Chapter 3 notes

Math 432

Elementary Counting Problems

Let S_n be all permutations of 1, 2, ..., n. (Symmetric group). (So if $n = 3$,

$$S_3 = \{123, 132, 213, 231, 312, 321\}$$

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Clearly $|S_n| = n(n-1)...(3)(2)(1)$. We know this as $n!$ (Take $0! = 1$).

Sterling's Formula

$n!$ is asympototic to $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

How many different sequences have three 1s, four 2s, and two 3s?

Any such sequence has length 9. Define a map:

$$f : S_n \mapsto \text{these sequences:}$$

|4, 1, 8, 2, 5, 3, 7, 6, 9|

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|2, 1, 3, 1, 2, 1, 2, 2, 3|

We:

- wrote down the 1s in the positions of 1, 2, 3
- Then the 2s in the positions of 4, 5, 6, 7
- Then the 3s in the positions of 8, 9

Note: the map f is onto

and the number of permutations mapping to a given seqeuence is

$$3! \times 4! \times 2!$$

The total number of sequences is

$$\frac{|S_9|}{3! \times 4! \times 2!} = \frac{9!}{3! \times 4! \times 2!}$$

More generally

Let $n, k, a_1, ..., a_k$ be non-negative integers such that

$$n = a_1 + a_2 + ... + a_k$$

Consider the length n sequences consisting of a_1 1s, a_2 2s, a_3 3s, ... etc. The total number of sequences is

$$\frac{n!}{a_1! \times a_2! \times a_3! ... \times a_k!}$$

Proof same as previous example

Ex

The number of distinct arrangements of the letters of the word “mississippi” is

$$\frac{11!}{1!2!4!4!}$$

(1 ‘m’, 2 ’p’s, 4 ’s’s, 4 ’i’s)

Let’s count permutations of $1, 2, ...n$ by “cycle structure”

Consider the permutation

$\Pi = 3, 4, 9, 2, 5, 6, 8, 7, 1$

1	2	3	4	5	6	7	8	9
3	4	9	2	5	6	8	7	1

can thing of these as “cycles”

- $1 \to 3 \to 9 \to 1...$
- $2 \to 4 \to 2...$
- $5 \to 5...$
- $6 \to 6...$
- $8 \to 7 \to 8...$

can write this as

$$(5)(6)(2,4)(7,8)(1,3,9)$$

let $n_i(\Pi) =$ the number of cycles of Π of length i

So $n_1(\Pi) = 2$, $n_2(\Pi) = 2$, $n_3(\Pi) = 3$.

Theorem) Let $n_1, n_2, n_3, ...$ etc be integers such that $\sum_i in_i = n$. Then the number of permutations in S_n with n_1 1-cycles, n_2 2-cycles, ... etc is equal to $\frac{n!}{\Pi i^{ni}(ni!)}$.

Proof Define a map $f : S_n \mapsto$ permutations with n_1 1-cycles, n_2 2-cycles, ... etc. Let's define this "by example".

Suppose $n = 14$, $n_1 = 2$, $n_2 = 1$, $n_3 = 2$, $n_4 = 1$.

$$4, 1, 14, 12, 9, 10, 8, 7, 13, 11, 2, 5, 6, 3$$

$\rightarrow$ ... (4)(1) - 1-cycles (14, 12) - 2-cycle (9,10, 8), (7, 13, 11) (2, 5, 6, 3)

This map is onto. And each permutation with n_1 1-cycles, n_2 2-cycles, ... etc is mapped to $\Pi_i i^{ni}(ni!)$ many times. You can swap any of the 3-cycles around (change (9, 10, 8) $\rightarrow$ (10, 8, 9)) and it's the same groupings. More generally, that's the term of i^{ni}. Then you can switch out two n-cycle permutations (can switch: $(4)(1) \rightarrow (1)(4)$) which is the $ni!$ term. We do this for every i (the Π_i) the number of permutations with n_1 1-cycles, n_2 2-cycles, ... etc, is $\frac{n!}{\Pi_i i^{ni}(ni!)}$.

Later when we talk about generating functions, we will give applications of this formula

Some examples

Example) The number of 6 digit strings on an alphabet a, b, ..., c

$$= 26 \times 26 \times ...26 = 26^6$$

Example) The number of subsets of an n-element set is 2^n. For each of the n elements, you have 2 choices: be in the subset, or not.

Example) Let n, k be positive integers with $n \geq k$. then the number of length k strings of an n-element alphabet where all symbols in the the string are different is

$$n \times (n-1) \times ... \times (n-k+1) = \frac{n!}{(n-k)!}$$

Proof Have n choices for first symbol. Then $n - 1$ choices for next symbol, etc.

Binomial Coefficients

Let $\binom{n}{k}$ = the number of k element subsets of $1, 2, ..., n$ where order does not matter.

For example, $\binom{4}{2} = 6$ since there are 6 2-element subsets of 1,2,3,4.

$$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$$

Theorem

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Proof) To pick a k element subset of 1,2,...,n, first select a k element string consisting of k distinct elements of $\{1, 2, ..., n\}$. This can be done in $\frac{n!}{(n-k)!}$ ways. Define a map $f:$ these sequences $\mapsto$ size k subsets of $\{1, 2, ..., n\}$ forgetting the order of the elements in the sequence. Each size k subset gets hit $k!$ times, so the total number of k size subsets

$$\frac{\frac{n!}{(n-k)!}}{k!} = \binom{n}{k}$$