Pigeonhole Principle

"combinatorics is an honest subject. You can count balls in boxes and either you have the right number or you haven't" - Gian-Carlo ROTA

Combinatorics is a concrete subject: you can do it with your hands. Construct an image, draw an example.

Pigeonhole Principle

k balls into n boxes and $k > n$

$\rightarrow$ at least 1 box has 2 balls

When we have a statement that is *painfully* obvious, we will use proof by contradiction.

Proof

Suppose not ($k > n$ s.t. no box has at least 2 balls *i.e. each box has* $\leq$ *1 ball*). That means the number of balls k must be at most the number of boxes n (i.e. $k \leq n$). Which is a contradiciton.

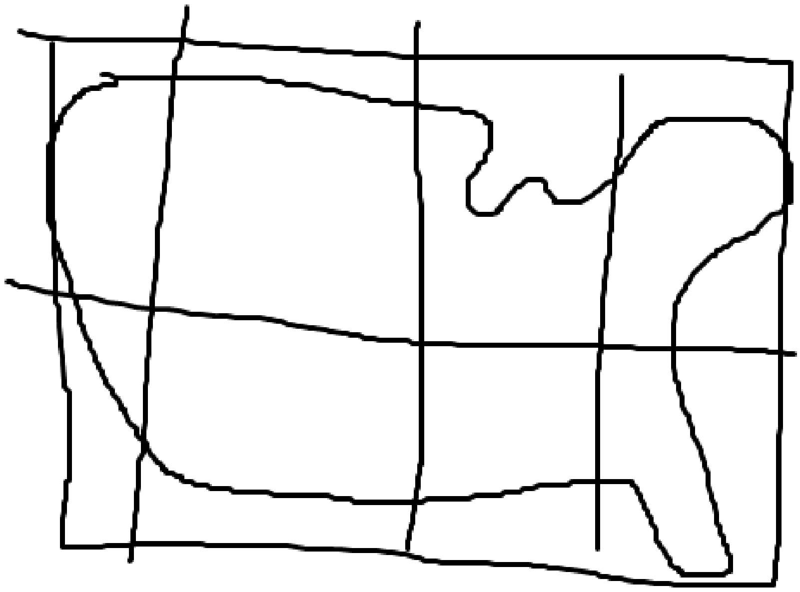
Example

Among any 7 people living in the contiguous U.S., at least 2 people live within 1500 miles of one another.

Ask: which are the balls? and which are the boxes?

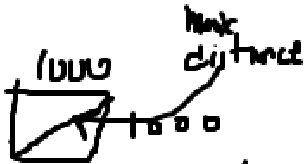
- You can think of "objects" as balls and "properties" as boxes.

So the balls are *people* and the boxes are *anything that has the property of being close together* This means I need 6 (or less) "boxes". Draw a picture: (figure 1)



W ↔ E $\leq$ 3000 mi

N ↕ S $\leq$ 2000 mi



1500 > 1414 ≈ $\sqrt{2}(1000)$

6 regions each is 1000 mi x 1000 mi

7 people → 2 in same region
2 within 1414 miles

Example

Pick any 6 numbers from 1 to 10. Then there are 2 such that one divides the other.

Think in terms of balls and boxes.

Object: numbers (which means we are looking at ≤ 5 boxes)

Property: they divide each other in the same box

$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

Approach: Group the numbers like so (in each group, the numbers divide each other)

#
1, 2, 4, 8
3, 6
5, 10
7
9

We can see that if we try to pick 6 numbers, we will end up picking 2 from the same group (that divide each other)

Why is this useful?

Think more generally. Rewrite the problem like so:

Pick any $m+1$ numbers from 1 to $2m$. Then there are 2 such that one divides the other.

$m+1$ balls (means m boxes)

By doing it specifically, it helps us solve the general problem.

General Solution

Make the boxes with the odd numbers

(i.e. |#| |—| |1| |3| |5| |...| |2m-1|)

On each box we can put 2^i times the first number.

(i.e. |#| |—| |1, 2, 4, ...| |3, $2 \cdot 3$, $2^2 \cdot 3$, ...| |5, $2 \cdot 5$, $2^2 \cdot 5$, ...| |...| |$2m-1$, $2 \cdot (2m-1)$, $2^2 \cdot (2m-1)$, ...|)

So each number is $2^j \cdot$some odd number. Which means every other number is in this list.