Chapter 2 - Induction

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Weak Induction

Want to prove a statement for all rational numbers n (starting at $n = 0$ or $n = 1$).

2 steps:

1. **Base Case** prove statement for smallest value of n where it's defined ($n = 0$ or $n = 1$).
2. **Induction Step** for each n, must show that if the statement is true for n, it's true for $n + 1$.

This proves the result for all n.

ex

Suppose you have infinite dominoes.

Base case: the first dominoes gets knocked over

Induction step: since we know that if nth domino gets knocked over, the $n + 1$th domino gets knocked over

The entire series of dominoes gets knocked over.

Proof

Suppose we completed both steps, but that statement is not true for all values. Let $m + 1$ be the smallest value where statement fails. Then since completed base case, the statement works for m. But the induction step implies that the statement works for $m + 1$, which is a contradiction.

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ex

Proof that for all positive integers m,

$$1 + 2 + 3 + ... + m = \frac{m(m + 1)}{2}$$

Step 1

Check for $m = 1$. $1 = 1\frac{2}{2}$. It works.

Step 2

Assume formula for m. Do it for $m+1$.

$$\begin{aligned} &1+2+3+...+m+(m+1) \\ &= \frac{m(m+1)}{2}+m+1 \\ &= \frac{(m+1)(m+2)}{2} \end{aligned}$$

ex

Let $a_0 = 0$ and let $a_{n+1} = 3a_n + 1$ for all $n \geq 0$. Derive a formula for a_m.

Let's look at sequence

$$a_0, a_1, a_2, ...$$

It begins with 1, 4, 13, 40, 121, ...

Look this up in Sloane's online encyclopedia of integer sequences. Conjecture: $a_m = \frac{3^m-1}{2}$.

Step 1

Check conjecture with $m = 0$.

$$a_0 = \frac{3^0-1}{2} = 0$$

Step 2

Assume the formula for m.

$$\begin{aligned} a_{m+1} &= 3a_m + 1 \\ &= 3\left[\frac{3^m-1}{2}\right]+1 \\ &= \frac{3^{m+1}-1}{2} \end{aligned}$$

Two things that can go wrong with induction:

1. *ex*) try to prove all numbers of the form $2m+1$ are even

Induction step works: Indeed suppose $2m+1$ is even. Then $2(m+1)+1 = 2m+3 = (2m+1)+2$ is even.

But the initial step $m = 1$ is false.

2. *ex*) Try to prove all horses are the same color.

The initial step: Have 1 horse. So initial step works

Induction step: Suppose any n horses have the same color. Consider horses 1, 2, ..., $n+1$. By assumption, horses 1, 2, ..., n have the same color *and* 2, 3, ..., $n+1$ have the same color (because it's still n horses). Now horse 2 is in both groups. So horses 1,2, ..., n+1 have all the same color. So it works.

note: if $n = 1$, the two groups are horse 1 and horse 2. So horse 2 isn't in both groups. So induction step fails when $n = 1$.

Strong Induction

Two steps:

1. Initial Step

Prove it for smallest value of n. Usually, $n = 0$ or $n = 1$.

2. Induction Step

Assume the result for 1, 2, ..., n. Need to prove it for $n + 1$.

Theorem Strong induction works

Proof

Let $m + 1$ be the smallest number that it fails. By the initial step, $m \geq 1$. So result holds for all of 1, 2, ..., m. So by induction step (strong), the result holds true for $m + 1$. Contradiction.

ex

Define a sequence

$$a_0, a_1, a_2, ...$$

by $a_0 = 0$

$$a_{n+1} = a_0 + a_1 + ... + a_n + (n + 1)$$

for all $n \geq 0$

Looking at a few terms, you guess that $a_n = 2^n - 1$. Weak induction doesn't work since a_{n+1} involves more than just a_n.

Solution

Initital step works for $n = 0$ since $2^0 - 1 = 0$.

Using weak induction

Assume $a_n = 2^n - 1$. Need to show $a_{n+1} = 2^{n+1} - 1$.

But

$$a_{n+1} = a_0 + a_1 + ... + a_{n-1} + a_n + (n+1)$$

We only know a_n, but we don't know all the other terms.

Using strong induction

Suppose is true for all of 1,2,..., n. Then,

$$\begin{aligned} a_{n+1} &= a_0 + a_1 + a_2 + ... + a_n + (n+1) \\ &= (2^0 - 1) + ... + (2^n - 1) + (n+1) \\ &= 1 + 2 + 4 + ... + 2^n \\ &= 2^{n+1} - 1 \end{aligned}$$

Then we're done.

ex

Theorem Any integer ≥ 2 is a product of primes.

Proof Use strong induction.

Initial Step

$n = 2$, True since 2 is prime.

Induction Step

Assume true for 2,3, ..., n.

2 Cases:

1. $n+1$ is prime. And we're done.
2. $n+1$ is not prime. Then

$$n + 1 = a \times b, \quad 2 \leq a, b \leq n$$

So we know a and b are both product of primes and we're done.