Algorithm Design and Implementation

Principle of Algorithms I

Introduction: Stable Matching

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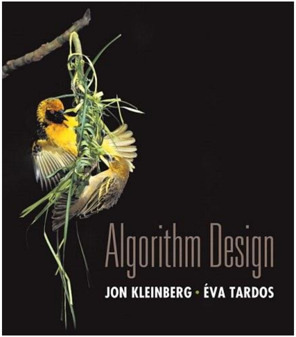
Reference Book

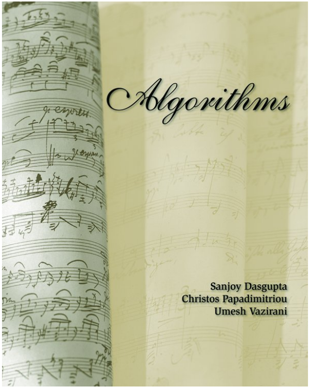
Text Book

- Algorithm Design
  - Jon Kleinberg
  - Éva Tardos
  - Addison-Wesley, 2005.



Reference Book

- Algorithms
  - Sanjoy Dasgupta
  - San Diego Christos Papadimitriou
  - Umesh Vazirani
  - McGraw-Hill, 2007.



Stable Matching Problem

Matching med-school students to hospitals

Goal: Given a set of preferences among hospitals and med-school students, design a self-reinforcing admissions process.

Unstable pair: Hospital h and student s form an unstable pair if both:

- h prefers s to one of its admitted students.
- s prefers h to assigned hospital.

Stable assignment: Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest prevents any hospital–student side deal.

Stable matching problem: input

Input: A set of n hospitals H and a set of n students S.

- Each hospital $h \in H$ ranks students.
- Each student $s \in S$ ranks hospitals.

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	Zeus
Boston	Yolanda	Xavier	Zeus
Chicago	Xavier	Yolanda	Zeus

	1st	2nd	3rd
Xavier	Boston	Atlanta	Chicago
Yolanda	Atlanta	Boston	Chicago
Zeus	Atlanta	Boston	Chicago

Perfect matching

Definition

A matching M is a set of ordered pairs h-s with $h \in H$ and $s \in S$ s.t.

- Each hospital $h \in H$ appears in at most one pair of M.
- Each student $s \in S$ appears in at most one pair of M.

Definition

A matching M is perfect if $| M |=| H |=| S |= n$.

	1st	2nd	3rd
Atlanta	Xavier	Yolanda	**Zeus**
Boston	**Yolanda**	Xavier	Zeus
Chicago	**Xavier**	Yolanda	Zeus

	1st	2nd	3rd
Xavier	Boston	Atlanta	**Chicago**
Yolanda	Atlanta	**Boston**	Chicago
Zeus	**Atlanta**	Boston	Chicago

Unstable pair

Definition

Given a perfect matching M, hospital h and student s form an unstable pair if both:

- h prefers s to matched student.
- s prefers h to matched hospital.

Key point An unstable pair h–s could each improve by joint action.

	1st	2nd	3rd
Atlanta	Xavier	**Yolanda**	**Zeus**
Boston	**Yolanda**	Xavier	Zeus
Chicago	**Xavier**	Yolanda	Zeus

	1st	2nd	3rd
Xavier	Boston	Atlanta	**Chicago**
Yolanda	**Atlanta**	**Boston**	Chicago
Zeus	**Atlanta**	Boston	Chicago

A-Y is an unstable pair for matching $M = \{A–Z, B–Y, C–X\}$

Quiz

Which pair is unstable in the matching $\{A–X, B–Z, C–Y\}$

	1st	2nd	3rd
Atlanta	**Xavier**	Yolanda	Zeus
Boston	Yolanda	Xavier	**Zeus**
Chicago	Xavier	**Yolanda**	Zeus

	1st	2nd	3rd
Xavier	Boston	**Atlanta**	Chicago
Yolanda	Atlanta	Boston	**Chicago**
Zeus	Atlanta	**Boston**	Chicago

Definition

A stable matching is a perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n hospitals and n students, find a stable matching (if one exists).

	1st	2nd	3rd
Atlanta	**Xavier**	Yolanda	Zeus
Boston	**Yolanda**	Xavier	Zeus
Chicago	Xavier	Yolanda	**Zeus**

	1st	2nd	3rd
Xavier	Boston	**Atlanta**	Chicago
Yolanda	Atlanta	**Boston**	Chicago
Zeus	Atlanta	Boston	**Chicago**

Stable roommate problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

Stable roommate problem.

- $2n$ people; each person ranks others from 1 to $2n-1$.
- Assign roommate pairs so that no unstable pairs

	1st	2nd	3rd
A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C

- A–B, C–D $\Rightarrow$ B–C unstable
- A–C, B–D $\Rightarrow$ A–B unstable
- A–D, B–C $\Rightarrow$ A–C unstable

Observation: Stable matchings need not exist.

Gale–Shapley Algorithm

Gale–Shapley deferred acceptance algorithm

Gale-Shapley(*preference lists for hospitals and students*)

Initialize M to empty matching;
while *some hospital h is unmatched and hasn't proposed to every student* **do**
    $s \leftarrow$ first student on h's list to whom h has not yet proposed;
    if *s is unmatched* **then**
        Add h–s to matching M;
    end
    else
        if *s prefers h to current partner h'* **then**
            Replace h'–s with h–s in matching M;
        end
        else
            s rejects h;
        end
    end
end
Return stable matching M;

Proof of correctness: termination

Observation 1

Hospitals propose to students in decreasing order of preference.

Observation 2

Once a student is matched, the student never becomes unmatched; only "trades up".

Claim

Algorithm terminates after at most n^2 iterations of **While** loop.

Proof. Each time through the **While** loop, a hospital proposes to a new student. Thus, there are at most n^2 possible proposals.

Proof of correctness: perfect matching

Claim

Gale–Shapley outputs a matching.

Proof.

Hospital proposes only if unmatched. $\Rightarrow$ matched to ≤ 1 student.

Student keeps only best hospital. $\Rightarrow$ matched to ≤ 1 hospital.

Proof of correctness: perfect matching

Claim

In Gale–Shapley matching, all hospitals get matched.

Proof. by contradiction

Suppose, for sake of contradiction, that some hospital $h \in H$ is unmatched upon termination of Gale–Shapley algorithm.

Then some student, say $s \in S$, is unmatched upon termination.

By Observation 2, s was never proposed to.

But, h proposes to every student, since h ends up unmatched.

Proof of correctness: perfect matching

Claim

In Gale–Shapley matching, all students get matched.

Proof. by counting

By previous claim, all n hospitals get matched.

Thus, all n students get matched.

Proof of correctness: stability

Claim

In Gale–Shapley matching M^*, there are no unstable pairs.

Proof.

Consider any pair h–s that is not in M^*.

- Case 1: h never proposed to s.
  - h prefers its Gale–Shapley partner s' to s.
  - h–s is not unstable.
- Case 2: h proposed to s.
  - s rejected h (either right away or later).
  - s prefers Gale–Shapley partner h' to h.
  - h–s is not unstable.
- In either case, the pair h–s is not unstable.

Summary

Stable matching problem. Given n hospitals and n students, and their preference lists, find a stable matching if one exists.

Theorem (Gale–Shapley 1962)

The Gale–Shapley algorithm guarantees to find a stable matching for any problem instance.

Quiz

Do all executions of Gale–Shapley lead to the same stable matching?

A. No, because the algorithm is nondeterministic.
B. No, because an instance can have several stable matchings.
C. Yes, because each instance has a unique stable matching.
D. Yes, even though an instance can have several stable matchings and the algorithm is nondeterministic.

Hospital Optimality

Understanding the solution

For a given problem instance, there may be several stable matchings.

	1st	2nd	3rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1st	2nd	3rd
X	B	A	C
Y	A	B	C
Z	A	B	C

an instance with two stable matchings:

- S = { A-X, B-Y, C-Z }
- S' = { A-Y, B-X, C-Z }

Understanding the solution

Definition

Student s is a valid partner for hospital h if there exists any stable matching in which h and s are matched.

Example

- Both X and Y are valid partners for A.
- Both X and Y are valid partners for B.
- Z is the only valid partner for C.

	1st	2nd	3rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1st	2nd	3rd
X	B	A	C
Y	A	B	C
Z	A	B	C

Quiz

Who is the best valid partner for W in the following instance?

- { A–W, B–X, C–Y, D–Z }
- { A–X, B–W, C–Y, D–Z }
- { A–X, B–Y, C–W, D–Z }
- { A–Z, B–W, C–Y, D–X }
- { A–Z, B–Y, C–W, D–X }
- { A–Y, B–Z, C–W, D–X }

	1st	2nd	3rd	4th
A	Y	Z	X	W
B	Z	Y	W	X
C	W	Y	X	Z
D	X	Z	W	Y

	1st	2nd	3rd	4th
X	D	A	B	C
Y	C	B	A	D
Z	C	B	A	D
W	D	A	B	C

Understanding the solution

Definition

Student s is a valid partner for hospital h if there exists any stable matching in which h and s are matched.

Hospital-optimal assignment. Each hospital receives best valid partner.

- Is it a perfect matching?
- Is it stable?

Claim

All executions of Gale–Shapley yield hospital-optimal assignment.

Corollary

Hospital-optimal assignment is a stable matching!

Hospital optimality

Claim

Gale–Shapley matching M^* is hospital-optimal.

Proof. by contradiction

Suppose a hospital is matched with student other than best valid partner.

Hospitals propose in decreasing order of preference.
$\Rightarrow$ some hospital is rejected by a valid partner during Gale–Shapley.

Let h be first such hospital, and let s be the first valid partner that rejects h.

Let M be a stable matching where h and s are matched.

When s rejects h in Gale–Shapley, s forms (or re-affirms) commitment to a hospital, say h'.
$\Rightarrow$ s prefers h' to h.

Hospital optimality

Claim

Gale–Shapley matching M^* is hospital-optimal.

Proof. by contradiction

Let s' be partner of h' in M.

h' had not been rejected by any valid partner (including s') at the point when h is rejected by s.

Thus, h' had not yet proposed to s' when h' proposed to s
$\Rightarrow$ h' prefers s to s'.

Thus, h'–s is unstable in M, a contradiction.

Student pessimality

Q. Does hospital-optimality come at the expense of the students?

A. Yes.

Student-pessimal assignment. Each student receives worst valid partner.

Claim

Gale–Shapley finds student-pessimal stable matching M^*.

Student pessimality

Claim

Gale–Shapley finds student-pessimal stable matching M^*.

Proof. by contradiction

Suppose h–s matched in M^* but h is not the worst valid partner for h.

There exists stable matching M in which s is paired with a hospital, say h', whom s prefers less than h.
$\Rightarrow$ s prefers h to h'.

Let s' be the partner of h in M.

By hospital-optimality, s is the best valid partner for h.
$\Rightarrow$ h prefers s to s'.

Thus, h–s is an unstable pair in M, a contradiction.

Quiz

Suppose each agent knows the preference lists of every other agent before the hospital propose-and-reject algorithm is executed. Which is true?

A No hospital can improve by falsifying its preference list.

B No student can improve by falsifying their preference list.

C Both A and B.

D Neither A nor B.

Extensions

Extension

Extension 1. Some agents declare others as unacceptable.

Extension 2. Some hospitals have more than one position.

Extension 3. Unequal number of positions and students.

Definition

Matching M is unstable if there is a hospital h and student s such that:

- h and s are acceptable to each other; and
- Either s is unmatched, or s prefers h to assigned hospital; and
- Either h does not have all its places filled, or h prefers s to at least one of its assigned students.

Extensions

Theorem

There exists a stable matching.

Proof. Straightforward generalization of Gale–Shapley algorithm.