Algorithm Design (XV)

Extending Tractability

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Coping with NP-completeness

Q. Suppose I need to solve an **NP**-complete problem. What should I do?
A. Theory says you're unlikely to find poly-time algorithm.

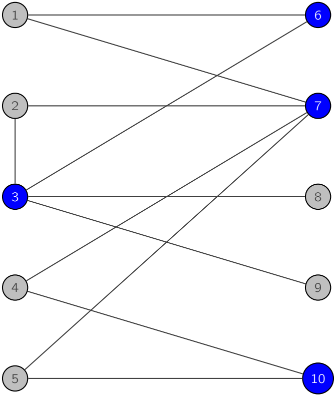
Must sacrifice one of three desired features.

- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve arbitrary instances of the problem.

This lecture. Solve some special cases of **NP**-complete problems.

Vertex cover

Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge (u, v) either $u \in S$ or $v \in S$ or both?



$S = \{3, 6, 7, 10\}$ is a vertex cover of size $k = 4$

Finding small vertex covers

Q. Vertex cover is **NP**-complete. But what if k is small?

Brute force. $O(kn^{k+1})$.

- Try all $C(n, k) = O(n^k)$ subsets of size k.
- Take $O(kn)$ time to check whether a subset is a vertex order.

Goal. Limit exponential dependency on k, say to $O(2^k kn)$.

Example. $n = 1,000, k = 10.$
Brute. $kn^{k+1} = 10^{34} \Rightarrow$ infeasible.
Better $2^k kn = 10^7 \Rightarrow$ feasible.

Remark

If k is a constant, then the algorithm is poly-time; if k is a small constant, then it's also practical.

Finding small vertex covers

Claim

Let (u, v) be an edge of G. G has a vertex cover of size $\leq k$ iff at least one of $G - \{u\}$ and $G - \{v\}$ has a vertex cover of size $\leq k - 1$.

Proof.

$\Rightarrow$

- Suppose G has a vertex cover S of size $\leq k$.
- S contains either u or v (or both). Assume it contains u.
- $S - \{u\}$ is a vertex cover of $G - \{u\}$.

$\Leftarrow$

- Suppose S is a vertex cover of $G - \{u\}$ of size $\leq k - 1$.
- Then $S \cup \{u\}$ is a vertex cover of G.

Claim

If G has a vertex cover of size k, it has $\leq k(n-1)$ edges.

Proof. Each vertex covers at most $n - 1$ edges.

Finding small vertex covers: algorithm

Claim

The following algorithm determines if G has a vertex cover of size $\leq k$ in $O(2^k kn)$ time.

VertexCover(G, k)

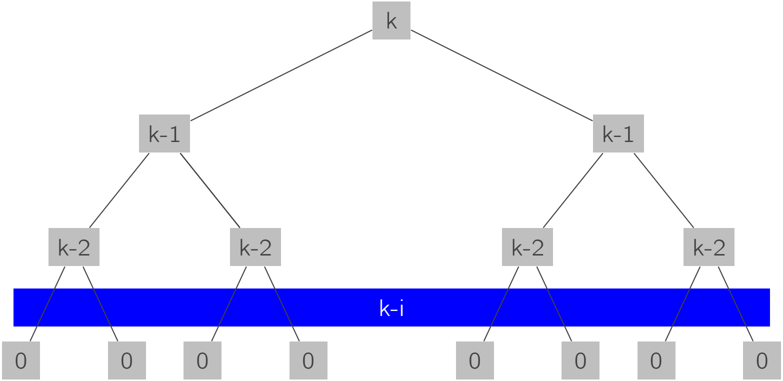
if *G contains no edges* **then** Return true;
if *G contains $\geq kn$ edges* **then** Return false;
let (u, v) be any edge of G;
$a = \text{VertexCover}(G - \{u\}, k - 1)$;
$b = \text{VertexCover}(G - \{v\}, k - 1)$;
Return a or b;

Proof.

- Correctness follows from previous two claims.
- There are $\leq 2^{k+1}$ nodes in the recursion tree; each invocation takes $O(kn)$ time.

Finding small vertex covers: recursion tree

$$T(n,k) \le \begin{cases} c & \text{if } k = 0 \\ cn & \text{if } k = 1 \\ 2T(n,k-1) + ckn & \text{if } k > 1 \end{cases} \quad \Rightarrow \quad T(n,k) \le 2^k ckn$$



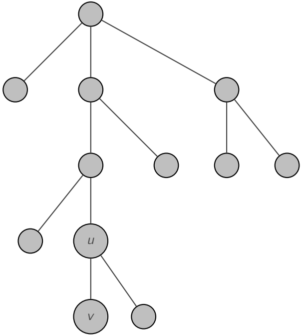
Solving NP-Hard Problems on Trees

Independent set on trees

Independent set on trees. Given a tree, find a maximum cardinality subset of nodes such that no two share an edge.

Fact. A tree on at least two nodes has at least two leaf nodes.

Key observation. If v is a leaf, there exists a maximum size independent set containing v.



Proof. (exchange argument)

- Consider a max cardinality independent set S.
- If $v \in S$, we're done.
- If $u \notin S$ and $v \notin S$, then $S \cup \{v\}$ is independent $\Rightarrow$ S not maximum.
- If $u \notin S$ and $v \notin S$, then $S \cup \{v\} - \{u\}$ is independent.

Independent set on trees: greedy algorithm

Theorem

The following greedy algorithm finds a maximum cardinality independent set in forests (and hence trees).

IndependentSetInForest(F)

$S \leftarrow \phi$;
while *F has at least one edge* **do**
  Let $e = (u, v)$ be an edge such that v is a leaf;
  Add v to S;
  Delete from F nodes u and v, and all edges incident to them;
end
Return S;

Remark Can implement in $O(n)$ time by considering nodes in postorder.

Weighted independent set on trees

Weighted independent set on trees. Given a tree and node weights $w_v > 0$, find an independent set S that maximizes $\Sigma_{v \in S} w_v$.

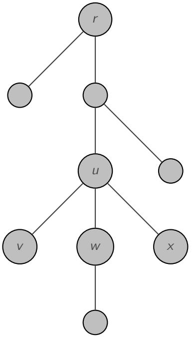
Observation. If (u, v) is an edge such that v is a leaf node, then either OPT includes u or OPT includes all leaf nodes incident to u.

Dynamic programming solution. Root tree at some node, say r.

- $OPT_{in}(u)$: max weight independent set of subtree rooted at u, containing u.
- $OPT_{out}(u)$: max weight independent set of subtree rooted at u, not containing u.

$$OPT_{in}(u) = w_u + \sum_{v \in \text{ children } (u)} OPT_{\text{out}}(v)$$

$$OPT_{\text{out}}(u) = \sum_{v \in \text{ children } (u)} \max\{OPT_{\text{in}}(v), OPT_{\text{out}}(v)\}$$



$children(u) = \{\text{v}, \text{w}, \text{x}\}$

Weighted independent set on trees: dynamic programming algorithm

Theorem

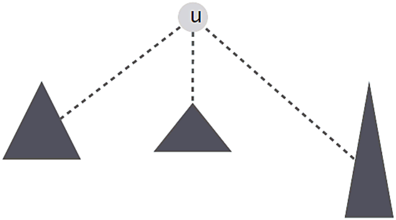
The dynamic programming algorithm finds a maximum weighted independent set in a tree in $O(n)$ time.

WeightedIndependentSetInTree(T)

Root the tree at a node r;
for *each node u of T in postorder* **do**
  if *u is a leaf* **then**
    $M_{in}[u] = w_u$;
    $M_{out}[u] = 0$;
  end
  else
    $M_{in}[u] = w_u + \Sigma_{v \in children(u)} M_{out}[v]$;
    $M_{out}[u] = \Sigma_{v \in children(u)} \max(M_{in}[v], M_{out}[v])$;
  end
end
Return $\max(M_{in}[r], M_{out}[r])$;

Context

Independent set on trees. This structured special case is tractable because we can find a node that breaks the communication among the subproblems in different subtrees.



Graphs of bounded tree width. Elegant generalization of trees that:

- Captures a rich class of graphs that arise in practice.
- Enables decomposition into independent pieces.

Circular Arc Coverings

Wavelength-division multiplexing

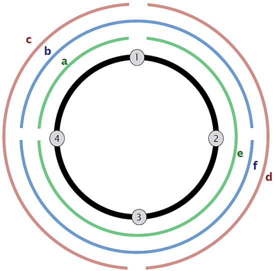
Wavelength-division multiplexing (WDM). Allows m communication streams (arcs) to share a portion of a fiber optic cable, provided they are transmitted using different wavelengths.

Ring topology. Special case is when network is a cycle on n nodes.

Bad news. **NP**-complete, even on rings.

Brute force. Can determine if k colors suffice in $O(k^m)$ time by trying all k-colorings.

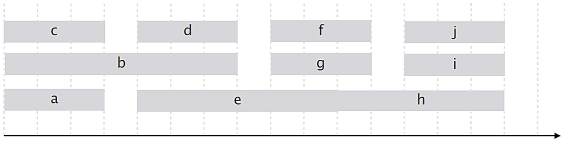
Goal. $O(f(k)) \cdot poly(m, n)$ on rings.



$n = 4, m = 6 \quad \{c, d\}, \{b, f\}, \{a, e\}$

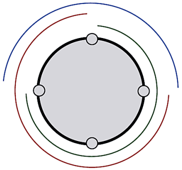
Review: interval coloring

Interval coloring. Greedy algorithm finds coloring such that number of colors equals depth of schedule.



Circular arc coloring.

- Weak duality: number of colors $\geq$ depth.
- Strong duality does not hold.

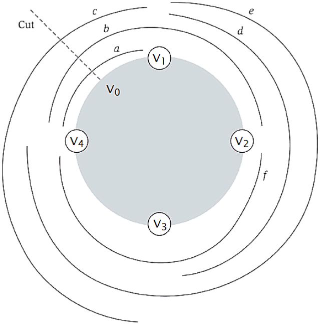


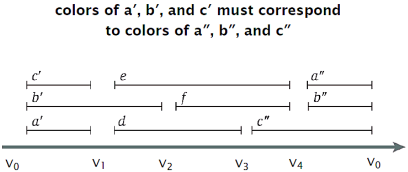
max depth $= 2$
min colors $= 3$

(Almost) transforming circular arc coloring to interval coloring

Circular arc coloring. Given a set of n arcs with depth $d \leq k$, can the arcs be colored with k colors?

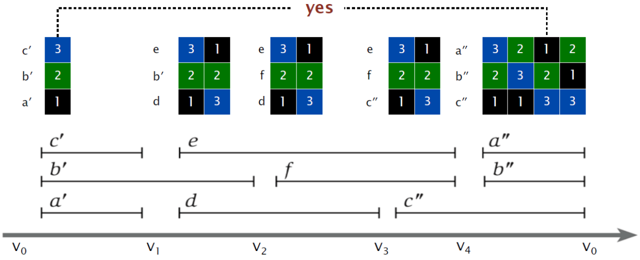
Equivalent problem. Cut the network between nodes v_1 and v_n. The arcs can be colored with k colors iff the intervals can be colored with k colors in such a way that "sliced" arcs have the same color.





Dynamic programming algorithm.

- Assign distinct color to each interval which begins at cut node v_0.
- At each node v_i, some intervals may finish, and others may begin.
- Enumerate all k-colorings of the intervals through v_i that are consistent with the colorings of the intervals through v_{i-1}.
- The arcs are k-colorable iff some coloring of intervals ending at cut node v_0 is consistent with original coloring of the same intervals.



Circular arc coloring: running time

Running time. $O(k! \cdot n)$.

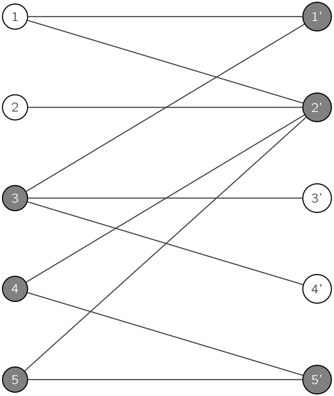
- The algorithm has n phases.
- Bottleneck in each phase is enumerating all consistent colorings.
- There are at most k intervals through v_i, so there are at most $k!$ colorings to consider.

Remark. This algorithm is practical for small values of k (say $k = 10$) even if the number of nodes n (or paths) is large.

Vertex Cover in Bipartite Graphs

Vertex cover

Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $| S | \leq k$, and for each edge (u, v) either $u \in S$ or $v \in S$ or both?

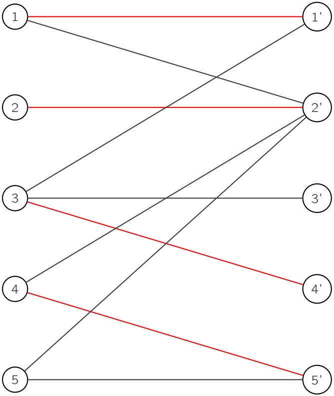


vertex cover $S = \{3, 4, 5, 1', 2'\}$

Vertex cover and matching

Weak duality. Let M be a matching, and let S be a vertex cover. Then, $| M | \leq | S |$.

Proof. Each vertex can cover at most one edge in any matching.

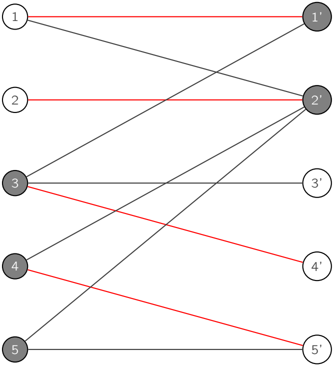


matching M: 1-1', 2-2', 3-4', 4-5'

Vertex cover in bipartite graphs: König-Egerváry Theorem

Theorem (König-Egerváry)

In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.



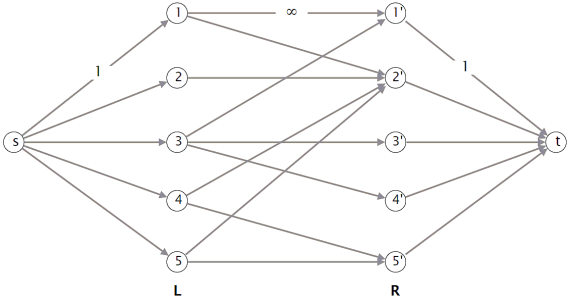
matching M: 1-1', 2-2', 3-4', 4-5'

vertex cover $S = \{3, 4, 5, 1', 2'\}$

Theorem (König-Egerváry)

In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find matching M and cover S such that $|M| = |S|$.
- Formulate max flow problem as for bipartite matching.
- Let M be max cardinality matching and let (A, B) be min cut.



Proof of König-Egerváry Theorem

Theorem (König-Egerváry)

In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find matching M and cover S such that $\mid M \mid = \mid S \mid$.
- Formulate max flow problem as for bipartite matching.
- Let M be max cardinality matching and let (A, B) be min cut.
- Define $L_A = L \cap A, L_B = L \cap B, R_A = R \cap A, R_B = R \cap B$

- Claim 1. $S = L_B \cup R_A$ is a vertex cover.
  - consider $(u, v) \in E$
  - $u \in L_A, v \in R_B$ impossible since infinite capacity
  - thus, either $u \in L_B$ or $v \in R_A$ or both

- Claim 2. $\mid M \mid = \mid S \mid$.
  - max-flow min-cut theorem $\Rightarrow M = \mathbf{cap}(A, B)$
  - only edges of form (s, u) or (v, t) contribute to $\mathbf{cap}(A, B)$
  - $|M| = \mathbf{cap}(A, B) = |L_B| + |R_A| = |\mathrm{S}|$.