Homework 2

Math 436

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3.17

a)

$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$

$$\begin{aligned} &= -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{bmatrix} 1 & 1 \\ 1 & -\lambda \end{bmatrix} + 1 \begin{vmatrix} 1 & -\lambda \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda^2(\lambda^2 - 1) - 1(-\lambda - 1) + (1 + \lambda) \\ &= -\lambda^3 + \lambda + 2\lambda + 2 \\ &= -\lambda^3 + 3\lambda + 2 \\ &\to \lambda^3 - 3\lambda - 2 = 0 \end{aligned}$$

By using Zero Remainder Theorem, we can deduce that -1 is a root so, doing some long division (not in this sheet)

$$\lambda^3 - 3\lambda - 2 = (\lambda + 1)^2(\lambda - 2) = 0$$

Which we can see since $\lambda = -1, 2$, this is **indefinite**.

b)

Computing the eigenvectors, for $\lambda_1 = -1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \to v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$$

And for $\lambda_2 = 2$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\to x_2 = x_3; 2x_1 = x_2 + x_3 \to v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We can see that for $v_2 \notin M$. In fact v_1 spans M so we get that v_2 is orthogonal to M. We can disregard the eigenvalue λ_2, so we get that Q is negative definite.e

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3.22

Frist we note that

$$||A||_\infty = \max_{||x||_\infty=1} ||Ax||_\infty$$

$$= \max_{||x||=1} ||\sum_{j=1}^{n} A_j x_j||_\infty \quad \text{(where } A_j \text{is the } j\text{th column of } A)$$

$$= \max_{||x||_\infty=1} \left(\max_{1\le i\le n} \left| \sum_{j=1}^{n} A_j x_j \right| \right)$$

As a kind of illustration, we can see

$$||Ax||_\infty = \max_{1\le j\le n} \left[\begin{pmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ ... \\ a_{m2} \end{pmatrix} x_2 + ... + \begin{pmatrix} a_{1n} \\ a_{2n} \\ ... \\ a_{mn} \end{pmatrix} x_n \right]$$

If $||x||_\infty = 1$, we can maximize each x_j to be ± 1 such that the maximum row sum can be made. Thus we can make

$$\max_{||x||_\infty=1} \left(\max_{1\le i\le n} \left| \sum_{j=1}^{n} A_j x_j \right| \right) = \max_{1\le i\le n} \sum_{j=1}^{n} |A_{ij}|$$

5.3

a)

If we write $f(x) = (a^T x)(b^T x) = (a_1 x_1 + ... + a_n x_n)(b_1 x_1 + ... + b_n x_n)$, we can calculate each partial derivative.

$$
\begin{aligned}
\frac{\partial f}{\partial x_1} = a_1(b_1x_1 + ... + b_nx_n) + b_1(a_1x_1 + ... + a_nx_n) \\
= a_1(b^Tx) + b_1(a^Tx) \\
... \\
\frac{\partial f}{\partial x_k} = a_k(b^Tx) + b_k(a^Tx)
\end{aligned}
$$

And that gives us the gradient:

$$
\nabla f(x) = \begin{bmatrix} a_1(b^Tx) + b_1(a^Tx) \\ . \\ . \\ . \\ a_k(b^Tx) + b_k(a^Tx) \\ . \\ . \\ . \\ a_n(b^Tx) + b_n(a^Tx) \end{bmatrix}
$$

b)

If we take each kth term: $\frac{\partial f}{\partial x_k} = a_k(b_1x_1 + ... + b_nx_n) + b_k(a_1x_1 + ... + a_nx_n)$, we can treat k as the "row" and l as the "column" for the Jacobian:

$$
\begin{aligned}
\frac{\partial f}{\partial x_k} = a_k(b_1x_1 + ... + b_nx_n) + b_k(a_1x_1 + ... + a_nx_n) \\
\frac{\partial^2 f}{\partial x_l \partial x_k} = a_kb_l + b_ka_l \\
\rightarrow D^2 f = \begin{bmatrix} 2a_1b_1 & a_1b_1 + b_1a_1 & ... & a_1b_n + b_1a_n \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_nb_1 + b_na_1 & a_nb_2 + b_na_2 & ... & 2a_nb_n \end{bmatrix}
\end{aligned}
$$

5.6

Let $f(x(t)) = (e^t + t^3)^3t^2(t+1)^2 + (e^t + t^3)t^2 + t + 1$.

$$
\begin{aligned}
\frac{d}{dt}f(x(t)) = 3(e^t + t^3)^2t^2(t+1)^2(e^t + 3t^2) + (e^t + t^3)^3(2t)(t+1)^2 \\
+(e^t + t^3)^3t^2(2(t+1)) + (e^t + 3t^2)(t^2) + (e^t + t^3)(2t) + 1
\end{aligned}
$$

With a bit of simplification:

$$= (e^t + t^3)^3[4t^3 + 6t^2 + 2t] + t^2(e^t + 3t^2)[3(e^t + t^3)^2 t^2 (t+1)^2 + 1] + (e^t + t^3)2t + 1$$

5.10

a)

$$\begin{aligned} f(x_0) &= 1 + 1 = 2 \\ f'(x_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} &= [e^{-x_2} \quad - x_1 e^{-x_2} + 1] \\ &= [1 \quad 0] \\ f''(x_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} &= \begin{bmatrix} 0 & -e^{-x_2} \\ -e^{-x_2} & x_1 e^{-x_2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

With all of these calculated, we can get the taylor's expansion:

$$\begin{aligned} f(x) = 2 + [1 \quad 0] \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} &+ \frac{1}{2} [x_1 - 1 \quad x_2] \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} \\ &= 2 + x_1 - 1 + \frac{1}{2} [-x_2 \quad -x_1 + 1 + x_2] \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} \\ &= 1 + x_1 + \frac{1}{2}(-x_1 x_2 + x_2 - x_1 x_2 + x_2 + x_2^2) \\ &= 1 + x_1 + \frac{1}{2} x_2^2 + x_2 - x_1 x_2 \end{aligned}$$

b)

$$\begin{aligned} f(x_0) &= 1 + 2 + 1 = 4 \\ f'(x_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} &= [3x_1^3 + 4x_1 x_2^2 \quad 4x_1^2 x_2 + 3x_2^3] \\ &= [7 \quad 7] \\ f''(x_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} &= \begin{bmatrix} 9x_1^2 + 4x_2^2 & 8x_1 x_2 \\ 8x_1 x_2 & 4x_1^2 + 9x_2^2 \end{bmatrix} \\ &= \begin{bmatrix} 13 & 8 \\ 8 & 13 \end{bmatrix} \end{aligned}$$

With all of these calculated, we can get the taylor's expansion:

$$
\begin{aligned}
f(x) &= 4 + \begin{bmatrix} 7 & 7 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 - 1 & x_2 - 1 \end{bmatrix} \begin{bmatrix} 13 & 8 \\ 8 & 13 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \\
&= 4 + 7x_1 - 7 + 7x_2 - 7 + \frac{1}{2} \begin{bmatrix} 13x_1 - 13 + 8x_2 - 8 & 8x_1 - 8 + 13x_2 - 13 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \\
&= -10 + 7x_1 + 7x_2 + \frac{1}{2}((13x_1 + 8x_2 - 21)(x_1 - 1) + (8x_1 + 13x_2 - 21)(x_2 - 1)) \\
&= -10 + 7x_1 + 7x_2 + \frac{1}{2}(13x_1^2 + 13x_2^2 + 16x_1x_2 - 42x_1 - 42x_2 + 42) \\
&= 11 - 14x_1 - 14x_2 + 8x_1x_2 + \frac{13}{2}x_1^2 + \frac{13}{2}x_2^2
\end{aligned}
$$

c)

$$
\begin{aligned}
f(x_0) &= e + 1 + e + 1 = 2e + 2 \\
f'(x_0) &= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} e^{x_1 - x_2} + e^{x_1 + x_2} + 1 & -e^{x_1 - x_2} + e^{x_1 + x_2} + 1 \end{bmatrix} \\
&= \begin{bmatrix} e + e + 1 & -e + e + 1 \end{bmatrix} = \begin{bmatrix} 2e + 1 & 1 \end{bmatrix} \\
f''(x_0) &= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} e^{x_1 - x_2} + e^{x_1 + x_2} & -e^{x_1 - x_2} + e^{x_1 + x_2} \\ -e^{x_1 - x_2} + e^{x_1 + x_2} & e^{x_1 - x_2} + e^{x_1 + x_2} \end{bmatrix} \\
&= \begin{bmatrix} e + e & -e + e \\ -e + e & e + e \end{bmatrix} = \begin{bmatrix} 2e & 0 \\ 0 & 2e \end{bmatrix}
\end{aligned}
$$

With all of these calculated, we can get the taylor's expansion:

$$
\begin{aligned}
f(x) &= 2e + 2 + \begin{bmatrix} 2e + 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 - 1 & x_2 \end{bmatrix} \begin{bmatrix} 2e & 0 \\ 0 & 2e \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} \\
&= 2e + 2 + (2e + 1)x_2 - 2e - 1 + x_2 + \frac{1}{2} \begin{bmatrix} 2e(x_1 - 1) & 2ex_2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} \\
&= 1 + (2e + 1)x_1 + x_2 + e((x_1 - 1)^2 + x_2^2) \\
&= 1 + (2e + 1)x_1 + x_2 + ex_1^2 - 2ex_1 + e + ex_2^2 \\
&= 1 + e + x_1 + x_2 + e(x_1^2 + x_2^e)
\end{aligned}
$$
