Lecture 4

Math 467

Quadratic Forms

$Q \in \mathbb{R}^{n\times n}$

define the quadratic form:

$$f(x) = x^T Q x = \sum_{i,j=1}^{n} x_i Q_{ij} x_j$$

that is $f : \mathbb{R}^n \mapsto \mathbb{R}$

Remark:

We can assume that $Q^T = Q$. *Note*: $x^T Q x = \sum_{i,j=1}^{n} x_i Q_{ij} x_j = \sum_{i,j}^{n} x_i Q_{ji} x_j = x^T Q^T x$.

Then put

$$\tilde{Q} = \frac{1}{2}(Q + Q^T)$$

evidently $\tilde{Q} = \tilde{Q}^T$ and

$$x^T \tilde{Q} x = \frac{1}{2}(x^T Q x + x^T Q^T x) = \frac{1}{2}(x^T Q x + x^T Q x) = x^T Q x$$

Definition

1. We say f is positive definite if $f(x) > 0$ for all $x \neq 0$.
2. f is positive semidefinite if $f(x) \geq 0$ for all $x \in R^n$
3. Similarly for negative (semi)definite

When is a quadratic form positive definite?

1. **Sylvestri's Criterion**:

Q is positive definite if determinants of all minors are positive (the minor in this $Q = \begin{bmatrix} A & b & c & ... \\ d & e & f & ... \end{bmatrix}$ is A)

2. *Theorem*) $f(x) = x^T Q x$ is positive definite if and only if all eigenvalues are positive.

Proof) $Q = Q^T \to$ all eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$. For simplicity, they are distinct. Then if $v_1, v_2, ..., v_n$ are associated eigenvectors, they are orthogonal. Assume $||v_j|| = 1$. $\to v_1, v_2, ...v_n$ are orthonormal basis. If

$$T = (v_1, v_2, ...v_n) \in \mathbb{R}^{n \times n}$$

T is orthogonal basis $\to T^T = T^{-1}$ (becase $T^T T = I$). In the basis $(v_1, v_2, ..., v_n)$

$$\tilde{Q} = T^T Q T = \begin{bmatrix} \lambda_1 & ... & 0 \\ 0 & \lambda_2 & ... \\ ... & ... & ... \\ 0 & 0 & \lambda_n \end{bmatrix}$$

let $x \in \mathbb{R}^n$. put $y = T^{-1}x = T^T x$ (i.e $x = \sum_{j=1}^n y_j v_j$). Then $x = Ty$ and $x^T = y^T T^T$.

$$\begin{aligned} f(x) = x^T Q x = y^T T^T Q T y = \tilde{Q} \\ = y^T \tilde{Q} y = \sum_{i,j}^n y_i \tilde{Q}_{ij} y_j \\ = \lambda_i \delta_i j \\ = \sum_{i=1}^n y_i y_i \lambda_i = \sum_{i=1}^n y_i^2 \lambda_i \end{aligned}$$

Now f is positive definite if and only if all $\lambda_i > 0$. Similarly f is postitive semidefinite if and only if $\lambda_i \geq 0$ for all i.

Ex)

$$Q = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Find eigenvalues:

$$P(\lambda) = \lambda^2 - 2\lambda + \frac{3}{4}$$

$$\lambda_1, 2 = 1 \pm \sqrt{\frac{1}{4}} = 1 \pm \frac{1}{2} = \begin{cases} \frac{3}{2} \\ \frac{1}{2} \end{cases}$$

$\lambda_1, \lambda_2 > 0 \to f(x) = x^T Q x$ is positive definite. In fact, for $f(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 + \frac{1}{2}x_1 x_2 + \frac{1}{2}x_1 x_2$. $f(x) = x_1^2 + x_2^2 + x_1 x_2$.

Ex 2 (trivial)

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Ex 3(trivial)

$$Q = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

eigenvalues: one of them is (trivially) 3. The other 2 are the same as the 1st example.

Measuring Sizes of Matrices

Norms of Matrices

$$A \in \mathbb{R}^{n \times n}$$

(or $\in \mathbb{C}^n \times n$). $||\cdot||$ is a matrix norm if

1. $||A|| \geq 0$ and $||A|| = 0$ if and only if $A = 0$
2. $||\alpha A|| = |\alpha| \cdot ||A||$ for $\alpha \in \mathbb{R}$ (or $\mathbb{C}$)
3. $||A + B|| \leq ||A|| + ||B||$ (triangle inequality)
4. $||A \cdot B|| \leq ||A|| \cdot ||B||$

Frobenius norm

in $\mathbb{R}^{n \times n}$

$$||A||_F = \left(\sum a_{ij}^2\right)^{\frac{1}{2}}$$

Induced Norm

$$||A|| = \max_{||x||=1} ||Ax||$$

which means "if $||x|| = 1$, then $||Ax|| \leq ||A||$".

if $v \in \mathbb{R}^n$, $||v|| > 0$, then put $x = \frac{v}{||v||}$. now $||x|| = 1$. $||A\frac{v}{||v||}|| = \frac{1}{||v||}||Av|| \leq ||A|| \rightarrow ||Av|| \leq ||A|| \cdot ||v||$.

$$||A|| = \max_{v \neq 0} \frac{||Av||}{||v||}$$

$A \in \mathbb{R}^{n \times n}$, $A : \mathbb{R}^n \mapsto \mathbb{R}^n$.

Theorem) The induced norm is a norm that also satisfies 4.

Proof

1. $||A|| \geq 0$ as $||Ax|| \geq 0$ for all $x \neq 0$. $||A|| = 0$ if and only if $||Ax|| = 0$ for all $||x|| = 1$ (or $A = 0$)
2. $||\alpha A|| = \max_{||x||=1} ||\alpha Ax|| = |\alpha| \max_{||x||=1} ||Ax|| = |\alpha| ||A||$

Homework

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