Chapter 5

Math 432

Partitions

Definition

A sequence $(a_1, a_2, ..., a_k)$ of non-negative integers which sum to n is called a *weak composition* of n. If we require that the a_is are all positive, we get a *composition* of n. We call $a_1, a_2, ..., a_k$ the *parts* of the composition. Earlier we proved

1. For $n, k > 0$ the number of weak compositions of n into k parts is $\binom{n+k-1}{k-1}$.
2. For the number of compositions of n into k parts is $\binom{n-1}{k-1}$.

Corollary

For all positive integers n, the total number of compositions is 2^{n-1}.

Ex: Let $n = 3$*, the possible compositions are* $\{3\}, \{2+1\}, \{1+2\}, \{1+1+1\}$.

Exercise: *Try to prove the corollary*

Proof

A composition of n has at least 1 and at most n parts. So the number of compositions of n

$$= \sum_{k=1}^{n} \text{number of composition with } k \text{ parts}$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1}$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$$

(The final line from the binomial theorem)

Set Partitions

A partition of the set $\{1, ..., n\}$ is a collection of non-empty blocks such that each of $1, ..., n$ belongs to exactly one of the blocks.

Let $S(n, k) =$ the number of set partitions of $\{1, ..., n\}$ into k blocks. Thus $S(n, k) = 0$ if $n < k$ (you can't partition n things if you have too many blocks). Set $S(0, 0) = 0$. Call $S(n, k)$ the "Sterling numbers of the second kind". *(Order of the blocks don't matter)*

_Ex: $S(4, 2) = 7$. Since the set partitions of $\{1, 2, 3, 4\}$ into 2 blocks:

$$\{1,2,3\}\{4\}, \{1,2,4\}\{3\}, \{1,3,4\}\{2\}, \{2,3,4\}\{1\}$$
$$\{1,2\}\{3,4\}, \{1,3\}\{2,4\}, \{1,4\}\{2,3\}$$

_Ex: $S(n, n-1) = \binom{n}{2}$. A set partition of $\{1, ..., n\}$ into $n-1$ blocks has $n-2$ blocks of size 1 and 1 block of size 2. And you can pick the block of size 2 in $\binom{n}{2}$.

Theorem

For all postitive integers $k < n$.

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

(Allows you to compute quickly on a computer)

Proof

Consider a set partition of $\{1, 2, ..., n\}$ into k blocks. There are 2 cases.

1. n is a block of size 1. Then $\{1, 2, ..., n-1\}$ are in $k-1$ blocks. So this occurs in $S(n-1, k-1)$ ways.
2. n is not in its own block. Then $\{1, 2, ..., n-1\}$ are partitioned into k blocks. Then you chose one of the k blocks to choose which to insert into n. Case 2 occurs in $S(n-1, k)$ multiply by k.

Theorem (Sort of Neat)

The number of onto functions

$$f : \{1, 2, ..., n\} \mapsto \{1, 2, ..., k\}$$

is equal to

$$k!S(n, k)$$

Proof

Onto functions correspond to set partitions of $\{1, ..., n\}$ into k blocks with an ordering on the blocks. Why? The 1st block $= f^{-1}(1) =$ things that f sends to 1. The 2nd block $= f^{-1}(2)$... kth block $= f^{-1}(k)$.

Define

$$B_n$$

The number of set partitions of $\{1, ..., n\}$. The "**Bell number**". So $B(3) = 5$ since

$$\{1,2,3\}, \{1,2\}\{3\}, \{1,3\}\{2\}, \{2,3\}\{1\}, \{1\}\{2\}\{3\}$$

Note $B(5) = 52$ (And there's a card trick regarding that)

Remark

1. $B(n) = \frac{1}{e}\sum_{k\geq 0} \frac{k^n}{n!}$

For example, if $n = 1$,

$$\begin{aligned} B(1) &= \frac{1}{e}\sum_{k\geq 0} \frac{k}{k!} \\ &= \frac{1}{e}\sum_{j\geq 0} \frac{1}{j!} \\ &= \frac{1}{e}e = 1 \end{aligned}$$

(using

$$e^z = \sum_{j\geq 0} \frac{z^j}{j!}$$

)

2. $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$.

Proof

Let's show that the right hand side enumerates all set partitions of $\{1, 2, ..., n + 1\}$. Suppose $n+1$ is a block of size $n-i+1$. Then there are i elements not in same block as $n+1$. You can choose these i elements in $\binom{n}{i}$ ways and make them into a set partition in $B(i)$ ways. Summing over i proves the theorem.

Integer Partitions

Definition

Let $a_1 \geq a_2 \geq ... \geq a_k \geq 1$ be integers so that $a_1 + a_2 + ... + a_k = n$. Call the sequence

$$(a_1, ..., a_k)$$

a partition of n.

Let $p(n)$ = the number of partitions of n. $p_k(n)$ = the number of partitions of n into parts.

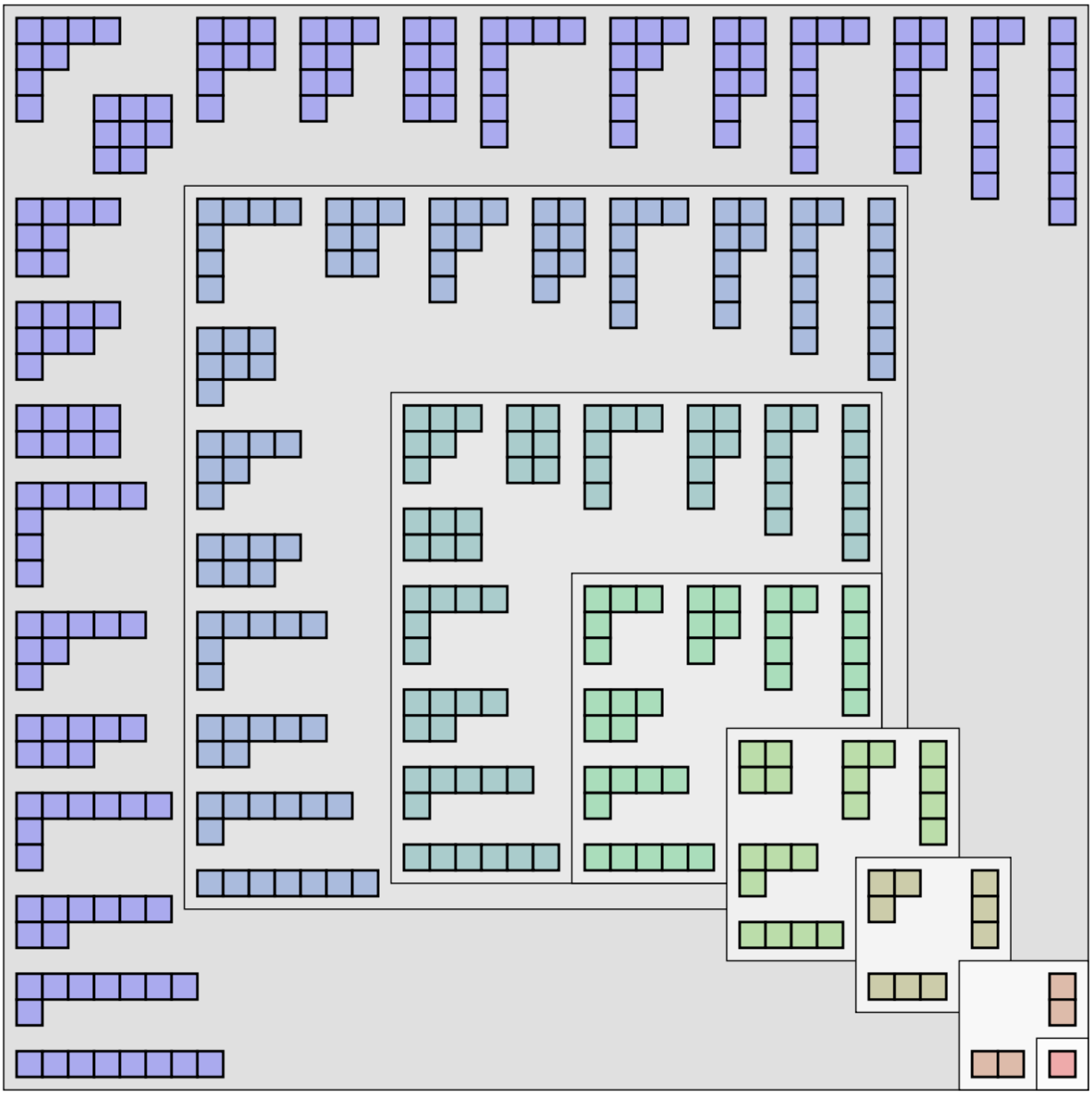

Figure 1: How to view partitions

Example

$p(4) = 5$, since the partitions of 4 are

$$\begin{aligned} (4) \\ (3,1) \\ (2,2) \\ (2,1,1) \\ (1,1,1,1) \end{aligned}$$

Ferrers Diagram

Remark

$p(n) \approx \frac{1}{4\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$

Theorem

The number of partitions of n into at most k parts = the number of partitions of n with all parts at most k.

Proof

A partition of n has at most k parts if and only if Ferrers diagram has at most k rows. Also, a partition has all parts at most k if and only if the ferrers diarram has at most k columns. The *conjugate* of a Ferrers diagram is given by "flipping the diagram".

So by taking conjugates, we see that the number of Ferrers diagram with at most n rows is equal to the number of Ferrers diagram with with at most k columns.

Another Theorem

The number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts

example

$n = 5$. partitions into distinct odd parts $= (5)$. Self-conjugate partitions:

Proof

Let's define f : self conjugate partitions of $n \to 2$ partitions of n into distinct odd parts.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & & \\ 1 & 2 & & & \\ 1 & 2 & & & \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & & \\ 3 & & & & & & & & \end{bmatrix}$$

Can invert f so f is a bijections and theorem is proved

Next theorem

Let $p(n)$ = number of partitions of n. Then $p(n) - p(n-1)$ = the number of partitions of n with no parts of size 1.

Proof

It's enough to show that $p(n-1)$ = number of partitions of n with at least 1 part of size 1.

Define a map: f : partitions of $n-1 \mapsto$ partitions of n with at least 1 part of size 1. f adds one row of size 1

Example

f :

$$\begin{matrix} X & X & X \\ X & X & X \end{matrix} \quad \mapsto \quad \begin{matrix} X & X & X \\ X & X & X \\ X & & \end{matrix}$$

Clearly f is a bijection so the result follows.

Theorem

$$\sum_{n\geq 0} p(n) z^n = \Pi_{i\geq 1} \frac{1}{1-z^i}$$

Proof

Need to show $p(n)$ = coefficient of z^n in $\Pi_{i\geq 1}\frac{1}{1-z^i}$.

$$\Pi_{i\geq 1} \frac{1}{1-z^i} = (1+z+z^2+z^3+...)(1+z^2+z^4+...)(1+z^3+z^6+...)...$$

take z^2, z^6, z^3 terms. partition of 11 into 2 partitions of size 1, 3 partitions of size 2, and 1 partitions of size 3.

Theorem

Let $p_o(n)$ be the number of partition of partitions into odd parts. Let $p_d(n)$ be the number of partitions of n into distinct parts. In fact, $p_o(n) = p_d(n)$.

Example

$n = 6$

partitions of odd parts:

$$\begin{gathered} (5,1) \\ (3,3) \\ (3,1,1) \\ (1,1,1,1,1) \end{gathered}$$

partitions of distinct parts:

$$
\begin{gathered}
(6) \\
(5,1) \\
(4,2) \\
(3,2,1)
\end{gathered}
$$

Proof

Arguing as in previous theorem,

$$p_o(n) = \text{ Coefficient of } z^n \text{ in } \Pi_{i \text{ odd}} \frac{1}{1-z^i}$$

$$
\begin{aligned}
p_d(n) &= \text{ Coefficient of } z^n \text{ in } \Pi_{i \geq 1}(1+z^i) \\
&= \text{ Coefficient of } z^n \text{ in } \Pi_{i \geq 1} \frac{1-z^2 i}{1-z^i} \\
&= \text{ Coefficient of } z^n \text{ in } \frac{\Pi_{i \text{ even}}(1-z^i)}{\Pi_{i \geq 1}(1-z^i)} \\
&= \text{ Coefficient of } z^n \text{ in } \frac{1}{\Pi_{i \text{ odd}}(1-z^i)}
\end{aligned}
$$

The result follows.

Book

"Paritions" by George Andrews