Chapter 6

Math 432

Cycles in Permutations

(Not quite in the book, but similar)

Recall that a permutation on n symbols can be decomposed into cycles (see Chapter 3). For example

$$\pi = 3, 1, 4, 6, 5, 2$$
$$(1, 3, 4, 6, 2), (2)$$

Let $n_i(\pi)$ = number of cycles of π of length i. So $n_1(\pi) = 1$, $n_5(\pi) = 1$, all other n_is are 0. Ask: pick a random π in S_n (the permutations on n symbols). How do the ns behave?

Clearly, if $\sum in_i \neq n$, the number of $\pi \in S_n$ with n_1 1-cycles, n_2 2-cycles, ... = 0. We proved earlier that if $\sum in_i \neq n$, then the number of $\pi \in S_n$ with n_1 1-cycles, n_2 2-cycles, ...$= \frac{n!}{\Pi_i (i^{ni})(ni!)}$.

Example

How many $\pi \in S_n$ consist of one cycles of length n?

Sol: $n_n(\pi) = 1$, all other $n_i(\pi) = 0$. Let $\frac{n!}{n^1 \times 1!} = (n-1)!$

Example 2

How many $\pi \in S_n$ consist of 1 2-cycle and $n-2$ cycles of size 1? (called fixed points) ((Such π are called "transpositions")).

Sol: By "box", answer is $\frac{n!}{1^{n-2}(n-2)!2^1(1)!} = \frac{(n)(n-1)}{2}$. Alternatively, answer is $\binom{n}{2}$ since π swaps two symbols and fixes the rest of them.

Example 3

Def: Let $c(n, k)$ = number of $\pi \in S_n$ with k many cycles (Sterling number of the first kind). Set $c(0, 0) = 1$, $c(n, k) = 0$ if $n < k$.

Theorem

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k)$$

Proof:

Let's show that the RHS counts the number of permutations with exactly k cycles. There's two cases:

1. n is in a cycle of length 1. Then there's $k-1$ cycles for the $n-1$ remaining elements, which contributes to $c(n-1, k-1)$
2. n is not in a cycle of length 1. Then take an element of S_{n-1} symbols with k cycles, and insert n into 1 of $n-1$ positions. Example: $n = 6$: $(1,4),(2,3,5)$. You can insert the 6 into $n-1$ positions. So this case contributes to $(n-1)c(n-1,k)$.

Theorem

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)...(x+n-1)$$

Test it out: $n = 3$

$$\begin{aligned} c(3,1) &= 2 \\ c(3,2) &= 3 \\ c(3,3) &= 1 \end{aligned}$$

$$x(2) + x^2(3) + x^3 = x(x+1)(x+2)$$

Proof:

Define $F_n(x) = x(x+1)...(x+n-1) = \sum_{k=0}^{n} b(n,k)x^k$. We'll show $b(n,k) = c(n,k)$ by showing they satisfy the same recursion. Since $F_n(x) = (x+n-1)F_{n-1}(x)$

$$\begin{aligned} &= xF_{n-1}(x) + (n-1)F_{n-1}(x) \\ = \sum_{k=1}^{n} b(n-1,k-1)x^k &+ (n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k \\ \to b(n,k) = b(n-1,&k-1) + (n-1)b(n-1,k) \end{aligned}$$

Since $b(n,k), c(n,k)$ satisfy the same recursion $b(n,k) = c(n,k)$ and the theorem is proved

Cycle Index Theorem

"The Product Symbol"

$$1 + \sum_{n\geq 1} \frac{u^n}{n!} \sum_{\pi \in S_n} \Pi_{i\geq 1} x_i^{n_i(\pi)} = \Pi_{m\geq 1} e^{\frac{x_m u^m}{m}}$$

Here $n_i(\pi)$ = number of cycles of π of length i.

Proof:

Note, if $\sum in_i \neq n$, then coeffircient of $u^n x_1^{n_1} x_2^{n_1}...$ on both sides $= 0$.

So suppose $\sum in_i = n$. Then coefficient of $u^n x_1^{n_1} x_2^{n_2}...$ on left side $= \frac{1}{n!}\times$ (the number of $\pi in S_n$ with n_1 1-cycles, n_2 2-cycles ...)

$$\begin{aligned} &= \frac{1}{n!}\frac{n!}{\Pi_{i\geq 1} i^{n_i}(n_i!)} \\ &= \frac{1}{\Pi_{i\geq 1} i_i^n (n_i!)} \end{aligned}$$

By the Taylor Expansion

$$e^z = 1 + \frac{z^1}{1!} + \frac{z^2}{2!} + ...$$

the right hand side $= \Pi_{m\geq 1}(1 + (\frac{x_m u^m}{m})^1 + \frac{(\frac{x_m u^m}{m})^2}{2!} + ...)$.

So coefficient of $u^m x_1^{n_1} x_2^{n_2} ... = \Pi_{i\geq 1}\frac{1}{i^{n_i}(n_i!)}$. These are equal, so we're done.

So What? Applications

1. Count the number of permutations in S_n with no fixed points ($n_1(\pi) = 0$). These are called *derangements*.

In cycle index, set $x_1 = 0, x_2 = x_3 = ... = 1$. Get

$$\begin{aligned} 1 + \sum_{n\geq 1}\frac{u^n}{n!}(\text{number of derangements}) = \Pi_{m\geq 2} e^{\frac{u^m}{m}} &= \frac{1}{e^u}\Pi_{m\geq 1} e^{\frac{u^m}{m}} \\ &= \frac{1}{e^u} e^{\sum_{m\geq 1}\frac{u^m}{m}} \\ &= \frac{1}{e^u} e^{-(\log(1-u))} \\ &= \frac{1}{e^u}\frac{1}{1-u} \\ &= (1 + u + u^2 + ...)\frac{1}{e^u} \end{aligned}$$

Take the coefficient of u^n on both sides. The proportion of derangements in S_n = The coefficient of u^n in $(1 + u + u^2 + ...)e^{-u} = \sum_{i=0}^n$coefficient of u^i in $e^{-u} = \sum_{i=0}^n \frac{(-1)^i}{i!}$.

Note: as $n \to \infty$, this converges to $e^{-1} = \frac{1}{e}$. (Say how many permutations of 100 without any fixed points? we can approximate with $\frac{1}{e}$).

2. Let's compute the expected value of n_1

$$E[n_i] = \text{ the average number of fixed points (cycles of size 1) of } \pi \in S_n$$

Example

$n = 3$

π	$n_1(\pi)$
123	3
132	1
213	1
231	0
312	0
321	1

So the average of $n_1(\pi)$ is

$$\frac{1}{6}(3+1+1+0+0+1) = 1$$

In cycle index, set $x_1 = x$, other $x_i = 1$. Get

$$\begin{aligned} 1 + \sum_{n\geq 1} \frac{u^n}{n!} \sum_{\pi \in S_n} &= e^{xu} \Pi_{m\geq 2} e^{\frac{u^m}{m}} \\ &= \frac{e^{xu}}{e^u} \Pi_{m\geq 1} e^{\frac{u^m}{m}} \\ &= \frac{e^{xu}}{e^u(1-u)} \end{aligned}$$

Now differentiate with respect to x and set $x = 1$. Get

$$1 + \sum_{n\geq 1} u^n (\text{average number of fixed points of } \pi \in S_n) = \frac{u}{1-u}$$

Take coefficient of u^n on both sides. Get

$$(\text{average number of fixed points of } \pi \in S_n) = 1$$

A Few Remarks

1. While $n \to \infty$, the distribution of number of fixed points of random $\pi \in S_n$ converges to Poisson(1) random variable.

Equivalently, for j-fixed

$$\lim_{n\to\infty} P(\pi \in S_n \text{ has } j\text{-fixed points}) = \frac{1}{e}\frac{1}{j!}$$

(Prove this using cycle index)

2. More generally, if $n \geq i$, then the average of the number of i-cycles of $\pi \in S_n$ is quald to $\frac{1}{i}$. As $n \to \infty$, the distribution of the number of i-cycles converges to Poisson($\frac{1}{i}$) random variable. This means

$$\lim_{n\to\infty} P(\pi \text{ has } j \text{ } i\text{-cycles}) = \frac{\left(\frac{1}{i}\right)^j}{e^{\frac{1}{i}}(j!)}$$

(Prove this using cycle index)

Let's resist counting permutations with a given number of cycles

In cycle index, set all $x_i = x$. We get

$$\begin{aligned} 1 + \sum_{n\geq 1} \frac{u^n}{n!} \sum_{\pi\in S_n} x^{c(\pi)} \\ = \Pi_{m\geq 1} e^{\frac{xu^m}{m}} \\ = \left(\Pi_{m\geq 1} e^{\frac{u^m}{m}}\right)^x \\ = \left(e^{\sum_{m\geq 1}\frac{u^m}{m}}\right)^x \\ = \left(e^{-\log(1-u)}\right)^x \\ = (1-u)^{-x} \end{aligned}$$

where $c(\pi)$ = the number of cycles of π.

Take coefficient of u^n on both sides. On left, get $\frac{1}{n!}\sum_{\pi\in S_n} x^{c(\pi)}$. On the right, get $\frac{1}{n!}x(x+1)(x+2)...(x+n-1)$.

Why?

Recall if

$$f(u) = \sum_{n\geq 0} a_n u^n$$

then

$$a_n = \frac{1}{n!} f^{(n)}(0)$$

Thus

$$\frac{1}{n!}\sum_{\pi\in S_n} x^{c(\pi)} = \frac{1}{n!}(x)(x+1)(x+2)...(x+n-1)$$

(We proved this earlier)

Remark

In "box", differentiate with respect to x and set $x = 1$. We get

$$\text{average number of cycles of a random permutation } = \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n} \sim \log(n)$$

Cycles to Records Bijection

Given a permutation $\pi \in S_n$, there's a unique way to write it as a product of cycles where each cycle starts with its largest element and then you order cycles in increasing order of their largest elements.

Example