Lecture 6

Math 467

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Taylor Expansions

2 cases, $f : \mathbb{R}^n \mapsto \mathbb{R}^m s$:

1. n = m = 1

if $f \in C^l$ (l continuous derivatives)

$$f(x) = \sum_{j=0}^{l-1} \frac{f^{(i)}(x^*)}{i!}(x - x^*) + R_l(x)$$

where

$$R_l(x) = \frac{f^{(l)}\tilde{x}}{l!}(x - x^*)^l$$

where $\tilde{x} = x + \eta(x - x^*)$ for some $\eta \in [0, 1]$.

Remark:

if $||x - x^*|| < 1$, then $R_l(x) = O(||x - x^*||^l)$.

2. m=1.

Take a base point $x^* \in \Omega \subset \mathbb{R}^n$.

$$f(x) = f(x^*) + Df(x^*)(x - x^*) + \frac{1}{2!}(x - x^*)^T D^2 f(x^*)(x - x^*) + R_3$$

where $Df = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right) = \nabla f^T$ and

$$D^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 x_1} & \cdots & \frac{\partial^2 f}{\partial x_n x_1} \\ \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_1 x_m} & \frac{\partial^2 f}{\partial x_2 x_m} & \cdots & \frac{\partial^2 f}{\partial x_m^2} \end{bmatrix}$$

$R_3 = o(||x - x^*||^2)$ and if $f \in C^3$, $R_3 = O(||x - x^*||^3)$.

Optimization

Given a function $f : \Omega \mapsto \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$. Problem is to minimize f. There are 2 cases:

1. unconstrained problem: $\Omega = \mathbb{R}^n$
2. constraint problem: $\Omega \neq \mathbb{R}^n$

Ω could be of the form

$$\Omega = \{x \in \mathbb{R}^n : h(x) = 0\}$$

(hypersurface)

or

$$\Omega = \{x \in \mathbb{R}^n : h(x) \leq 0\}$$

Definition:

1. $x^* \in \Omega$ is a *local* minimizer (you don't go too far) on $f(\Omega)$ if $\exists \epsilon > 0$ such that $f(x) \geq f(x^*)$ for all

$$\begin{cases} x \in \Omega : \{x^*\} \\ ||x - x^*|| < \epsilon \end{cases}$$

2. x^* is a global minimizer of f in Ω such that $f(x) \geq f(x^*)$ for all $x \in \Omega : \{x^*\}$.

strict local, gloal minimise if $f(x) > f(x^*)$.

Notation

if minimiser $x^* \in \Omega$ is unique then

$$x^* = \min_{x \in \Omega} f(x)$$

Example

$$x(x-2) = f(x), \Omega = \mathbb{R}$$

has $\min_{x \in \mathbb{R}} f(x)$. $x^* = 1$ is a local minimum. $f(x) = x^2 - 2x + 1 - 1 = (x-1)^2 - 1$, but since $(x-1)^2 \geq 0$, $f(x) \geq -1$.

Conditions for Minimizers (FONC, SONC)

Def: Let $\Omega \in \mathbb{R}^n$, Let $x \in \Omega$. We say $d \in \mathbb{R}^n$ (vector) is feasible direction if

$$\exists \alpha_0 > 0 \text{ s.t. } x + \alpha d \in \Omega \quad \forall \quad 0 \leq \alpha \geq \alpha_0$$

Remark:

1. One can assume that $||d|| = 1$.
2. If x is interior point of Ω.

(Interior point: $x \in \Omega$ if $\exists \epsilon > 0$ so that $B_\epsilon(x) \subset \Omega$.

(The ball)

$$B_\epsilon(x) = \{y \in \mathbb{R}^n : ||x - y|| < \epsilon\}$$

)

then $\exists \epsilon > 0$ so that for every $d \in \mathbb{R}$, $||d|| = 1$. $x + \alpha d \in B_\epsilon(x) \subset \Omega \quad \forall \quad 0 \leq \alpha < \epsilon$. so take $\alpha_0 = \frac{\epsilon}{2}$. Which means all $d \in \mathbb{R}^n$ are feasible directions.

3. $f : \Omega \in \mathbb{R}$ (differentiable). Let $d \in \mathbb{R}^n$ be a feasible direction. Then $\frac{\partial}{\partial d} f(x) = \frac{d}{d\alpha} f(x + \alpha d)|_{\alpha=0} = d^T Df(x)$. (In Calc 3, you assume $||d|| = 1$. Then $\frac{\partial}{\partial d} f = (Df)(x) \cdot d = \langle d, Df(x) \rangle = d^T Df(x) \to \frac{\partial}{\partial p} f(x)$ - n directional derivative m direction of d.)

Homework