Red-Black Tree

8.高级搜索树

（xa1）红黑树：动机

As she looks at the blood on the snow, she says to herself, "Oh, how I wish that I had a daughter that had skin WHITE as snow, lips RED as blood, and hair BLACK as ebony".

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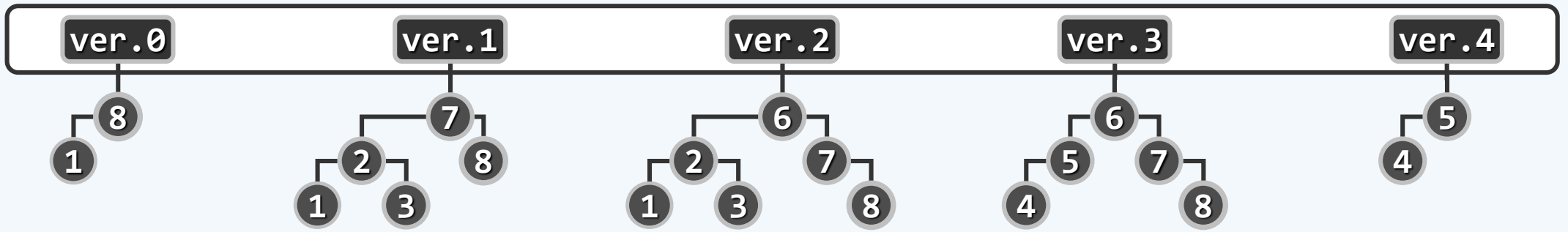
一致性结构

❖ Persistent structure：支持对历史版本的访问 //ephemeral

T.search(ver, key); T.insert(ver, key); T.remove(ver, key)

❖ 蛮力实现：每个版本独立保存；各版本入口自成一个搜索结构



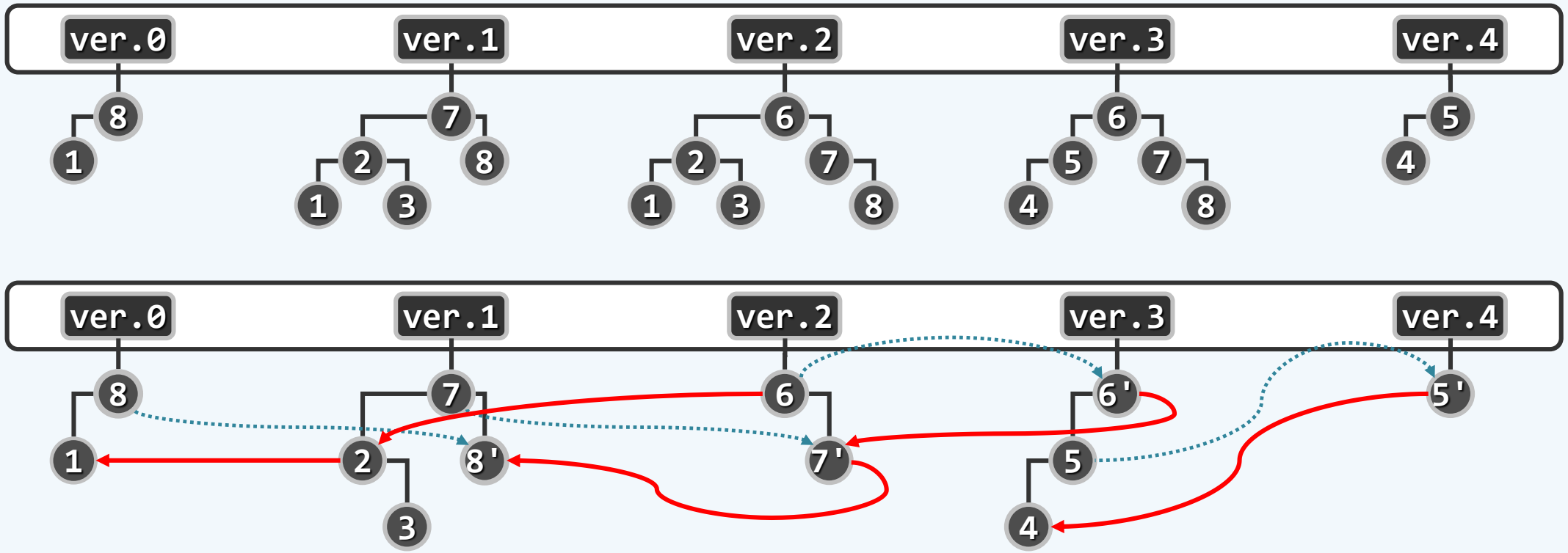
❖ 单次操作$O(\log h + \log n)$，累计$O(h*n)$时间/空间 //h = |history|

❖ 挑战：可否将复杂度控制在$O(n + h*\log n)$内？

❖ 可以！为此需利用相邻版本之间的关联性...

O(1)重构

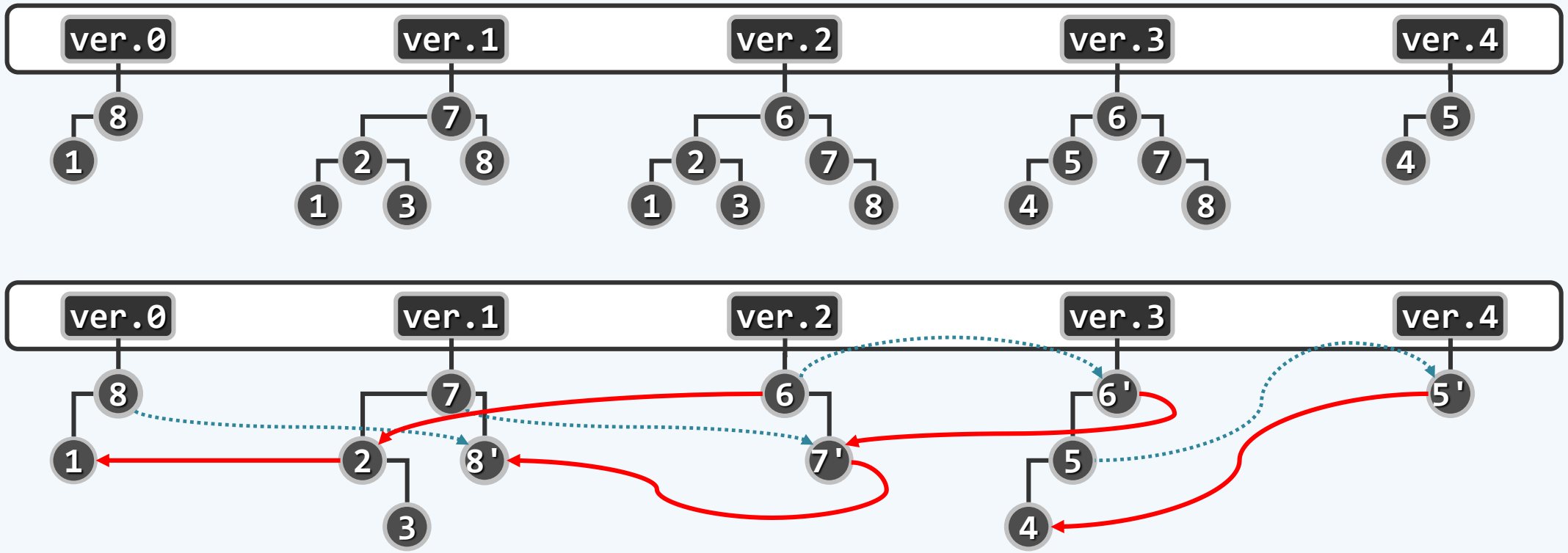
❖ 大量共享，少量更新：每个版本的新增复杂度，仅为O(logn)



❖ 能否进一步提高，比如总体O(n + h)、单版本O(1)？可以！

O(1)重构

❖ 为此，就树形结构的拓扑而言，相邻版本之间的差异不能超过O(1)



❖ 很遗憾，AVL、Splay等BBST均不具备这一性质；须另辟蹊径...

java.util.TreeMap

```
import java.util.*;

public class TestTreeMap {
   public static void main( String[] args ) {
      TreeMap scarborough = new TreeMap();
      scarborough.put( "P", "parsley" );
      scarborough.put( "S", "sage" );
      scarborough.put( "R", "rosemary" );
      scarborough.put( "T", "thyme" );
      System.out.println( scarborough );
   }
}
```

8.高级搜索树

（xa2）红黑树：结构

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红与黑

- 1972, R. Bayer, "symmetric binary B-tree"

  1978, L. Guibas & R. Sedgewick, "red-black tree"

  1982, H. Olivie, "half-balanced binary search tree"

- 由红、黑两类节点组成的BST //亦可给边染色

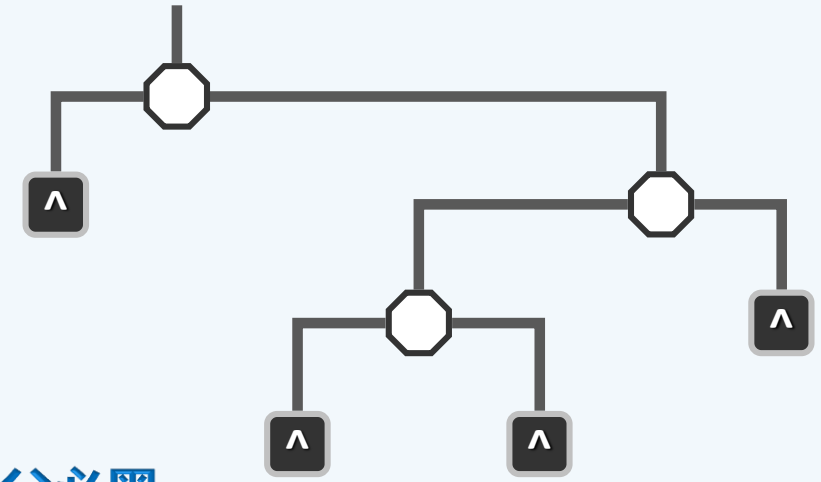
  (统一增设外部节点NULL，使之成为真二叉树)

  (1) 树根：必为黑色

  (2) 外部节点：均为黑色

  (3) 其余节点：若为红，则只能有黑孩子 //红之子、之父必黑

  (4) 外部节点到根：途中黑节点数目相等 //黑深度



- 以上定义颇为费解，有直观解释吗？如此定义的BST，也是BBST？

Acta Informatica 1, 290—306 (1972)

Symmetric Binary B-Trees: Data Structure and Maintenance Algorithms*

R. Bayer

Received January 24, 1972

Summary. A class of binary trees is described for maintaining ordered sets of data. Random insertions, deletions, and retrievals of keys can be done in time proportional to $\log N$ where N is the cardinality of the data-set. Symmetric B-Trees are a modification of B-trees described previously by Bayer and McCreight. This class of trees properly contains the balanced trees.

Example

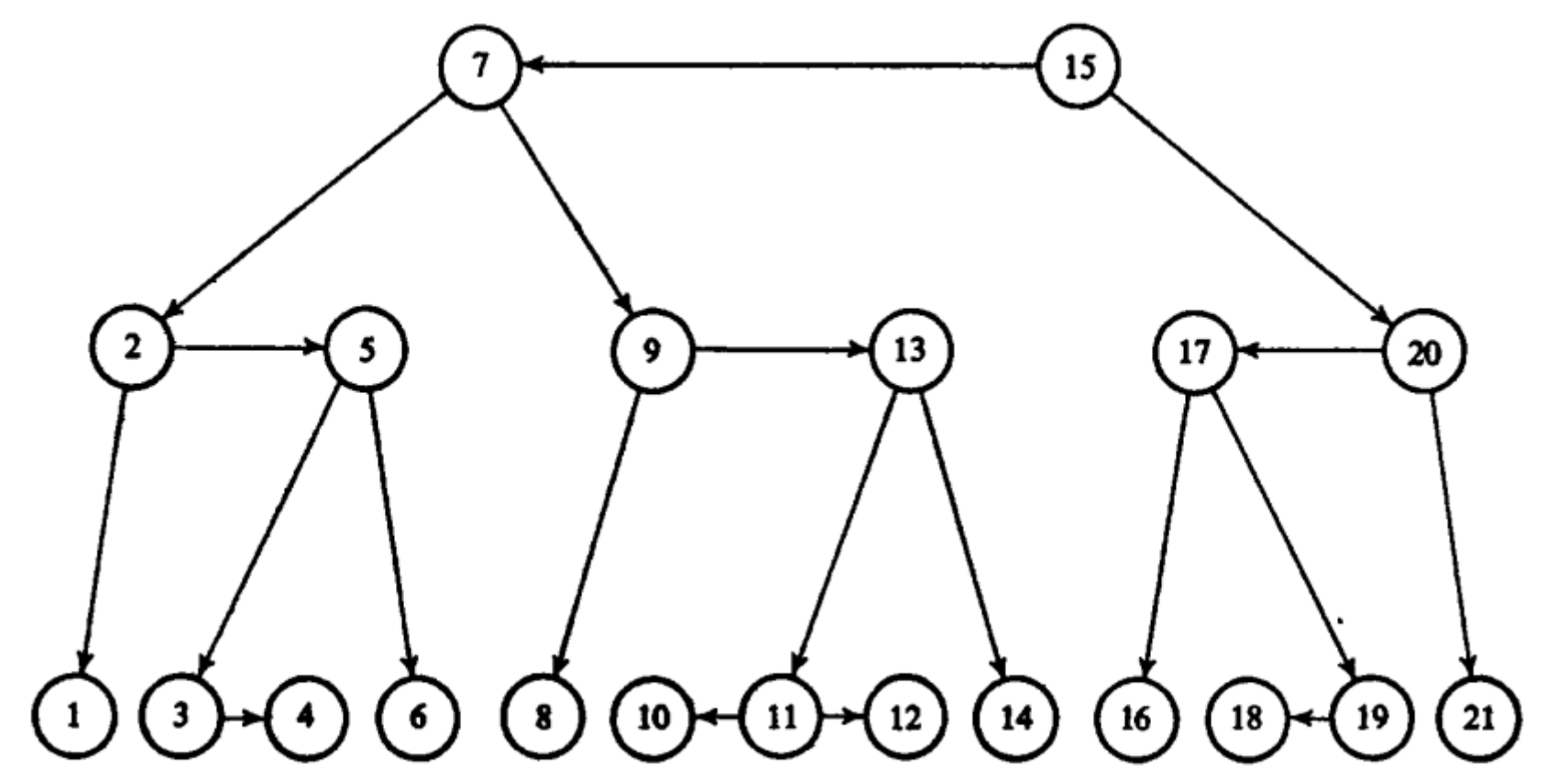
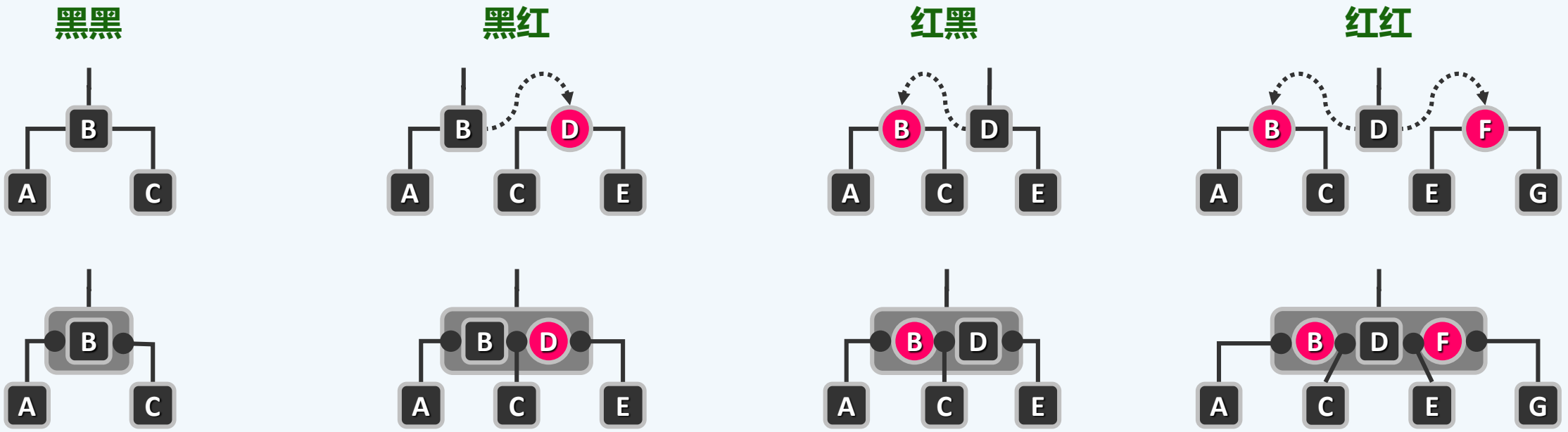


Fig. 1. Example of a symmetric binary B-tree

(2，4)树 == 红黑树

- 提升各红节点，使之与其（黑）父亲等高——于是每棵红黑树，都对应于一棵(2，4)-树
- 将黑节点与其红孩子视作（关键码并合并为）超级节点...
- 无非四种组合，分别对应于4阶B-树的一类内部节点 //反过来呢?



红黑树 ∈ BBST

- 由等价性，既然B-树是平衡的，红黑树自然也应是 //更严谨地...
- 定理：包含n个内部节点的红黑树T，高度 h = O(logn) //n + 1个外部节点

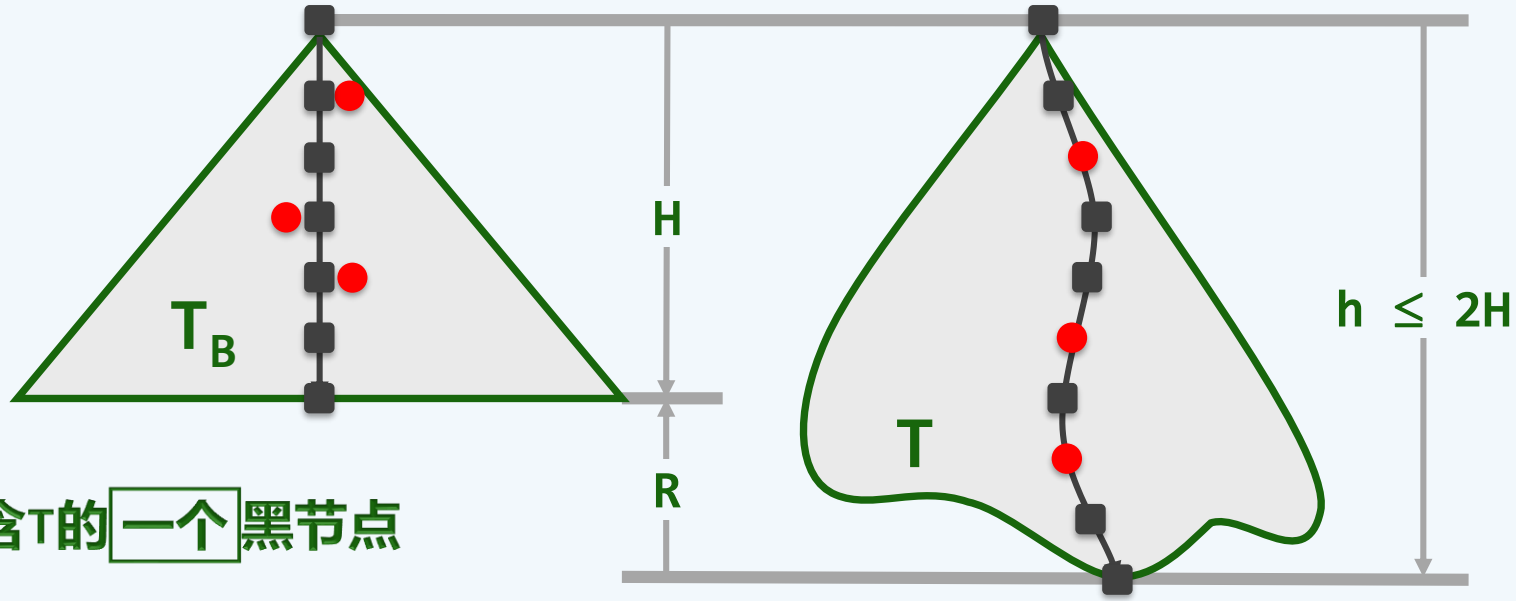
$$log_2(n+1) \leq h \leq 2 * log_2(n+1)$$

- 若：T高度为h，黑高度为H

  则：h = R + H ≤ 2H

- 若T所对应的B-树为T_B

  则H即是T_B的高度



- T_B的每个节点，包含且仅包含T的一个黑节点

- 于是，$H \leq log_{\lceil 4/2 \rceil} \frac{n+1}{2} + 1 \leq log_2(n+1)$

RedBlack

```
❖ template <typename T> class RedBlack : public BST<T> { //红黑树
  public:     //BST::search()等其余接口可直接沿用
              BinNodePosi(T) insert( const T & e ); //插入（重写）
              bool remove( const T & e ); //删除（重写）
  protected:  void solveDoubleRed( BinNodePosi(T) x ); //双红修正
              void solveDoubleBlack( BinNodePosi(T) x ); //双黑修正
              int updateHeight( BinNodePosi(T) x ); //更新节点x的高度
  };

❖ template <typename T> int RedBlack<T>::updateHeight( BinNodePosi(T) x ) {
     x->height = max( stature( x->lc ), stature( x->rc ) );
     if ( IsBlack( x ) ) x->height++; return x->height; //只计黑节点
  }
```

8.高级搜索树

（xa3）红黑树：插入

莫赤匪狐，莫黑匪乌

惠而好我，携手同车

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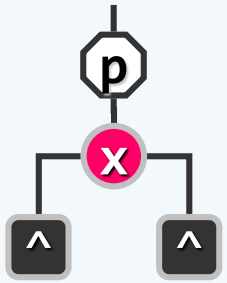
算法

- 现拟插入关键码e //不妨设T中本不含e
- 按BST的常规算法，插入之 //x = insert(e)必为末端节点

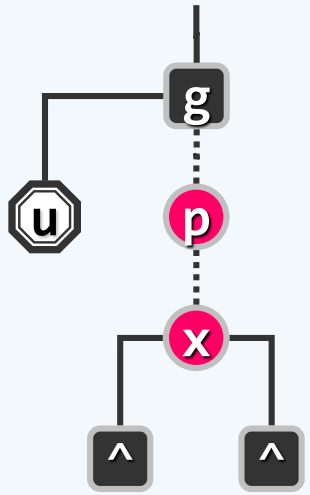
  不妨设x的父亲 `p = x->parent` 存在 //否则，即平凡的首次插入



- 将x染红（除非它是根） `//x->color = isRoot(x) ? B : R`

  条件1 + 2 + 4依然满足；但3不见得，有可能...

- 双红 `double-red` `//p->color == x->color == R`



- 考查： x的祖父 `g = p->parent` `//g != null && g->color == B`

  p的兄弟 `u = p == g->lc ? g->rc : g->lc` //即x的叔父

- 视u的颜色，分两种情况处理...

实现

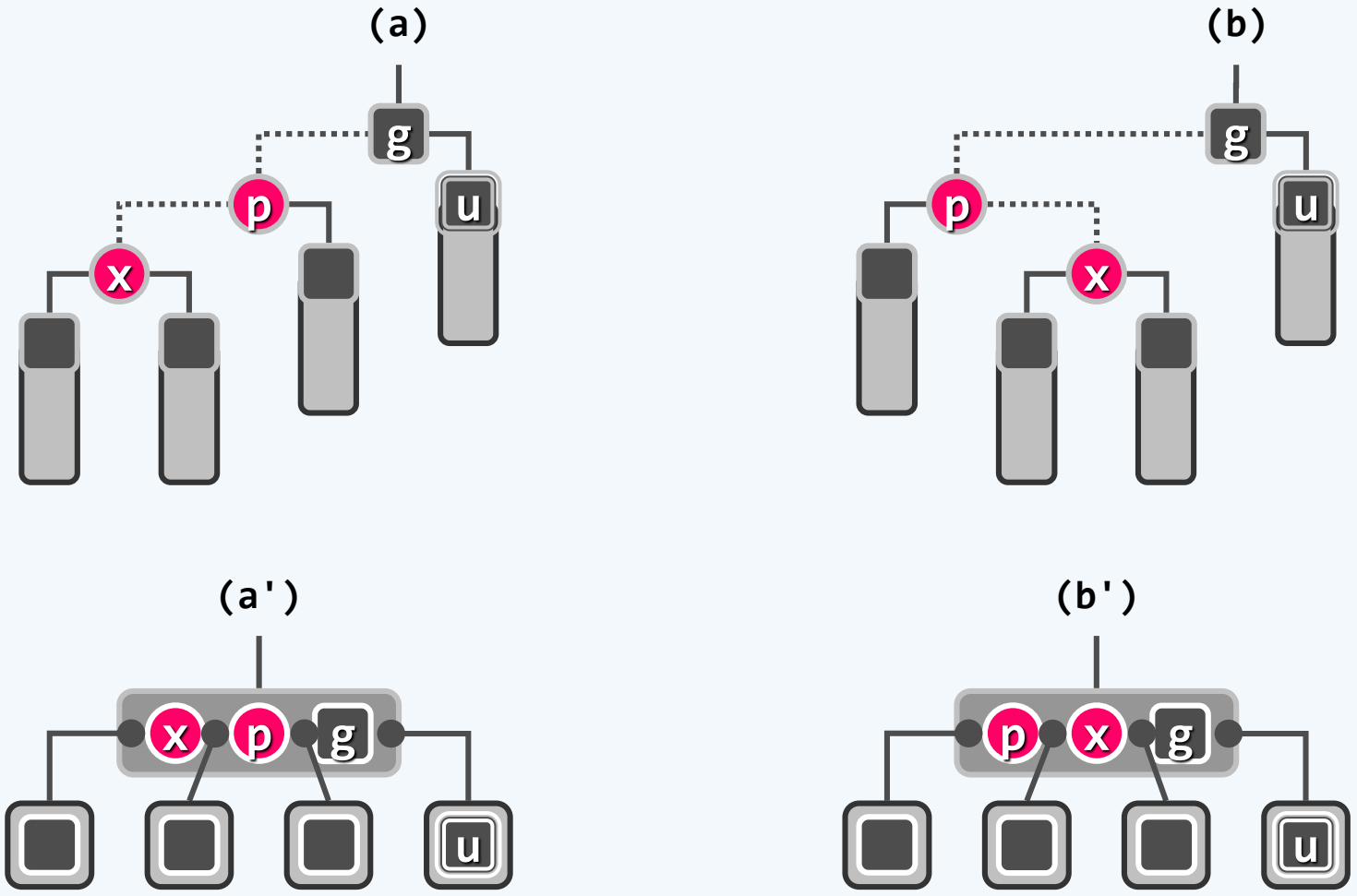
```
❖ template <typename T> BinNodePosi(T) RedBlack<T>::insert( const T & e ) {
  // 确认目标节点不存在（留意对_hot的设置）
     BinNodePosi(T) & x = search( e ); if ( x ) return x;
  // 创建红节点x，以_hot为父，黑高度-1
     x = new BinNode<T>( e, _hot, NULL, NULL, -1 ); _size++;
  // 如有必要，需做双红修正
     solveDoubleRed( x );
  // 返回插入的节点
     return x ? x : _hot->parent;
  } //无论原树中是否存有e，返回时总有x->data == e
```

双红修正

```
❖ template <typename T> void RedBlack<T>::solveDoubleRed( BinNodePosi(T) x ) {
    if ( IsRoot( *x ) ) { //若已（递归）转至树根，则将其转黑，整树黑高度也随之递增
       { _root->color = RB_BLACK; _root->height++; return; } //否则...
    BinNodePosi(T) p = x->parent; //考查x的父亲p（必存在）
    if ( IsBlack( p ) ) return; //若p为黑，则可终止调整；否则
    BinNodePosi(T) g = p->parent; //x祖父g必存在，且必黑
    BinNodePosi(T) u = uncle( x ); //以下视叔父u的颜色分别处理
    if ( IsBlack( u ) ) { /* ... u为黑（或NULL） ... */ }
    else                { /* ... u为红 ... */ }
  }
```

RR-1: u->color == B

- 此时：
  - x、p、g的四个孩子（可能是外部节点）
    - 全为黑，且
    - 黑高度相同
- 另两种对称情况
  - 自行补充

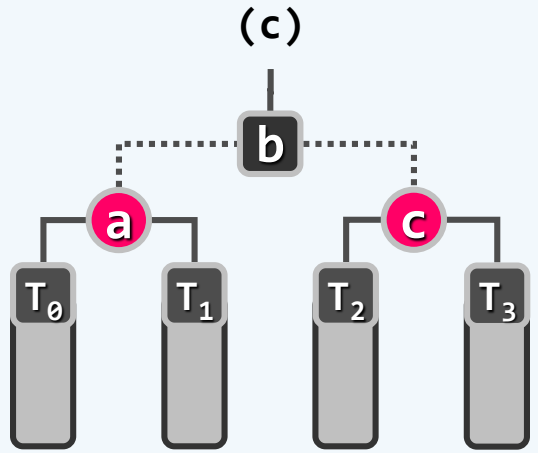


RR-1: u->color == B

1．参照AVL树算法，做局部3+4重构

将节点x、p、g及其四棵子树，按中序组合为：

T_0 < a < T_1 < b < T_2 < c < T_3



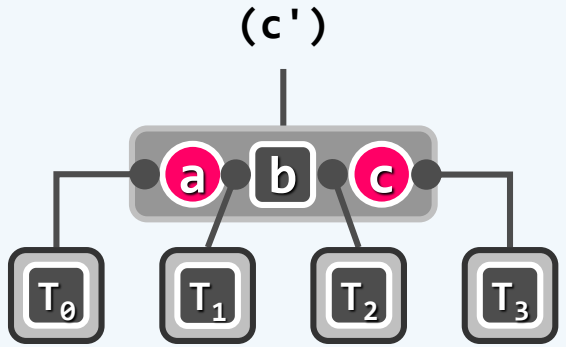
2．染色：b转黑，a或c转红

❖ 从B-树的角度，如何理解这一情况？

1．调整前之所以非法，是因为

在某个三叉节点中插入红关键码，使得

原黑关键码不再居中 //RRB或BRR，出现相邻的红关键码



2．调整之后的效果相当于 //B-树的拓扑结构不变，但

在新的四叉节点中，三个关键码的颜色改为RBR

RR-1：实现

```
❖ template <typename T> void RedBlack<T>::solveDoubleRed( BinNodePosi(T) x ) {
    /* ..... */
    if ( IsBlack( u ) ) { //u为黑或NULL
    // 若x与p同侧，则p由红转黑，x保持红；否则，x由红转黑，p保持红
       if ( IsLChild( *x ) == IsLChild( *p ) ) p->color = RB_BLACK;
       else                                    x->color = RB_BLACK;
       g->color = RB_RED; //g必定由黑转红
       BinNodePosi(T) gg = g->parent; //great-grand parent
       BinNodePosi(T) r = FromParentTo( *g ) = rotateAt( x );
       r->parent = gg; //调整之后的新子树，需与原曾祖父联接
    } else { /* ... u为红 ... */ }
  }
```

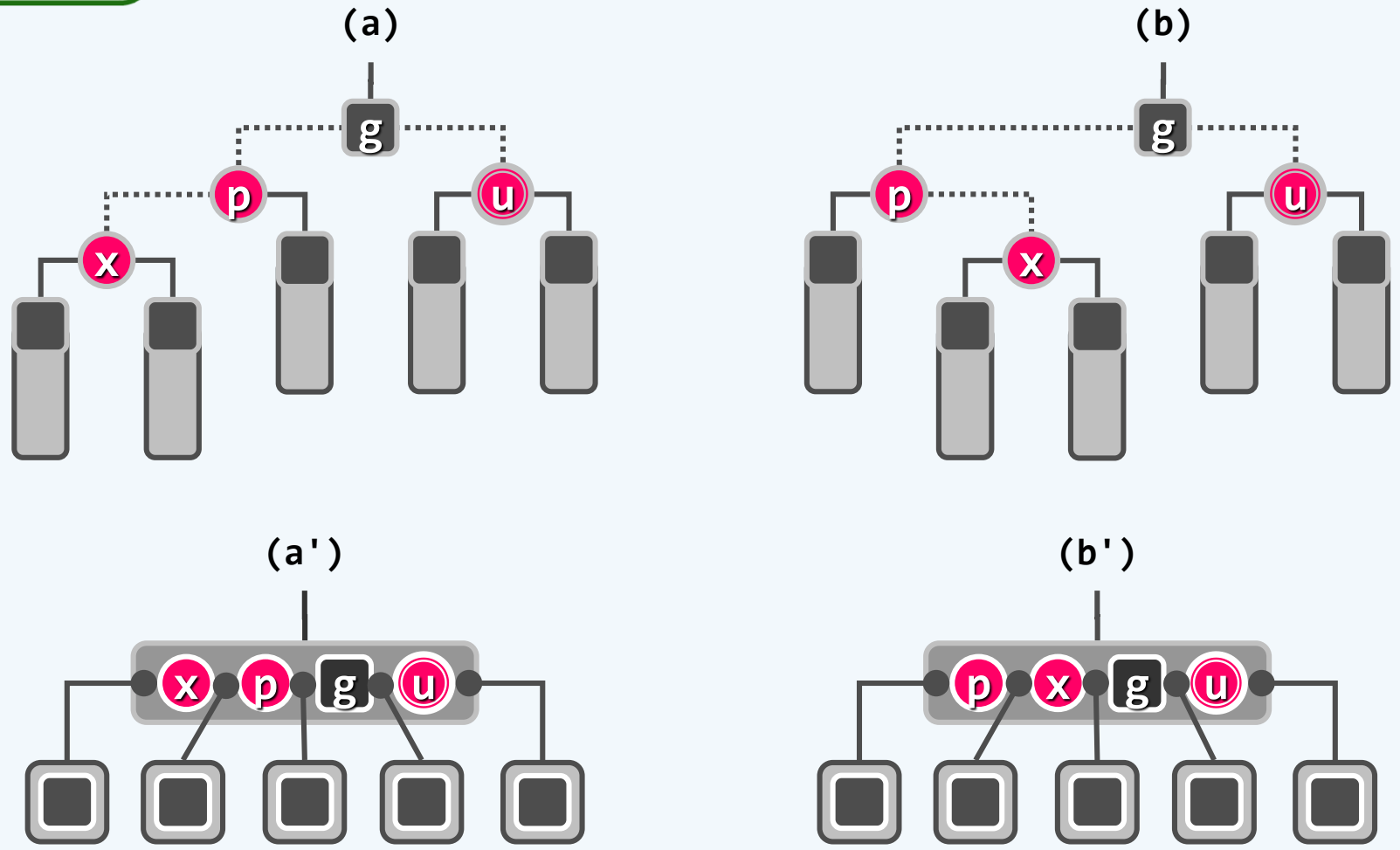
RR-2: u->color == R

❖ 在B-树中，等效于

超级节点发生上溢

//另两种对称情况

//请自行补充

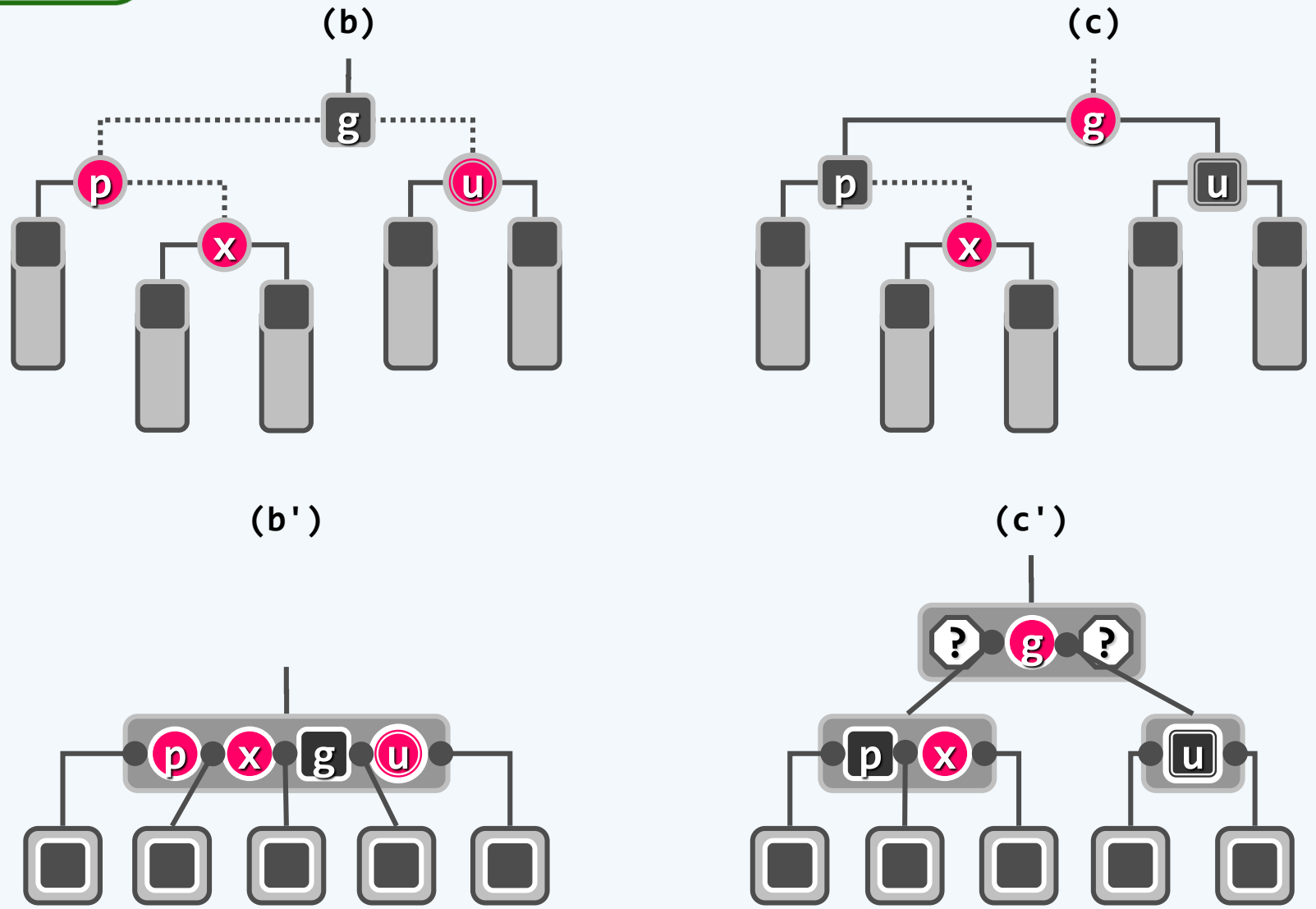


RR-2: u->color == R

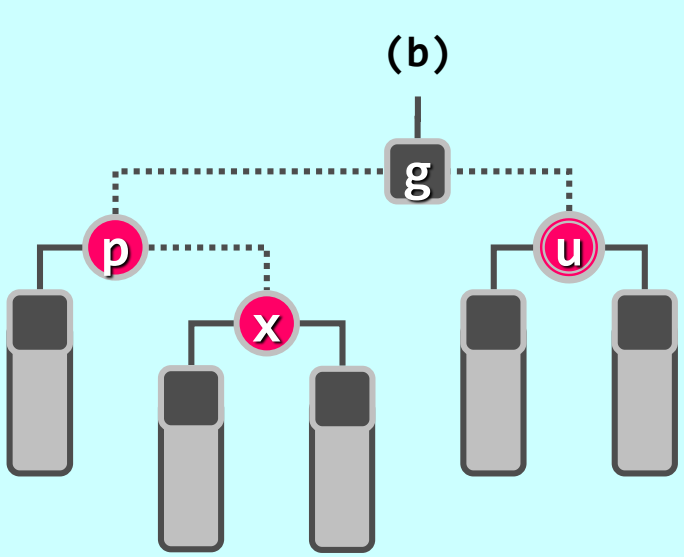
- p与u转黑，g转红
- 在B-树中，等效于

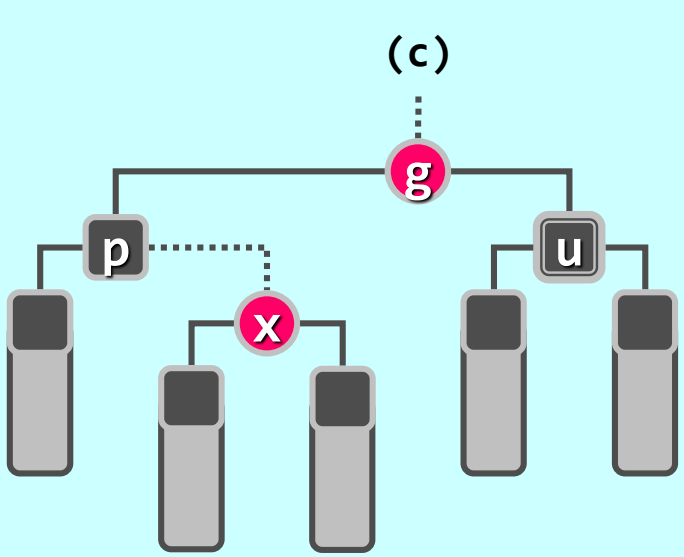
节点分裂

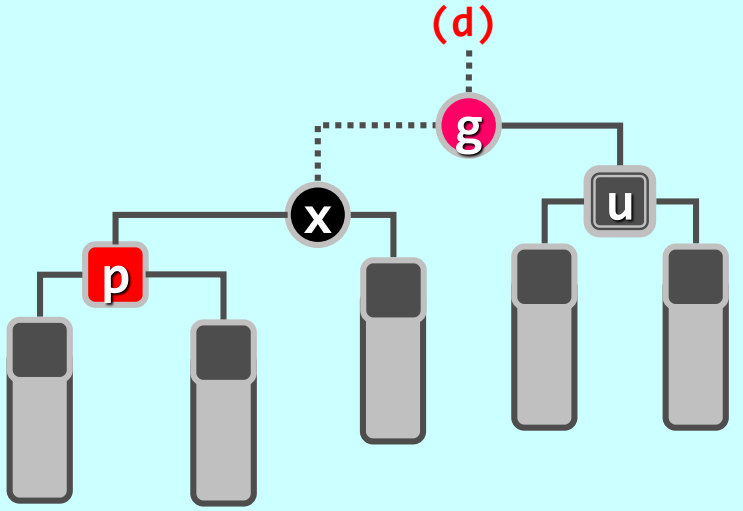
关键码g上升一层

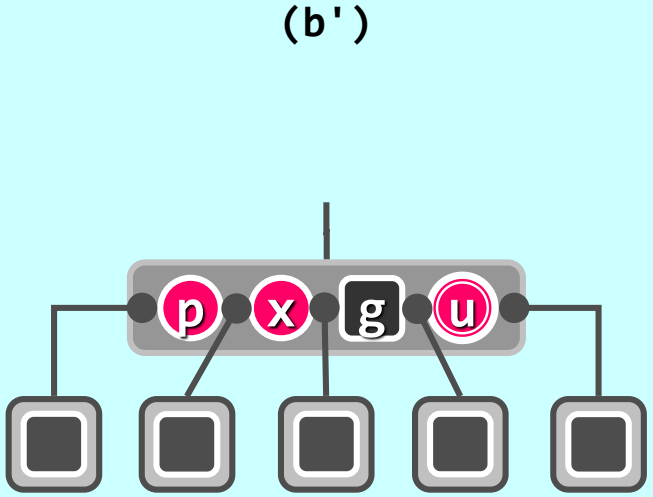


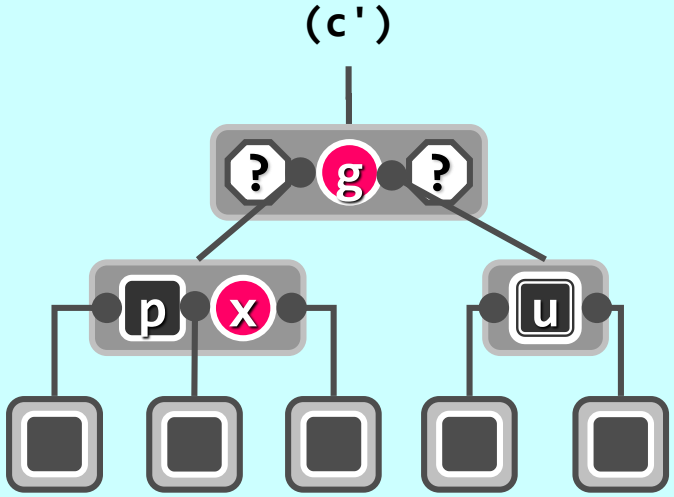
RR-2：u->color == R 另外一种情况?

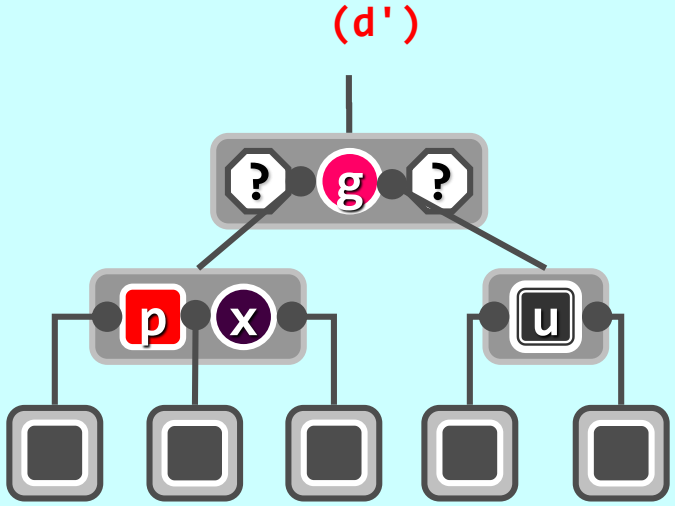












RR-2: u->color == R

- 既然是分裂，也应有可能继续向上传递

  亦即，g与parent(g)再次构成双红

- 果真如此，可

  等效地将g视作新插入的节点

  区分以上两种情况，如法处置

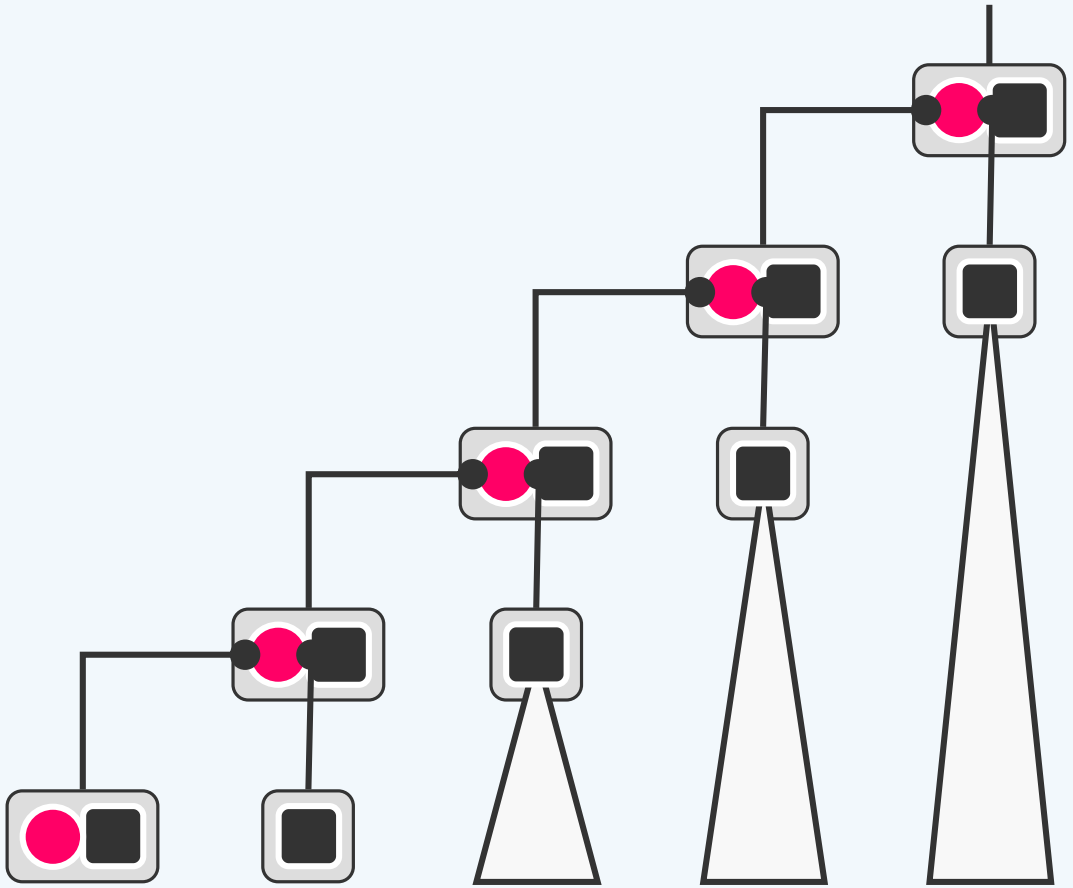
- 直到所有条件满足（即不再双红）

  或者抵达树根

- g若果真到达树根，则

  1.强行将g转为黑色

  2.整树（黑）高度加一



RR-2：实现

```
❖ template <typename T> void RedBlack<T>::solveDoubleRed( BinNodePosi(T) x ) {
    /* ..... */
    if ( IsBlack(u) ) { /* ... u为黑（或NULL） ... */ }
    else { //u为红色
       p->color = RB_BLACK; p->height++; //p由红转黑，增高
       u->color = RB_BLACK; u->height++; //u由红转黑，增高
       if ( !IsRoot( *g ) ) g->color = RB_RED; //g若非根则转红
       solveDoubleRed( g ); //继续调整g（类似于尾递归，可优化）
    }
 }
```

双红修正：复杂度

- 重构、染色均属常数时间的局部操作

  故只需统计其总次数

- 红黑树的每一次插入操作

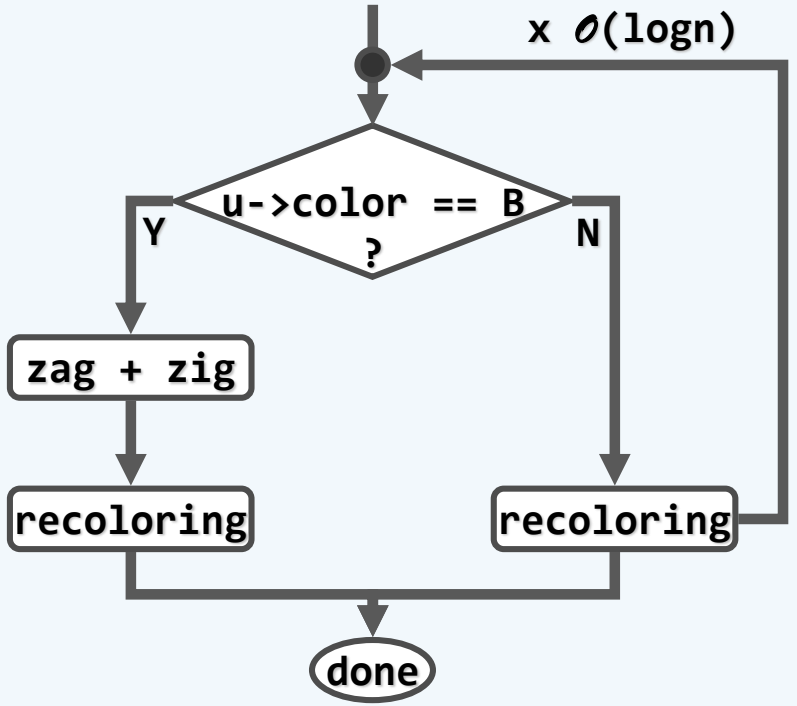
  都可在O(logn)时间内完成

- 其中至多做：

  1. O(logn)次节点染色

  2. 一次“3+4”重构

情况	旋转次数	染色次数	此后
u为黑	1~2	2	调整随即完成
u为红	0	3	可能再次双红 但必上升两层



Review Bottom Up Insertion

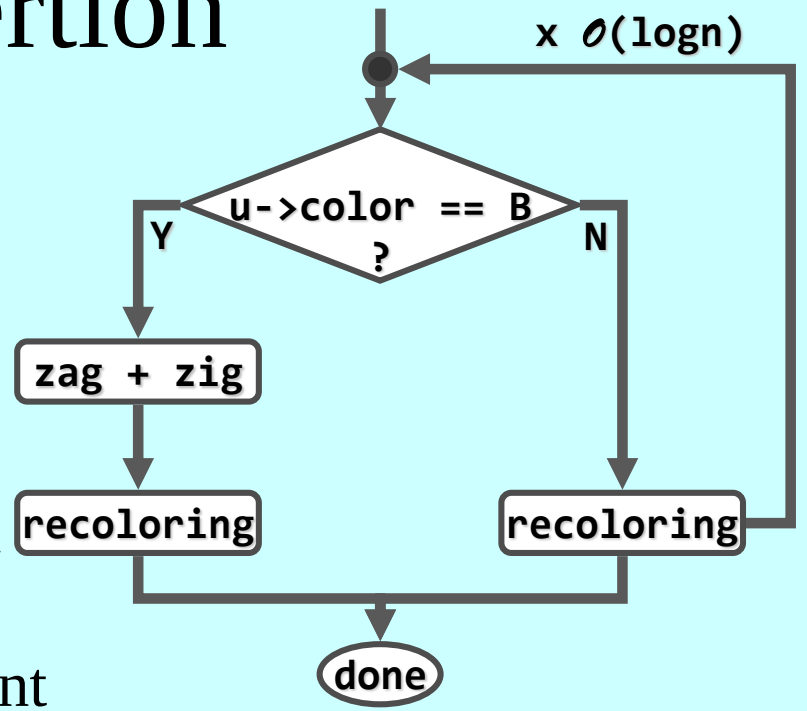
- Insert node; Color it Red; X is pointer to it
- Cases

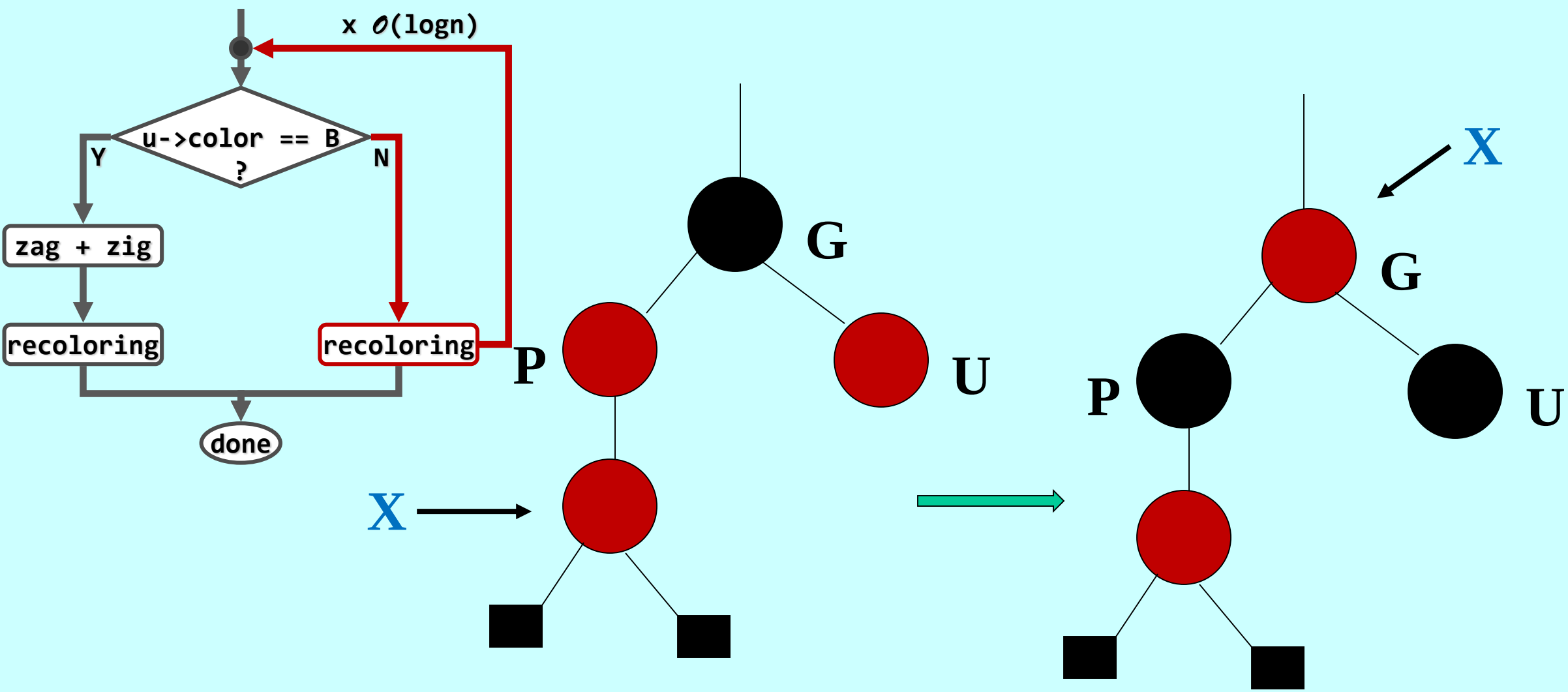
  0: X is the root -- color it Black

  1: Both parent and uncle are Red -- color parent and uncle Black, color grandparent Red. Point X to grandparent and check new situation.

  2 (zig-zag): Parent is Red, but uncle is Black. X and its parent are opposite type children -- color grandparent Red, color X Black, rotate left(right) on parent, rotate right(left) on grandparent

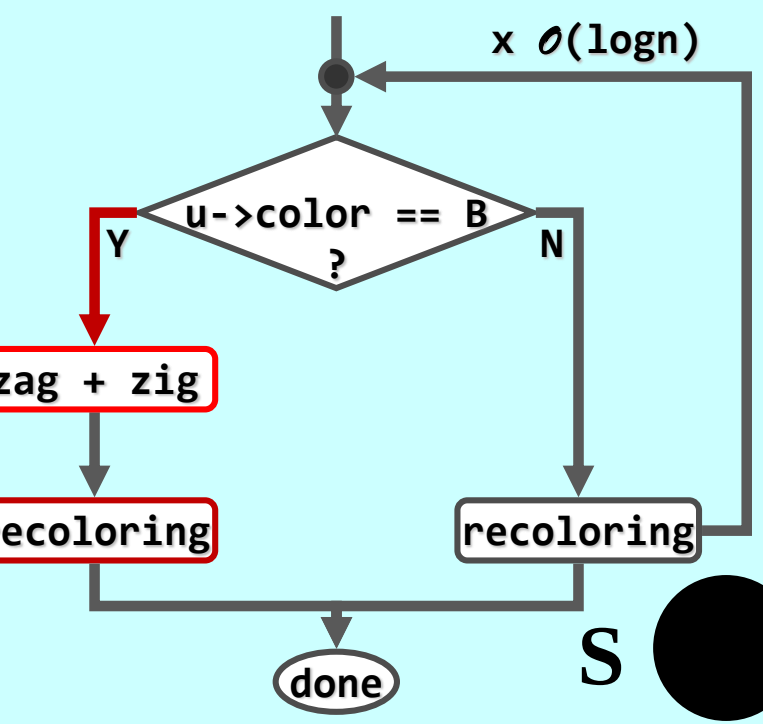
  3 (zig-zig): Parent is Red, but uncle is Black. X and its parent are both left (right) children -- color parent Black, color grandparent Red, rotate right(left) on grandparent

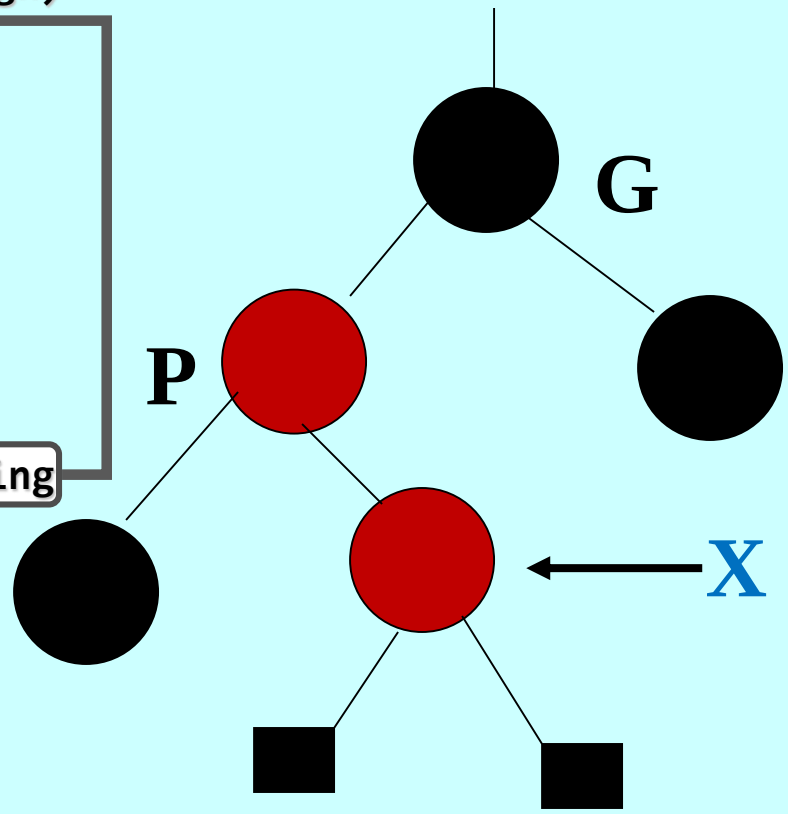


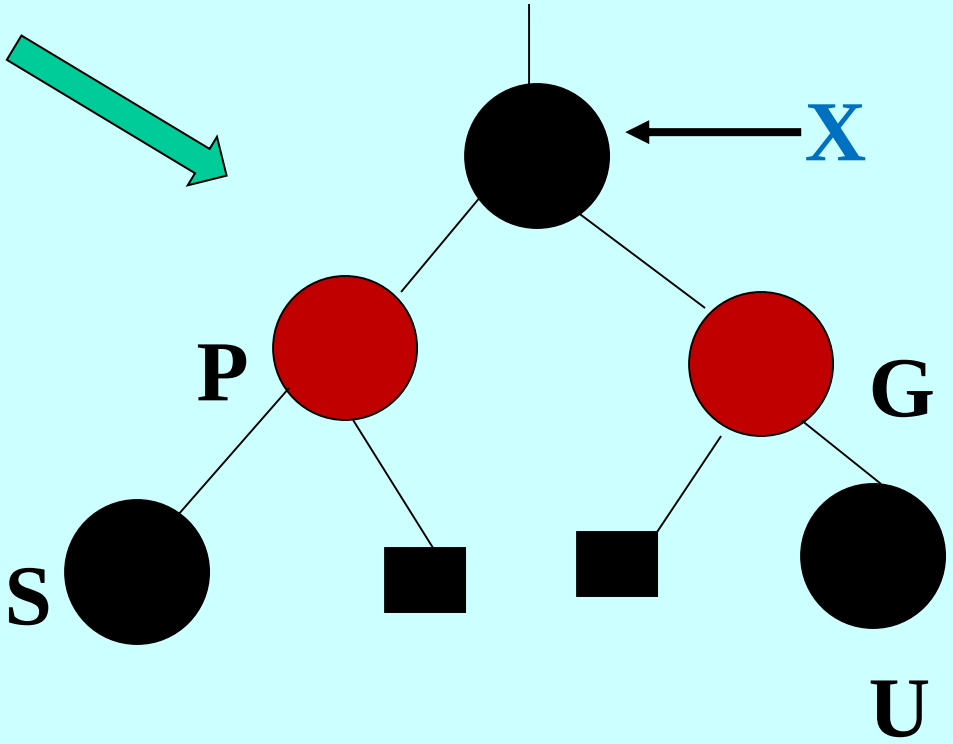


Case 1 – U is Red

Just Recolor and move up





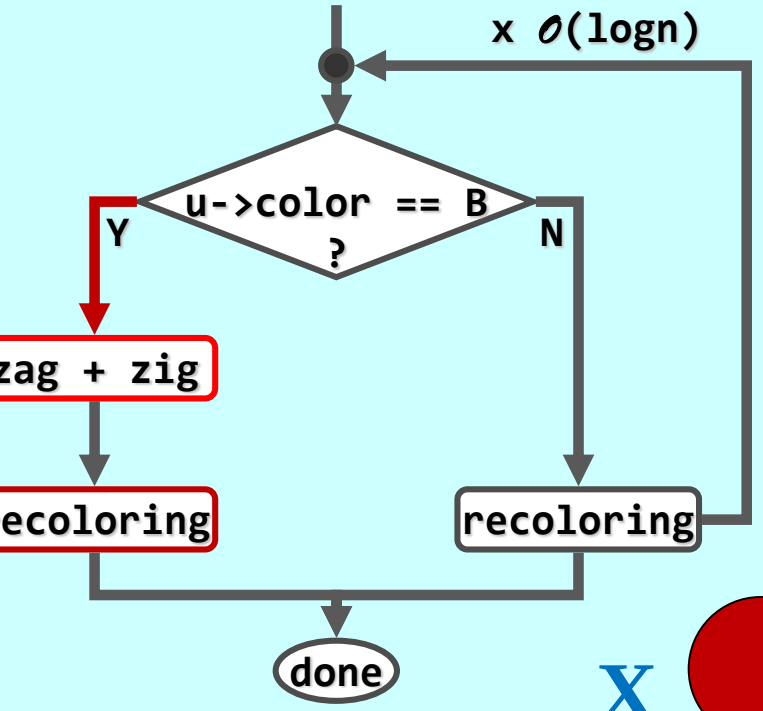


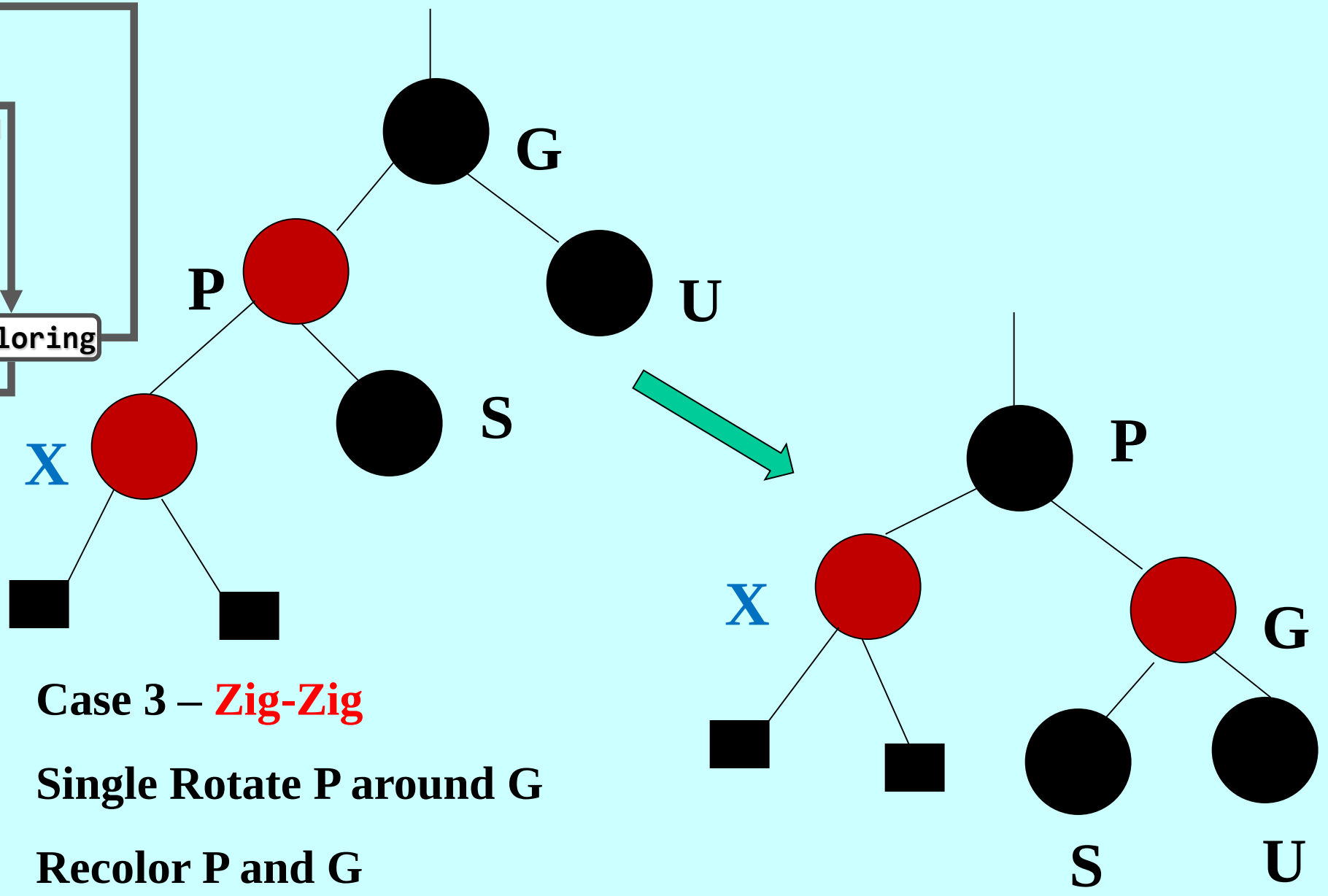
Case 2 – Zig-Zag

Double Rotate

X around P; X around G

Recolor G and X

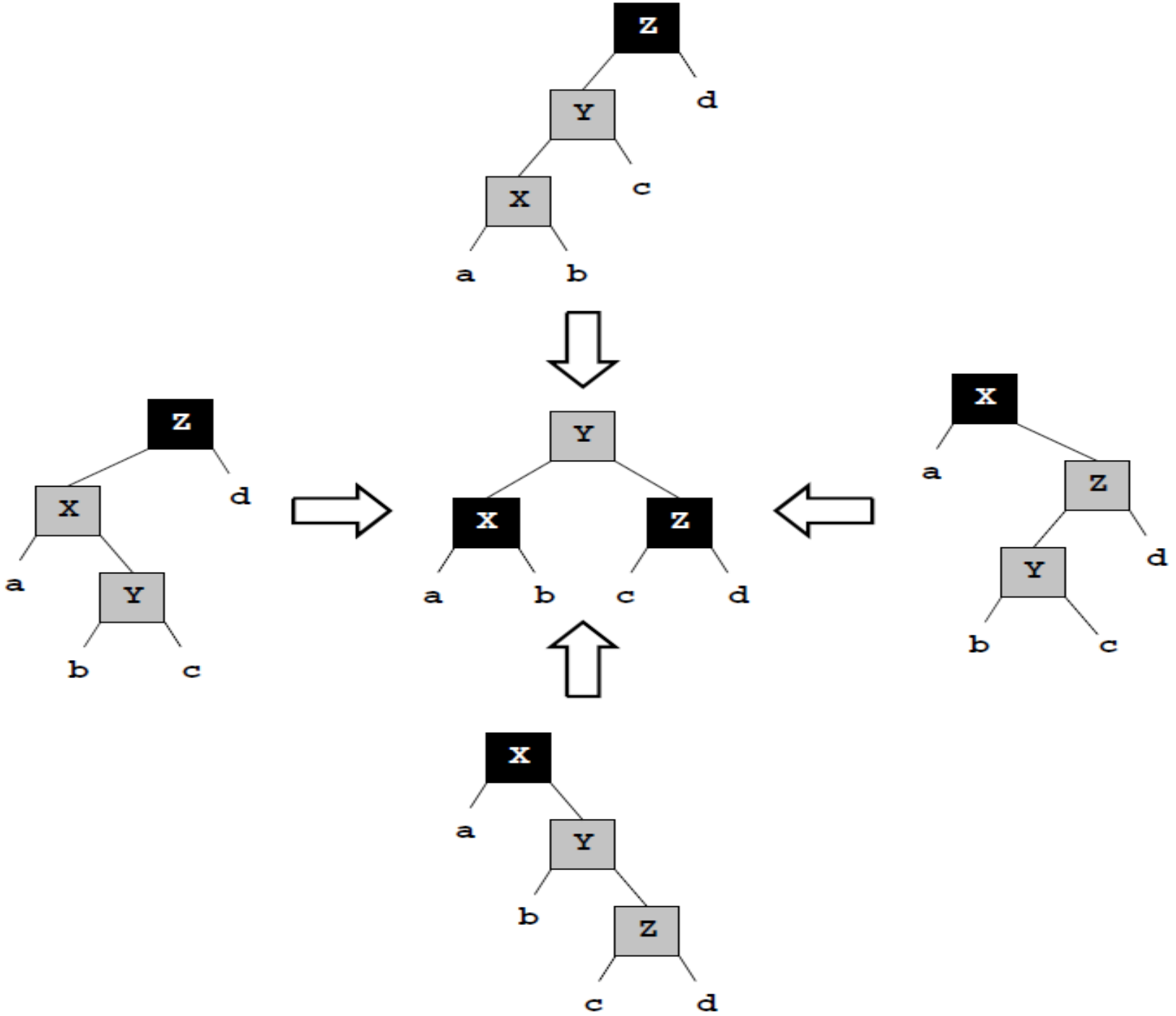




Case 3 – Zig-Zig

Single Rotate P around G

Recolor P and G


Z
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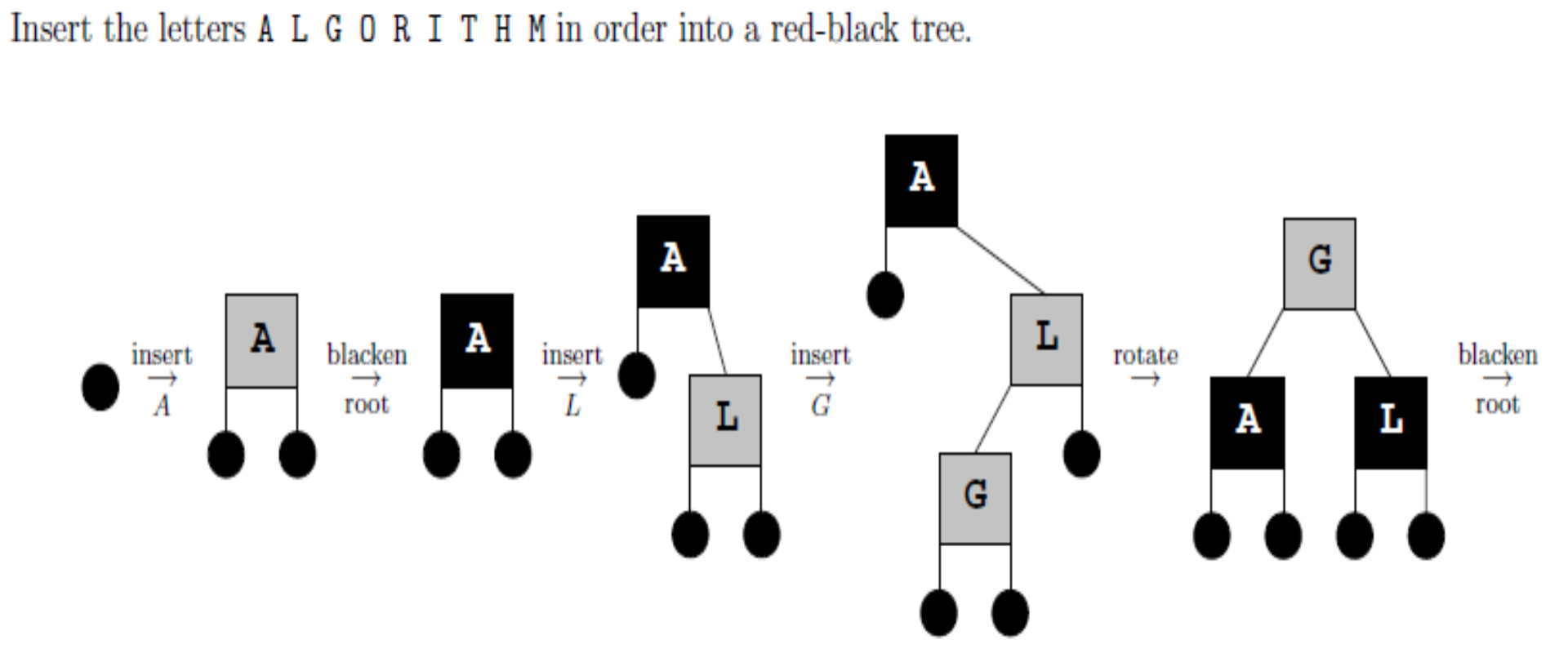
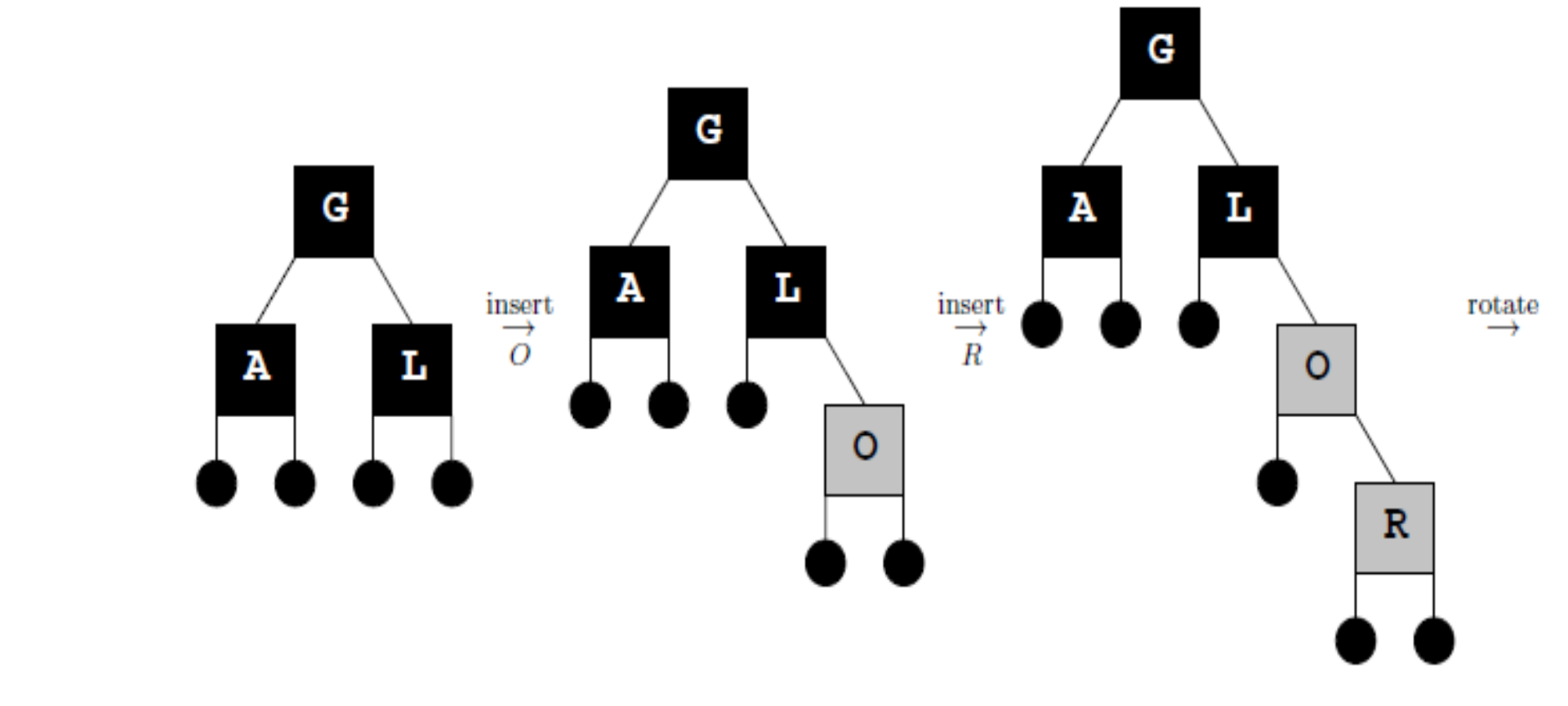
Bottom Up Insertion Example

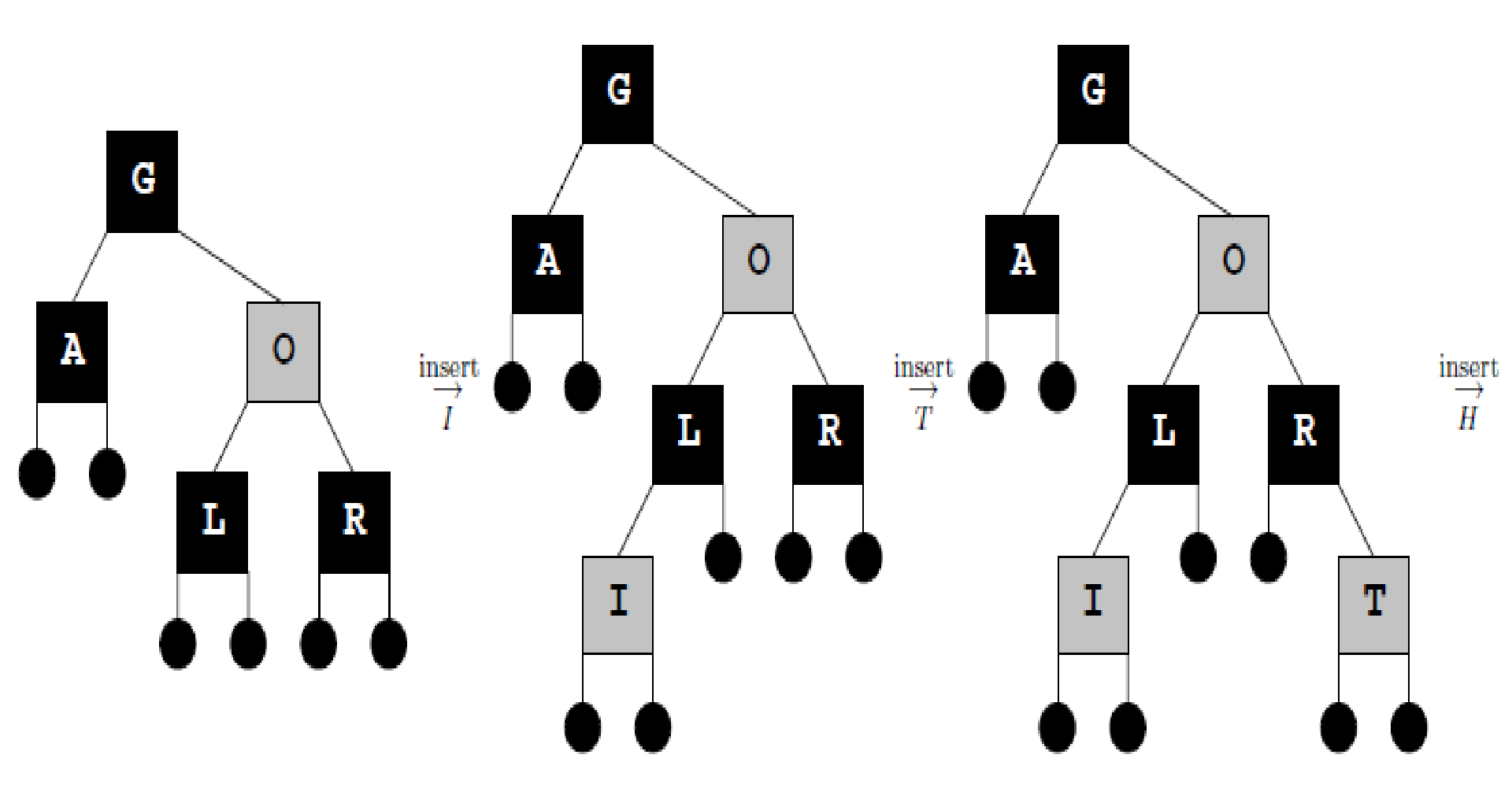


Bottom Up Insertion Example



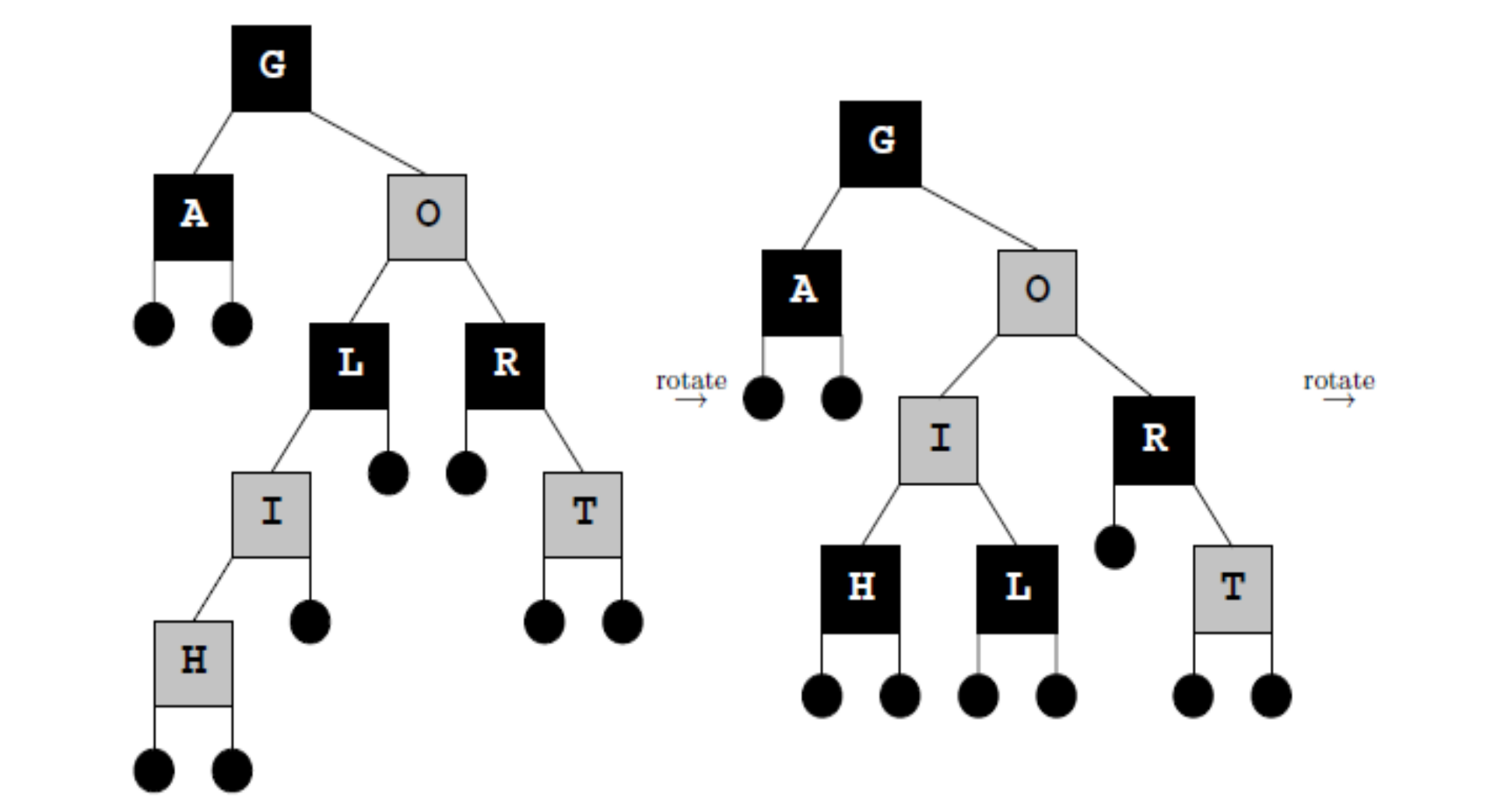
Example: Insert A L G O R I T H M in order into a red-black tree

Bottom Up Insertion Example



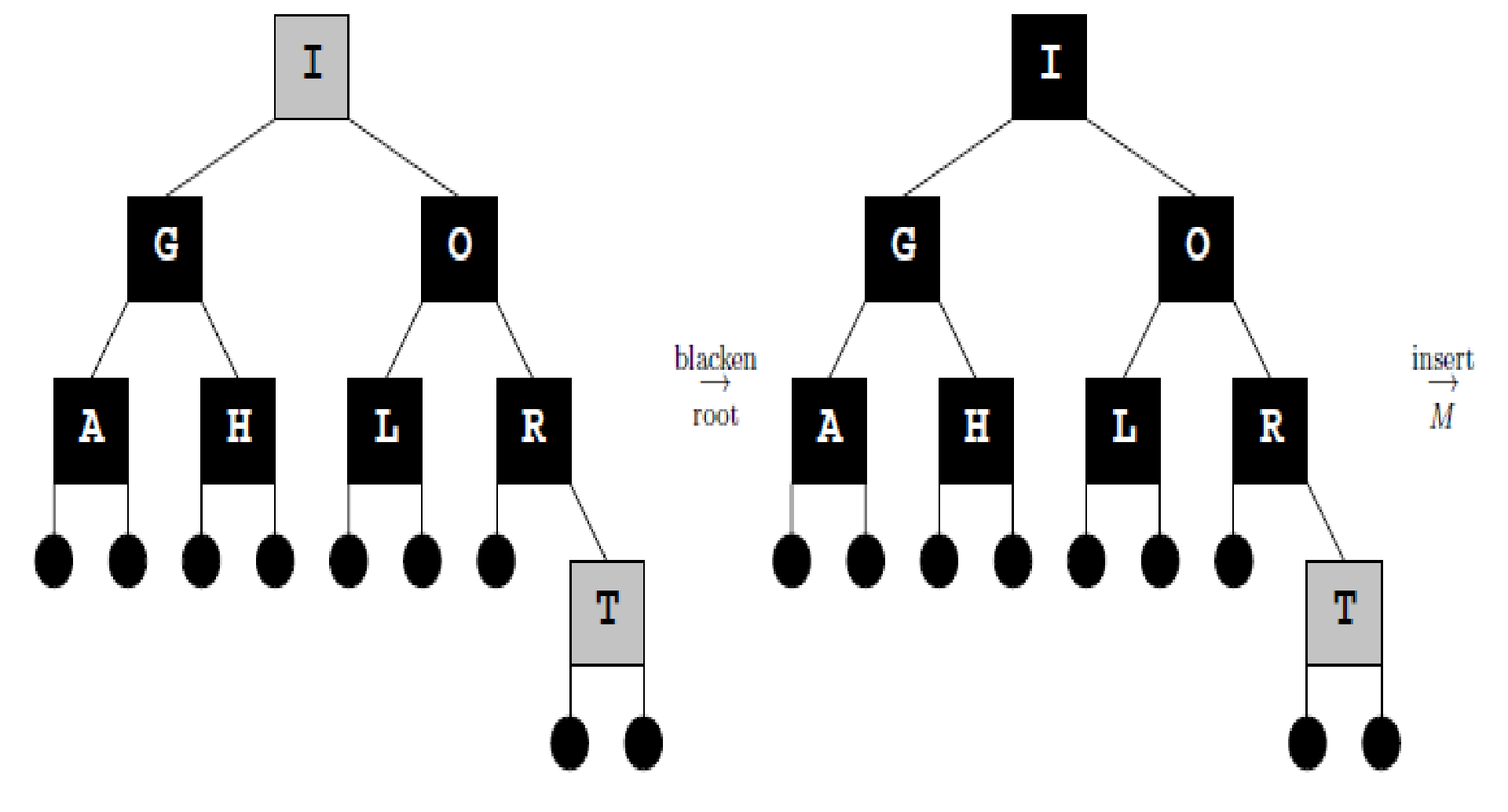
Example: Insert A L G O R I T H M in order into a red-black tree

Bottom Up Insertion Example



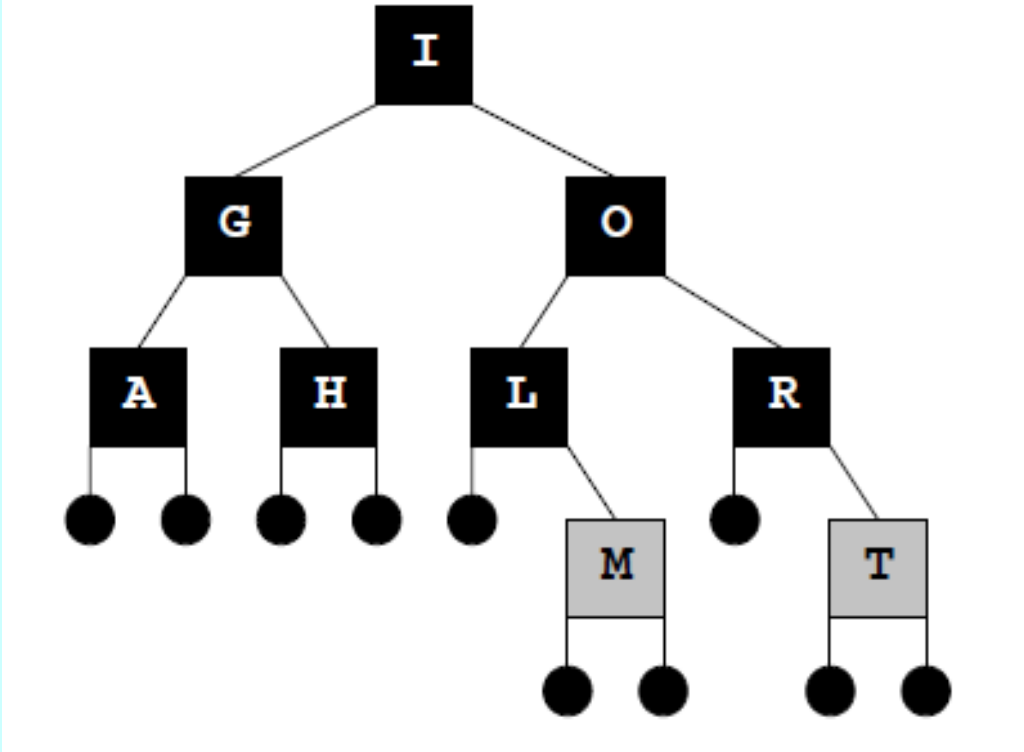
Example: Insert A L G O R I T H M in order into a red-black tree

Bottom Up Insertion Example



Example: Insert A L G O R I T H M in order into a red-black tree

Bottom Up Insertion Example



Example: Insert A L G O R I T H M in order into a red-black tree

Review of Bottom-Up Insertion

- In Bottom-Up insertion, “ordinary” BST insertion was used, followed by correction of the tree on the way back up to the root
- This is most easily done recursively
  - Insert winds up the recursion on the way down the tree to the insertion point
  - Fixing the tree occurs as the recursion unwinds

8.高级搜索树

（xa4）红黑树：删除

变白以为黑兮，倒上以为下

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算法

- 首先按照BST常规算法，执行：

  r = removeAt(x, _hot)

- x由孩子r接替

  //另一孩子记作w（即黑的NULL）

- 条件1和2依然满足

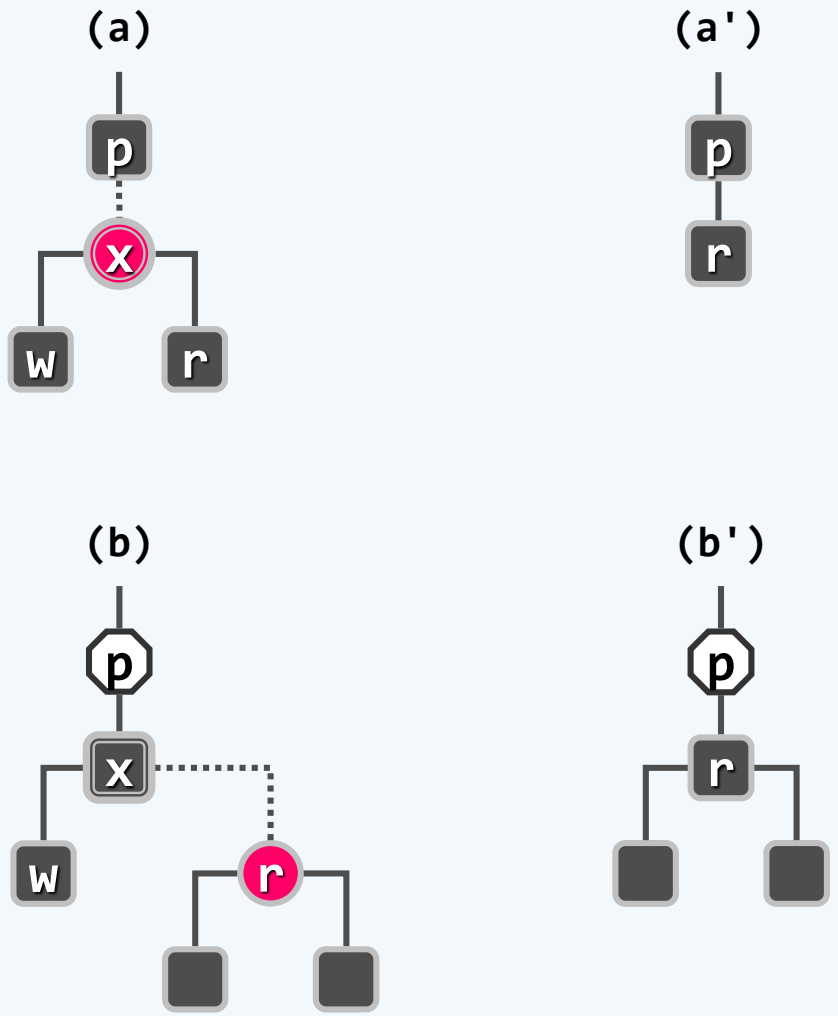
  但3和4不见得

  //在原树中，考查x与r...

- 若二者之一为红

  则3和4不难满足

  //删除遂告完成！



算法

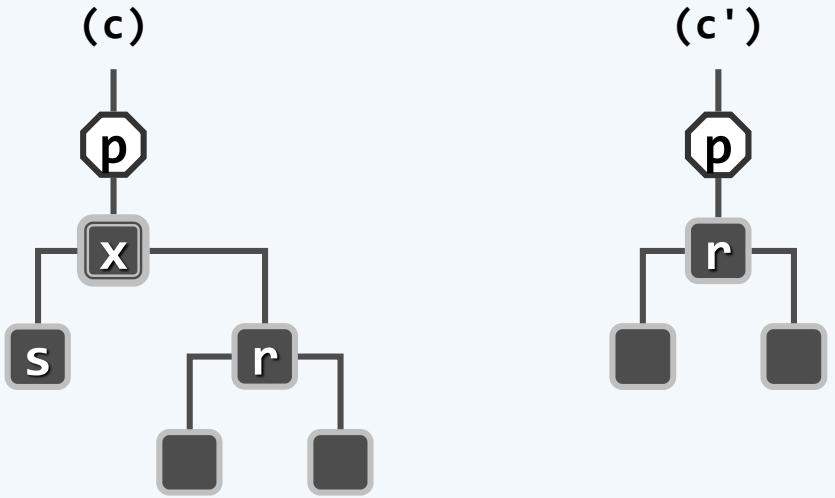
- 若x与r均黑double-black

  则不然...

- 摘除x并代之以r后

  全树黑深度不再统一

  原B-树中x所属节点下溢



- 在新树中，考查

  r的父亲 `p = r->parent` //亦即原树中x的父亲

  r的兄弟 `s = r == p->lc ? p->rc : p->lc`

- 以下分四种情况处理...

实现

```
❖template <typename T> bool RedBlack<T>::remove( const T & e ) {
    BinNodePosi(T) & x = search( e ); if ( !x ) return false; //查找定位
    BinNodePosi(T) r = removeAt( x, _hot ); //删除_hot的某孩子，r指向其接替者
    if ( ! ( -- _size ) ) return true; //若删除后为空树，可直接返回
    if ( ! _hot ) { //若被删除的是根，则
       _root->color = RB_BLACK; //将其置黑，并
       updateHeight( _root ); //更新（全树）黑高度
       return true;
    } //至此，原x（现r）必非根
```

实现

```
❖ // 若父亲（及祖先）依然平衡，则无需调整
    if ( BlackHeightUpdated( * _hot ) ) return true;
  // 至此，必失衡
  // 若替代节点r为红，则只需简单地翻转其颜色
    if ( IsRed( r ) ) { r->color = RB_BLACK; r->height++; return true; }
  // 至此，r以及被其替代的x均为黑色
    solveDoubleBlack( r ); //双黑调整（入口处必有 r == NULL）
    return true;
  }
```

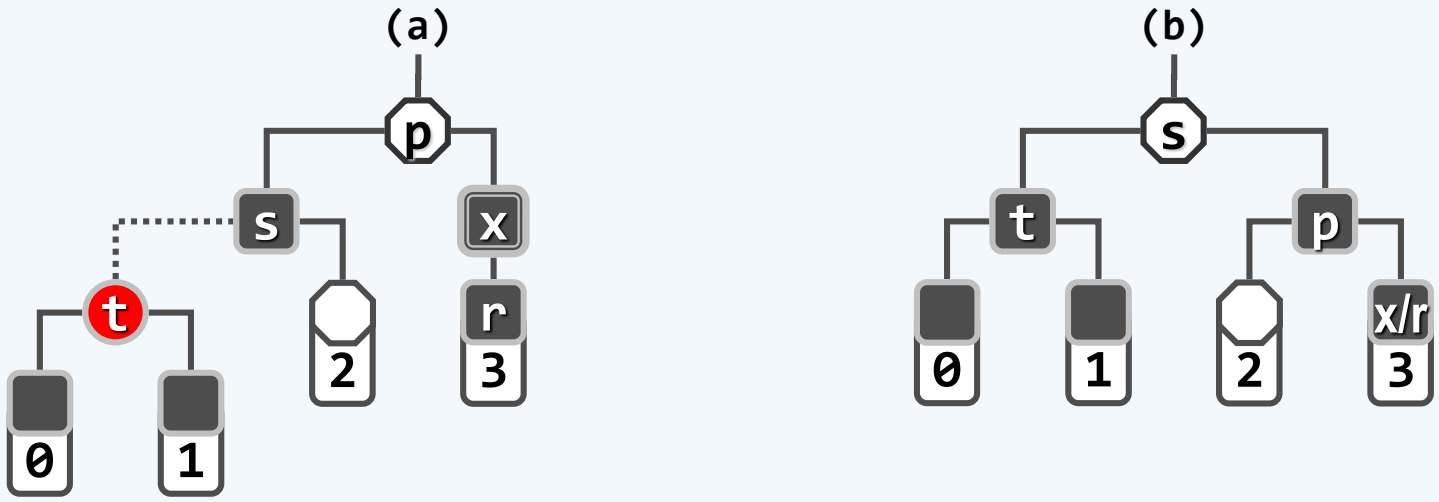
双黑修正

```
❖ template <typename T> void RedBlack<T>::solveDoubleBlack( BinNodePosi(T) r ) {
    BinNodePosi(T) p = r ? r->parent : _hot; if ( !p ) return; //r的父亲
    BinNodePosi(T) s = (r == p->lc) ? p->rc : p->lc; //r的兄弟
    if ( IsBlack( s ) ) { //兄弟s为黑
       BinNodePosi(T) t = NULL; //以下将t取作s的红孩子
       if      ( HasLChild( *s ) && IsRed( s->lc ) ) t = s->lc;
       else if ( HasRChild( *s ) && IsRed( s->rc ) ) t = s->rc;
       if ( t ) { /* ... 黑s有红孩子: BB-1 ... */ }
       else { /* ... 黑s无红孩子: BB-2R或BB-2B ... */ }
    } else { /* ... 兄弟s为红: BB-3 ... */ }
  }
```

BB-1：s为黑，且至少有一个红孩子t

❖ 3+4重构：t、s、p重命名为a、b、c

r保持黑；a和c染黑；b继承p的原色



❖ 如此，红黑树性质在全局得以恢复——删除完成！ //zig-zag等类似

❖ 在对应的B-树中，以上操作等效于...

BB-1：s为黑，且至少有一个红孩子t

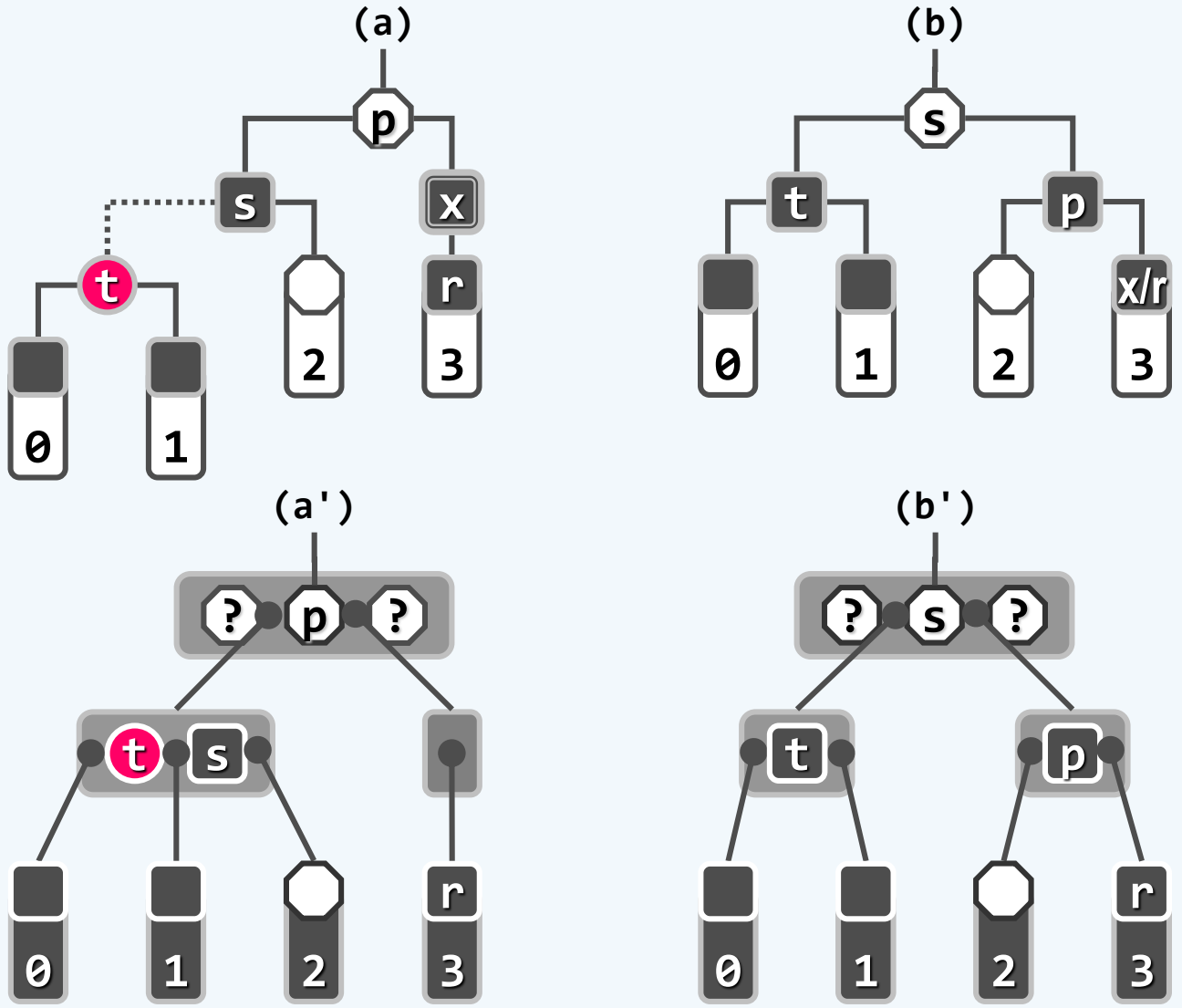
- 通过关键码的旋转

  消除超级节点的下溢

- 问号节点

  可同时存在

  颜色不定

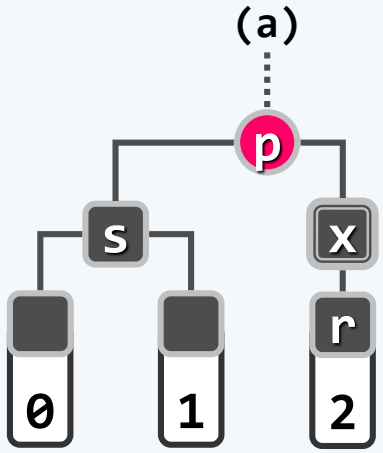


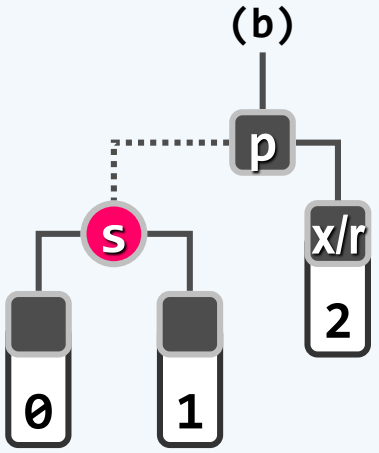
BB-1：实现

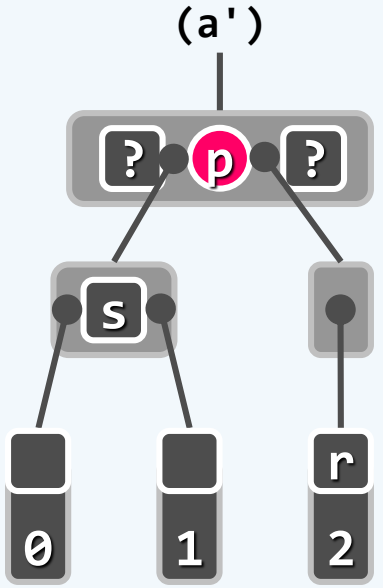
```
❖if ( IsBlack( s ) ) { //兄弟s为黑
    if ( t ) { //黑s有红孩子：BB-1
        RBColor oldColor = p->color; //备份p颜色，并对t、父亲、祖父
        BinNodePosi(T) b = FromParentTo( *p ) = rotateAt( t ); //旋转
        if ( HasLChild( *b ) ) b->lChild->color = RB_BLACK; //新子树之左子染黑
        if ( HasRChild( *b ) ) b->rChild->color = RB_BLACK; //新子树之右子染黑
        updateHeight( b->lc ); updateHeight( b->rc );
        b->color = oldColor; updateHeight( b ); //新根继承原根的颜色
    else { /* ... 黑s无红孩子：BB-2R或BB-2B ... */ }
  } else { /* ... 兄弟s为红：BB-3 ... */ }
```

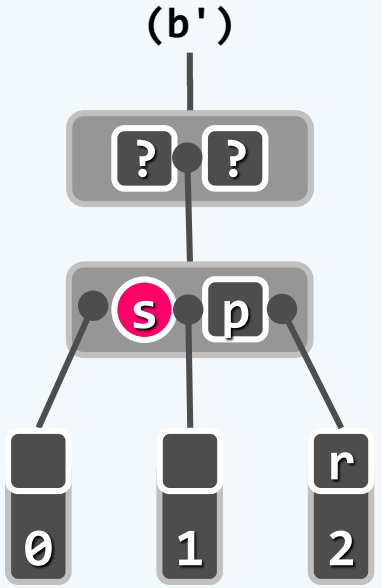
BB-2R：s为黑，且两个孩子均为黑；p为红

- r保持黑；s转红；p转黑
- 在对应的B-树中，等效于
  下溢节点与兄弟合并
- 红黑树性质在全局得以恢复
- 失去关键码p后，上层节点
  会否继而下溢？不会！
- 合并之前，在p之左或右侧
  还应有（问号）关键码
  必为黑色
  有且仅有一个



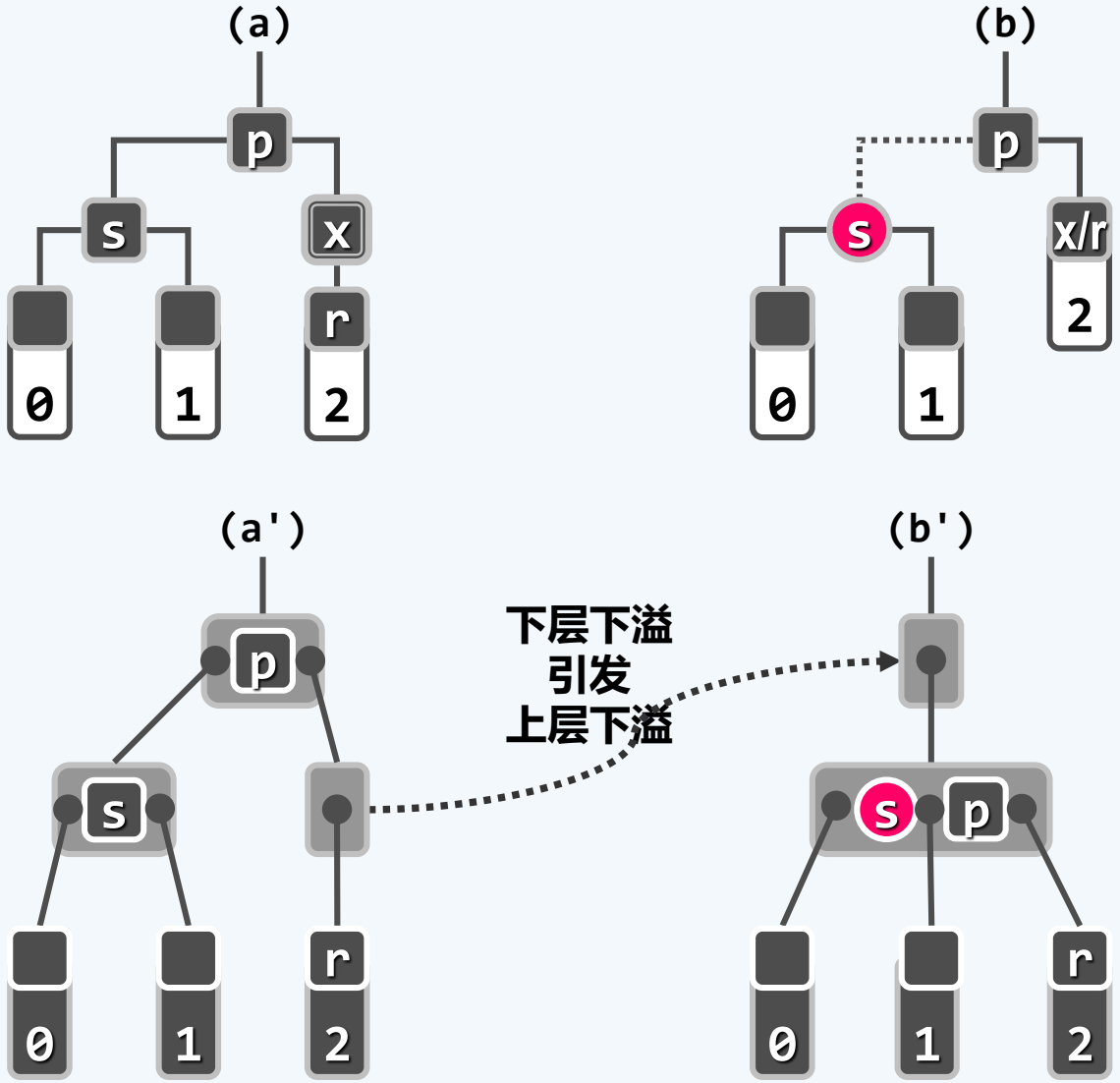






BB-2B：s为黑，且两个孩子均为黑；p为黑

- s转红；r与p保持黑
- 红黑树性质在局部得以恢复
- 在对应的B-树中，等效于
  下溢节点与兄弟合并
- 合并之前，p和s均对应于单关键码节点
- 失去关键码p后
  上层节点必然继而下溢
- 好在可继续分情况处理
  高度递增，至多$O(\log n)$步



BB-(2R+2B)：实现

```
❖if ( IsBlack( s ) ) { //兄弟s为黑
    if ( t ) { /* ... 黑s有红孩子：BB-1 ... */ }
    else { /* 黑s无红孩子 */
       s->color = RB_RED; s->height--; //s转红
       if ( IsRed( p ) ) //BB-2R：p转黑，但黑高度不变
          { p->color = RB_BLACK; }
       else //BB-2B：p保持黑，但黑高度下降；递归修正
          { p->height--; solveDoubleBlack( p ); }
    }
  } else { /* ... 兄弟s为红：BB-3 ... */ }
```

BB-3：s为红（其孩子均为黑）

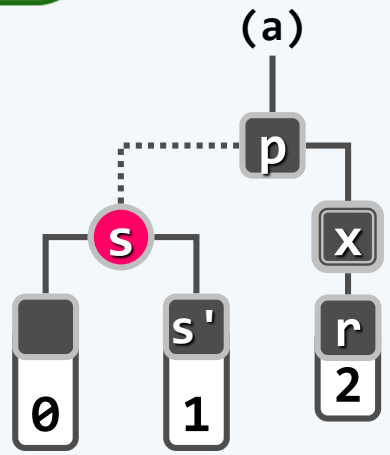
- zag(p)或zig(p)；红s转黑，黑p转红
- 黑高度依然异常，但...
- r有了一个新的黑兄弟s'

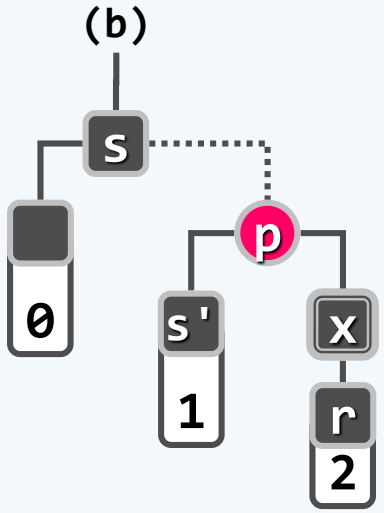
  故转化为前述情况，而且...
- 既然p已转红，接下来

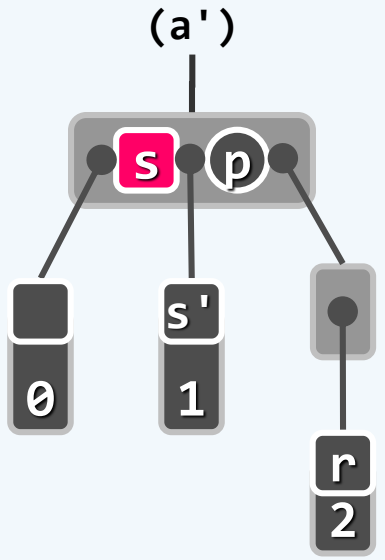
  绝不会是情况BB-2B

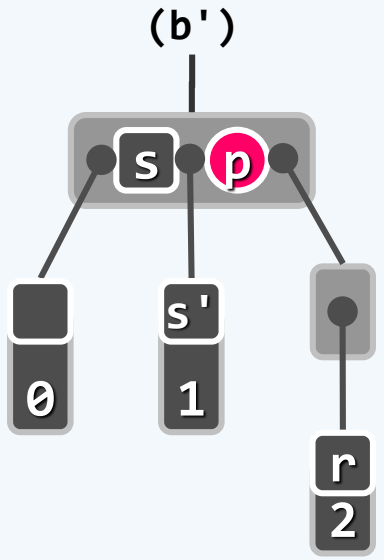
  而只能是BB-1或BB-2R
- 于是，再经一轮调整之后

  红黑树性质必然全局恢复









BB-3：实现

```
❖if ( IsBlack( s ) ) { //兄弟s为黑
    if ( t ) { /* ... 黑s有红孩子：BB-1 ... */ }
    else { /* ... 黑s无红孩子：BB-2R或BB-2B ... */ }
  } else { //兄弟s为红：BB-3
    s->color = RB_BLACK; p->color = RB_RED; //s转黑，p转红
    BinNodePosi(T) t = IsLChild( *s ) ? s->lc : s->rc; //取t与其父s同侧
    _hot = p; FromParentTo( *p ) = rotateAt( t ); //对t及其父亲、祖父做平衡调整
    solveDoubleBlack( r ); //继续修正r——此时p已转红，故后续只能是BB-1或BB-2R
  }
```

复杂度

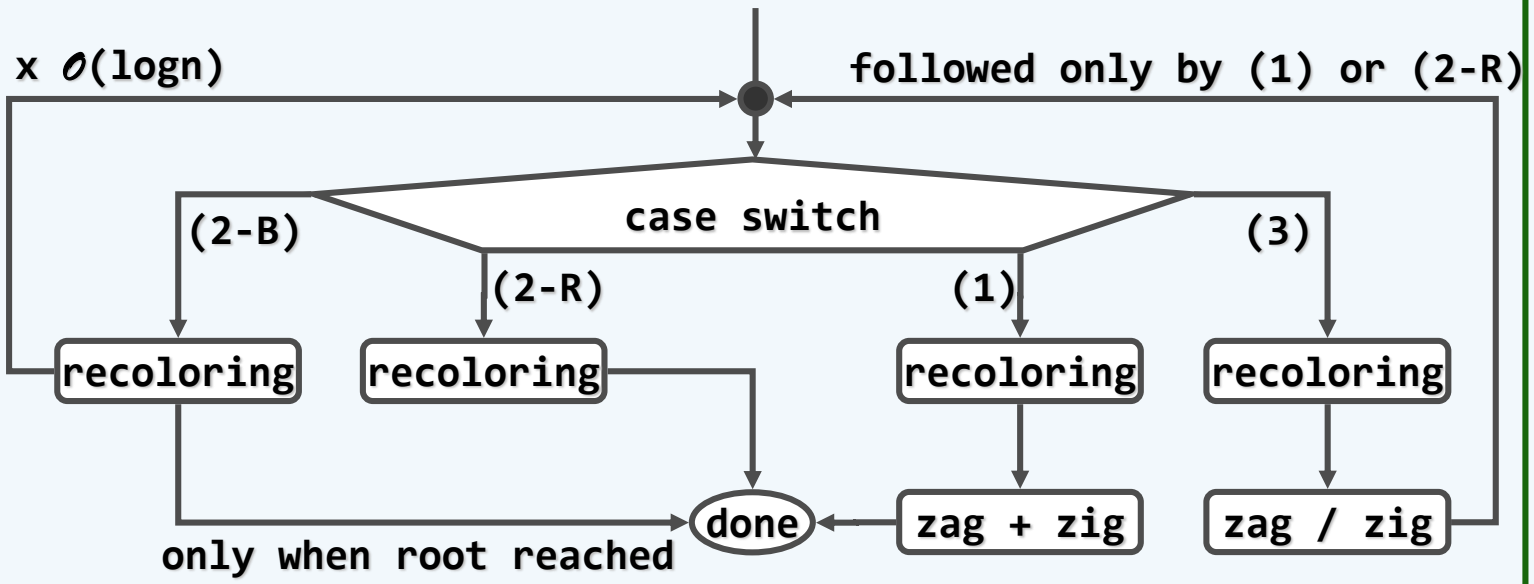
- 红黑树的每一删除操作

  都可在$O(\log n)$时间内完成

- 其中，至多做

  1. $O(\log n)$次重染色
  2. 一次“3+4”重构
  3. 一次单旋

情况	旋转次数	染色次数	此后
(1) 黑s有红子t	1~2	3	调整随即完成
(2R) 黑s无红子，p红	0	2	调整随即完成
(2B) 黑s无红子，p黑	0	1	必然再次双黑 但将上升一层
(3) 红s	1	2	转为(1)或(2R)



思考题

- Bottom-Up is recursive
  - BST deletion going down the tree (winding up the recursion)
  - Fixing the RB properties coming back up the tree (unwinding the recursion)
- Top-Down is iterative
  - Restructure the tree on the way down so we don't have to go back up

类似于Spray树， 是否有Top-down的删除算法呢?

Reference

- *Rudolf Bayer (1972). "Symmetric binary B-Trees: Data structure and maintenance algorithms". Acta Informatica. 1 (4): 290–306*
- *Data Structures and Algorithm Analysis in C++* Section 12.2
- http://users.cis.fiu.edu/~weiss/dsaa_c++3/code/RedBlackTree.h

Next

- kd- 树
- 数据结构（C++语言版）第三版 Chapter 8.4