part2:

$$\min_{\begin{bmatrix}\Delta u\\ \Delta T\end{bmatrix}} a\|u+\Delta u\|^2 + b\|T+\Delta T\|^2 + \lambda \left\|\begin{matrix}\Delta u\\ \Delta T\end{matrix}\right\|^2$$

$$\text{s.t} \quad x_N + H\begin{bmatrix}\Delta u\\ \Delta T\end{bmatrix} = x_f$$

$$T + \Delta T \geq 0$$

where $T = \begin{bmatrix} t_0 \\ \vdots \\ t_{N-1}\end{bmatrix}$, $\Delta T = \begin{bmatrix} dt_0 \\ \vdots \\ dt_{N-1}\end{bmatrix}$, t_i: time duration (time step)

H is computed as follows:

$$x_{k+1} = \phi(x_k, u_k, t_k) \Rightarrow dx_{k+1} = \underbrace{\frac{\partial \phi}{\partial x}\Big|_{(x_k,u_k,t_k)}}\, dx_k + \underbrace{\frac{\partial \phi}{\partial u}\Big|_{x_k,u_k,t_k}}\, du_k + \underbrace{\frac{\partial \phi}{\partial t}\Big|_{(x_k,u_k,t_k)}}\, dt_k$$

$$= A_k\, dx_k + B_k\, du_k + C_k\, dt_k$$

Rolling out the procedure: $dx_1 = A_0 dx_0 + B_0 du_0 + C_0 dt_0$

$$\vdots$$

$$dx_N = \underbrace{A_{N-1} \cdots A_0\, dx_0} + [H_u]\begin{bmatrix} du_0 \\ \vdots \\ du_{N-1}\end{bmatrix} + [H_t]\begin{bmatrix} dt_0 \\ \vdots \\ dt_{N-1}\end{bmatrix}$$

$$\Rightarrow dx_N = 0 + \underbrace{\begin{bmatrix} H_u \,\vdots\, H_t\end{bmatrix}}_{:=H}\begin{bmatrix}\Delta u\\ \Delta T\end{bmatrix}$$