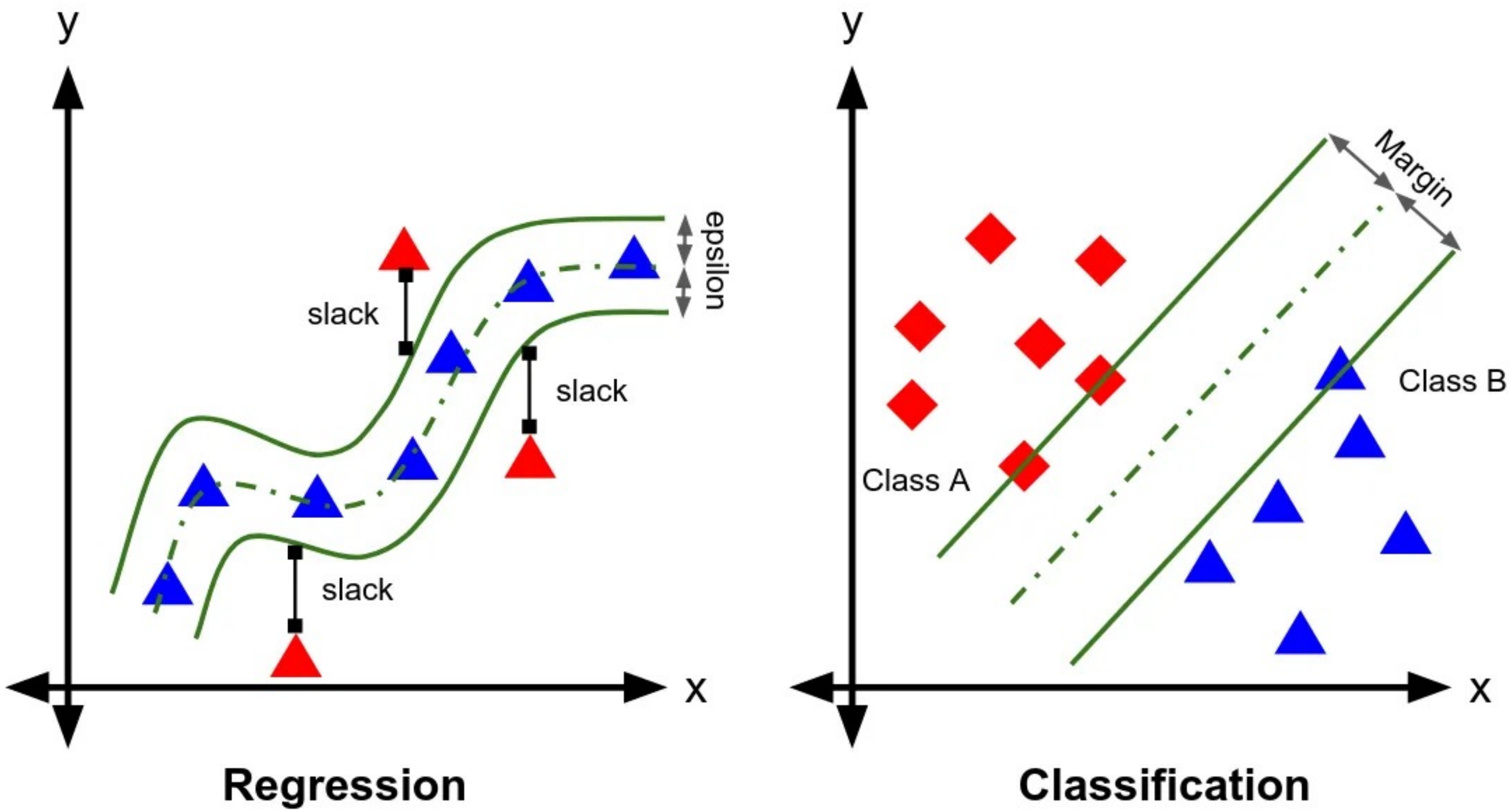
Machine Learning
SVM Regression

Jian Liu



Linear Regression

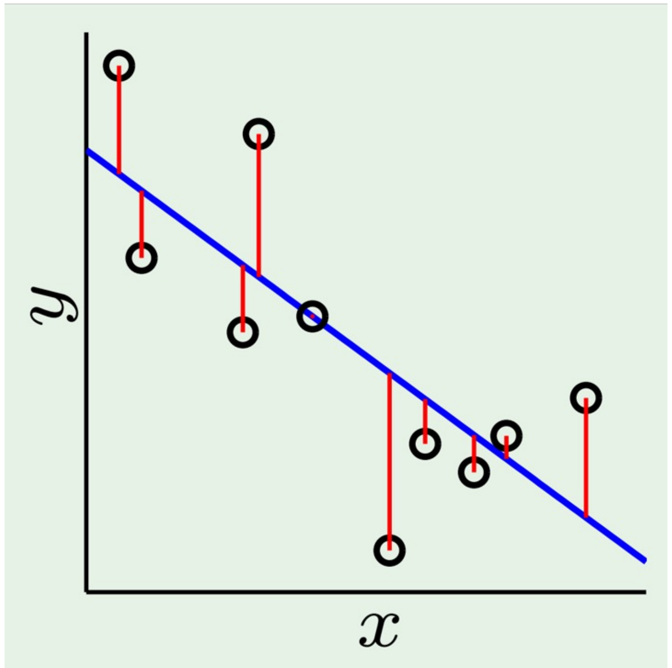
- Given data **X** and target **Y**
- The objective: Find a function that returns the best fit.
- Assume that the relationship between **X** and **Y** is approximately linear. The model can be represented as (**W** represents coefficients and **b** is an intercept)

$$f(w_1,\ldots,w_n,b) = y = \mathbf{w}\cdot\mathbf{x} + b + \varepsilon$$

Linear Regression

- To find the best fit, we minimize the sum of squared errors -> Least square estimation

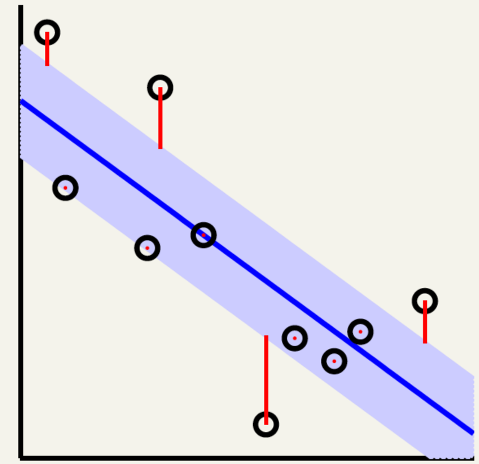
$$\min \sum_{i=1}^{m} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{m} (y_i - (\mathbf{w} \cdot \mathbf{x}_i + b))^2$$



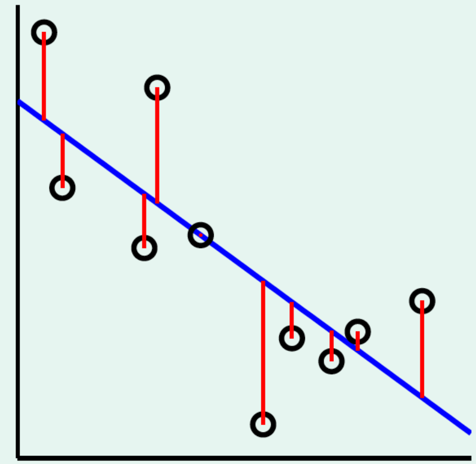
Evaluating Regression Models

- Common metrics for evaluating regression models:
  - Coefficient of determination or $R^2 = 1 - \frac{\sum_i (y_i - \hat{y}_i)^2}{\sum_i (y_i - \bar{y})^2}$; $\bar{y}$ is the mean of the observed targets
  - Mean absolute error (MAE) $= \frac{1}{N} \sum_i^N \mid y_i - \hat{y}_i \mid$
  - Mean squared error (MSE) $= \frac{1}{N} \sum_i^N (y_i - \hat{y}_i)^2$
  - Root mean squared error (RMSE) $= \sqrt{\frac{1}{N} \sum_i^N (y_i - \hat{y}_i)^2}$
  - $\cdots$ and several others!

Support Vector Regression



tube: $\mathrm{err}(y, s) = \max(0, |s - y| - \epsilon)$

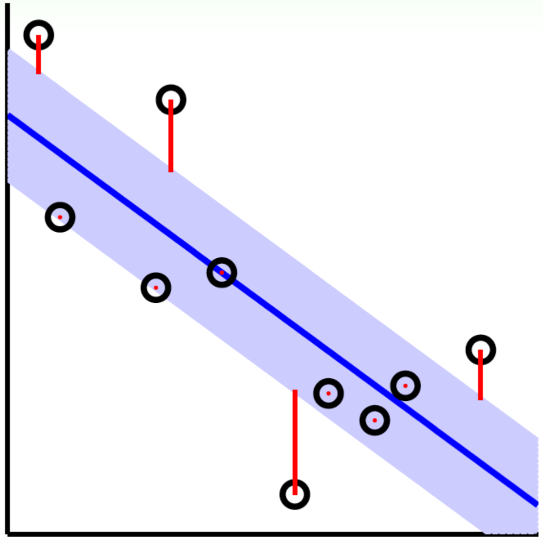


squared: $\mathrm{err}(y, s) = (s - y)^2$

Support Vector Regression

will consider **tube regression**

- within a tube: **no error**
- outside a tube: **error** by distance to tube



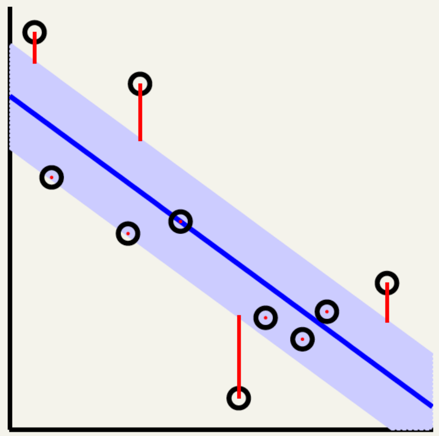
error measure:

$$\text{err}(y, s) = \max(0, |s - y| - \epsilon)$$

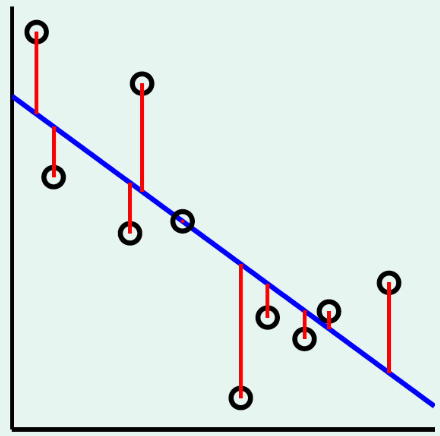
- $|s - y| \leq \epsilon$: 0
- $|s - y| > \epsilon$: $|s - y| - \epsilon$

—usually called ϵ**-insensitive error** with $\epsilon > 0$

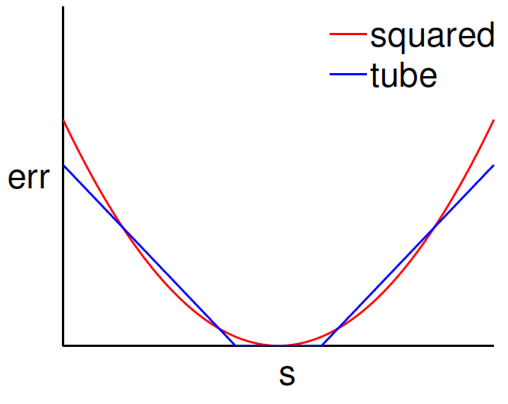
Support Vector Regression



tube: $\text{err}(y, s) = \max(0, |s - y| - \epsilon)$



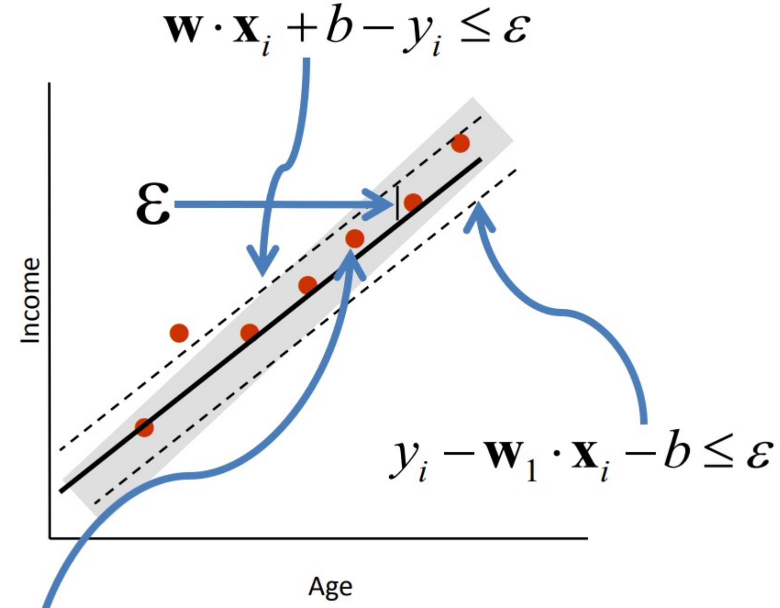
squared: $\text{err}(y, s) = (s - y)^2$



tube $\approx$ squared when $|s - y|$ small
& **less affected by outliers**

Support Vector Regression

- Find a function, f(x),
  with at most ε-deviation from the target y



We do not care about errors as long as they are less than ε

Support Vector Regression

- Find a function, f(x),
  with at most ε-deviation from the target y

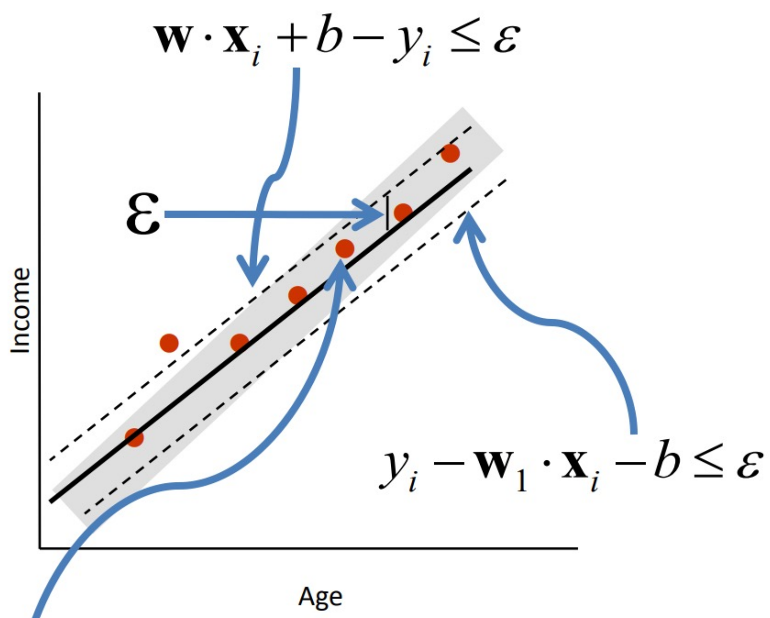
The problem can be written as a convex optimization problem

L2-Regularized
(more details in later lectures)

$$\boxed{\min \frac{1}{2}\|\mathbf{w}\|^2}$$

$$s.t.\ y_i - \mathbf{w}_1 \cdot \mathbf{x}_i - b \le \varepsilon;$$

$$\mathbf{w}_1 \cdot \mathbf{x}_i + b - y_i \le \varepsilon;$$



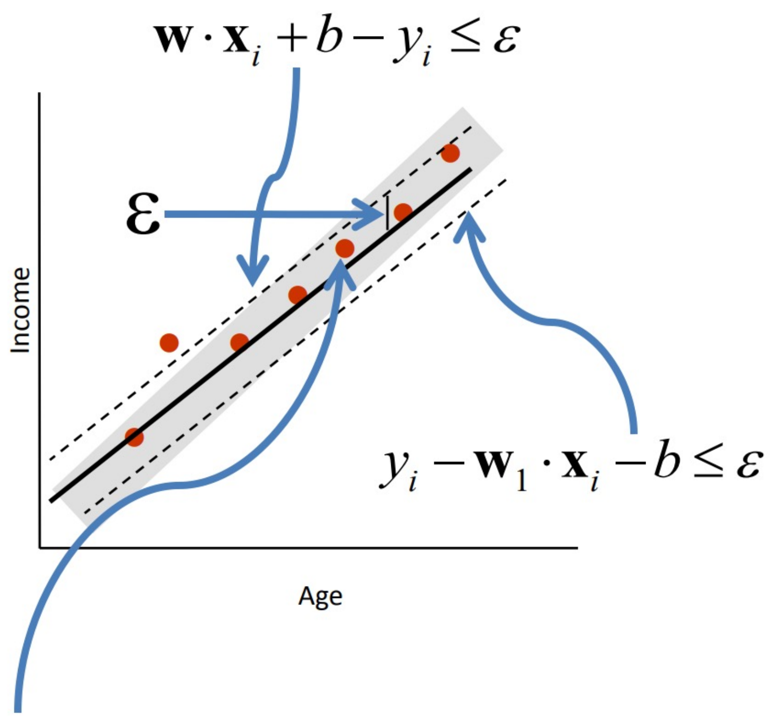
We do not care about errors as long as they are less than ε

Support Vector Regression

- Find a function, f(x),
  with at most ε-deviation from the target y

What if the problem is not feasible?

We can introduce slack variables
(similar to soft margin loss function).

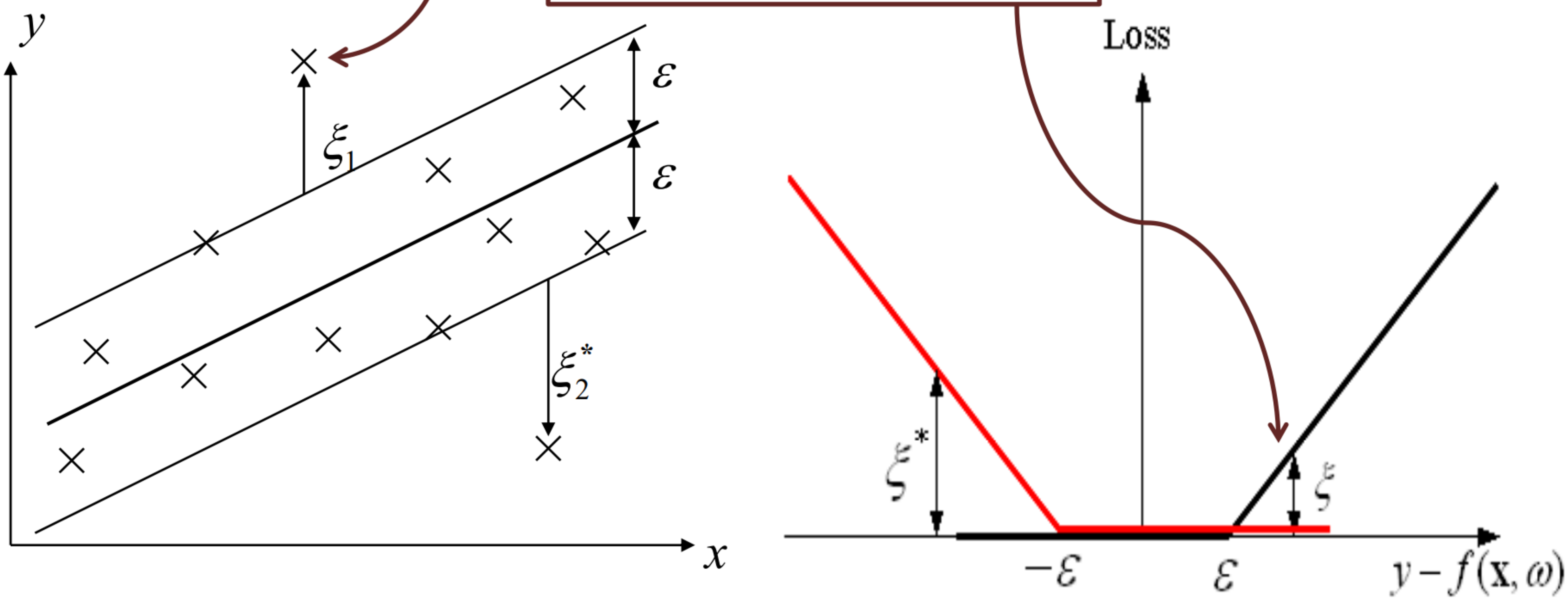


We do not care about errors as long as they are less than ε

Support Vector Regression

We can introduce slack variables (similar to soft margin loss function).

Only the point outside the ε-region contribute to the final cost



Support Vector Regression

We can introduce slack variables (similar to soft margin loss function).

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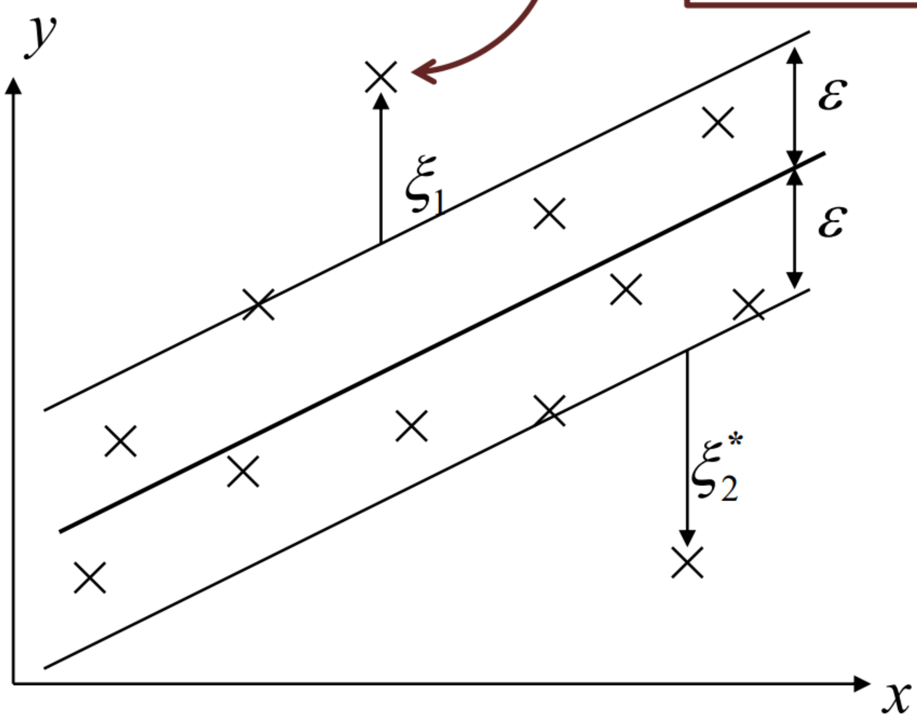

$$J(w) = \frac{1}{2} w' w + C \sum_{1}^{N} (\xi + \xi^*);$$

$$y_i - (x_i w + b) \leq \epsilon + \xi_i$$
$$(x_i w + b) - y_i \leq \epsilon + \xi_i^*$$
$$\xi^* \geq 0$$
$$\xi_i \geq 0$$

Optimizing the Lagrangian

$$
\begin{aligned}
L := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\xi_i + \xi_i^*) - \sum_{i=1}^{\ell}(\eta_i\xi_i + \eta_i^*\xi_i^*) \\
- \sum_{i=1}^{\ell}\alpha_i(\varepsilon + \xi_i - y_i + \langle w, x_i\rangle + b) \\
- \sum_{i=1}^{\ell}\alpha_i^*(\varepsilon + \xi_i^* + y_i - \langle w, x_i\rangle - b)
\end{aligned}
$$

Lagrange multipliers $\alpha_i^{(*)}, \eta_i^{(*)} \geq 0.$

Optimizing the Lagrangian

The partial derivatives of L with respect to the variables

$$\partial_b L = \sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0$$

$$\partial_w L = w - \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) x_i = 0$$

$$\partial_{\xi_i^{(*)}} L = C - \alpha_i^{(*)} - \eta_i^{(*)} = 0$$

Optimizing the Lagrangian

$$\partial_b L = \sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0$$

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$$- \sum_{i=1}^{\ell} \alpha_i (\varepsilon + \xi_i - y_i + \langle w, x_i \rangle + b)$$

$$- \sum_{i=1}^{\ell} \alpha_i^* (\varepsilon + \xi_i^* + y_i - \langle w, x_i \rangle - b)$$

Optimizing the Lagrangian

$$\partial_b L = \sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0$$

$$\partial_w L = w - \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) x_i = 0$$

$$\boxed{\partial_{\xi_i^{(*)}} L = C - \alpha_i^{(*)} - \eta_i^{(*)} = 0}$$

$$L := \frac{1}{2}\|w\|^2 + C \sum_{i=1}^{\ell} (\xi_i + \xi_i^*) - \boxed{\sum_{i=1}^{\ell} (\eta_i \xi_i + \eta_i^* \xi_i^*)}$$

$$- \sum_{i=1}^{\ell} \alpha_i (\varepsilon + \xi_i - y_i + \langle w, x_i \rangle + b)$$

$$- \sum_{i=1}^{\ell} \alpha_i^* (\varepsilon + \xi_i^* + y_i - \langle w, x_i \rangle - b)$$

Optimizing the Lagrangian

$$\partial_b L = \sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0$$

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$$L := \frac{1}{2}\|w\|^2 + C \sum_{i=1}^{\ell} (\xi_i + \xi_i^*) - \sum_{i=1}^{\ell} (\boxed{\eta_i} \xi_i + \eta_i^* \xi_i^*)$$

$$- \sum_{i=1}^{\ell} \alpha_i (\varepsilon + \xi_i - y_i + \langle w, x_i \rangle + b)$$

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Optimizing the Lagrangian

$C - \alpha_i^{(*)}$

$$L := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\xi_i + \xi_i^*) - \sum_{i=1}^{\ell}(\boxed{\eta_i}\xi_i + \eta_i^*\xi_i^*)$$

$$- \sum_{i=1}^{\ell}\alpha_i(\varepsilon + \xi_i - y_i + \langle w, x_i\rangle + b)$$

$$- \sum_{i=1}^{\ell}\alpha_i^*(\varepsilon + \xi_i^* + y_i - \langle w, x_i\rangle - b)$$

Optimizing the Lagrangian

$$\boxed{C - \alpha_i^{(*)}}$$

$$L := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell}(\boxed{\eta_i}\cancel{\xi_i} + \eta_i^*\cancel{\xi_i^*})$$

$$- \sum_{i=1}^{\ell}\alpha_i(\varepsilon + \cancel{\xi_i} - y_i + \langle w, x_i\rangle + b)$$

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Optimizing the Lagrangian

$$\partial_b L = \sum_{i=1}^{\ell} (\alpha_i^* - \alpha_i) = 0$$

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Optimizing the Lagrangian

$$\partial_b L = \boxed{\sum_{i=1}^{\ell}(\alpha_i^* - \alpha_i) = 0}$$

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Optimizing the Lagrangian

$$\partial_b L = \boxed{\sum_{i=1}^{\ell}(\alpha_i^* - \alpha_i) = 0}$$

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Optimizing the Lagrangian

$$\partial_w L = w - \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) x_i = 0$$

$$L := \frac{1}{2}\|w\|^2 + C \sum_{i=1}^{\ell} (\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell} (\eta_i \cancel{\xi_i} + \eta_i^* \cancel{\xi_i^*})$$

$$- \sum_{i=1}^{\ell} \alpha_i (\varepsilon + \cancel{\xi_i} - y_i + \langle \cancel{w}, x_i \rangle + \cancel{b})$$

$$- \sum_{i=1}^{\ell} \alpha_i^* (\varepsilon + \cancel{\xi_i^*} + y_i - \langle \cancel{w}, x_i \rangle - \cancel{b})$$

Optimizing the Lagrangian

$$\partial_w L = w - \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i = 0 \longrightarrow w = \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i$$

$$\begin{aligned}
L := \boxed{\frac{1}{2}\|w\|^2} &+ C\sum_{i=1}^{\ell}(\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell}(\eta_i \cancel{\xi_i} + \eta_i^* \cancel{\xi_i^*}) \\
&- \sum_{i=1}^{\ell}\alpha_i(\varepsilon + \cancel{\xi_i} - y_i + \cancel{\langle w, x_i\rangle + b}) \\
&- \sum_{i=1}^{\ell}\alpha_i^*(\varepsilon + \cancel{\xi_i^*} + y_i - \cancel{\langle w, x_i\rangle - b})
\end{aligned}$$

Optimizing the Lagrangian

$$\partial_w L = w - \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i = 0 \longrightarrow w = \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i$$

$$L := \boxed{\frac{1}{2}\|w\|^2} + C\sum_{i=1}^{\ell}(\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell}(\eta_i \cancel{\xi_i} + \eta_i^* \cancel{\xi_i^*})$$

$$- \sum_{i=1}^{\ell} \boxed{\alpha_i(\varepsilon} + \cancel{\xi_i} \boxed{- y_i} \cancel{+ \langle w, x_i \rangle + b})$$

$$- \sum_{i=1}^{\ell} \boxed{\alpha_i^*(\varepsilon} + \cancel{\xi_i^*} \boxed{+ y_i} \cancel{- \langle w, x_i \rangle - b})$$

Optimizing the Lagrangian

$$\boxed{\partial_w L = w - \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i = 0 \quad \longrightarrow \quad w = \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*)x_i}$$

maximize $\begin{cases} \boxed{-\frac{1}{2}\sum_{i,j=1}^{\ell}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j\rangle} \\ \boxed{-\varepsilon\sum_{i=1}^{\ell}(\alpha_i + \alpha_i^*)} + \boxed{\sum_{i=1}^{\ell} y_i(\alpha_i - \alpha_i^*)} \end{cases}$

subject to $\sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*) = 0$ and $\alpha_i, \alpha_i^* \in [0, C]$

$$L := \boxed{\frac{1}{2}\|w\|^2} + C\sum_{i=1}^{\ell}(\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell}(\eta_i\cancel{\xi_i} + \eta_i^*\cancel{\xi_i^*})$$

$$- \sum_{i=1}^{\ell}\boxed{\alpha_i(\varepsilon} + \cancel{\xi_i}\boxed{- y_i} + \cancel{\langle w, x_i\rangle + b})$$

$$- \sum_{i=1}^{\ell}\boxed{\alpha_i^*(\varepsilon} + \cancel{\xi_i^*}\boxed{+ y_i} \cancel{- \langle w, x_i\rangle - b})$$

Optimizing the Lagrangian

$$\text{maximize} \quad \begin{cases} -\frac{1}{2} \sum_{i,j=1}^{\ell} (\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j \rangle \\ -\varepsilon \sum_{i=1}^{\ell} (\alpha_i + \alpha_i^*) + \sum_{i=1}^{\ell} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

$$\begin{aligned} L := {} & \frac{1}{2}\|w\|^2 + C \sum_{i=1}^{\ell} (\cancel{\xi_i} + \cancel{\xi_i^*}) - \sum_{i=1}^{\ell} (\eta_i \cancel{\xi_i} + \eta_i^* \cancel{\xi_i^*}) \\ & - \sum_{i=1}^{\ell} \alpha_i(\varepsilon + \cancel{\xi_i} - y_i + \cancel{\langle w, x_i \rangle + b}) \\ & - \sum_{i=1}^{\ell} \alpha_i^*(\varepsilon + \cancel{\xi_i^*} + y_i - \cancel{\langle w, x_i \rangle - b}) \end{aligned}$$

Optimizing the Lagrangian

$$\text{maximize} \quad \begin{cases} -\frac{1}{2} \sum_{i,j=1}^{\ell} (\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j \rangle \\ -\varepsilon \sum_{i=1}^{\ell} (\alpha_i + \alpha_i^*) + \sum_{i=1}^{\ell} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

Optimizing the Lagrangian

↓

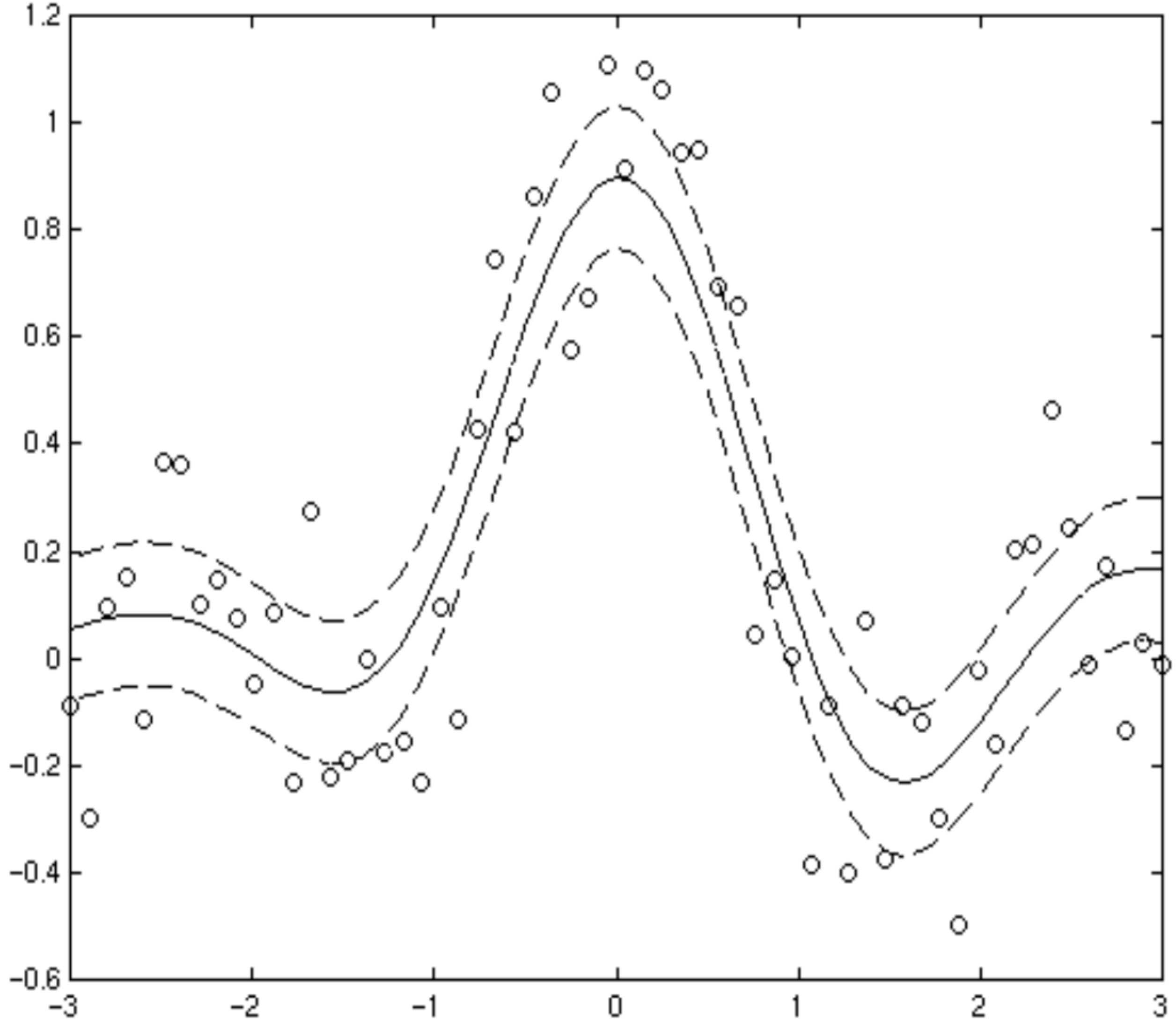
Dual optimization

$$\text{maximize} \quad \begin{cases} -\frac{1}{2} \sum_{i,j=1}^{\ell} (\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*) \boxed{\langle x_i, x_j \rangle} \\ -\varepsilon \sum_{i=1}^{\ell} (\alpha_i + \alpha_i^*) + \sum_{i=1}^{\ell} y_i (\alpha_i - \alpha_i^*) \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell} (\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

Now we can use the similar tricks as in SVM !
(recall what you have in SVM lectures)

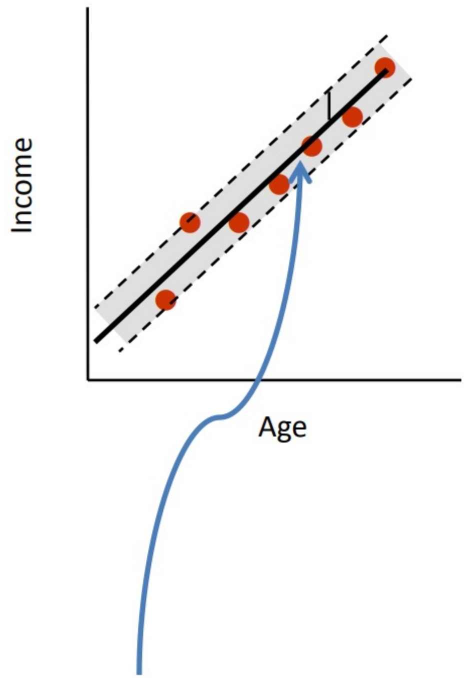
How about a non-linear case?



How about a non-linear case?

- **Linear case**

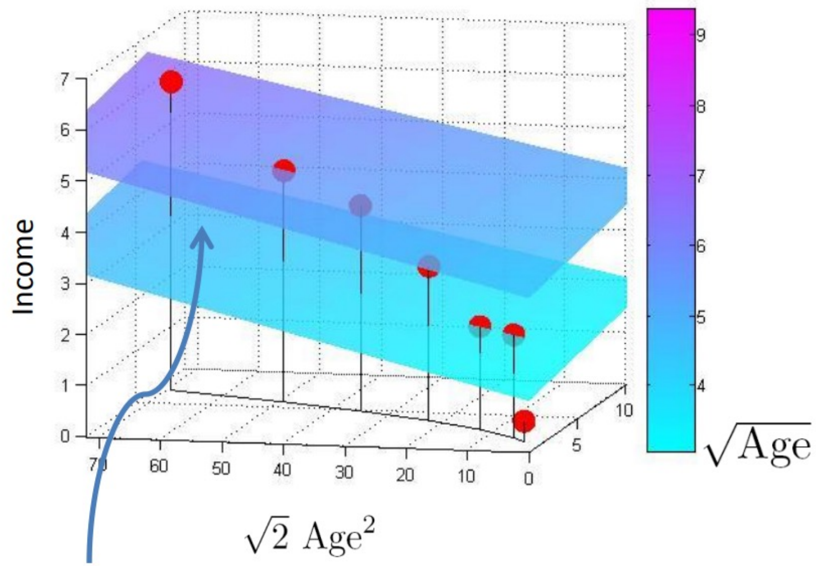
$$f : age \rightarrow income$$



$$y_i = \mathbf{w}_1 \cdot \mathbf{x}_i + b$$

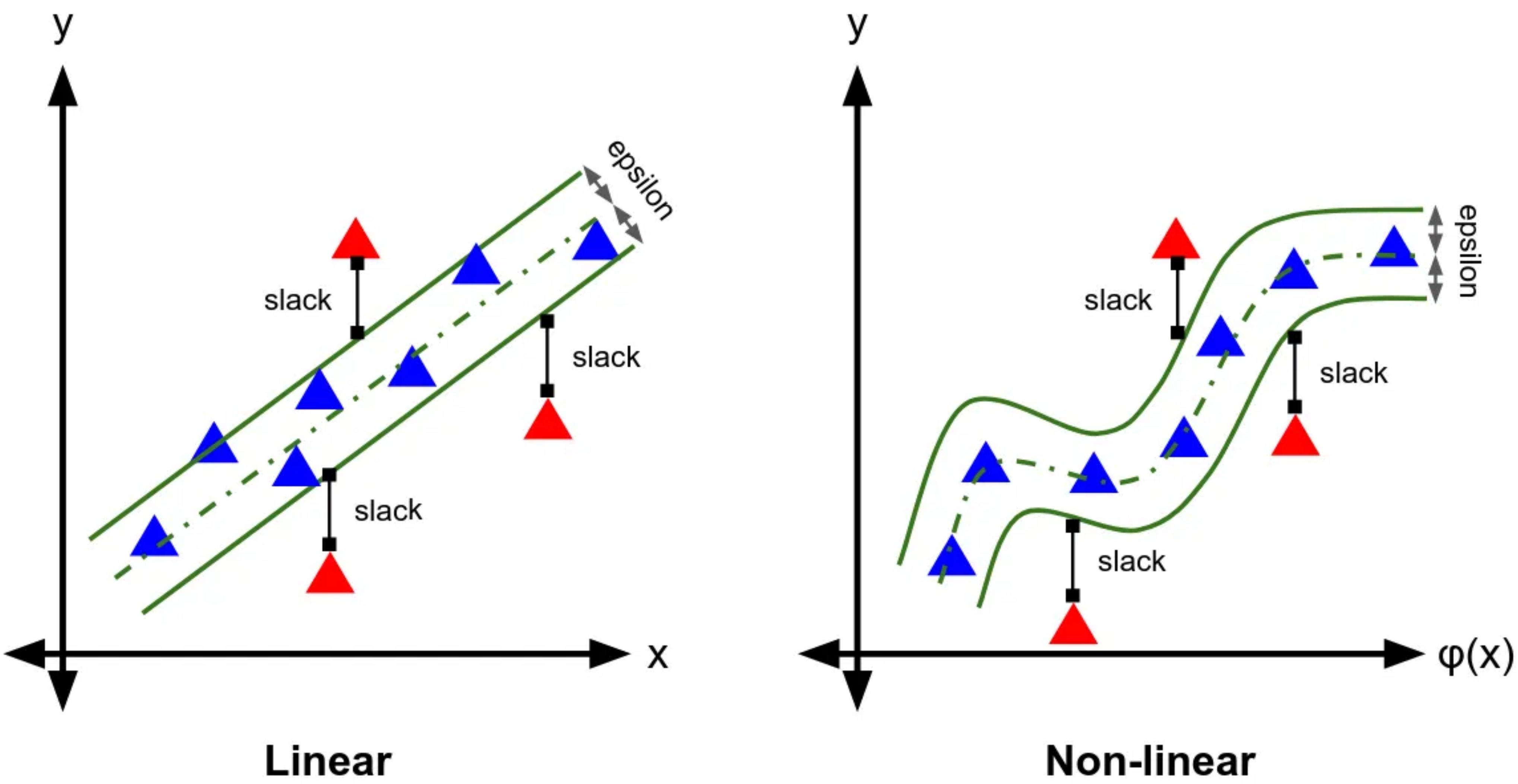
- **Non-linear case**
  - Map data into a higher dimensional space, e.g.,

$$f : (\sqrt{age}, \sqrt{2}age^2) \rightarrow income$$



$$y_i = \mathbf{w}_1 \sqrt{\mathbf{x}_i} + \mathbf{w}_2 \sqrt{2}\mathbf{x}_i^2 + b$$

Linear vs Non-linear



Dual problem

$$\min \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_{i=1}^{m}(\xi_i + \xi_i^*)$$

$$s.t.\begin{cases} y_i - (\mathbf{w}\cdot\mathbf{x}_i) - b \le \varepsilon + \xi_i \\ (\mathbf{w}\cdot\mathbf{x}_i) + b - y_i \le \varepsilon + \xi_i^* \\ \xi_i, \xi_i^* \ge 0, i = 1,\ldots,m \end{cases}$$

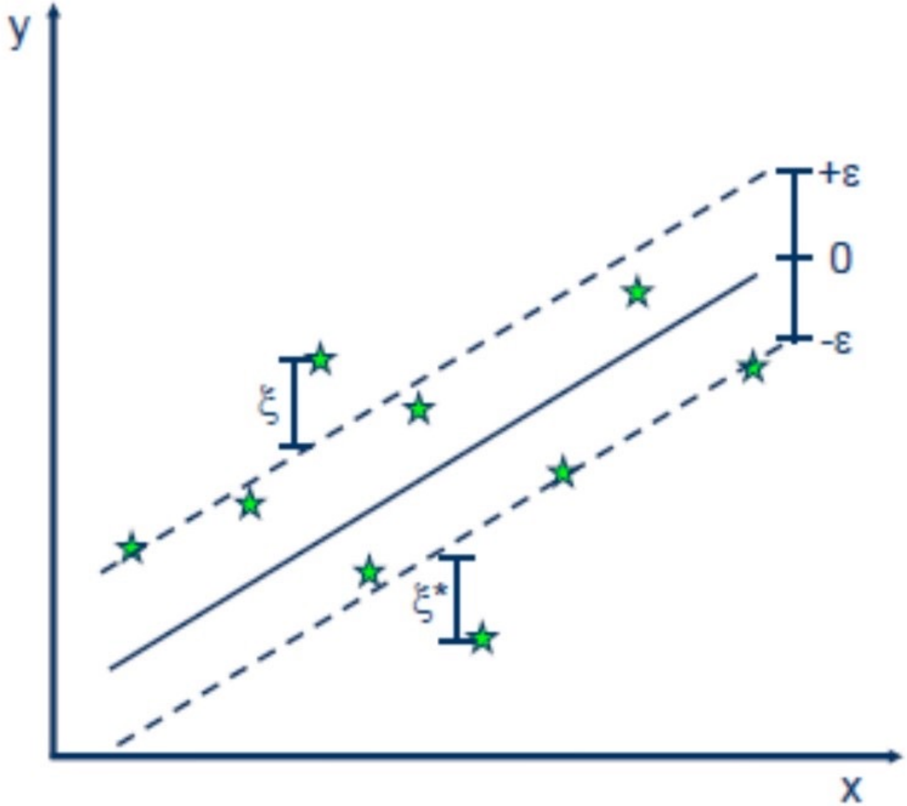
$$\text{maximize} \begin{cases} -\frac{1}{2}\sum_{i,j=1}^{\ell}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j\rangle \\ -\varepsilon\sum_{i=1}^{\ell}(\alpha_i + \alpha_i^*) + \sum_{i=1}^{\ell} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

Primal variables: w for each feature dim	Dual variables: α, α* for each data point
Complexity: the dim of the input space	Complexity: Number of support vectors

Primal **Dual**

Dual problem



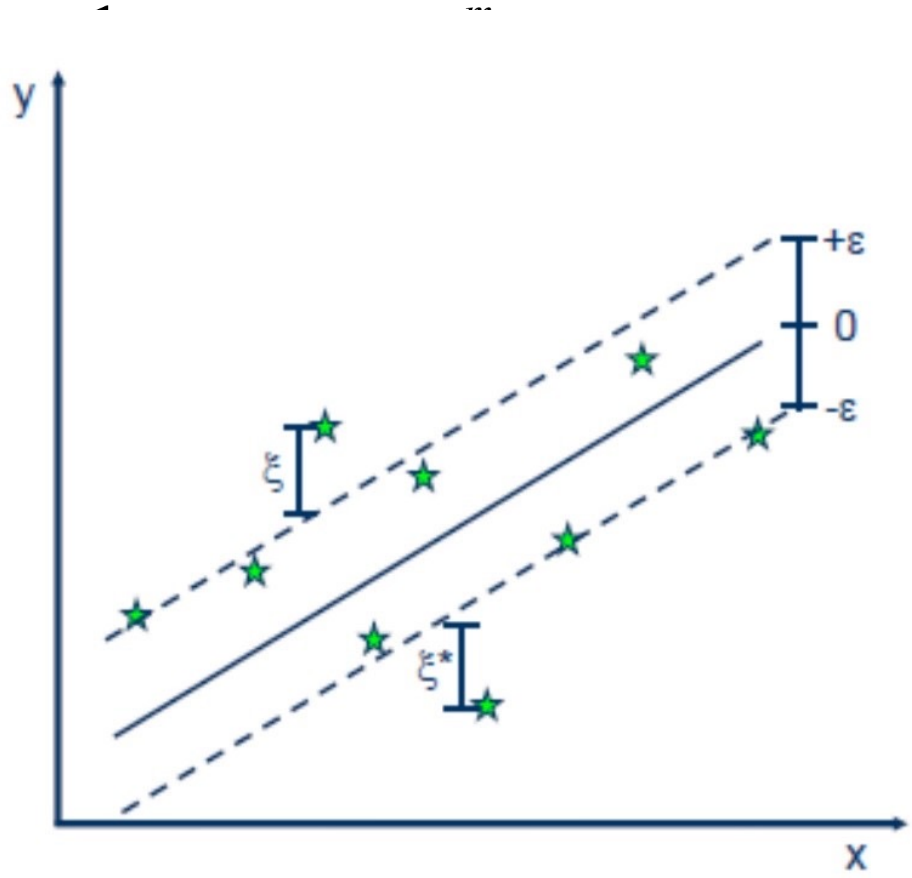
$$\text{maximize} \quad \begin{cases} -\frac{1}{2}\sum_{i,j=1}^{\ell}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j\rangle \\ -\varepsilon\sum_{i=1}^{\ell}(\alpha_i + \alpha_i^*) + \sum_{i=1}^{\ell} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

Dual variables: α, α* for each data point
Complexity: Number of support vectors

Dual

Dual problem



Kernel trick

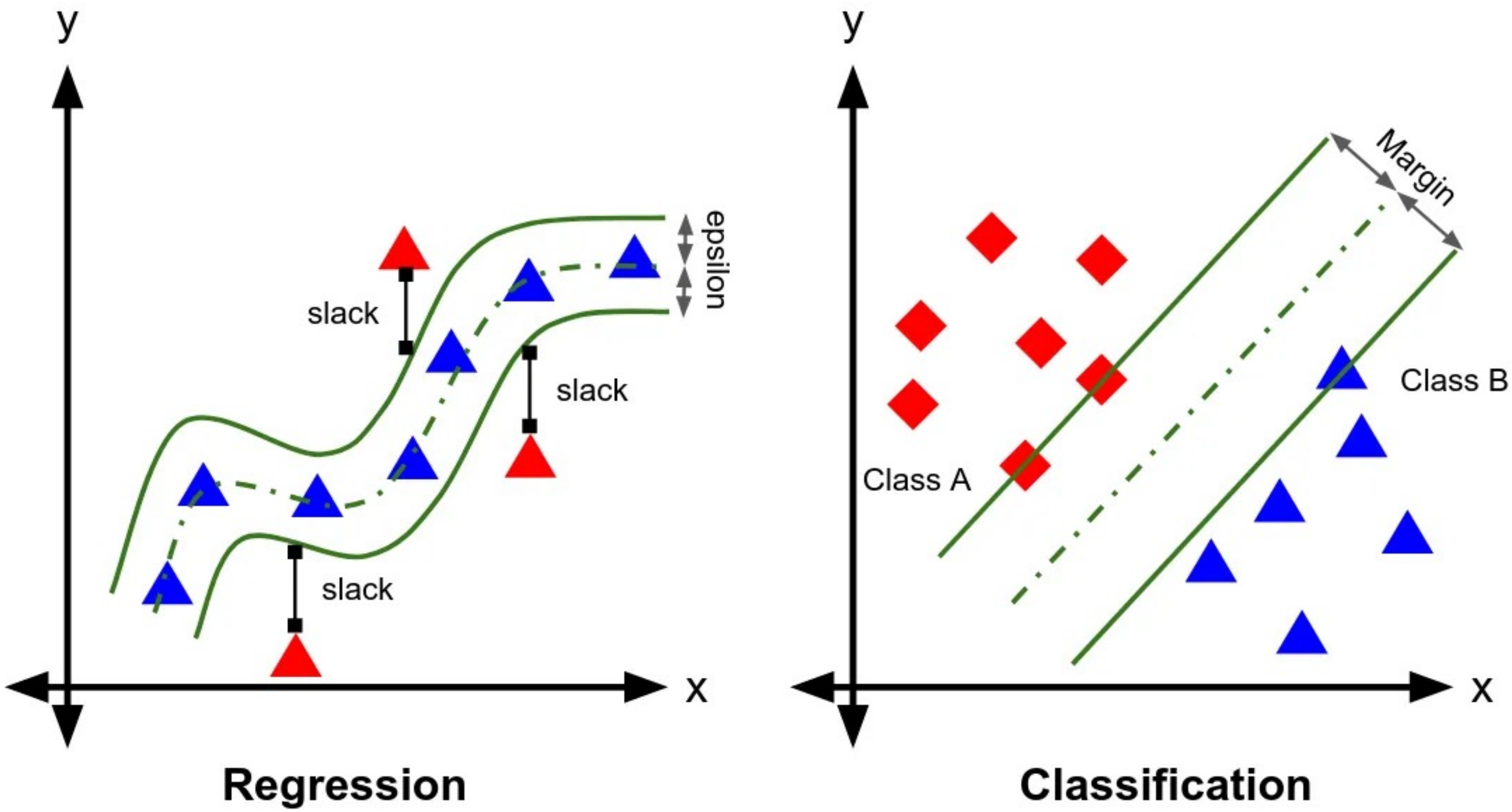
$$\text{maximize} \quad \begin{cases} -\frac{1}{2}\sum_{i,j=1}^{\ell}(\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\langle x_i, x_j\rangle \\ -\varepsilon\sum_{i=1}^{\ell}(\alpha_i + \alpha_i^*) + \sum_{i=1}^{\ell} y_i(\alpha_i - \alpha_i^*) \end{cases}$$

$$\text{subject to} \quad \sum_{i=1}^{\ell}(\alpha_i - \alpha_i^*) = 0 \quad \text{and} \quad \alpha_i, \alpha_i^* \in [0, C]$$

Dual variables: α, α^* for each data point
Complexity: Number of support vectors

Dual

SVM: Regression vs Classification



Summary

- Linear regression tries to minimize the error between the real and predicted value.

- SVR tries to fit the best line within a threshold value (a tube).

- The threshold value is the distance between the hyperplane and boundary line.

- Observations within the threshold of epsilon produce no error, only the observation outside of the epsilon range produce error – sparse kernel machines