Linear Algebra. 송빈호

Mid 40%	Final 40%	HW 10%	Att 10%
~Sec 4.3	4.4 ~ 7.4	Webwork	전자출결 ($\frac{1}{4} \Rightarrow F$)

2주차부터 공지 (i campus)

- A $\leq 35\%$, B $\leq 70\%$

엄밀한 디테일. ⟵——————————⟶ 직관적. 큰 그림.

- 보강 = 녹화 강의 업로드.
- Office hour = { 시간: 화 15:00 – 17:00
  장소: 51304

- 공부, 왜 재미없나? 안 궁금하니까!
- 불편하지만 감내해야 하는 것 = 용어에 익숙해지기.

소통을 위한 "새로운 언어"   "복습의 중요성"

- 수업, 위기의 순간 = ① 용어를 몰라서 이해가 안될 때.

방심은 금물. ← ② 이미 아는 게 나올 때.

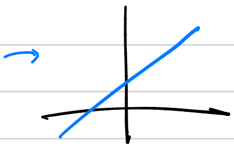
§0. Preview

- 선형 대수? 행렬?

Linear Algebra   Matrix

$$y = ax + b$$

의미? →



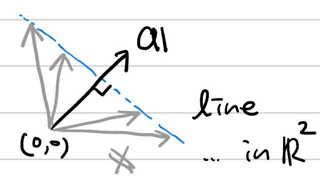
⇝ $a_1x_1 + a_2x_2 = b$ ⇝ line!

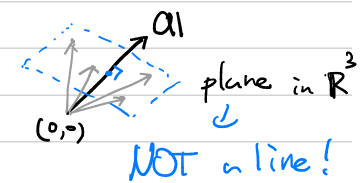
⇝ $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. ⇝ line?

$\begin{cases} a_1, \dots, a_n \Rightarrow \text{계수, coefficients} \\ x_1, \dots, x_n \Rightarrow \text{변수, variables.} \end{cases}$

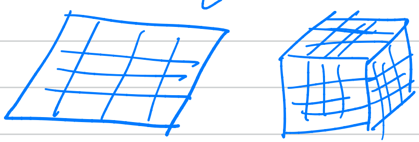
$\boxed{\text{Jump}}$ $\mathbf{a} \cdot \mathbf{x} = b$, where

$\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{x} = (x_1, \dots, x_n)$.





But...



"Frame" is a set of lines!

≈ Spaces are not "curved".

linear (equation, function, ...) ≈ every monomial is of degree 1.

차수

linearity ≈ the property that it is written in a linear "form".

- monomial = 항.

  c.f. polynomial = sum of 항's.

- degree of a monomial

$$\deg x_1^{a_1} \cdots x_n^{a_n} = a_1 + \cdots + a_n .$$

power of x_1.

ex) $x_1 + 2x_2 = 1$ ⇝ linear

$x_1^2 + 2x_2 = 1$ ⇝ non-linear

$x_1 + x_1 x_2 = 1$ ⇝ non-linear.

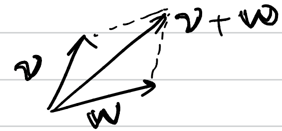
- linear equation.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b ,$$

where $a_i, b \in \mathbb{R}$ or $\mathbb{C}$.

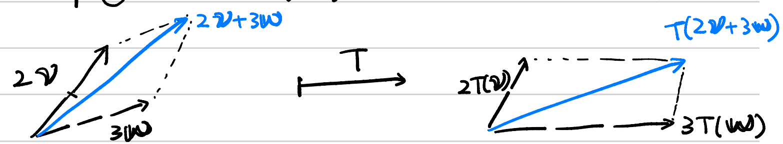
the set of real numbers complex

$\boxed{\text{Jump}}$ v, w : "vector"



T: vector를 vector로 보내는 함수

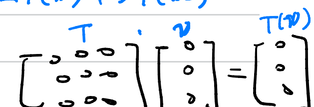
$$T(av + bw) = aT(v) + bT(w)$$



$= 2T(v) + 3T(w)$

∴ T is a linear map.

⟺ T = "matrix"



☆ $T_1 \circ T_2$ ⇝ (행렬 T_1) × (행렬 T_2) Algebra!

§1.1. System of linear equations.

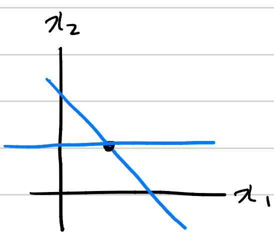
- Linear system (System of linear equations)

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \quad\vdots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{cases}$$

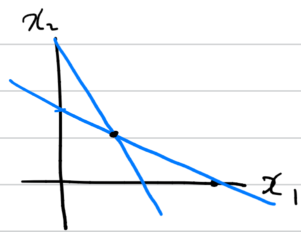
- 해(집합) : solution (set)

- Two linear systems are equivalent if they have the same solution set.

ex) $\begin{cases} x_1 + x_2 = 2 \\ 2x_1 \qquad = 2 \end{cases} \overset{\text{equiv.}}{\Longleftrightarrow} \begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 + x_2 = 3 \end{cases}$

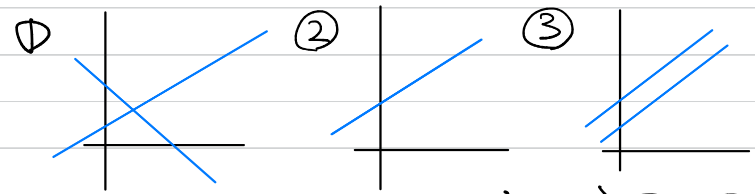


&



- Possible solution sets?

(책)



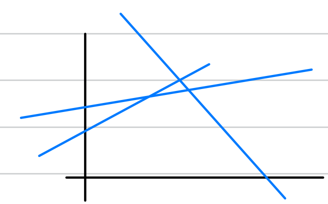
consistent　　inconsistent

~~Q. Inconsistent ⇒ parallel?~~

질문을 명확하게!

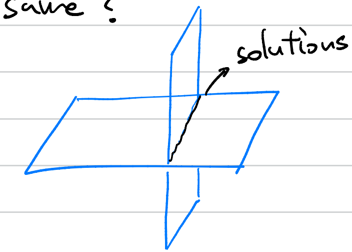
↳ Q. If a linear system is inconsistent, then can we say that linear equations in the system are parallel to each other?

A. No!



Q2. If a linear system of two equations has infinitely many solutions, then can we say that the two eq. are the same?

A2. No!



Q3. What is the necessary condition for a linear system to have exactly one solution?

A3. # variables = # equations.

└ the number of

↳ Q4. Sufficient & necessary condition?

"필요충분 조건"

• Matrix 행렬. "숫자의 나열"

ex) row

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix},$$

column

2x3 matrix

"two by three"

$$[1, 2, 3], \quad [3].$$

r x c

• $$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Coefficient matrix   augmented matrix

- **Elementary Row Operations**
  "연립 방정식을 푸는 과정에서의 각 스텝"

Let R_i = the i^{th} row of the augmented matrix.

① (Replacement) $R_i \Leftarrow R_i + aR_j$

② (Interchange) $R_i \Leftrightarrow R_j$

③ (Scaling) $R_i \Leftarrow aR_i$ $\quad a \in \mathbb{R}$ (or $\mathbb{C}$)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ 2x_2 - 8x_3 = 8 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 0 & 2 & -8 & 8 \end{bmatrix}$$

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ +)\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline -3x_2 + 13x_3 = -9 \end{array} \qquad \begin{array}{r} 4 \cdot [1 \ \ -2 \ \ 1 \ \ 0] \\ +)\ [-4 \ \ 5 \ \ 9 \ \ -9] \\ \hline [0 \ \ -3 \ \ 13 \ \ -9] \end{array}$$

$$\Downarrow$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ -3x_2 + 13x_3 = -9 \\ 2x_2 - 8x_3 = 8 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 2 & -8 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 2 & -8 & 8 \end{bmatrix} \xrightarrow{R_2 \Leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\xrightarrow{R_2 \Leftarrow \frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix} \xrightarrow{R_3 \Leftarrow 3 \cdot R_2 + R_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{\cdots} \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned}$$

- ERO : operations that produce another equiv. linear systems.

  = A map that sends a matrix to another matrix of the same size.
  $m \times n$

- Two matrices A and B are row equivalent if

$$A \xrightarrow{ERO} \rightarrow \rightarrow B$$

- ERO : "reversible". ☆

$$A \xrightarrow{ERO} \rightarrow \rightarrow B \Rightarrow A \leftarrow \leftarrow \leftarrow B$$

each operation is reversible.

- 왜 이런 쉬운걸 하나. → 방심하지말 것.

preview

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ 2x_2 - 8x_3 = 8 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ -4 & 5 & 9 \\ 0 & 2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 8 \end{bmatrix}$$

$$\Rightarrow A \cdot X = b \quad \text{ERO}$$

$$\Rightarrow \cancel{A^{-1} \cdot A} \cdot X = A^{-1} b \quad \Updownarrow$$

$$\Rightarrow A^{-1} = \text{product of elementary matrix}$$

알고 싶은 것 : matrix multiplication.
"Inverse" of a matrix.

하고 있는 것 : 알고 싶은 것들의 기원, 배경.

- Possible solution sets?
(Augmented matrix version)

Let A = the augmented matrix of a given linear system.

⇓ ERO

$$\left[\begin{array}{ccc|c} 1 & * \cdots & * & b_1 \\ 0 & 1 \cdots & * & b_2 \\ & \vdots & & \vdots \\ 0 & \cdots 0 & 1 & b_n \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & * & * & * & b_1 \\ 0 & 1 & * & * & b_2 \\ & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & * & * & * & b_1 \\ 0 & 1 & * & * & b_2 \\ & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & b_n \end{array}\right] \; b_n \neq 0$$

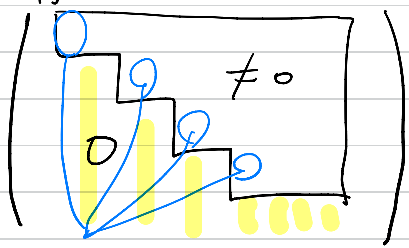
$$\underbrace{\text{① unique sol.} \quad \text{② infitely many sol.}}_{\text{consistent}} \quad \underbrace{\text{③ no sol.}}_{\text{inconsistent.}}$$

§1.2. Row reduction & echelon forms

respectively. 각각.

• nonzero row (resp. column) $\neq [0, \dots, 0]$ (resp. $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$)

• A matrix M is in (row) echelon form (REF) if

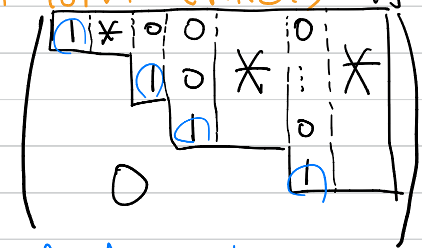
M =  .

leading entry = the leftmost nonzero entry

i.e. l_i = the leading entry of
that is, the i^{th} row.

즉, ① l_i 밑으로 전부 0.
② $i<j \Rightarrow l_i$는 l_j 보다 오른쪽 column.
③ $[0, \dots, 0]$는 맨 아래.

• A matrix M is in reduced (row) echelon form (RREF) if

M = 

○ : leading entry = 1.

↳ called a reduced echelon matrix.

• [Thm] M $\xrightarrow{\text{ERO}}$ REF, REF, REF but
theorem

M $\xrightarrow{\text{ERO}}$ RREF (unique)

※ 어떤 M이든 REF, RREF로 변환 가능!
(i.e. M is row equiv. to an echelon matrix or a reduced —)

- Solving a linear system
  = row reduce the given augmented matrix to obtain an REF / the RREF.
  (do ERO

- Algorithm for solving a linear system?
  = Algorithm for producing the RREF from the given augmented matrix.

Step 1. Find the leftmost nonzero column,

pivot position
= the topmost position in the pivot column

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 0 & 2 & -8 & 8 \end{bmatrix}$$

pivot
= leading entry

say a ↑ pivot column.

Pick the topmost nonzero entry in the chosen pivot column.
If it is not in the pivot position, then interchange rows to move this entry to the pivot position.

Step 2. Use ERO to transform the pivot column into the form $\begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 0 & 2 & -8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 2 & -8 & 8 \end{bmatrix}$$

Step 3. Ignore all rows above the one containing the pivot position.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 2 & -8 & 8 \end{bmatrix}$$

: ignore

Repeat the process with the remaining submatrix until there are no more nonzero rows to modify.
⇒ You'll get a REF!

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 2 & -8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & \frac{2}{3} & 2 \end{bmatrix}$$

$\boxed{\text{Ex}}$ [RREF]

Step 4. For all pivots, make them all 1's by scaling.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & \frac{2}{3} & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{13}{3} & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 5. Make all entries above pivots zero, starting with the right most pivot to the left.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{13}{3} & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- Step 1 ~ 3 : forward phase.
  Scan columns from left to right $\longrightarrow$
- Step 4 ~ 5 : backward phase $\longleftarrow$

- For a given matrix, the leading entries are always in the same positions in any (reduced) echelon form. $\boxed{\text{Why?}}$