Counting in 2 ways

How many dots are in a $(n+1) \times (n+1)$ grid?

Method 1 (Count by Area)

$(n+1)^2$ dots.

ex $n = 4$, $(n+1 = 5)$

Method 2 (by diagonals)

$(n+1)$ - middle

$2(1+2+...+n)$ - diagonals

$$\begin{aligned}
(n+1)^2 &= (n+1) + 2(1+2+...+n) \\
(n+1)^2 - (n+1) &= 2(1+2+...+n) \\
(n+1)(n+1-1) &= 2(1+2+...+n) \\
\frac{(n+1)n}{2} &= 1+2+...n
\end{aligned}$$

Some more

Square #s

1, 4, 9, 16, 25, 36, ...

Cube #s

1, 8, 27, 64, 125

Triangle #s

1, 3, 6, 10, 15, 21,...

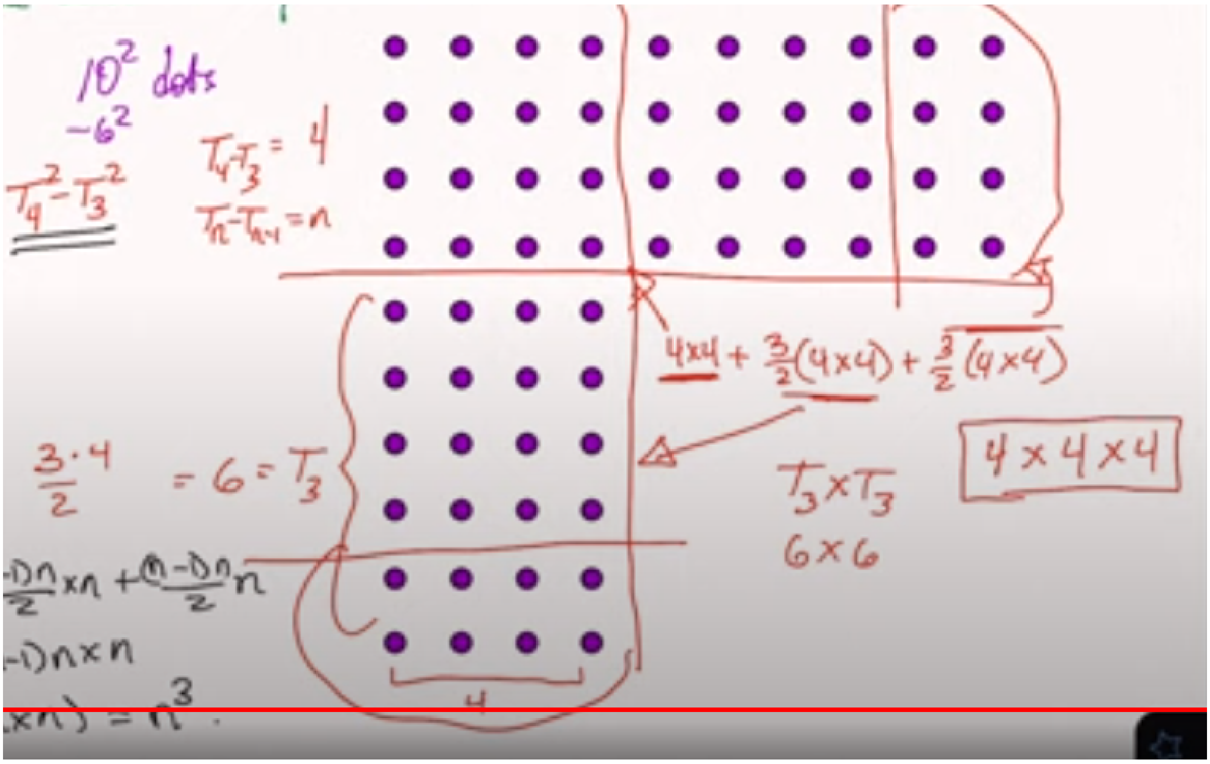
$1+2+...+n = \frac{n(n+1)}{2}$

Example

Show the difference between the squares of two consecutive triangle numbers is a cube.

$3^2 - 1^2 = 2^3$, $6^2 - 3^2 = 3^3$, $10^2 - 6^2 = 4^3$, $15^2 - 10^2 = 5^3$

Why is this true?



For the case $10^2 - 6^2 = 4^3$, we can see that $T_4^2 - T_3^2 = 4$, and this (removing them from the grid) leaves $T_3 = 6$ (using $\frac{n(n+1)}{2}$) twice. In other words $T_n^2 - T_{n-1}^2 = n \times n + \frac{(n-1)n}{2} \times n + \frac{(n-1)n}{2} \times n = (1 + (n-1)) \times n \times n$ or n^3.

Some more examples

Show the average number of divisors of $1, 2, ..., n$ is approximately $\log(n)$.

Divisors:	
1	1
2	1, 2
3	1,3
4	1, 2, 4
5	1, 5
6	1, 2, 3, 6

$\frac{1+2+2+3+2+4}{6} = 2.3$

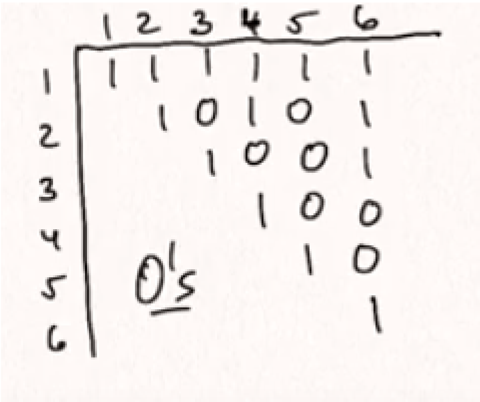
$\log(6) = 1.8$

$\text{div}(100) = 4.82$

$\log(100) = 4.61$

Make an array $(n \times n)$

$$a_{ij} = \begin{cases} 1 & \text{if i divides j} \\ 0 & \text{else} \end{cases}$$



Count by columns:

$$\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}\right) = \frac{1}{n}\sum_{j=1}^{n}(\text{number of divisors of j})$$

= average # of divisors of 1,2, ..., n

Count by rows: (row i, every ith entry is 1, others are 0).

$$\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\right) = \frac{1}{2}\sum_{i=1}^{n}\left(\lfloor\frac{n}{i}\rfloor\right)$$

$$\approx= \frac{1}{n}\sum_{i=1}^{n}\frac{n}{i} = \sum_{i=1}^{n}\frac{1}{i} \approx \log(n) \text{ (using calculus: Reimann sums)}$$