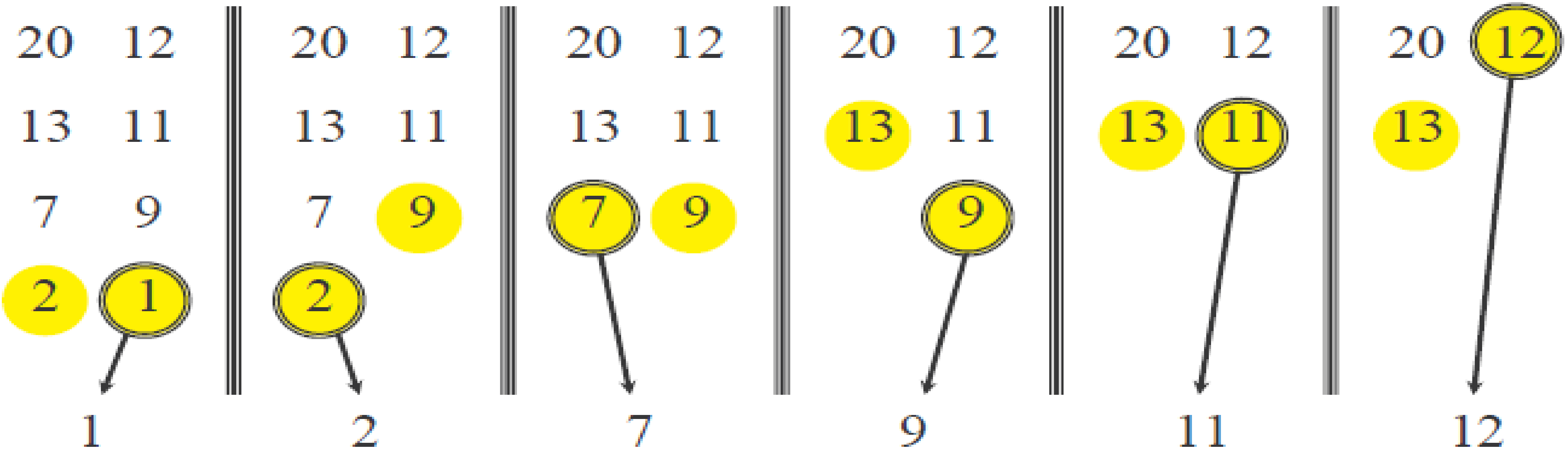


Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

$T(n)$	MERGE-SORT $A[1 \, . \, . \, n]$
$\Theta(1)$	1. If $n = 1$, done.
$2\,T(n/2)$	2. Recursively sort $A[\,1 \, . \, . \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \, . \, . \, n\,]$.
$\Theta(n)$	3. *"Merge"* the 2 sorted lists

Return

Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.

Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

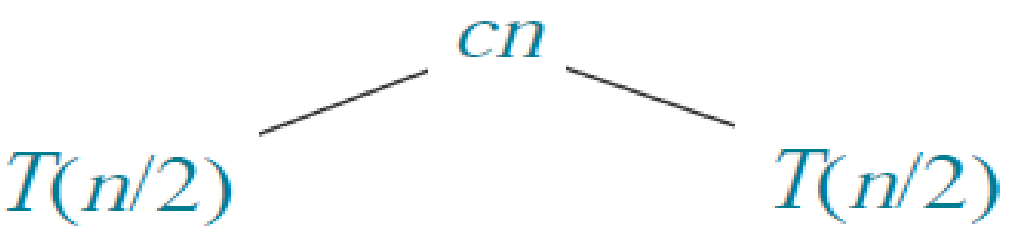
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$T(n)$$

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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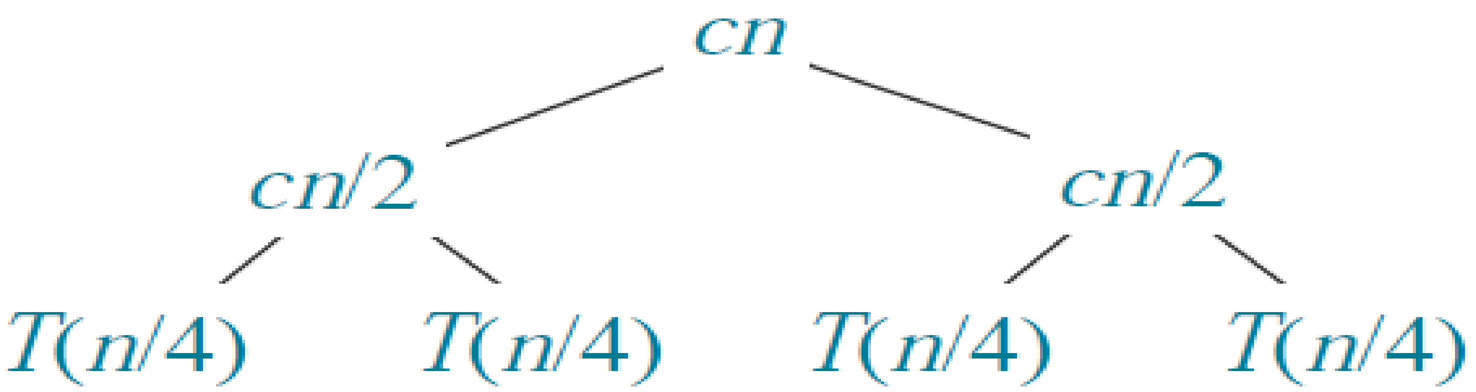

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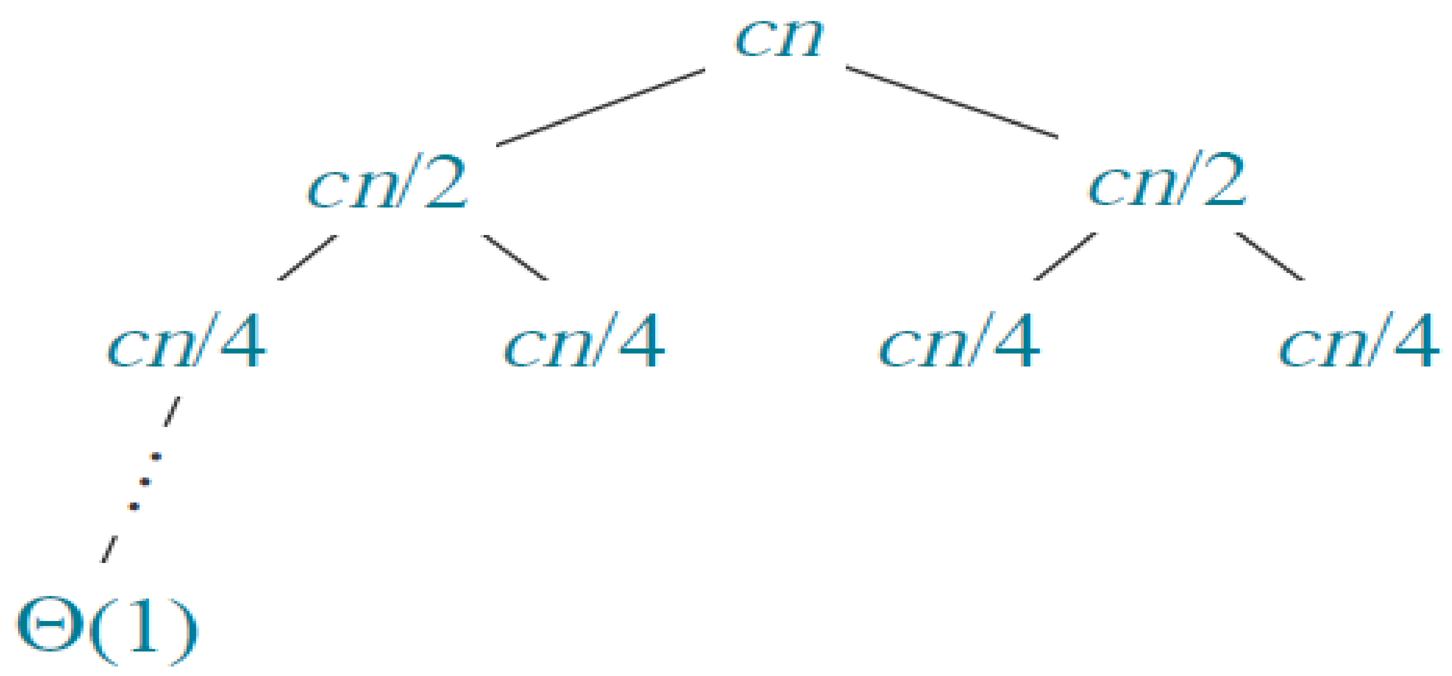

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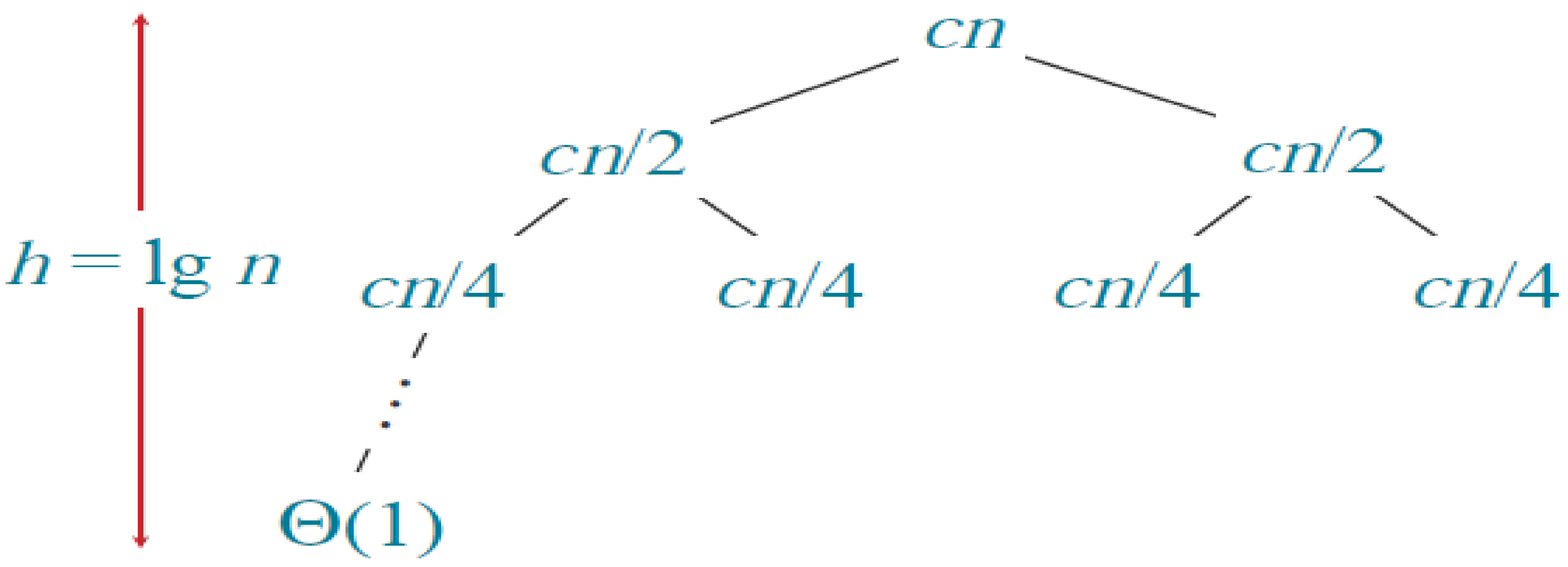

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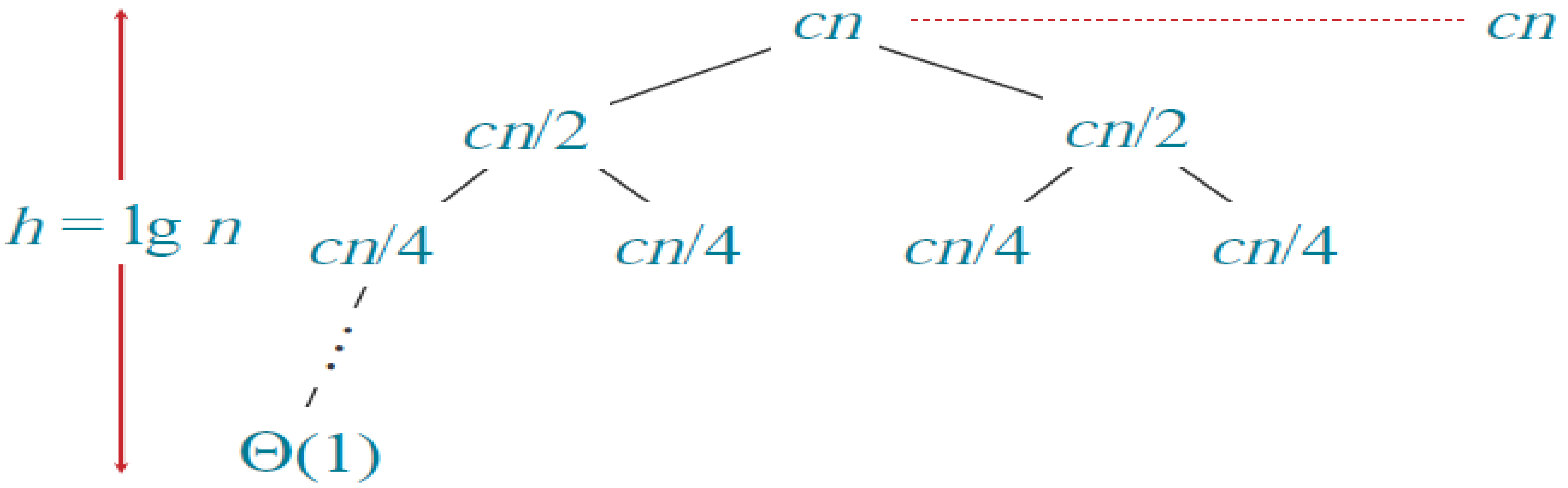

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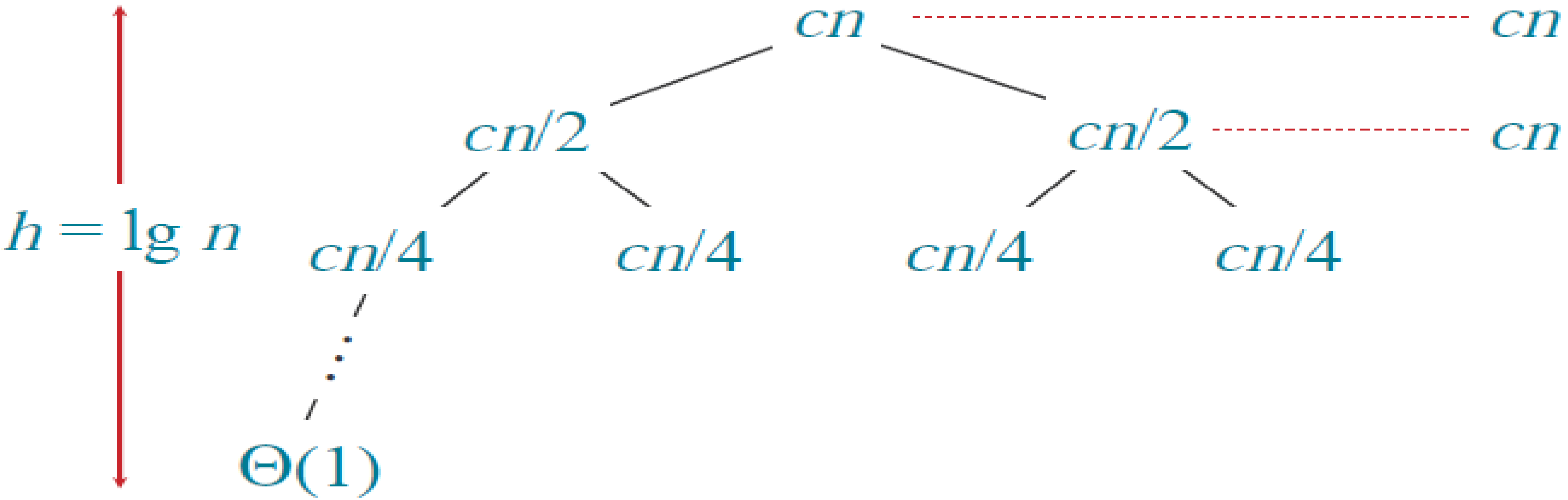

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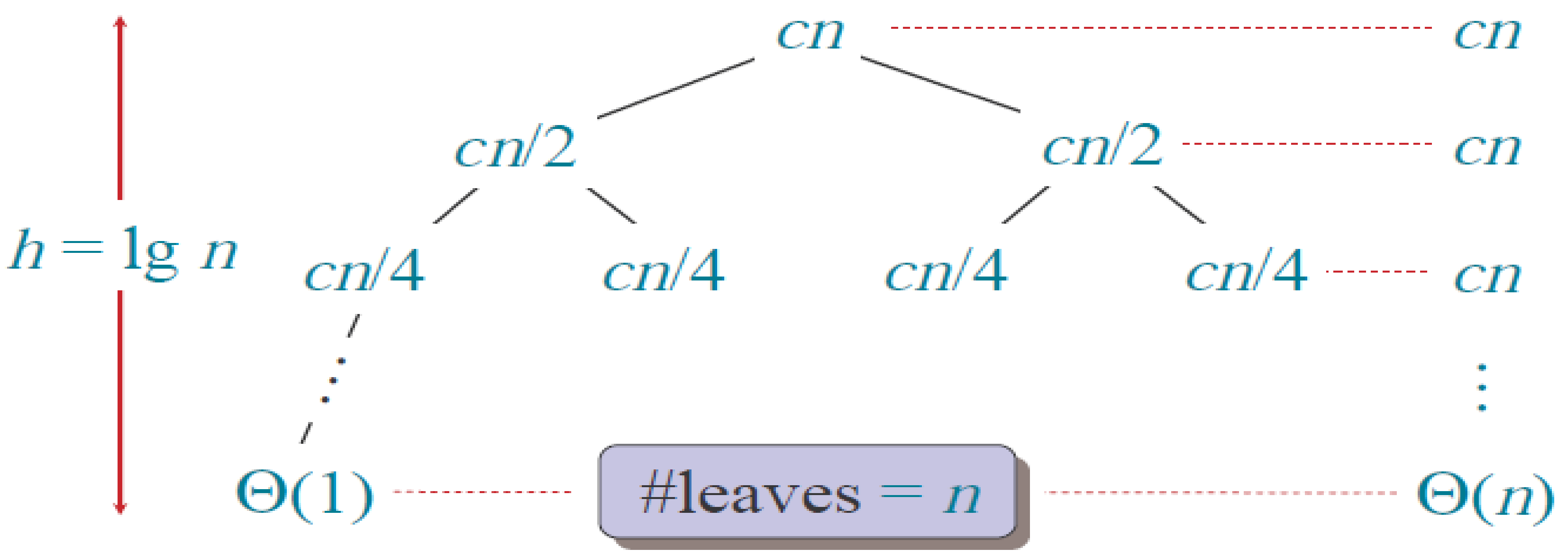

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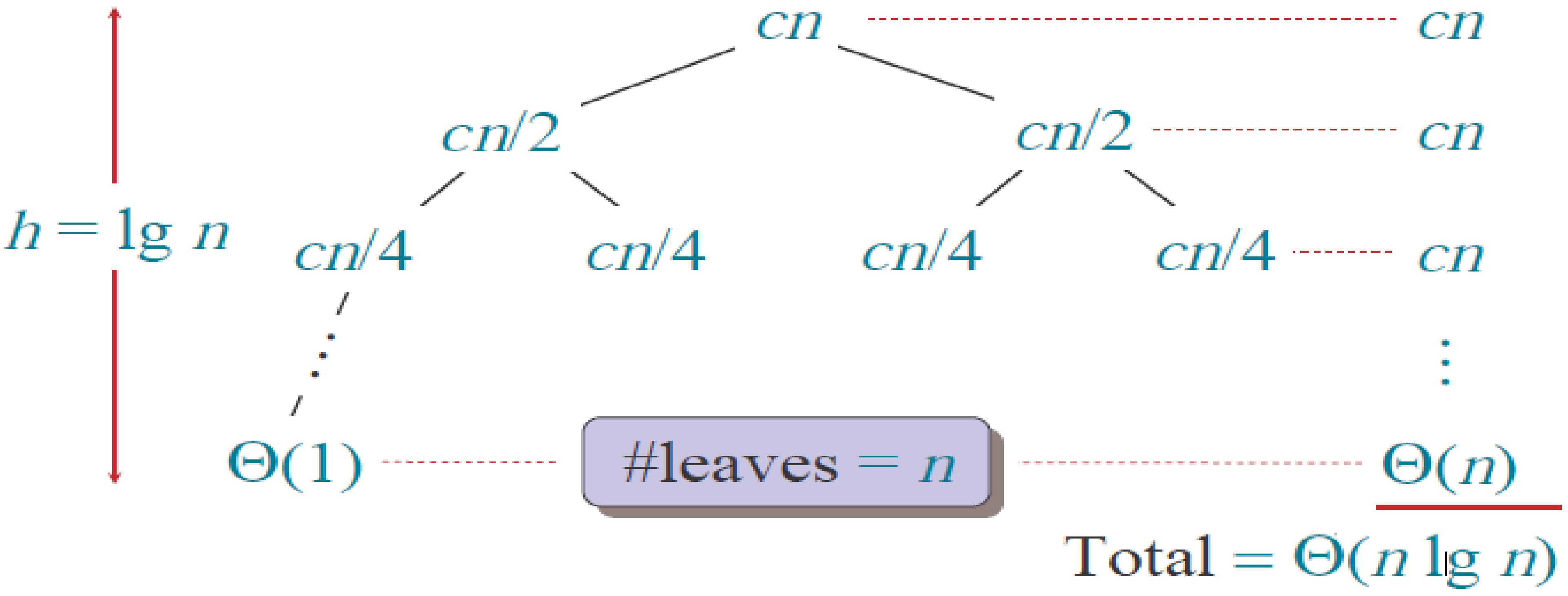

Conclusions

$\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.

Therefore, merge sort asymptotically beats insertion sort in the worst case.

In practice, merge sort beats insertion sort for $n > 30$ or so.

Implementation

Implementation of merge sort

MERGE-SORT $A[1\,..\,n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1\,..\,\lceil n/2\rceil\,]$ and $A[\,\lceil n/2\rceil+1\,..\,n\,]$.
3. "*Merge*" the 2 sorted lists.

Key subroutine: MERGE

Implementation of merge sort

MERGE-SORT $A[1 \,..\, n]$

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Key subroutine: MERGE

```
[1]  # Definition of merge sort
     def mergeSort(A, p:int, r:int):
       if p < r:
         q = (p+r) // 2
         mergeSort(A, p, q)
         mergeSort(A, q+1, r)
         merge(A, p, q, r)

     def merge(A, p:int, q:int, r:int):
       i = p; j = q+1; t =0
       tmp = [0 for i in range(len(A))]
       while i <= q and j <= r:
         if A[i] <= A[j]:
           tmp[t] = A[i]; t += 1; i += 1
         else:
           tmp[t] = A[j]; t += 1; j += 1
       while i <= q:
         tmp[t] = A[i]; t += 1; i += 1
       while j <= r:
         tmp[t] = A[j]; t += 1; j += 1
       i = p; t = 0
       while i <= r:
         A[i] = tmp[t]; t+=1; i+=1
```

```
[2]  # List
     input_list1 = [8, 2, 4, 9, 3, 6]
     print(input_list1)

     [8, 2, 4, 9, 3, 6]
```

```
[3]  # Sorting
     mergeSort(input_list1, 0, len(input_list1)-1)
     print(input_list1)

     [2, 3, 4, 6, 8, 9]
```

```
[4]  # random list
     import random
     input_list2 = random.sample(range(100),10)
     print(input_list2)

     [42, 72, 31, 30, 96, 35, 49, 3, 20, 12]
```

```
[5]  # Sorting
     mergeSort(input_list2, 0, len(input_list2) -1 )
     print(input_list2)

     [3, 12, 20, 30, 31, 35, 42, 49, 72, 96]
```

Example code test

- Code test: https://www.acmicpc.net/problem/2751
- Solving the problem using merge sort
- Example result of submission

2751	aikiho	모든 언어	모든 결과	검색

제출 번호	아이디	문제	결과	메모리	시간	언어	코드 길이	제출한 시간
48335930	aikiho	**2751**	**맞았습니다!!**	139560 KB	1444 ms	PyPy3 / 수정	806 B	2분 전

THANK YOU