GENERAL ASSEMBLY

INTRO TO DATA SCIENCE
LECTURE 5: REGRESSION & REGULARIZATION

LAST TIME:

- INTRO TO PROBABILITY & BAYESIAN INFERENCE
- THE NAÏVE BAYESIAN CLASSIFIER
- DOCUMENT VECTORS & SPAM FILTER

QUESTIONS?

I. INTRO TO REGRESSION
II. REGULARIZATION

EXERCISES:
III. IMPLEMENTING A REGULARIZED FIT IN R

I. LINEAR REGRESSION

	continuous	*categorical*
supervised	???	???
unsupervised	???	???

	continuous	categorical
supervised	regression	classification
unsupervised	dimension reduction	clustering

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ε = **residual** *(the prediction error)*

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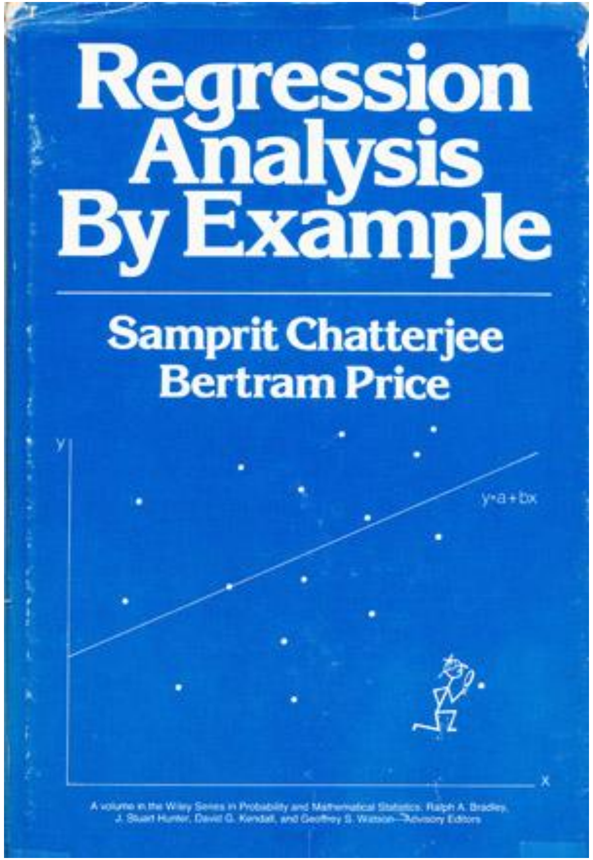
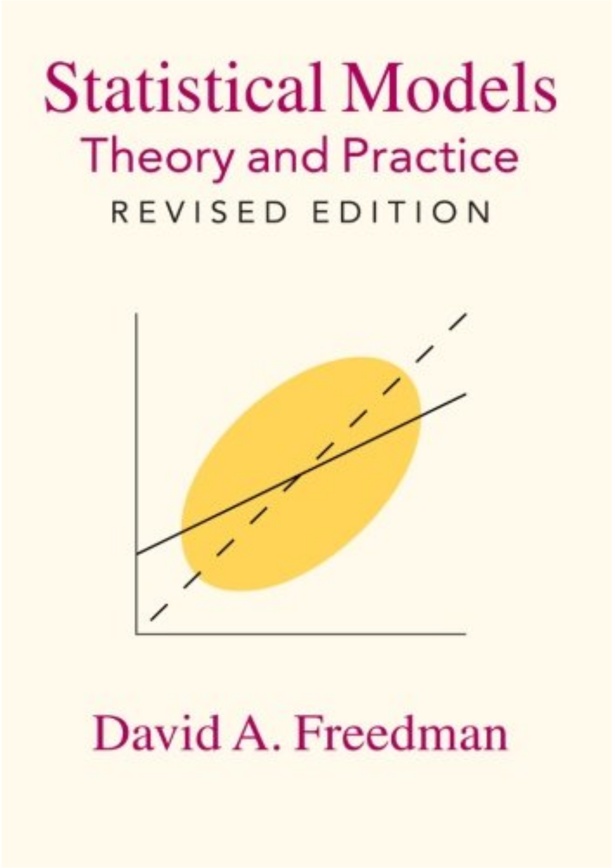
Linear regression involves several technical assumptions and is often presented with lots of mathematical formality.

The math is not very important for our purposes, but you should check it out if you get serious about solving regression problems.



Regression Analysis By Example
Samprit Chatterjee
Bertram Price
y
y=a+bx
x
A volume in the Wiley Series in Probability and Mathematical Statistics. Ralph A. Bradley, J. Stuart Hunter, David G. Kendall, and Geoffrey S. Watson—Advisory Editors



Statistical Models
Theory and Practice
REVISED EDITION
David A. Freedman

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But again, if you get serious about regression, you should learn how this works!

II: POLYNOMIAL REGRESSION

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$$y = \alpha + \beta_1 x + \beta_2 x^2 + \varepsilon$$

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"Although polynomial regression fits a nonlinear model to the data, as a statistical estimation problem it is linear, in the sense that the regression function E(y|x) is linear in the unknown parameters that are estimated from the data. For this reason, polynomial regression is considered to be a special case of multiple linear regression." *-- Wikipedia*

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Q: Does anyone know what it is?

A: This model violates one of the assumptions of linear regression!

This model displays **multicollinearity***, which means the predictor variables are highly correlated with each other.*

$$y = \alpha + \beta_1 x + \beta_2 x^2 + \ldots + \beta_n x^n + \varepsilon$$

```
> x <- seq(1, 10, 0.1)
> cor(x^9, x^10)
[1] 0.9987608
```

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NOTE

This results in a *singularity*. We will see an example of this in just a minute!

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OPTIONAL NOTE

These polynomial functions form an *orthogonal basis* of the function space.

So far, we've seen how polynomial regression allows us to fit complex nonlinear relationships, and even to avoid multicollinearity (by using basis functions).

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Q: Can a regression model be too complex?

III: REGULARIZATION

Recall our earlier discussion of **overfitting.**

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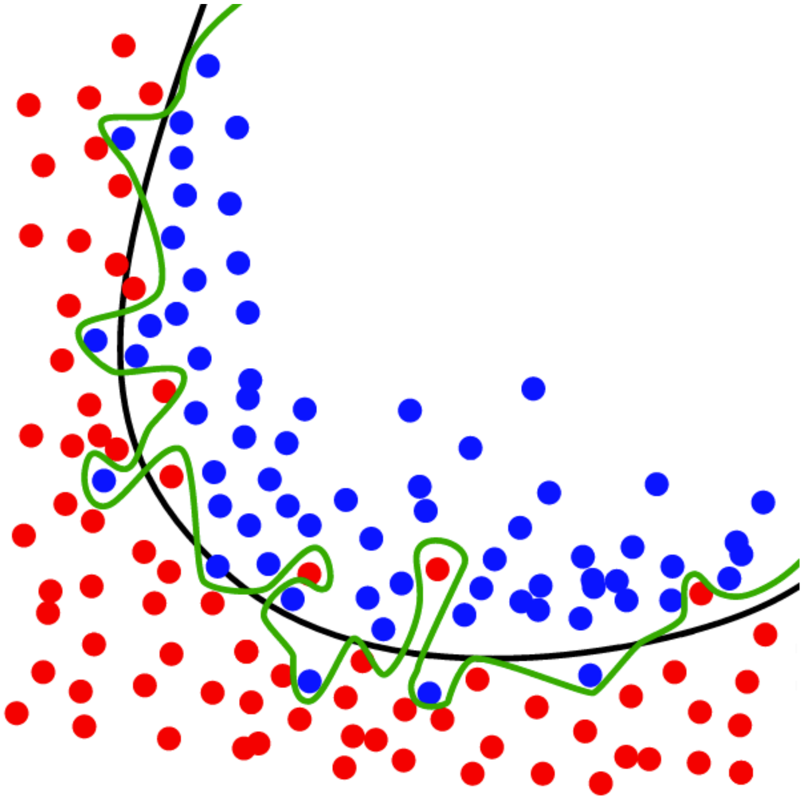
In other words, an overfit model matches the **noise** *in the dataset instead of the* **signal***.*

OVERFITTING EXAMPLE (CLASSIFICATION)

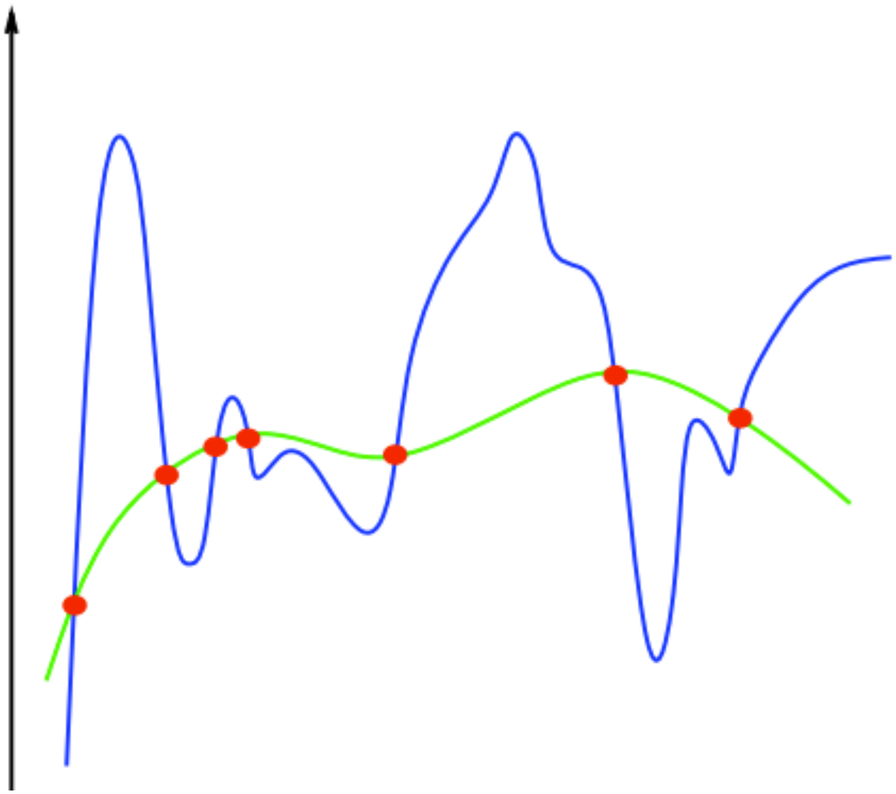


source: http://upload.wikimedia.org/wikipedia/commons/1/19/Overfitting.svg

The same thing can happen in regression.

It's possible to design a regression model that matches the noise in the data instead of the signal.

This happens when our model becomes too complex *for the data to support.*



source: http://www.mit.edu/~9.520/spring12/slides/class02/class02.pdf

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Ex 1: $\Sigma\,|\beta_i|$

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A: One method is to define complexity as a function of the size of the coefficients.

Ex 1: $\Sigma\,|\beta_i|$ *this is called the* **L1-norm**
Ex 2: $\Sigma\,{\beta_i}^2$ *this is called the* **L2-norm**

These measures of complexity lead to the following **regularization** *techniques:*

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Regularization *refers to the method of preventing* **overfitting** *by explicitly controlling model* **complexity***.*

These regularization problems can also be expressed as:

L1 regularization: $min(||y - x\beta||^2 + \lambda||x||)$

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Variance refers to predictions that are generally *inaccurate.*

BIAS AND VARIANCE

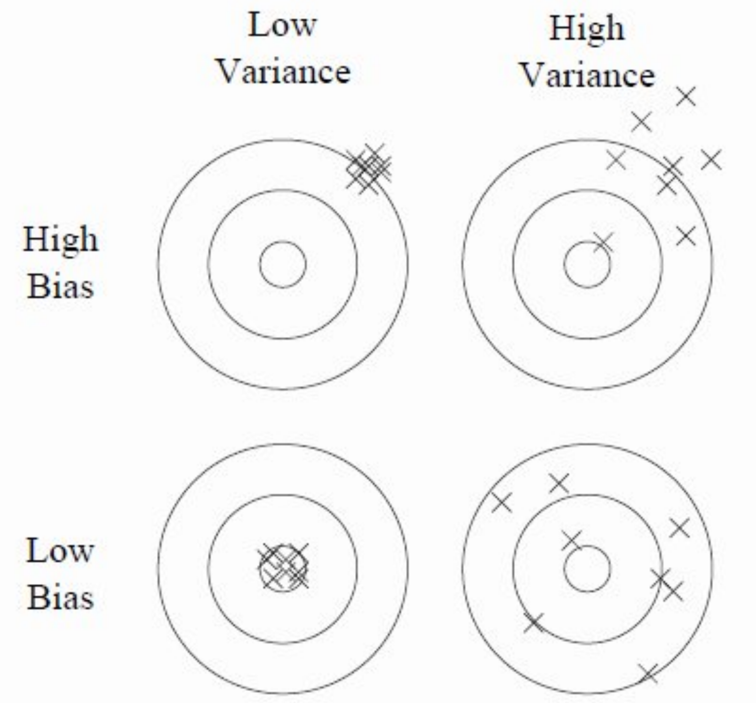


Figure 1: Bias and variance in dart-throwing.

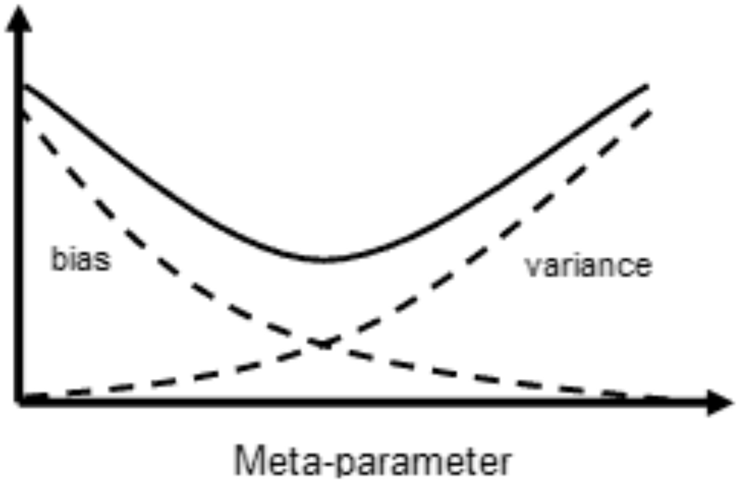
source: http://homes.cs.washington.edu/~pedrod/papers/cacm12.pdf

Q: What are bias and variance?

A: Bias refers to predictions that are systematically inaccurate.
Variance refers to predictions that are generally inaccurate.

It turns out (after some math) that the generalization error in our model can be decomposed into a bias component and variance component.

This is another example of the **bias-variance tradeoff**.



source: http://www.isu.edu/chem/images/kalivasmeta.gif

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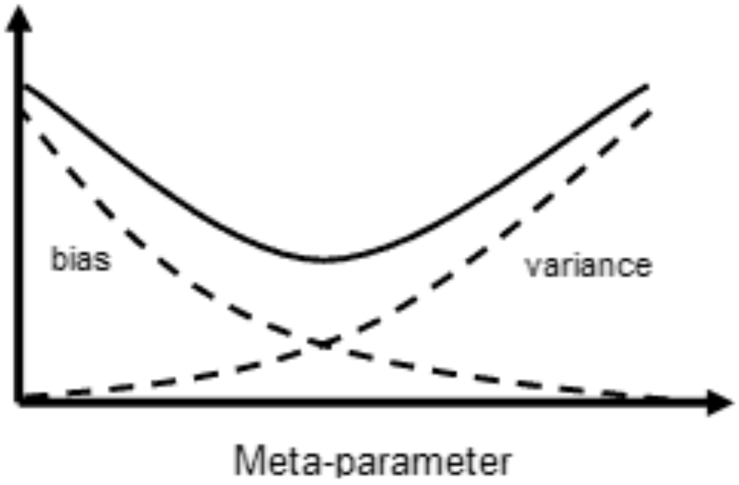

NOTE

The "meta-parameter" here is the lambda we saw above.

A more typical term is "hyperparameter".

source: http://www.isu.edu/chem/images/kalivasmeta.gif

This tradeoff is regulated by a **hyperparameter** λ*, which we've already seen:*

L1 regularization*:* $y = \Sigma\, \beta_i x_i + \varepsilon \quad st. \quad \Sigma\, |\beta_i| < \lambda$
L2 regularization*:* $y = \Sigma\, \beta_i x_i + \varepsilon \quad st. \quad \Sigma\, \beta_i^2 < \lambda$

So regularization represents a method to trade away some variance for a little bias in our model, thus achieving a better overall fit.

EX: POLYNOMIAL REGRESSION & REGULARIZATION

EXERCISE – POLYNOMIAL REGRESSION & REGULARIZATION

KEY OBJECTIVES

- *observe multicollinearity in naïve polynomial fit*
- *perform polynomial fit using orthogonal basis functions*
- *observe overfitting in polynomial fit of high degree*
- *perform regularized fit to control overfitting*

TOOLS

- *lm*
- *poly*
- *poly*
- *glmnet*