Algorithm Design and Implementation

Principle of Algorithms IV

Strongly Connected Components

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Depth-First Search in Graphs

Exploring Graphs

EXPLORE(G, v)

input : $G = (V, E)$ is a graph; $v \in V$

output: $visited(u)$ to *true* for all nodes u reachable from v

$visited(v) = true$;

PREVISIT(v);

for *each edge* $(v, u) \in E$ **do**

    if *not* $visited(u)$ **then** EXPLORE(G, u);

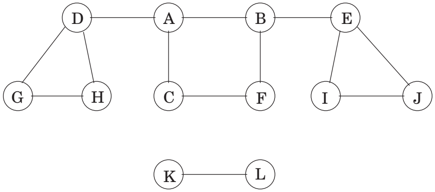
end

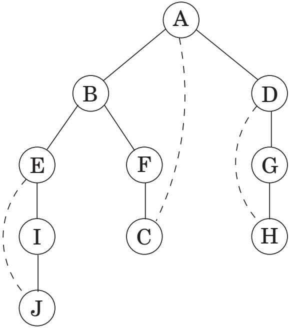
POSTVISIT(v);

Types of Edges in Undirected Graphs

Those edges in G that are traversed by EXPLORE are tree edges.

The rest are back edges.





Depth-First Search

```
DFS(G)

for all v ∈ V do
    visited(v) = false;
end
for all v ∈ V do
    if not visited(v) then Explore(G, v);
end
```

Previsit and Postvisit Orderings

For each node, we will note down the times of two important events:

- the moment of first discovery (corresponding to `PREVISIT`);
- and the moment of final departure (`POSTVISIT`).

`PREVISIT(`v`)`

$pre[v] = clock;$
$clock++;$

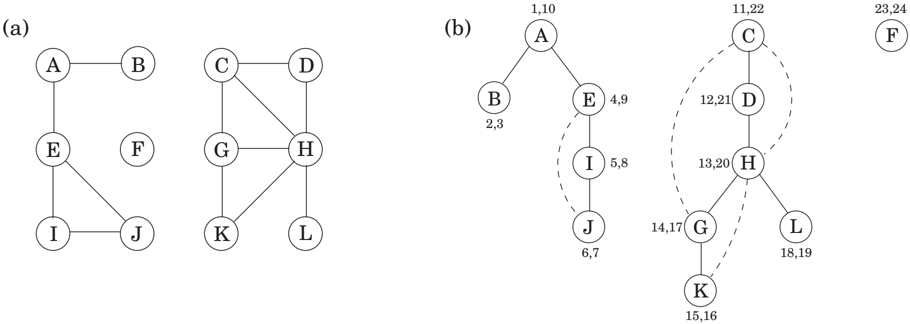
`POSTVISIT(`v`)`

$post[v] = clock;$
$clock++;$

Lemma

For any nodes u and v, the two intervals $[pre(u), post(u)]$ and $[pre(u), post(u)]$ are either disjoint or one is contained within the other.

Previsit and Postvisit Orderings



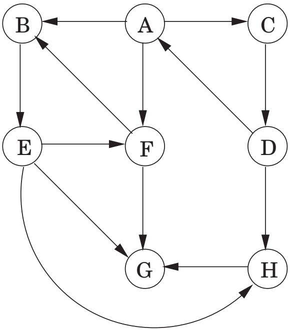
Types of Edges in Directed Graphs

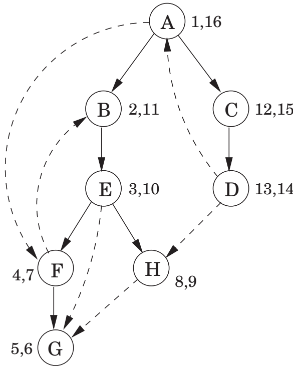
DFS yields a search tree/forests.

- root.
- descendant and ancestor.
- parent and child.

- Tree edges are actually part of the DFS forest.
- Forward edges lead from a node to a nonchild descendant in the DFS tree.
- Back edges lead to an ancestor in the DFS tree.
- Cross edges lead to neither descendant nor ancestor.

Directed Graphs





Types of Edges

pre/post ordering for (u, v)				Edge type
$[_u$	$[_v$	$]_v$	$]_u$	Tree/forward
$[_v$	$[_u$	$]_u$	$]_v$	Back
$[_v$	$]_v$	$[_u$	$]_u$	Cross

Q: Is that all?

Directed Acyclic Graphs (DAG)

Definition

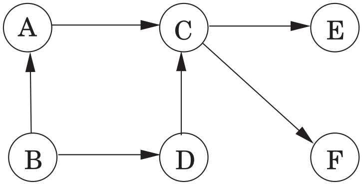
A cycle in a directed graph is a circular path

$$v_0 \to v_1 \to v_2 \to \ldots v_k \to v_0$$

Lemma

A directed graph has a cycle if and only if its depth-first search reveals a back edge.

Directed Acyclic Graphs (DAG)



Directed Acyclic Graphs (DAG)

Linearization/Topologically Sort: Order the vertices such that every edge goes from a earlier vertex to a later one.

Q: What types of dags can be linearized?

A: All of them.

DFS tells us exactly how to do it: perform tasks in decreasing order of their post numbers.

The only edges (u, v) in a graph for which $post(u) < post(v)$ are back edges, and we have seen that a DAG cannot have back edges.

Directed Acyclic Graphs (DAG)

Lemma

In a DAG, every edge leads to a vertex with a lower post number.

Directed Acyclic Graphs (DAG)

There is a linear-time algorithm for ordering the nodes of a DAG.

Acyclicity, linearizability, and the absence of back edges during a depth-first search - are the same thing.

The vertex with the smallest post number comes last in this linearization, and it must be a sink - no outgoing edges.

Symmetrically, the one with the highest post is a source, a node with no incoming edges.

Directed Acyclic Graphs (DAG)

Lemma

Every DAG has at least one source and at least one sink.

Algorithm

The guaranteed existence of a source suggests an alternative approach to linearization:

1. Find a source, output it, and delete it from the graph.
2. Repeat until the graph is empty.

Strongly Connected Components

Defining Connectivity for Directed Graphs

Definition

Two nodes u and v of a directed graph are connected if there is a path from u to v and a path from v to u.

Definition

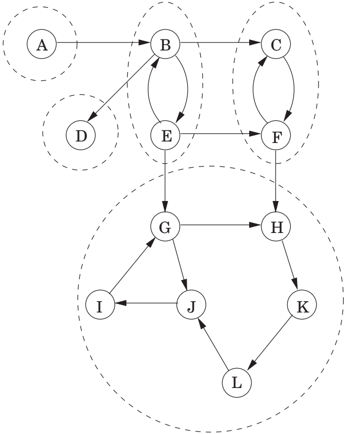
SCC This relation partitions V into disjoint sets that we call strongly connected components (SCC).

Lemma

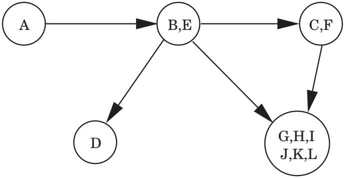
Every directed graph is a DAG of its SCC.

Strongly Connected Components

(a)



(b)



An Efficient Algorithm

Lemma

If the `EXPLORE` *subroutine at node u, then it will terminate precisely when all nodes reachable from u have been visited.*

If we call explore on a node that lies somewhere in a sink SCC, then we will retrieve exactly that component.

We have two problems:

1. How do we find a node that we know for sure lies in a sink SCC?
2. How do we continue once this first component has been discovered?

An Efficient Algorithm

Lemma

The node that receives the highest post number in a depth-first search must lie in a source SCC.

Lemma

If C and C' are SCC, and there is an edge from a node in C to a node in C', then the highest post number in C is bigger than the highest post number in C'.

Hence the SCCs can be linearized by arranging them in decreasing order of their highest post numbers.

Solving Problem A

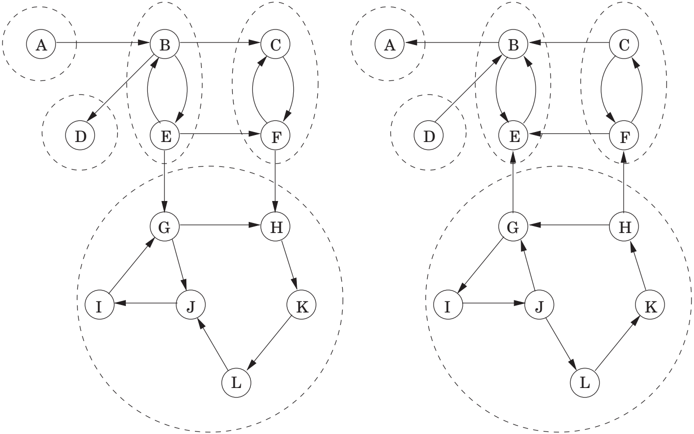
Consider the reverse graph G^R, the same as G but with all edges reversed.

G^R has exactly the same SCCs as G.

If we do a depth-first search of G^R, the node with the highest post number will come from a source SCC in G^R.

It is a sink SCC in G.

Strongly Connected Components



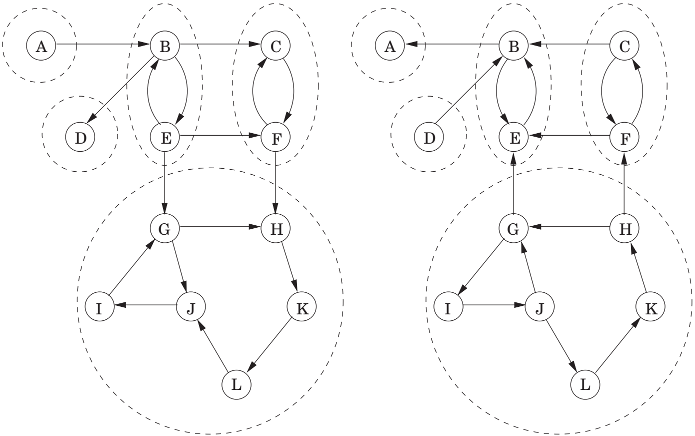
Solving Problem B

Once we have found the first SCC and deleted it from the graph, the node with the highest post number among those remaining will belong to a sink SCC of whatever remains of G.

Therefore we can keep using the post numbering from our initial depth-first search on G^R to successively output the second strongly connected component, the third SCC, and so on.

1. Run depth-first search on G^R.
2. Run the EXPLORE algorithm on G, and during the depth-first search, process the vertices in decreasing order of their post numbers from step 1.

Strongly Connected Components



Thinking About

How the SCC algorithm works when the graph is very, very huge?

An Easter egg: A report or a solution will earn extra scores!

A Question to Think About

How about edges instead of paths?

Exercises

Exercises 1

Suppose a CS curriculum consists of n courses, all of them mandatory. The prerequisite graph G has a node for each course, and an edge from course v to course w if and only if v is a prerequisite for w. Find an algorithm that works directly with this graph representation, and computes the minimum number of semesters necessary to complete the curriculum (assume that a student can take any number of courses in one semester). The running time of your algorithm should be linear.

Exercises 2

Give an efficient algorithm which takes as input a directed graph $G = (V, E)$, and determines whether or not there is a vertex $s \in V$ from which all other vertices are reachable.