Notes

Minyoung Heo

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Inner Products

vectorspace $\mathbb{R}^n$

inner product:

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R} \\ \mathbf{v}, \mathbf{w} \int \mathbb{R}^n \\ \text{then, } \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i w_i \end{aligned}$$

Cauchy-Schwarz:

$| \langle \mathbf{v}, \mathbf{w} \rangle \leq ||\mathbf{v}|| \cdot ||\mathbf{w}||$

where $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{w} \rangle}$

Theorem (triangle inequality)

$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$

Proof

$$\begin{aligned} ||\mathbf{v} + \mathbf{w}||^2 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle \\ = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \\ = ||\mathbf{v}||^2 + 2 \langle \mathbf{v}, \mathbf{w} \rangle + ||\mathbf{w}||^2 \\ \rightarrow ||\mathbf{v} + \mathbf{w}||^2 \leq ||\mathbf{v}||^2 + 2||\mathbf{v}|| \cdot ||\mathbf{w}|| + ||\mathbf{w}||^2 \\ = (||\mathbf{v}|| + ||\mathbf{w}||)^2 \end{aligned}$$

Example of other norms

$|| \cdot ||$ is a norm if

Example $x \in \mathbb{R}^n$,

$$||x|| = \max_{1 \leq i \leq n} |x_i|$$

in $n = 2$. $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $||x|| = \max(|x_1|, |x_2|)$

Ball of radius r

$$B_r(x) = \{y \in \mathbb{R}^2 : ||y - x|| < r\}$$

It's a square:

$$\leq \{y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : |y_1|, |y_2| < r\}$$

Example $x \in \mathbb{R}^n$

$$||x|| = \sum_{i=1}^{n} |x_i|$$

in $n = 2$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to ||x|| = |x_1| + |x_2|$.

Ball of radius r again. It's a box but looks diagonal. It's called a "*taxicab norm*".

Example $x \in \mathbb{R}^n$

$$||x|| = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{n}}$$

(*exercise: verify with triangle inequality*)

With the same ball of radius r again, it's like the first norm but the corners are curved.

In the complex:

Innerproduct: $x, y \in \mathbb{C}^n$,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix}$$

$$x_i, y_i \in \mathbb{C}$$

for instance $y_j = u_j + iv_j$, $i^2 = -1$

$u_j = \text{Re } y_i$ the real part

$v_j = \text{Im } y_i$ the imaginary part

then $y_j \overline{y_j} = (u_j + iv_j)(u_j - iv_j) = (u_j)^2 - (iv_j)^2 = u_j^2 + v_j^2$

therefore, $y_j \overline{y_j} = |y_j|^2$

then,

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$$
$$y_j = \overline{y_j}$$

only if $y_j \in \mathbb{R}$.

$u_j + iv_j = u_j - iv_j \to v_j = 0$

note:

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$$
$$\langle y, x \rangle = \sum_{j=1}^{n} y_j \overline{x_j}$$
$$\overline{\langle y, x \rangle} = \overline{\sum_{j=1}^{n} y_j \overline{x_j}}$$
$$= \sum_{j=1}^{n} \overline{y_j \overline{x_j}}$$
$$= \sum j = 1^n \overline{y_j} x_j$$
$$= \langle x, y \rangle$$

Eigenvalues

A is a $n \times n$ real matrix

$A \in \mathbb{R}^{n \times n}$, $A = (a_{ij})_{1 \leq i,j \leq n}$, $a_{ij} \in \mathbb{R}$

Characteristic Polynomial

$P(\lambda) = \det(A - \lambda)$

degree $= n \to P(\lambda) = 0$ has n roots: $\lambda_1, \lambda_2, ...\lambda_n$

λ_i are eigenvalues of A.

if λ_j is eigenvalue of A $\to \det(A - \lambda_j) = 0$ (i.e. $A - \lambda_j$ has rank $\leq n - 1$)

$\to$ it has nontrivial null space

if $\lambda_j \in \mathbb{C} \to$ there exists $v \in \mathbb{C}^n$ such that

$$Av = \lambda_i v$$

if $\lambda_j \in \mathbb{R} \to$ there exists $v \in \mathbb{R}^n$ such that

$$Av = \lambda_j v$$

then v is called an eigenvector of A to eigenvalue λ_j.

Theorem Assume $A \in \mathbb{R}^{n\times n}$ has n distinct real eigenvalues $\lambda_1, ..., \lambda_n$, then it has n linearly independent eigenvectors.

Proof λ_j eigenvalue $\to$ there exists $v_j \in \mathbb{R}^n$ so that $Av_j = \lambda_j v_j$, $j = 1, ..., n$.

We have to know $v_1, ..., v_n$ are linearly independent (if

$$\sum_{i=1}^{n} c_j v_j = 0$$

for coefficients $c_1, c_2, ..., c_n$ then one must have $c_j = 0$ for all i.)

Use the trick: Put $z_i = \Pi_{k\neq j}(\lambda_k I - A)$ so

$z_j = (\lambda_n I - A)...(\lambda_{j+1} I - A)(\lambda_{j-1} I - A)...(\lambda_2 I - A)(\lambda_1 I - A)$ (notice no j term)

then,

$$\begin{aligned}(\lambda_k I - A)v_j = \lambda_k v_j - \lambda_j v_j = (\lambda_k - \lambda_j)v_j \\ \to z_j v_j = \Pi_{k\neq j}(\lambda_k - \lambda_j)v_j \neq 0\end{aligned}$$

Also

$$z_j v_k = 0$$

Now apply z_i to $\sum_{k=1}^{n} c_k v_k$.

$$\begin{aligned} z_i\left(\sum_{k=1}^{n} c_k v_k\right) &= \sum_{k=1}^{n} z_i(c_k v_k) \\ &= \sum_{k=1}^{n} c_k z_j v_k \\ &= c_j z_j v_j \end{aligned}$$

But $z_j v_j \neq 0$ so if these were to be 0, $c_j = 0$. This is true for all $j = 1, 2, ..., n$. So $c_j = 0$ for all j.

$\to v_1, v_2, ...v_n$ are linearly independent.

Homework

pg 40: 3.8, 3.9