

Notes

Minyoung Heo

Jan 10 2024

Inner Products

vectorspace \mathbb{R}^n

inner product:

$$\begin{aligned}\mathbb{R}^n \times \mathbb{R}^n &\mapsto \mathbb{R} \\ \mathbf{v}, \mathbf{w} &\int \mathbb{R}^n \\ \text{then, } \langle \mathbf{v}, \mathbf{w} \rangle &= \sum_{i=1}^n v_i w_i\end{aligned}$$

Cauchy-Schwarz:

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \cdot \|\mathbf{w}\|$$

where $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$

Theorem (triangle inequality)

$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

Proof

$$\begin{aligned}\|\mathbf{v} + \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle \\ &= \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \\ &= \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 \\ \rightarrow \|\mathbf{v} + \mathbf{w}\|^2 &\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\| \cdot \|\mathbf{w}\| + \|\mathbf{w}\|^2 \\ &= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2\end{aligned}$$

Example of other norms

$\|\cdot\|$ is a norm if

Example $x \in \mathbb{R}^n$,

$$\|x\| = \max_{1 \leq i \leq n} |x_i|$$

in $n = 2$. $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\|x\| = \max(|x_1|, |x_2|)$

Ball of radius r

$$B_r(x) = \{y \in \mathbb{R}^2 : \|y - x\| < r\}$$

It's a square:

$$\leq \{y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : |y_1|, |y_2| < r\}$$

Example $x \in \mathbb{R}^n$

$$\|x\| = \sum_{i=1}^n |x_i|$$

in $n = 2$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \|x\| = |x_1| + |x_2|$.

Ball of radius r again. It's a box but looks diagonal. It's called a "*taxicab norm*".

Example $x \in \mathbb{R}^n$

$$\|x\| = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

(exercise: verify with triangle inequality)

With the same ball of radius r again, it's like the first norm but the corners are curved.

In the complex:

Innerproduct: $x, y \in \mathbb{C}^n$,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}$$

$x_i, y_i \in \mathbb{C}$

for instance $y_j = u_j + iv_j$, $i^2 = -1$

$u_j = \operatorname{Re} y_j$ the real part

$v_j = \operatorname{Im} y_j$ the imaginary part

then $y_j \overline{y_j} = (u_j + iv_j)(u_j - iv_j) = (u_j)^2 - (iv_j)^2 = u_j^2 + v_j^2$

therefore, $y_j \overline{y_j} = |y_j|^2$

then,

$$\begin{aligned}\langle x, y \rangle &= \sum_{j=1}^n x_j \overline{y_j} \\ y_j &= \overline{y_j}\end{aligned}$$

only if $y_j \in \mathbb{R}$.

$$u_j + iv_j = u_j - iv_j \rightarrow v_j = 0$$

note:

$$\begin{aligned}\langle x, y \rangle &= \sum_{j=1}^n x_j \overline{y_j} \\ \langle y, x \rangle &= \sum_{j=1}^n y_j \overline{x_j} \\ \overline{\langle y, x \rangle} &= \overline{\sum_{j=1}^n y_j \overline{x_j}} \\ &= \sum_{j=1}^n \overline{y_j \overline{x_j}} \\ &= \sum_{j=1}^n \overline{y_j} x_j \\ &= \langle x, y \rangle\end{aligned}$$

Eigenvalues

A is a $n \times n$ real matrix

$$A \in \mathbb{R}^{n \times n}, A = (a_{ij})_{1 \leq i, j \leq n}, a_{ij} \in \mathbb{R}$$

Characteristic Polynomial

$$P(\lambda) = \det(A - \lambda I)$$

degree = $n \rightarrow P(\lambda) = 0$ has n roots: $\lambda_1, \lambda_2, \dots, \lambda_n$

λ_i are eigenvalues of A .

if λ_j is eigenvalue of $A \rightarrow \det(A - \lambda_j I) = 0$ (i.e. $A - \lambda_j I$ has rank $\leq n - 1$)

\rightarrow it has nontrivial null space

if $\lambda_j \in \mathbb{C} \rightarrow$ there exists $v \in \mathbb{C}^n$ such that

$$Av = \lambda_j v$$

if $\lambda_j \in \mathbb{R} \rightarrow$ there exists $v \in \mathbb{R}^n$ such that

$$Av = \lambda_j v$$

then v is called an eigenvector of A to eigenvalue λ_j .

Theorem Assume $A \in \mathbb{R}^{n \times n}$ has n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$, then it has n linearly independent eigenvectors.

Proof λ_j eigenvalue \rightarrow there exists $v_j \in \mathbb{R}^n$ so that $Av_j = \lambda_j v_j$, $j = 1, \dots, n$.

We have to know v_1, \dots, v_n are linearly independent (if

$$\sum_{i=1}^n c_i v_i = 0$$

for coefficients c_1, c_2, \dots, c_n then one must have $c_i = 0$ for all i .)

Use the trick: Put $z_i = \prod_{k \neq i} (\lambda_k I - A)$ so

$$z_j = (\lambda_n I - A) \dots (\lambda_{j+1} I - A) (\lambda_{j-1} I - A) \dots (\lambda_2 I - A) (\lambda_1 I - A) \quad (\text{notice no } j \text{ term})$$

then,

$$\begin{aligned} (\lambda_k I - A)v_j &= \lambda_k v_j - \lambda_j v_j = (\lambda_k - \lambda_j)v_j \\ &\rightarrow z_j v_j = \prod_{k \neq j} (\lambda_k - \lambda_j)v_j \neq 0 \end{aligned}$$

Also

$$z_j v_k = 0$$

Now apply z_i to $\sum_{k=1}^n c_k v_k$.

$$\begin{aligned} z_i \left(\sum_{k=1}^n c_k v_k \right) &= \sum_{k=1}^n z_i (c_k v_k) \\ &= \sum_{k=1}^n c_k z_i v_k \\ &= c_j z_j v_j \end{aligned}$$

But $z_j v_j \neq 0$ so if these were to be 0, $c_j = 0$. This is true for all $j = 1, 2, \dots, n$. So $c_j = 0$ for all j .

$\rightarrow v_1, v_2, \dots, v_n$ are linearly independent.

Homework

pg 40: 3.8, 3.9