Lecture 7

Math 467

Jan 24 2024

$f : \Omega \mapsto \mathbb{R}, \Omega \subset \mathbb{R}^n$

restricted problem if $\Omega \neq \mathbb{R}^n$ unrestricted problem if $\Omega = \mathbb{R}^n$

Assume if differentiable, $d \neq 0$, $d \in \mathbb{R}^n$ is a feasible direction at $x \in \Omega$, if $\exists \alpha_0 > 0$ so that $x + \alpha d \in \Omega \forall 0 \leq \alpha \leq \alpha_0$.

Remark

1. x interior point if and only if every $d \in \mathbb{R}^n$ is a feasible direction
2. (directional derivative, $||d|| = 1$)

$$\frac{\partial}{\partial d} = d^T Df(x)$$

(*Example*) Given $n = 3$, $f : \mathbb{R}^3 \mapsto f$

$$f(x_1, x_2, x_3) = 2(x_1 + x_2 + x_2 x_3)$$

$$= 2x_1 + 2x_2 + 2x_2 x_3$$

Our direction:

$$d = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \frac{1}{\sqrt{14}}$$

Gradient

$$Df(x) = \begin{pmatrix} 2 \\ 2 + 2x_3 \\ 2 + 2x_2 \end{pmatrix}$$

$x = (x_1, x_2, x_3)$.

$$
\begin{aligned}
\frac{\partial}{\partial d} f = d^T Df(x) &= \frac{1}{\sqrt{14}}(1,2,3)\begin{pmatrix} 2 \\ 2+2x_3 \\ 2+2x_2 \end{pmatrix} \\
&= \frac{2}{\sqrt{14}}(1+2+2x_3+3x_2)
\end{aligned}
$$

at $\tilde{x} = (6,8,4)$

$$
\frac{\partial}{\partial p} f(\tilde{x}) = \frac{2}{\sqrt{14}}(3+2\cdot 4+3\cdot 8) = \frac{70}{\sqrt{14}}
$$

First Order Necessary Condition (FONC)

Let $f : \Omega \mapsto \mathbb{R}$, $\Omega \subset \mathbb{R}^3$, be differentiable. Assume $\begin{cases} x* & \text{is a local minimizer} \\ d & \text{feasible direction} \end{cases}$. Then

$$
d^T Df(x*) \geq 0
$$

What it means

Corollary

If x^* is an interior point of Ω, then $\nabla f(x^*) = 0$.

Proof

x interior point of Ω means that d is feasible and $-d$ is also feasible. d being feasible implies $d^T Df(x^*) \geq 0$. $-d$ being feasible implies $(-d)^T D(f(x^*)) \geq 0$ or $-(d^T Df(x^*)) \geq 0$. Thus $d^T Df(x^*) = 0 \forall d$.

Proof of FONC

Let x^* be a local minimizer and d a feasible direction. $x(\alpha) = x^* + \alpha d \in \Omega$. $\forall \quad \alpha \in [0, \alpha_0]$ for some $\alpha_0 > 0$. $\phi(\alpha) = f(x(\alpha)) = f(x^* + \alpha d)$. $\phi(0) = f(x^*)$.

x^* is a local minimizer implies $\epsilon > 0$ so that $\phi(\alpha) \geq \phi(0)$ for all $\alpha < \epsilon$. Taylor's Approximation

$$
\phi(\alpha) = \phi(0) + \alpha\phi'(0) + r(\alpha)
$$

with remainder $r(\alpha) = o(\alpha)$. where $\phi'(0) = d^T Df(x^*)$. We have to show $\phi'(0) \geq 0$ (by contradiction). Assume $\phi'(0) < 0$. Now, there exists an $\epsilon' > 0$, $\epsilon' < \epsilon$ so that $\left|\frac{r(\alpha)}{2}\right| < \frac{|\phi'(0)|}{2} \forall 0 < \alpha < \epsilon'$. This implies that $\phi(\alpha) \leq \phi(0) + \alpha\phi'(0) + \alpha\frac{|\phi'(0)|}{2} \forall \alpha \in (0, \epsilon')$.

$$
\begin{gathered}
\to \phi(\alpha) + \frac{\alpha}{2}\phi'(0) < \phi(0) \\
f(x^* + \alpha d) < f(x^*) \forall \alpha \in (0, \epsilon')
\end{gathered}
$$

This contradicts the assumption that x^* is a local minimizer. So $\phi'(0) \geq 0$.

Example

$n = 2,\ f(x_1, x_2) = x_1^2 + \frac{1}{2}x_2^2 + 3x_2 + \frac{9}{2}$

$$\Omega = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \geq 0, x_2 \geq 0\}$$

(it's the first quadrant)

For convenience, the gradient

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 \\ x_2 + 3 \end{pmatrix}$$

1. $x^* = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Is this a local minimizer?

Feasible directions are all, x^* is interior point.

$$\nabla f(1, 3) = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \neq 0$$

$\rightarrow x^* = (1, 3)$ is not a local minimizer.

2. Is $x^* = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ is local minimizer? $x^* \in \partial\Omega$ (the boundary of Ω). Feasible directions:

$d = \begin{pmatrix} a \\ b \end{pmatrix}$, $a, b \in \mathbb{R}$. d is feasible if $a \geq 0$.

$$d^T Df(0, 3) = (a, b) \begin{pmatrix} 0 \\ 6 \end{pmatrix} = 6b$$

Therefore, a could be positive or negative. So $(0, 3)$ is not a local minimizer.

3. Is $x^* \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ a local minimizer?, $x \in \partial\Omega$ Feasible directions:

$$d = \begin{pmatrix} a \\ b \end{pmatrix}, a, b \in \mathbb{R}$$

$b \geq 0$ (because it must go up)

$$d^T D(f(1, 0)) = (a, b) \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2a + 3b$$

$\rightarrow$ (1, 0) is not a Local Minimizer

4. Is $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a local minimizer? Feasible directions

$d = \begin{pmatrix} a \\ b \end{pmatrix}$ $a \geq 0, b \geq 0$

$\rightarrow$ $d^T Df(0,0) = (a,b) \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3b \geq 0$ for all feasible directions $\rightarrow$ (0, 0) is a local minimizer.

Second Order Necessary Condition (SONC)

Let $f : \Omega \mapsto \mathbb{R}$ be C^2 assume $\begin{cases} x^* & \text{is local minimizer} \\ d & \text{feasible direction} \\ d^T Df(x^*) = 0 & \end{cases}$. Then

$$d^T F d \geq 0$$

Where F is the Hessian matrix.

Homework

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