Lecture 1

vector space $\mathbf{R}^n$

vectors $v \in \mathbf{R}^n$

transposed vectors $v^T = |v_1, v_2, ..., v_n|$

$n \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{22} & ... & a_{1n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} = (a_{ij})_{1 \leq i,j \leq n}$

$n \times m$ matrix $A = \begin{bmatrix} a_{11} & a_{22} & ... & a_{1m} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nm} \end{bmatrix} = (a_{ij})_{1 \leq i,j \leq n}$

$n \times n$ matrix defines a linear map on $\mathbf{R}^n$

$v = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{pmatrix} \mapsto Av \in \mathbf{R}^n$ where $Av = \begin{pmatrix} (Av)_1 \\ (Av)_2 \\ ... \\ (Av)_n \end{pmatrix}$, $(Av)_i = \sum_1^n A_{ij} v_i$

Example $n = 2$

$A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$

$Av = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 + 4v_2 \\ v_1 + 2v_2 \end{pmatrix}$

Maps are linear:

1. if $v, w \in R^n$, then $A(v + w) = Av + Aw$
2. if $\alpha \in \mathbf{R}^n$, then $A(\alpha v) = \alpha(Av)$

Definition (linear product) $\quad (\cdot, \cdot) : \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R}$

that is if $x, y \in \mathbf{R}^n$, then

$(x, y) \in \mathbf{R}$

if

Linear Product Properties	Property
$\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = 0$	Positivity
$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$	Symmetry
$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$	Additivity
$\langle r\mathbf{x}, \mathbf{y} \rangle = r \langle \mathbf{x}, \mathbf{y} \rangle$ for every $r \in \mathbf{R}$	Homogeneity

Remark

1. Have $(\cdot, \cdot) \in \mathbf{R}^n$ is the dot product:

$$x^T = (x_1, x_2, ..., x_n), y^T = (y_1, y_2, ...y_n)$$

$$(x, y) = \sum_i^n x_i y_i$$

2. *Def*: $x \in \mathbf{R}^n$, then $||x|| = \sqrt{(x, x)}$ in Euclidean norm of x. also, $x, y \in \mathbf{R}n$, if $(x, y) = 0$ then x, y are orthogonal
3. $(z, x + y) = (x + y, z) = (x, z) + (y, z) = (z, x) + (z, y)$
4. $(x, \alpha y) = (\alpha y, x) = \alpha(y, x) = \alpha(x, y)$
5. $x, y \in \mathbf{R}^n$, $A = (a_{ij})_{1 \leq ij \leq n}$ then,

$$(Ax, y) = \sum_i^n (Ax)_i y_i$$

$$= \sum_i^n \left(\sum_j^i a_{ij} x_j \right) y_i$$

$$= \sum_i^n x_i a_{ij} y_i$$

$$= \sum_i^n x_i \left(a_{ij} y_i \right) = (x, A^T y)$$

$$\rightarrow (Ax, y) = (x, A^T y)$$

$$\rightarrow (x, Ay) = (A^T x, y)$$

Example take vectorspace of functions $f(t)$ on $[0, 2\pi]$

let f,g be the transformation defined inner product

$$(f, g) = \int_0^{2\pi} f(t)g(t)dt$$

then for instance, $n, m \in \mathbf{N}$

$$f(t) = \sin(nt), g(t) = \sin(mt)$$

then if $n \neq m$

$$(f,g)=\int_0^{2\pi}\sin(nt)\sin(mt)dt$$
$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$
$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$$
$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin(\alpha)\sin(\beta)$$
$$\alpha=nt,\beta=mt,n\neq m$$
$$\to\sin(nt)\sin(mt)=\frac{1}{2}(\cos((n-m)t)-\cos((n+m)t))$$

so

$$(f,g)=\int_0^{2\pi}\sin(nt)\sin(mt)dt$$

$$=\frac{1}{2}\int_0^{2\pi}\cos((n-m)t)-\cos((n+m)t)dt=0$$

that is f,g are orthogonal

Theorem (Cauchy-Schwarz Inequality)

$x,y\in\mathbf{R}^n$, then

$$|(x,y)|\leq||x||\cdot||y||$$

Proof

let $v,w\in\mathbf{R}^n$ so that $||v||=||w||=1$, then

$$0\leq||v+w||^2=(v+w,v+w)$$
$$\to 0\leq(v,v)+(v,w)+(w,v)+(w,w)$$
$$=||v||^2+2(v,w)+||w||^2$$
$$=2(1+(v,w))$$
$$\to 0\leq 1+(v,w)$$
$$\to(v,w)\leq 1=||v||\cdot||w||$$

also $0\leq||v-w||^2$ given $-(v,-w)\leq 1$

in general, $x,y\in\mathbf{R}^n$ let

$$
\begin{aligned}
v = \frac{x}{||x||} &\rightarrow ||v|| = 1 \\
w = \frac{y}{||y||} &\rightarrow = 1 \\
\rightarrow \left(\frac{x}{||x||}, \frac{y}{||y||} \right) &\leq 1 \\
\rightarrow \frac{1}{||x|| \cdot ||y||} |(x, y)| &\leq 1 \\
\rightarrow |(x, y)| &\leq ||x|| \cdot ||y||
\end{aligned}
$$

Triangle Inequality

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$$

Homework

p22f 2.8, 2.9, 2.10