Homework 1

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2.8

$$\langle \mathbf{x}, \mathbf{y} \rangle_2 = 2x_1y_1 + 3x_2y_1 + 3x_1y_1 + 5x_2y_2$$

1. Positivity

$$\begin{aligned}\langle \mathbf{x}, \mathbf{x} \rangle_2 &= 2x_1x_1 + 3x_1x_2 + 3x_1x_2 + 5x_2^2 \\ &= 2x_1^2 + 6x_1x_2 + 5x_2^2 \\ &= 2(x_1^2 + 3x_1x_2 + \frac{5}{2}x_2^2) \\ &= 2(x_1^2 + 3x_1x_2 + \frac{9}{4}x_2^2 + \frac{1}{4}x_2^2) \\ &= 2(x_1 + \frac{3}{2}x_2)^2 + \frac{1}{2}x_2^2 \geq 0\end{aligned}$$

It's easy to see that if $x = 0$, then $\langle \mathbf{x}, \mathbf{x} \rangle_2 = 0$

Now suppose $\langle \mathbf{x}, \mathbf{x} \rangle_2 = 2(x_1 + \frac{3}{2}x_2)^2 + \frac{1}{2}x_2^2 = 0$. That means

$2(x_1 + \frac{3}{2}x_2)^2 = -\frac{1}{2}x_2^2$ where $-\frac{1}{2}x_2^2 \leq 0$ and its equality is only when $x_2 = 0$.

Plugging it back in, $2(x_1 + \frac{3}{2}x_2)^2 \leq 0$ can only be true when $2(x_1 + \frac{3}{2}x_2^2)^2 = 0$ so $\mathbf{x} = 0$

2. Symmetry

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle_2 &= 2x_1y_1 + 3x_2y_1 + 3x_1y_2 + 5x_2y_2 \\ &= 2y_1x_1 + 3y_1x_2 + 3y_2x_1 + 5y_2x_2 \\ &= 2y_1x_1 + 3y_2x_1 + 3y_1x_2 + 5y_2x_2 \\ &= \langle \mathbf{y}, \mathbf{x} \rangle_2\end{aligned}$$

3. Additivity

$$\begin{aligned}
\langle \mathbf{x}+\mathbf{y},\mathbf{z}\rangle_2 &= 2(x_1+y_1)z_1+3(x_2+y_2)z_1+3(x_1+y_1)z_2+5(x_2+y_2)z_2 \\
&= 2x_1z_2+3x_2z_1+3x_1z_2+5x_2z_2+2y_1z_1+3y_2z_1+3y_1z_2+5y_2z_2 \\
&= \langle \mathbf{x},\mathbf{z}\rangle + \langle \mathbf{y},\mathbf{z}\rangle
\end{aligned}$$

4. Homogeneity

$$\begin{aligned}
\langle r\mathbf{x},\mathbf{y}\rangle &= 2(rx_1)y_1+3(rx_2)(y_1)+3(rx_1)y_2+5(rx_2)y_2 \\
&= r(2x_1y_1+3x_2y_1+3x_1y_2+5x_2y_2) \\
&= r\,\langle \mathbf{x},\mathbf{y}\rangle
\end{aligned}$$

2.9

$$\begin{aligned}
\mathbf{x} &= (\mathbf{x}-\mathbf{y})+\mathbf{y} \\
||\mathbf{x}|| = ||(\mathbf{x}-\mathbf{y})+\mathbf{y}|| &\leq ||\mathbf{x}-\mathbf{y}|| + ||\mathbf{y}|| \\
\to ||\mathbf{x}-\mathbf{y}|| &\geq ||\mathbf{x}|| - ||\mathbf{y}||
\end{aligned}$$

2.10

Since $||\mathbf{x}-\mathbf{y}|| < \delta$,

$$|||\mathbf{x}|| - ||\mathbf{y}||| < \delta$$

And we can let $\delta = \epsilon$.
