Chapter 4

Math 432

Binomial Theorem + Related Identities

The Theorem

For all non-negative integers n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof 1(Combinatorial)

To get a term $x^k y^{n-k}$ when you expand

$$(x+y)(x+y)...(x+y)$$

you choose k terms that contribute x and the other $n-k$ terms contribute y. We can do this in $\binom{n}{k}$ ways so the theorem follows.

Proof 2 (Induction)

When $n=1$, $x+y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0$ so the case for $n=1$ checks.

Let's assume it works for $n-1$, we need to prove it for n.

$$\begin{aligned}(x+y)^n &= (x+y)(x+y)^{n-1} \\ &= (x+y)\sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}\end{aligned}$$

Letting $i = k+1$ in first sum and $i = k$ in the second sum, we get

$$\sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i}$$

$$= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] x^i y^{n-i} + y^n$$

Recall

$$\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$$

So we get

$$x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i + y^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Corollaries

Corollary

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof

Set $x = -1, y = 1$ in the binomial theorem. $0^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$. (We'lll use this later when proving principle of inclusion-exclusion).

Another Corollary

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof

Set $x = 1, y = 1$ in the binomial theorem.

Trickier Corollary

For all non-negative intgers $n \geq 1$,

$$\sum_{k=1}^{n} k \binom{n}{k} = n 2^{n-1}$$

Proof 1 (Combinatorial)

Claim that both sides count the number of ways to form a committee from some non-empty subset of n people and then choose a president for the committee.

The left hand side have $\binom{n}{k}$ choices for a size k committee and then k choices for the president of the committee. For the right hand side, we can pick the president in n ways. Then you choose a subset of the other $n-1$ people to serve as a committee for the president, which you can do 2^{n-1} ways. And the theorem follows.

Proof 2

By the binomial theorem, $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Differentiate both sides with respect to x.

$$n(x+1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} kx^{k-1}$$

and then set $x = 1$.

$$n2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}$$

Remarks:

The theorem is useful for computing the mean of a binomial(n, $\frac{1}{2}$) random variable. Indeed, the mean is

$$\sum_{k=1}^{n} \frac{k\binom{n}{k}}{2} = \frac{n2^{n-1}}{2^n} = \frac{n}{2}$$

Next Theorem

For positive integers n, m, k,

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}$$

Exercise: Prove this combinatorially

Algebraic Proof

right hand side

$$\begin{aligned} &= \text{coefficient of } x^k \text{ in } [\sum_{i \geq 0} \binom{n}{i}][\sum_{j \geq 0} \binom{m}{j}] \\ &= \text{coeff of } x^k \text{ in } (x+1)^n (x+1)^m \\ &= \text{coeff of } x^k \text{ in } (x+1)^{n+m} \\ &= \binom{n+m}{k} \end{aligned}$$

Multinomial Theorem

when $a_1, a_2, ..., a_n$ are non-negative integers which sums to n, degine the "multinomial coefficient"

$$\binom{n}{a_1, a_2, ..., a_n} - = \frac{n!}{a_1!a_2!...a_n!}$$

when $k = 2$, this is a binomial coefficient.

Theorem

$a_1, a_2, ..., a_n$ is nonnegative and sum to n

$$(x_1 + ... + x_k)^n = \sum_{a_1, a_2, ..., a_n} \binom{n}{a_1, a_2, ..., a_n} x_1^{a_1} x_2^{a_2} ... x_k^{a_k}$$

When $k = 2$, this is the binomial theorem

Proof

Coefficient of $x_1^{a_1} x_2^{a_2} ... x_k^{a_k}$ in $(x_1 + ... + x_n)^n$

$=$ the number of length n strings with a_1 1s, a_2 2s,...a_k ks.

$$= \binom{n}{a_1, a_2, ..., a_n}$$

Remark

Let's give another proof that the number of length n strings with a_1 1s, a_2 2s, ..., a_k ks $= \binom{n}{a_1, a_2, ..., a_n}$.

Count these strings. Choose the positions for the 1s in $\binom{n}{a_1}$ ways. Then choose position for 2s in $\binom{n - a_1}{a_2}$ ways. Etc. So we get

$$\binom{n}{a_1}\binom{n - a_1}{a_2}\binom{n - a_1 - a_2}{a_3}...$$
$$= \frac{n!}{a_1!(n - a_1)!} \frac{(n - a_1)!}{a_2!(n - a_1 - a_2)!} \frac{(n_1 - a_1 - a_2)!}{a_3!(n - a_1 - a_2 - a_3)!}...$$
$$= \frac{n!}{a_1!a_2!a_3!...} = \binom{n}{a_1, a_2, ..., a_n}$$

What can we say about $(1 + x)^m$ when m is not a positive integer?

Let m be any real number, and k a non-negative integer. Define $\binom{m}{0} = 1$ and $\binom{m}{k} = \frac{m(m-1)...(m-k+1)}{k!}$ for $k \geq 1$.

Theorem

$$(1+x)^m = \sum_{n\geq 0} \binom{m}{n} x^n$$

_Note: if m is an integer, this is a finite sum since $\binom{m}{n} = 0$ if $n > m$.

Proof

if $f(x) = \sum_{n\geq 0} c_n x^n$, $c_n = \frac{1}{n!} f^{(n)}(0)$ (using calculus). So apply this to $f(x) = (1+x)^m$ nth derivative of $f(x) = m(m-1)...(m-n+1)(1+x)^{m-n}$. Evaluating this at $x = 0$ gives

$$m(m-1)...(m-n+1)$$

so coeffiveint of x^n in $(1+x)^m$ is equal to

$$\frac{m(m-1)...(m-n+1)}{n!} = \binom{m}{n}$$

Example

Find power series of $\sqrt{1-4x}$.

Note

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2} - \frac{1}{2} - \frac{3}{2} ... - \frac{(2n+3)}{2}}{n!} = \frac{(-1)^{n-1}(2n-3)!!}{2^n n!}$$

By the theorem

$$\begin{aligned} \sqrt{1-4x} &= (1-4x)^{\frac{1}{2}} \\ &= \sum_{n>0} \binom{\frac{1}{2}}{-4x^n} \\ &= 1 - 2x - \sum_{n\geq 2} \frac{(2)^{n-1}(2n-3)!!}{n!} x^n \\ &= 1 - 2x - \sum_{n\geq 2} \frac{2^n (2n-3)!!(n-1)!}{n!(n-1)!} x^n \end{aligned}$$

*(Double factorial is **not** factorial done twice)*

Note $2^{n-1}(n-1)!$ is equal to product of all even integers from 2 to $2n-2$. so we get

$$1 - 2x - \sum_{n\geq 2} \frac{2(2n-2)!}{n!(n-1)!} = 1 - 2x - \sum_{n\geq 2} \frac{2}{n} \binom{2n-2}{n-1}$$