

# Homework 1

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## 2.8

$$\langle \mathbf{x}, \mathbf{y} \rangle_2 = 2x_1y_1 + 3x_2y_1 + 3x_1y_2 + 5x_2y_2$$

### 1. Positivity

$$\begin{aligned}\langle \mathbf{x}, \mathbf{x} \rangle_2 &= 2x_1x_1 + 3x_1x_2 + 3x_1x_2 + 5x_2^2 \\ &= 2x_1^2 + 6x_1x_2 + 5x_2^2 \\ &= 2(x_1^2 + 3x_1x_2 + \frac{5}{2}x_2^2) \\ &= 2(x_1^2 + 3x_1x_2 + \frac{9}{4}x_2^2 + \frac{1}{4}x_2^2) \\ &= 2(x_1 + \frac{3}{2}x_2)^2 + \frac{1}{2}x_2^2 \geq 0\end{aligned}$$

It's easy to see that if  $x = 0$ , then  $\langle \mathbf{x}, \mathbf{x} \rangle_2 = 0$

Now suppose  $\langle \mathbf{x}, \mathbf{x} \rangle_2 = 2(x_1 + \frac{3}{2}x_2)^2 + \frac{1}{2}x_2^2 = 0$ . That means

$2(x_1 + \frac{3}{2}x_2)^2 = -\frac{1}{2}x_2^2$  where  $-\frac{1}{2}x_2^2 \leq 0$  and its equality is only when  $x_2 = 0$ .

Plugging it back in,  $2(x_1 + \frac{3}{2}x_2)^2 \leq 0$  can only be true when  $2(x_1 + \frac{3}{2}x_2)^2 = 0$  so  $\mathbf{x} = 0$

### 2. Symmetry

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle_2 &= 2x_1y_1 + 3x_2y_1 + 3x_1y_2 + 5x_2y_2 \\ &= 2y_1x_1 + 3y_1x_2 + 3y_2x_1 + 5y_2x_2 \\ &= 2y_1x_1 + 3y_2x_1 + 3y_1x_2 + 5y_2x_2 \\ &= \langle \mathbf{y}, \mathbf{x} \rangle_2\end{aligned}$$

### 3. Additivity

$$\begin{aligned}\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle_2 &= 2(x_1 + y_1)z_1 + 3(x_2 + y_2)z_1 + 3(x_1 + y_1)z_2 + 5(x_2 + y_2)z_2 \\ &= 2x_1z_1 + 3x_2z_1 + 3x_1z_2 + 5x_2z_2 + 2y_1z_1 + 3y_2z_1 + 3y_1z_2 + 5y_2z_2 \\ &= \langle \mathbf{x}, \mathbf{z} \rangle_2 + \langle \mathbf{y}, \mathbf{z} \rangle_2\end{aligned}$$

#### 4. Homogeneity

$$\begin{aligned}
\langle r\mathbf{x}, \mathbf{y} \rangle &= 2(rx_1)y_1 + 3(rx_2)(y_1) + 3(rx_1)y_2 + 5(rx_2)y_2 \\
&= r(2x_1y_1 + 3x_2y_1 + 3x_1y_2 + 5x_2y_2) \\
&= r\langle \mathbf{x}, \mathbf{y} \rangle
\end{aligned}$$


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### 2.9

$$\begin{aligned}
\mathbf{x} &= (\mathbf{x} - \mathbf{y}) + \mathbf{y} \\
\|\mathbf{x}\| &= \|(\mathbf{x} - \mathbf{y}) + \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| \\
&\rightarrow \|\mathbf{x} - \mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|
\end{aligned}$$


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### 2.10

Since  $\|\mathbf{x} - \mathbf{y}\| < \delta$ ,

$$||\|\mathbf{x}\| - \|\mathbf{y}\|| < \delta$$

And we can let  $\delta = \epsilon$ .

### 3.8

$$A = \begin{bmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2x_2 = x_3 \rightarrow x_2 = \frac{1}{2}x_3$$

$$4x_1 = x_3 \rightarrow x_1 = \frac{1}{4}x_3$$

$$\rightarrow N(A) = \text{span of } \left\{ \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$


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### 3.9

$R(A)$  is a subspace of  $\mathbb{R}^m$ :

**Existence of 0:**

$R(\mathbf{0}) = \mathbf{0}$  for any  $m \times n$  zero matrix  $\mathbf{0}$

**Closure under vector addition:**

Let  $R(A) \in \mathbb{R}^m$  and  $R(B) \in \mathbb{R}^m$  for  $m \times n$  matrices  $A$  and  $B$ .

For  $R(A) + R(B) = \{Ax + By : x, y \in \mathbb{R}^n\}$ , we can see that the span  $\{Ax + By \text{ for any } x, y \in \mathbb{R}^n\} \in \mathbb{R}^m$ . Therefore,  $R(A) + R(B) \in \mathbb{R}^m$  and thus the range is closed under vector addition.

**Closure under scalar multiplication:**

Let  $R(A) \in \mathbb{R}^m$  and  $\alpha \in \mathbb{R}$  for some  $m \times n$  matrix  $A$ .

$$\alpha \cdot R(A) = \{\alpha Ax : x \in \mathbb{R}^n\}$$

$\alpha A$  is still a  $m \times n$  matrix and projects the  $x$  onto the space  $\mathbb{R}^m$ .  $\rightarrow \alpha Ax \in \mathbb{R}^m$  for any  $x \in \mathbb{R}^n$  so  $\alpha \cdot R(A) \in \mathbb{R}^m$  and thus the range is closed under scalar multiplication.

$N(A)$  *is a subspace of*  $\mathbb{R}^n$ :

**Existence of 0:**

$N(A)$  for some  $n \times n$  matrix  $A$  such that its cols are linearly independent,  $N(A) = 0$ .

**Closure under vector addition:**

Let  $N(A), N(B) \in \mathbb{R}^n$  for  $m \times n$  matrices  $A$  and  $B$ .

$$N(A) + N(B) = \{x + y \in \mathbb{R}^n : Ax = 0, By = 0\}$$

We can see that such vector  $z = x + y \in \mathbb{R}^n$ , thus  $N(A) + N(B) \in \mathbb{R}^n$ , and is closed under vector addition.

**Closure under scalar multiplication**

Let  $N(A) \in \mathbb{R}^n$  for some  $m \times n$  matrix  $A$  and  $\alpha \in \mathbb{R}$ .

$$\alpha \cdot N(A) = \{\alpha x : A(\alpha x) = 0, x \in \mathbb{R}^n\}$$

Both  $x$  and  $\alpha x$  are  $\in \mathbb{R}^n$ , so  $\alpha \cdot N(A) \in \mathbb{R}^n$ , so the nullspace is closed under scalar multiplication.

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## 3.10

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### 3.15

$$\begin{aligned}
& (x_1 \ x_2) \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= [x_1 + x_2 \quad -8x_1 + x_2] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= x_1^2 + x_1x_2 - 8x_1x_2 + x_2^2 \\
&= x_1^2 - 7x_1x_2 + x_2^2 \quad (\text{Use this one for calculations}) \\
&= x_1^2 - 7x_1x_2 + \frac{49}{4}x_2^2 - \frac{45}{4}x_2^2 \\
&= (x_1 - \frac{7}{2}x_2)^2 - \frac{45}{4}x_2^2
\end{aligned}$$

All we need to do is first check for cases where  $(x_1 - \frac{7}{2}x_2)^2 - \frac{45}{4}x_2^2 < 0$  and then check for  $(x_1 - \frac{7}{2}x_2)^2 - \frac{45}{4}x_2^2 > 0$ .

For  $(x_1 - \frac{7}{2}x_2)^2 - \frac{45}{4}x_2^2 < 0$  :

$$\begin{aligned}
& (x_1 - \frac{7}{2}x_2)^2 < \frac{45}{4}x_2^2, \text{ so} \\
& -\sqrt{\frac{45}{4}}x_2 < x_1 - \frac{7}{2}x_2 < \sqrt{\frac{45}{4}}x_2 \\
& \rightarrow \frac{7 - \sqrt{45}}{2}x_2 < x_1 < \frac{7 + \sqrt{45}}{2}x_2
\end{aligned}$$

An example of this is when  $x_1 = \frac{7}{2}x_2 \rightarrow x = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ . Plugging this into  $x_1^2 - 7x_1x_2 + x_2^2$ , we get  $-\frac{45}{4} < 0$ .

For  $(x_1 - \frac{7}{2}x_2)^2 - \frac{45}{4}x_2^2 > 0$  :

$$\begin{aligned}
& (x_1 - \frac{7}{2}x_2)^2 > \frac{45}{4}x_2^2, \text{ so} \\
& x_1 - \frac{7}{2}x_2 < -\sqrt{\frac{45}{4}}x_2 \cup x_1 - \frac{7}{2}x_2 > \sqrt{\frac{45}{4}}x_2 \\
& \rightarrow x_1 < \frac{7 - \sqrt{45}}{2}x_2 \cup x_1 > \frac{7 + \sqrt{45}}{2}x_2
\end{aligned}$$

An example of this is when  $x_1 = \frac{100}{2}x_2 \rightarrow x = \begin{pmatrix} 100 \\ 2 \end{pmatrix}$ . Plugging this into  $x_1^2 - 7x_1x_2 + x_2^2$ , we get  $9301 > 0$ .

Therefore this is **indefinite**.

## 3.16

$$\begin{aligned}\Delta_1 &= 2 \\ \Delta_2 &= \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0 \\ \Delta_3 &= \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{vmatrix} = 2 \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} + 2 \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0\end{aligned}$$

And all of them are nonnegative.

Looking at the quadratic form  $x^T \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} x$ :

$$\begin{aligned}x^T \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} x &= [2x_1 + 2x_2 + 2x_3 \quad 2x_1 + 2x_2 + 2x_3 \quad 2x_1 + 2x_2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= 2x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_2 + 2x_2^2 + 2x_2x_3 + 2x_1x_3 + 2x_2x_3 \\ &= 2x_1^2 + 2x_2^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 \\ &= 2(x_1 + x_2)^2 + 4x_3(x_1 + x_2)\end{aligned}$$

Now we try to make  $x_1 + x_2 < -2x_3$ , while making sure  $|x_1 + x_2| < |2x_3|$ . So by setting  $x_3 = 1$ , we can easily produce an example vector  $x = \begin{bmatrix} -\frac{1}{10} \\ -\frac{1}{10} \\ 1 \end{bmatrix}$ . If we plug this back into  $2(x_1 + x_2)^2 + 4x_3(x_1 + x_2)$ , we get

$$2(-0.2)^2 + 4(1)(0.2) = 0.08 - 0.8 < 0$$

As we can see, there exists a vector such that it's not positive definite, so  $A$  is not positive definite.