Chapter 1

Pigeonhole Principle

Some Resource: Sami Assaf (432 videos)

Theorem 1 Let n, k be positive integers with $n > k$, Put n balls into k boxes in some what. Then must have at least 1 box with 2 balls *Proof*: Proceed by contradiction: Assume that each of the k boxes has at least 1 ball. Then the number of balls is at most k But we assumed there there are n balls and $n > k$. Contradiction

Example **(doesn't use pigeonhole principle)** Consider the numbers $1, 2, ..., 2n$. Take any $n+1$ of them. Then among them there must be two which are relatively prime. *Proof (sketch)*: If have $n + 1$ numbers from $1, 2, 3, ..., 2n$, then must have two consecutive numbers these are relatively prime.

Example **(does use pigeonhole principle)** Let A be a set of $n+1$ numbers chosen from $1, 2, 3, ..., 2n$. Then there are always two numbers in A such that one divides the other. *Soln* Write each $a \in A$ as $a = 2^k m$ where m is odd and at most $2n - 1$. Call m the "odd part of a". Since A has $n + 1$ elements, and there are only m possible odd parts, the pigeonhole principle implies that there are 2 numbers in A with the same odd part, so one of these numbers divides the other.

Example Consider a sequence $a_1, a_2, ..., a_{mn+1}$ of $mn+1$ distinct real numbers. Then there exists an increasing subsequence $a_{i_1} < a_{i_2} < ... < a_{i_{m+1}}$ (here $i_1 < i_2 < ... < i_{m+1}$) of length $m+1$ ***OR*** A decreasing subsequence $a_{j_1} > a_{j_1} > ... > a_{j_{n+1}}$ (here $j_1 < j_2 < ... < j_{m+1}$) (or both) *Example* Consider the sequence $4, 1, 3, 5, 7, 8, 2, 6$ The $3, 7, 8$ is an increasing subsequence $4, 3, 2$ is a decreasing subsequence

Remarks For a permutation Π let $I(\Pi)$ = the longest increasing subsequence of Π

1. If Π is a random permutation, then average of $I(\Pi)$ is asymptotic to $2\sqrt{n}$
2. $I(\Pi)$ has applications:
    - patience sorting
    - airline boarding

$n = 3$	$I(\Pi)$
$1, 2, 3$	3
1, 3, 2	2

... So the average of $I(\Pi) = 2$ for $n = 3$

Claim $\lim_{n\to\infty} \frac{\text{average of } I(\Pi)}{2\sqrt{n}} = 1$

Proof of claim For each number a_i let t_i = the length of the longest increasing subsequence starting at a_i. If $t_i \geq m+1$ for some i, then have an increasing subsequence of length $m+1$ and we're done.

Claim So can assume $t_i \leq m$ for all i, so the function $f : a_i \to t_i$ maps $\{a_i, ..., a_{mn+1}\} \to \{1, ..., m\}$

By generalize pigeonhole principle, there is some $s \in \{1, ..., m\}$ so that $f(a_i) = s$ for $n+1$ numbers a_i. Let $a_{j_1}, a_{j_2}, ..., a_{j_n+1}$ $(j_1 < j_2 < ... < j_{m+1})$ be these numbers. Claim $a_{j_1} > a_{j_2} > ... > a_{j_{n+1}}$ is a decreasing subsequence of length $n+1$. To see this, assume to contrary that $a_{j_i} < a_{j_i+1}$. Then we'd have a length s increasing subsequence starting from $a_{j_{i+1}}$ so a length $s+1$ subsequence starting at a_{j_i}. Contradiction! Thus $a_{j_1} > a_{j_2} > ... > a_{j_{n+1}}$, and we're done

Remark The above proof used the "generalized pigeonholde principle" which states let n, m, r be positive integers such that $n > rm$. Put n balls into m boxes. Then some box must have at least $r+1$. (Pigeonhole principle is the special case where $r = 1$) *Proof* Assume the contrary that each box has at most r balls. Then total number of balls is at rm but $n > rm$, a contradiction.

Who Cares? n cards $1, 2, ...n$ in some order so say: $4, 2, 3, 6, 5, 1, 7$ We can cut into piles....