Homework 1

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2.8

$$\langle \mathbf{x}, \mathbf{y} \rangle_2 = 2x_1y_1 + 3x_2y_1 + 3x_1y_1 + 5x_2y_2$$

1. Positivity

$$\begin{aligned} \langle \mathbf{x}, \mathbf{x} \rangle_2 &= 2x_1x_1 + 3x_1x_2 + 3x_1x_2 + 5x_2^2 \\ &= 2x_1^2 + 6x_1x_2 + 5x_2^2 \\ &= 2(x_1^2 + 3x_1x_2 + \frac{5}{2}x_2^2) \\ &= 2(x_1^2 + 3x_1x_2 + \frac{9}{4}x_2^2 + \frac{1}{4}x_2^2) \\ &= 2(x_1 + \frac{3}{2}x_2)^2 + \frac{1}{2}x_2^2 \geq 0 \end{aligned}$$

It's easy to see that if $x = 0$, then $\langle \mathbf{x}, \mathbf{x} \rangle_2 = 0$

Now suppose $\langle \mathbf{x}, \mathbf{x} \rangle_2 = 2(x_1 + \frac{3}{2}x_2)^2 + \frac{1}{2}x_2^2 = 0$. That means

$2(x_1 + \frac{3}{2}x_2)^2 = -\frac{1}{2}x_2^2$ where $-\frac{1}{2}x_2^2 \leq 0$ and its equality is only when $x_2 = 0$.

Plugging it back in, $2(x_1 + \frac{3}{2}x_2)^2 \leq 0$ can only be true when $2(x_1 + \frac{3}{2}x_2^2)^2 = 0$ so $\mathbf{x} = 0$

2. Symmetry

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle_2 &= 2x_1y_1 + 3x_2y_1 + 3x_1y_2 + 5x_2y_2 \\ &= 2y_1x_1 + 3y_1x_2 + 3y_2x_1 + 5y_2x_2 \\ &= 2y_1x_1 + 3y_2x_1 + 3y_1x_2 + 5y_2x_2 \\ &= \langle \mathbf{y}, \mathbf{x} \rangle_2 \end{aligned}$$

3. Additivity

$$\begin{aligned} \langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle_2 &= 2(x_1 + y_1)z_1 + 3(x_2 + y_2)z_1 + 3(x_1 + y_1)z_2 + 5(x_2 + y_2)z_2 \\ &= 2x_1z_2 + 3x_2z_1 + 3x_1z_2 + 5x_2z_2 + 2y_1z_1 + 3y_2z_1 + 3y_1z_2 + 5y_2z_2 \\ &= \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle \end{aligned}$$

4. Homogeneity

$$\begin{aligned}\langle r\mathbf{x}, \mathbf{y}\rangle &= 2(rx_1)y_1 + 3(rx_2)(y_1) + 3(rx_1)y_2 + 5(rx_2)y_2 \\ &= r(2x_1y_1 + 3x_2y_1 + 3x_1y_2 + 5x_2y_2) \\ &= r\langle \mathbf{x}, \mathbf{y}\rangle\end{aligned}$$

2.9

$$\begin{aligned}\mathbf{x} &= (\mathbf{x}-\mathbf{y}) + \mathbf{y} \\ ||\mathbf{x}|| = ||(\mathbf{x}-\mathbf{y}) + \mathbf{y}|| &\leq ||\mathbf{x}-\mathbf{y}|| + ||\mathbf{y}|| \\ \rightarrow ||\mathbf{x}-\mathbf{y}|| &\geq ||\mathbf{x}|| - ||\mathbf{y}||\end{aligned}$$

But similarly, we can see that

$$||\mathbf{y}-\mathbf{x}|| \geq ||\mathbf{y}|| - ||\mathbf{x}||$$

Since $||\mathbf{y}-\mathbf{x}|| = ||\mathbf{x}-\mathbf{y}||$, we can deduce that these are greater than the greater of the two, namely

$$||\mathbf{x}-\mathbf{y}|| \geq | \; ||\mathbf{x}|| - ||\mathbf{y}|| \; |$$

2.10

Since $||\mathbf{x}-\mathbf{y}|| < \delta$,

$$|||\mathbf{x}|| - ||\mathbf{y}||| < \delta$$

And we can let $\delta = \epsilon$.

3.8

$$A = \begin{bmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}2x_2 = x_3 \rightarrow x_2 &= \frac{1}{2}x_3 \\ 4x_1 = x_3 \rightarrow x_1 &= \frac{1}{4}x_3 \\ \rightarrow N(A) = \text{span of } &\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix} \right\}\end{aligned}$$

3.9

_$R(A)$ **is a subspace of** $\mathbb{R}^m$-:

Existence of 0:

$R(\mathbf{0}) = 0$ for any $m \times n$ zero matrix $\mathbf{0}$

Closure under vector addition:

Let $R(A) \in \mathbb{R}^m$ and $R(B) \in \mathbb{R}^m$ for $m \times n$ matrices A and B.

For $R(A) + R(B) = \{Ax + By : x, y \in \mathbb{R}^m\}$, we can see that the span $\{Ax + By$ for any $x, y \in \mathbb{R}^m\} \in \mathbb{R}^m$. Therefore, $R(A) + R(B) \in \mathbb{R}^m$ and thus the range is closed under vector addition.

Closure under scalar multiplication:

Let $R(A) \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$ for some $m \times n$ matrix A.

$$\alpha \cdot R(A) = \{\alpha Ax : x \in \mathbb{R}^n\}$$

αA is still a $m \times n$ matrix and projects the x onto the space $\mathbb{R}^m$. $\rightarrow \alpha Ax \in \mathbb{R}^m$ for any $x \in \mathbb{R}^n$ so $\alpha \cdot R(A) \in \mathbb{R}^m$ and thus the range is closed under scalar multiplication.

$N(A)$ ***is a subspace of*** $\mathbb{R}^n$:

Existence of 0:

$N(A)$ for some $n \times n$ matrix A such that its cols are linearly independent, $N(A) = 0$.

Closure under vector addition:

Let $N(A), N(B) \in \mathbb{R}^n$ for $m \times n$ matrices A and B.

$$N(A) + N(B) = \{x + y \in \mathbb{R}^n : Ax = 0, By = 0\}$$

We can see that such vector $z = x + y \in \mathbb{R}^n$, thus $N(A) + N(B) \in \mathbb{R}^n$, and is closed under vector addition.

Closure under scalar multiplication

Let $N(A) \in R^n$ for some $m \times n$ matrix A and $\alpha \in \mathbb{R}$.

$$\alpha \cdot N(A) = \{\alpha x : A(\alpha x) = 0, x \in \mathbb{R}^n\}$$

Both x and αx are $\in \mathbb{R}^n$, so $\alpha \cdot N(A) \in \mathbb{R}^n$, so the nullspace is closed under scalar multiplication.

3.10

First we note that if we were to have $N(A^T) \subset N(B^T)$, we can deduce that

$$N(B^T)^\perp \subset N(A^T)^\perp$$

(This is analogous to the following: suppose we have two sets X and Y and their respective complements X' and Y'. If $X \subset Y$, then $Y' \subset X'$).

Then we use **Theorem 3.4** (namely $R(A)^\perp = N(A^T)$ or equivalently $R(A) = N(A^T)^\perp$) to derive the conclusion

$$R(B) \subset R(A)$$

3.15

$$\begin{aligned}
&\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{bmatrix} x_1 + x_2 & -8x_1 + x_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= x_1^2 + x_1 x_2 - 8x_1 x_2 + x_2^2 \\
&= x_1^2 - 7x_1 x_2 + x_2^2 \quad \text{(Use this one for calculations)} \\
&= x_1^2 - 7x_1 x_2 + \frac{49}{4} x_2^2 - \frac{45}{4} x_2^2 \\
&= (x_1 - \frac{7}{2} x_2)^2 - \frac{45}{4} x_2^2
\end{aligned}$$

All we need to do is first check for cases where $(x_1 - \frac{7}{2}x_2)^2 - \frac{45}{4}x_2^2 < 0$ and then check for $(x_1 - \frac{7}{2}x_2)^2 - \frac{45}{4}x_2^2 > 0$.

For $(x_1 - \frac{7}{2}x_2)^2 - \frac{45}{4}x_2^2 < 0$:

$$\begin{aligned}
(x_1 - \frac{7}{2} x_2)^2 &< \frac{45}{4} x^2, \text{so} \\
-\sqrt{\frac{45}{4}} x_2 < x_1 - \frac{7}{2} x_2 &< \sqrt{\frac{45}{4}} x_2 \\
\to \frac{7 - \sqrt{45}}{2} x_2 < x_1 &< \frac{7 + \sqrt{45}}{2} x_2
\end{aligned}$$

An example of this is when $x_1 = \frac{7}{2}x_2 \to x = \begin{pmatrix} \frac{7}{2} \\ 1 \end{pmatrix}$. Plugging this into $x_1^2 - 7x_1x_2 + x_2^2$, we get $-\frac{45}{4} < 0$.

For $(x_1 - \frac{7}{2}x_2)^2 - \frac{45}{4}x_2^2 > 0$:

$$
\begin{aligned}
(x_1 - \frac{7}{2}x_2)^2 &> \frac{45}{4}x^2, \text{so} \\
x_1 - \frac{7}{2}x_2 < -\sqrt{\frac{45}{4}}x_2 \cup x_1 - \frac{7}{2}x_2 &> \sqrt{\frac{45}{4}}x_2 \\
\rightarrow x_1 < \frac{7-\sqrt{45}}{2}x_2 \cup x_1 &> \frac{7+\sqrt{45}}{2}x_2
\end{aligned}
$$

An example of this is when $x_1 = \frac{100}{2}x_2 \rightarrow x = \begin{pmatrix} \frac{100}{2} \\ 1 \end{pmatrix}$. Plugging this into $x_1^2 - 7x_1x_2 + x_2^2$, we get $9301 > 0$.

Therefore this is **indefinite**.

3.16

$$
\begin{aligned}
\Delta_1 &= 2 \\
\Delta_2 = \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} &= 0 \\
\Delta_3 = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} &= 0
\end{aligned}
$$

And all of them are nonnegative.

Looking at the quadratic form $x^T \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} x$:

$$
\begin{aligned}
x^T \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} x = \begin{bmatrix} 2x_1 + 2x_2 + 2x_3 & 2x_1 + 2x_2 + 2x_3 & 2x_1 + 2x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
= 2x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_2 + 2x_2^2 + 2x_2x_3 + 2x_1x_3 + 2x_2x_3 \\
= 2x_1^2 + 2x_2^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 \\
= 2(x_1 + x_2)^2 + 4x_3(x_1 + x_2)
\end{aligned}
$$

Now we try to make $x_1 + x_2 < -2x_3$, while making sure $|x_1 + x_2| < |2x_3|$. So by setting $x_3 = 1$, we can easily produce an example vector $x = \begin{bmatrix} -\frac{1}{10} \\ -\frac{1}{10} \\ 1 \end{bmatrix}$. If we plug this back into $2(x_1 + x_2)^2 + 4x_3(x_1 + x_2)$, we get

$$
2(-0.2)^2 + 4(1)(0.2) = 0.08 - 0.8 < 0
$$

As we can see, there exists a vector such that it's not positive definite, so A is not positive definite.