Lecture 1

vector space $\mathbf{R}^n$

vectors $v \in \mathbf{R}^n$

transposed vectors $v^T = |v_1, v_2, ..., v_n|$

$n \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{22} & ... & a_{1n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} = (a_{ij})_{1 \le i,j \le n}$

$n \times m$ matrix $A = \begin{bmatrix} a_{11} & a_{22} & ... & a_{1m} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nm} \end{bmatrix} = (a_{ij})_{1 \le i,j \le n}$

$n \times n$ matrix defines a linear map on $\mathbf{R}^n$

$v = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix} \mapsto Av \in \mathbf{R}^n$ where $Av = \begin{pmatrix} (Av)_1 \\ (Av)_2 \\ ... \\ (Av)_n \end{pmatrix}$, $(Av)_i = \sum_1^n A_{ij} v_i$

Example $n = 2$

$A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$

$Av = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 + 4v_2 \\ v_1 + 2v_2 \end{pmatrix}$

Maps are linear:

1. if $v, w \in R^n$, then $A(v + w) = Av + Aw$
2. if $\alpha \in \mathbf{R}^n$, then $A(\alpha v) = \alpha(Av)$

Definition (linear product) $(\cdot, \cdot) : \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R}$

that is if $x, y \in \mathbf{R}^n$, then

$(x, y) \in \mathbf{R}$

if

1. $(x, x) \ge 0$ and $(x, x) = 0$ if and only if $x = 0$
2. $(x, y) = (y, x)$ (*Symmetry*)
3. $(x + y, z) = (x, z) + (y, z)$
4. $(\alpha x, y) = \alpha(x, y)$ (for $\alpha \in \mathbf{R}$)

Remark

1. Have $(\cdot, \cdot) \in \mathbf{R}^n$ is the dot product: $x^T = (x_1, x_2, ..., x_n), y^T = (y_1, y_2, ...y_n)$ $(x, y) = \sum_i^n x_i y_i$
2. *Def*: $x \in \mathbf{R}^n$, then $||x|| = \sqrt{(x, x)}$ in Euclidean norm of x also, $x, y \in \mathbf{R}n$, if $(x, y) = 0$ then x, y are orthogonal
3. $(z, x + y) = (x + y, z) = (x, z) + (y, z) = (z, x) + (z, y)$
4. $(x, \alpha y) = (\alpha y, x) = \alpha(y, x) = \alpha(x, y)$

5. $x, y \in \mathbf{R}^n$, $A = (a_{ij})_{1 \le ij \le n}$ then, $(Ax, y) = \sum_i^n (Ax)_i y_i = \sum_i^n \left(\sum_j^i a_{ij} x_j\right) y_i$ $= \sum_i^n x_i a_{ij} y_i = \sum_i^n x_i \left(a_{ij} y_i\right) = (x, A^T y) \to (Ax, y) = (x, A^T y)$ $\to (x, Ay) = (A^T x, y)$

Example take vectorspace of functions $f(t)$ on $[0, 2\pi]$ let f,g be the transformation defined inner product $(f, g) = \int_0^{2\pi} f(t)g(t)dt$ then for instance, $n, m \in \mathbf{N}$ $f(t) = \sin(nt), g(t) = \sin(mt)$ then if $n \ne m$ $(f, g) = \int_0^{2\pi} \sin(nt)\sin(mt)dt$ $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin(\alpha)\sin(\beta)$ $\alpha = nt, \beta = mt, n \ne m \to \sin(nt)\sin(mt) = \frac{1}{2}(\cos((n-m)t) - \cos((n+m)t))$ so $(f, g) = \int_0^{2\pi} \sin(nt)\sin(mt)dt = \frac{1}{2}\int_0^{2\pi} \cos((n-m)t) - \cos((n+m)t)dt = 0$ that is f, g are orthogonal

Theorem (Cauchy-Schwarz Inequality) $x, y \in \mathbf{R}^n$, then $|(x, y)| \le ||x|| \cdot ||y||$ *Proof* let $v, w \in \mathbf{R}^n$ so that $||v|| = ||w|| = 1$, then $0 \le ||v + w||^2 = (v + w, v + w) \to 0 \le (v, v) + (v, w) + (w, v) + (w, w) = ||v||^2 + 2(v, w) + ||w||^2$ $= 2(1 + (v, w)) \to 0 \le 1 + (v, w) \to (v, w) \le 1 = ||v|| \cdot ||w||$ also $0 \le ||v - w||^2$ given $-(v, -w) \le 1$

in general, $x, y \in \mathbf{R}^n$ let $v = \frac{x}{||x||} \to ||v|| = 1$ $w = \frac{y}{||y||} \to = 1$ then, $\left(\frac{x}{||x||}, \frac{y}{||y||}\right) \le 1$ $\to \frac{1}{||x|| \cdot ||y||}|(x, y)| \le 1 \to |(x, y)| \le ||x|| \cdot ||y||$

Homework p22f 2.8, 2.9, 2.10