ECON 210C Homework 1

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1. Questions from Romer

(1) Romer 5.8.

$$\max \sum \left(\frac{1}{1+\rho}\right)^t \cdot \left(C_t - \theta C_t^2\right) \qquad Y_t = C_t + \underbrace{K_{t+1} - K_t}_{\text{just } I_t}$$

$Y_t = AK_t + e_t, \quad K_{t+1} = K_t + Y_t - C_t$

A is interest rate. $A = \rho$.

$e_t = \phi e_{t-1} + \varepsilon_t, \quad -1 < \phi < 1$

(a) $$\mathcal{L} = E_0 \max_{C_t, K_{t+1}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \left\{ \left(C_t - \theta C_t^2\right) + \lambda_t \left(AK_t + e_t - C_t - K_{t+1} + K_t\right) \right\}$$

FOC

C_t: $1 - 2\theta C_t = \lambda_t$

K_{t+1}: $\lambda_t = E_t \cdot \cancel{\left(\frac{1}{1+\rho}\right)} \cancel{(1+A)}\, \lambda_{t+1}$

$\Rightarrow 1 - 2\theta C_t = E_t\left(1 - 2\theta C_{t+1}\right)$

$\therefore C_t = E_t C_{t+1}$

(b) Guess $C_t = \alpha + \beta K_t + \gamma e_t$

$K_{t+1} = K_t + Y_t - \alpha - \beta K_t - \gamma e_t$

$= K_t + \rho K_t + e_t - \alpha - \beta K_t - \gamma e_t$

$= (1+\rho-\beta) K_t + (1-\gamma) e_t - \alpha$.

$\therefore K_{t+1} = (1+\rho-\beta) K_t + (1-\gamma) e_t - \alpha$

(c) FOC in (a): $C_t = E_t C_{t+1}$

Plug in the guess.

$$\alpha + \beta K_t + \gamma e_t = E_t[\alpha + \beta K_{t+1} + \gamma e_{t+1}]$$

$$= E_t\left[\alpha + \beta\{(1+\rho-\beta)K_t + (1-\gamma)e_t - \alpha\} + \gamma(\phi e_t + \varepsilon_{t+1})\right]$$

$$= \alpha - \beta\alpha + \beta(1+\rho-\beta)K_t + (\beta(1-\gamma)+\gamma\phi)e_t$$

Method of undetermined coefficients

(1) $\alpha = \alpha - \beta\alpha \Rightarrow \alpha = \alpha(1-\beta)$. unless $\beta = 0$, $\alpha = 0$.

(2) $\beta = \beta(1+\rho-\beta) \Rightarrow 1 = 1+\rho-\beta$. $\therefore \beta = \rho$

(3) $\gamma = \beta(1-\gamma)+\gamma\phi \Rightarrow \gamma = \rho(1-\gamma)+\gamma\phi$

$$\gamma(1+\rho-\phi) = \rho$$

$$\Rightarrow \gamma = \frac{\rho}{1+\rho-\phi}$$

(d) $Y_t = \rho K_t + e_t$

$C_t = \rho K_t + \frac{\rho}{1+\rho-\phi} e_t$: only $\frac{\rho}{1+\rho-\phi}$ fraction goes to here

$K_{t+1} = K_t + \frac{1-\phi}{1+\rho-\phi} e_t$: the rest are invested. since there's no δ, it goes to K_{t+1} 100%.

(2) Romer 5.9.

(a)

$$\mathcal{L} = \mathbb{E}_0 \max_{C_t, K_{t+1}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \left\{ \begin{array}{l} (C_t - \theta(C_t^2 + 2C_t\nu_t + \nu_t^2) \\ + \lambda_t(AK_t - C_t - K_{t+1} + K_t) \end{array} \right\}$$

FOC.

C_t: $1 - 2\theta C_t - 2\theta \nu_t = \lambda_t$

K_{t+1}: $\lambda_t = \mathbb{E}_t \cdot \left(\frac{1}{1+\rho}\right) \lambda_{t+1}(A+1)$

$\Rightarrow \lambda_t = \mathbb{E}_t \lambda_{t+1}$

Combine them.

$\cancel{1} - 2\theta C_t - 2\theta \nu_t = \mathbb{E}_t(\cancel{1} - 2\theta C_{t+1} - 2\theta \nu_{t+1})$

$\Rightarrow C_t + \nu_t = \mathbb{E}_t C_{t+1}$

(since $\mathbb{E}_t \nu_{t+1} = 0$)

(b) Guess $C_t = \alpha + \beta K_t + \gamma \nu_t$.

$K_{t+1} = K_t + AK_t - C_t$

$= (1+A)K_t - \alpha - \beta K_t - \gamma \nu_t$

$= (1+A-\beta)K_t - \gamma \nu_t - \alpha$.

(c) Plug in the guess

$\alpha + \beta K_t + \gamma \nu_t + \nu_t = \mathbb{E}_t[\alpha + \beta K_{t+1} + \gamma \nu_{t+1}]$

$= \mathbb{E}_t[\alpha + \beta(1+A-\beta)K_t - \beta\gamma \nu_t - \beta\alpha]$

$= \alpha - \beta\alpha + \beta(1+A-\beta)K_t - \beta\gamma \nu_t$

$\alpha = \alpha - \beta\alpha \Rightarrow \alpha = 0$

$\beta = \beta(1+A-\beta) \Rightarrow \beta = A = \rho$

$(1+\gamma) = -\beta\gamma \Rightarrow 1 + \gamma + \rho\gamma = 0.\ \gamma = \frac{-1}{1+\rho}$

(d) Plugging in the solutions...

$$C_t = \rho K_t - \frac{1}{1+\rho} v_t$$

$$K_{t+1} = K_t + \frac{1}{1+\rho} v_t$$

Because ~~positive~~ v_t means distaste for consumer.

But why $\frac{1}{1+\rho}$? consumption smoothing

(3) Romer 5.11.

(a) Lifetime maximized value
= maximized value of this period
+ discounted maximized value from the next period.

(b) $V(K_t, A_t) = \max_{C_t, l_t} \left[\ln C_t + b\ln(1-l_t) + e^{-\rho}\{\beta_0 + \beta_K \ln(Y_t - C_t) + \beta_A \rho_A \ln A_t\} \right]$

s.t. $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$

FOC for C_t.

$$\frac{1}{C_t} = e^{-\rho} \cdot \beta_K \cdot \frac{1}{Y_t - C_t}.$$

$$C_t/Y_t = e^{\rho} \cdot \frac{1}{\beta_K} \cdot (Y_t - C_t)/Y_t$$

$$= e^{\rho} \cdot \frac{1}{\beta_K} \cdot \left(1 - \frac{C_t}{Y_t}\right)$$

$$\Rightarrow \frac{C_t}{Y_t}\left(1 + e^{\rho} \cdot \frac{1}{\beta_K}\right) = e^{\rho} \cdot \frac{1}{\beta_K} \Rightarrow \frac{C_t}{Y_t} = \frac{\frac{e^{\rho}}{\beta_K}}{1 + \frac{e^{\rho}}{\beta_K}} : \text{constant}$$

So consumption - output ratio is constant, which is less than 1.

(c) $$\frac{b}{1-l_t} = \beta_K \cdot \frac{(1-\alpha) K_t^{\alpha} (A_t l_t)^{-\alpha} \cdot A_t}{K_t^{\alpha}(A_t l_t)^{1-\alpha} - C_t}$$

$$= \beta_K \cdot \frac{(1-\alpha) \cdot l_t^{-1} \cdot Y_t}{Y_t - C_t}$$

$$= \beta_K \cdot \frac{(1-\alpha) l_t^{-1}}{1 - C_t/Y_t}$$

Hence l_t does not depend on A_t or K_t!

(d) Re-write the Bellman equation.

$$V(K_t, A_t) = \max_{C_t, l_t} \left[\ln C_t + b\ln(1-l_t) + e^{-\rho}\{\beta_0 + \beta_K \ln(K_t^{\alpha}(A_t L_t)^{1-\alpha} - C_t) + \beta_A \rho_A \ln A_t\} \right]$$

Results from (b) & (c) are

$$\text{(b)} \quad \frac{C_t}{Y_t} = \frac{\frac{e^{\rho}}{\beta_K}}{1+\frac{e^{\rho}}{\beta_K}} = \frac{e^{\rho}}{\beta_K + e^{\rho}} \Rightarrow C_t = \left(\frac{e^{\rho}}{\beta_K + e^{\rho}}\right) Y_t$$

$$\text{(c)} \quad \frac{b}{1-l_t} = \beta_K \cdot \frac{(1-\alpha) l_t^{-1}}{1 - C_t/Y_t}$$

$$\Rightarrow \frac{l_t}{1-l_t} = \frac{\beta_K}{b} \cdot (1-\alpha) \Big/ \frac{1}{1+\frac{e^{\rho}}{\beta_K}}$$

$$\Rightarrow b l_t = \beta_K (1-\alpha)(1-l_t)\left(1+\frac{e^{\rho}}{\beta_K}\right)$$

$$\Rightarrow l_t = \frac{(1-\alpha)}{(1-\alpha) + \frac{b}{1+\frac{\beta_K}{e^{\rho}}}}$$

Now we can substitute them for the optimal C_t and l_t ...

This is hideous

2. PIH and excess-smoothing

2-1.

$$\mathcal{L} = \max_{\{C_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(C_t) + \lambda \left(A_0 - \sum_{t=0}^{\infty} \beta^t (C_t - Y_t) \right)$$

FOC : $u'(C_t) = \lambda$

Budget constraint : $A_0 = E_t \sum_{t=0}^{\infty} \beta^t (C_t - Y_t)$

Now let's get the impulse response.

No shock at all	one-time shock at t
(a) $Y_t = \mu \cdot t + \phi Y_{t-1}$	$Y_t = \mu \cdot t + \phi Y_{t-1} + \varepsilon_t$
$Y_{t+1} = \mu \cdot (t+1) + \phi Y_t$	$Y_{t+1} = \mu \cdot (t+1) + \phi Y_t$
$= \mu(t+1) + \phi(\mu t + \phi Y_{t-1})$	$= \mu(t+1) + \phi(\mu t + \phi Y_{t-1} + \varepsilon_t)$
$= \mu[t+1+\phi t] + \phi^2 Y_{t-1}$	$= \mu[t+1+\phi t] + \phi^2 Y_{t-1} + \phi \varepsilon_t$
	$\Rightarrow \dfrac{\partial Y_{t+s}}{\partial \varepsilon_t} = \phi^s \cdot \varepsilon_t$
(b) $Y_t = Y_{t-1}$	$Y_t = Y_{t-1} + \varepsilon_t$
$Y_{t+1} = Y_t = Y_{t-1}$	$Y_{t+1} = Y_t = Y_{t-1} + \varepsilon_t$
	$\Rightarrow \dfrac{\partial Y_{t+s}}{\partial \varepsilon_t} = \varepsilon_t$ (shock is permanent)
(c) $\Delta Y_t = \phi \Delta Y_{t-1}$	$Y_t - Y_{t-1} = \phi(Y_{t-1} - Y_{t-2}) + \varepsilon_t$
$\Rightarrow Y_t - Y_{t-1} = \phi(Y_{t-1} - Y_{t-2})$	$\Rightarrow Y_t = (1+\phi) Y_{t-1} - \phi Y_{t-2} + \varepsilon_t$
$\Delta Y_{t+1} = \phi \Delta Y_t$	$Y_{t+1} - Y_t = \phi(Y_t - Y_{t-1})$
$Y_{t+1} - Y_t = \phi(Y_t - Y_{t-1})$	$\Rightarrow Y_{t+1} = (1+\phi) Y_t - \phi Y_{t-1}$
	$= (1+\phi)((1+\phi) Y_{t-1} - \phi Y_{t-2} + \varepsilon_t)$
	$\Rightarrow \dfrac{\partial Y_{t+s}}{\partial \varepsilon_t} = (1+\phi)^s \varepsilon_t$ shock is explosive.

Now revisit the FOC and the budget constraint.

FOC : $U'(C_t) = \lambda \Rightarrow C_1 = C_2 = \cdots = C^*$

Budget constraint : $A_0 = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t (C_t - Y_t)$

$$\Rightarrow \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t C_t = A_0 + \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t Y_t$$

Quadratic utility implies that consumption follows random walk. $\mathbb{E}C_{t+1} = C_t$

$$\Rightarrow \frac{1}{1-\beta} \cdot C^* = A_0 + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t Y_t$$

Thus, consumption is a function of lifetime income

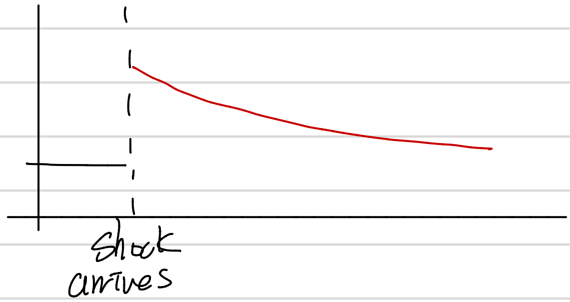
$$C^* = (1-\beta) \cdot A_0 + (1-\beta) \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t Y_t$$

Now we can solve each case analytically.

(a) In this case shocks are transitory. so consumption increases only small amount. precisely, ΔC is

$$(1-\beta)\sum_{t=0}^{\infty}\phi^t \varepsilon_t.$$

Saving's path is :



(b) In this case shocks are permanent.
Thus, consumption also jumps by ε_t amount.
Hence saving does not change

(c) In this case shocks are explosive.
Hence the lifetime income goes to infinity. Therefore
consumption goes to infinity as well

Saving's path is :

2-2.

Result from (a) implies that as long as shocks are transitory, consumption is always smoother than income. Following this argument, policies which affect income temporarily (ex) lump-sum transfer, temporary tax-cut) will be ineffective. However, if the shocks are permanent, result in (b) implies that consumption and income will move with the same magnitude.

Result in (c)?

2-3.

Yes. If income follows a random walk, (b) implies that consumption is no longer smoother than income. (excess-smoothing?)

Possible explanations would be...

- Households' precautionary saving (The result relies on quadratic utility, which eliminates this motives)

- Borrowing constraint
  Borrowing-constrainted consumers might want to accumulate a buffer stock savings for a rainy day.

3.

(1) $\mathcal{L} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} R_t \{ (1-\tau)F(K_t, L_t) - w_t L_t - I_t (1 + a(\frac{I_t}{K_t} - \delta)) + q_t ((1-\delta)K_t + I_t - K_{t+1}) \} \right]$

Choose K_{t+1}, L_t, I_t

$\frac{\partial \mathcal{L}}{\partial L_t}$: $w_t = (1-\tau)(1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}$

$\frac{\partial \mathcal{L}}{\partial K_{t+1}}$: $q_t = \mathbb{E}_t \frac{(1-\delta)q_{t+1} + (1-\tau)\alpha\left(\frac{K_{t+1}}{L_{t+1}}\right)^{\alpha-1} + a\left(\frac{I_{t+1}}{K_{t+1}}\right)^2}{1+r_{t+1}}$

$\frac{\partial \mathcal{L}}{\partial I_t}$: $q_t = 1 + a\left(\frac{I_t}{K_t} - \delta\right) + a \cdot \frac{I_t}{K_t}$

Log-linearization. in s.s, $q = 1 + 2a \cdot \frac{I}{K} - a\delta$
$= 1 + 2a \cdot \delta - a\delta = 1 + a\delta$

$\check{q}_t = \frac{2a\delta}{1+a\delta}(\check{I}_t - \check{K}_t)$

$\check{q}_t, \check{I}_t$ $\check{K}_t$ are observable, assume that we have an estimate for δ.

we can run a regression

$$\check{q}_t = \beta \cdot (\check{I}_t - \check{K}_t) + \varepsilon_t$$

and interpret $\hat{\beta}$ as $\frac{2\hat{a}\delta}{1+\hat{a}\delta} \Rightarrow \hat{\beta} + \hat{\beta}\hat{a}\delta = 2\hat{a}\delta$

$\Rightarrow \hat{\beta} = \hat{a}(2\delta - \hat{\beta}\delta) \Rightarrow \hat{a} = \frac{\hat{\beta}}{2\delta - \hat{\beta}\delta}$

error term includes unexplained parts of q_t
value of firm might not be explained solely by investment and capital stock they have.

(2) Euler equation is

$$q_t = E_t \frac{(1-\delta)q_{t+1} + (1-\tau)\alpha\left(\frac{K_{t+1}}{L_{t+1}}\right)^{\alpha-1} + a\left(\frac{I_{t+1}}{K_{t+1}}\right)^2}{1+r_{t+1}}$$

q_t is unobservable. substitute q_t using FOC for I

$$1+a\left(\frac{I_t}{K_t}-\delta\right)+a\cdot\frac{I_t}{K_t} = (1-\delta)E_t\frac{1}{1+r_{t+1}}\left(1+a\left(\frac{I_{t+1}}{K_{t+1}}-\delta\right)+a\frac{I_{t+1}}{K_{t+1}}\right)$$
$$+E_t\frac{(1-\tau)\alpha\left(\frac{K_{t+1}}{L_{t+1}}\right)^{\alpha-1}+a\left(\frac{L_{t+1}}{K_{t+1}}\right)^2}{1+r_{t+1}}$$

Log-linearize

LHS: $\frac{2a\delta}{1+a\delta}(\check{I}_t-\check{K}_t)$

RHS: $\overbrace{\left((1-\delta)q_{t+1}+(1-\tau)\alpha\left(\frac{K_{t+1}}{L_{t+1}}\right)^{\alpha-1}+a\left(\frac{I_{t+1}}{K_{t+1}}\right)^2\right)}^{①} - \overbrace{(1+r_{t+1})}^{②}$

In steady state, $q=\frac{1}{1+r}\cdot\left[(1-\delta)q+(1-\tau)\alpha\left(\frac{K}{L}\right)^{\alpha-1}+a\left(\frac{I}{K}\right)^2\right]$

$\Rightarrow q(1+r) = (1-\delta)q+(1-\tau)\alpha\left(\frac{K}{L}\right)^{\alpha-1}+a\left(\frac{I}{K}\right)^2$

①: $$\frac{(1-\delta)\cancel{q}}{\cancel{q}(1+r)}\check{q}_{t+1} + \frac{(1-\tau)\alpha}{q(1+r)}\left(\frac{K}{L}\right)^{\alpha-1}(\alpha-1)(\check{K}_{t+1}-\check{L}_{t+1})$$
$$+\frac{2a}{q(1+r)}\left(\frac{I}{K}\right)^2\cdot(\check{I}_{t+1}-\check{K}_{t+1})$$

$$= \frac{(1-\delta)}{(1+r)}\check{q}_{t+1} + \frac{(1-\tau)\alpha(\alpha-1)}{(1+a\delta)(1+r)}\left(\frac{K}{L}\right)^{\alpha-1}(\check{K}_{t+1}-\check{L}_{t+1})$$
$$+\frac{2a\delta^2}{(1+a\delta)(1+r)}(\check{I}_{t+1}-\check{K}_{t+1})$$

②: $\frac{r}{1+r}\check{r}_{t+1}$

Simplifying ...

$$\frac{2a\delta}{1+a\delta}(\check{I}_t-\check{K}_t)=\frac{(1-\delta)}{(1+r)}\cdot\frac{2a\delta}{(1+a\delta)}\mathbb{E}_t(\check{I}_{t+1}-\check{K}_{t+1})$$
$$+\frac{(1-\tau)\alpha(\alpha-1)}{(1+a\delta)(1+r)}\left(\frac{K}{L}\right)^{\alpha-1}\mathbb{E}_t(\check{K}_{t+1}-\check{L}_{t+1})$$
$$+\frac{2a\delta^2}{(1+a\delta)(1+r)}\mathbb{E}_t(\check{I}_{t+1}-\check{K}_{t+1})$$
$$-\frac{r}{1+r}\mathbb{E}_t\check{r}_{t+1} \quad //$$

$$=\frac{2a\delta}{(1+r)(1+a\delta)}\mathbb{E}_t(\check{I}_{t+1}-\check{K}_{t+1})$$
$$+\frac{(1-\tau)\alpha(\alpha-1)}{(1+r)(1+a\delta)}\left(\frac{K}{L}\right)^{\alpha-1}\mathbb{E}_t(\check{K}_{t+1}-\check{L}_{t+1})$$
$$-\frac{r}{1+r}\mathbb{E}_t\check{r}_{t+1}$$

Using expectation error, $\mathbb{E}_t x_{t+1}=x_{t+1}-(x_{t+1}-\mathbb{E}_t x_{t+1})$
Denote $(x_{t+1}-\mathbb{E}_t x_{t+1})=\tilde{x}_{t+1}$. Then

$$\mathbb{E}_t x_{t+1}=x_{t+1}-\tilde{x}_{t+1}$$

Also, Let $(1+r)(1+a\delta)=g$

$$\frac{1}{g}2a\delta(1+r)(\check{I}_t-\check{K}_t)=\frac{2a\delta}{g}\mathbb{E}_t(\check{I}_{t+1}-\check{K}_{t+1})$$
$$+\frac{1}{g}(1-\tau)\alpha(\alpha-1)\left(\frac{K}{L}\right)^{\alpha-1}\mathbb{E}_t(\check{K}_{t+1}-\check{L}_{t+1})-\frac{r}{1+r}\mathbb{E}_t\check{r}_{t+1}$$

(3)

Rearranging a little bit ...

$$\frac{r}{1+r}E_t\check{r}_{t+1} = \frac{1}{g}2a\delta(1+r)(\check{I}_t-\check{K}_t) - \frac{2a\delta}{g}E_t(\check{I}_{t+1}-\check{K}_{t+1})$$
$$-\frac{1}{g}(1-\tau)\alpha(\alpha-1)\left(\frac{K}{L}\right)^{\alpha-1}E_t(\check{K}_{t+1}-\check{L}_{t+1})$$

Let's denote ...

$$\frac{r}{1+r}(\check{r}_{t+1}-\hat{r}_{t+1}) = A(\check{I}_t-\check{K}_t)+B(\check{I}_{t+1}-\check{K}_{t+1}-\hat{I}_{t+1}+\hat{K}_{t+1})$$
$$+C(\check{K}_{t+1}-\check{L}_{t+1}-\hat{K}_{t+1}+\hat{L}_{t+1})$$

Stuck all expectation error into a single error term.

$$\frac{r}{1+r}\check{r}_{t+1} = A(\check{I}_t-\check{K}_t)+B(\check{I}_{t+1}-\check{K}_{t+1})+C(\check{K}_{t+1}-\check{L}_{t+1})$$
$$+\left(\frac{r}{1+r}\hat{r}_{t+1}-B(\hat{I}_{t+1}-\hat{K}_{t+1})-C(\hat{K}_{t+1}-\hat{L}_{t+1})\right)$$

$$\therefore r_{t+1} = \beta_0(\check{I}_t-\check{K}_t)+\beta_1(\check{I}_{t+1}-\check{K}_{t+1})+\beta_2(\check{K}_{t+1}-\check{L}_{t+1})+\varepsilon_{t+1}$$

Unknown parameter: a, δ, α

so we can back out all of them by $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$.

(4) Solve for FOC again w/ $K_t = (1-\delta)K_{t-1} + I_t$

$$\mathcal{L} = \mathbb{E}_0\left[\sum_{t=0}^{\infty} R_t\left\{(1-\tau)F(K_t, L_t) - W_t L_t - I_t\left(1+a\left(\frac{I_t}{K_t}-\delta\right)\right)\right. \right.$$
$$\left.\left. + q_t\left(K_t - (1-\delta)K_{t-1} - I_t\right)\right\}\right]$$

<FOC>

(1) $\frac{\partial \mathcal{L}}{\partial L_t}$: $W_t = (1-\tau)(1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}$

(2) $\frac{\partial \mathcal{L}}{\partial I_t}$: $1 + a\left(\frac{I_t}{K_t} - \delta\right) + a \cdot \frac{I_t}{K_t} = q_t$

(3) $\frac{\partial \mathcal{L}}{\partial K_t}$: $(1-\tau)\alpha\left(\frac{K_t}{L_t}\right)^{\alpha-1} + I_t \cdot a \cdot \left(\frac{I_t}{K_t^2}\right) + q_t = \frac{(1-\delta)}{1+r_{t+1}} q_{t+1}$

(1) & (2) are unchanged. Hence, $\check{q}_t = \frac{2a\delta}{1+a\delta}(\check{I}_t - \check{K}_t)$

Log-linearize (3).

In steady state, $\frac{1-\delta}{1+r} q = q + (1-\tau)\alpha\left(\frac{K}{L}\right)^{\alpha-1} + a\left(\frac{I}{K}\right)^2$

RHS: $\check{q}_{t+1} - \frac{r}{1+r}\check{r}_{t+1}$

LHS: Note that $q = 1+a\delta$. Let $\frac{1}{1+r}(1-\delta)(1+a\delta) = g$

$$\frac{\cancel{q}}{\frac{(1-\delta)}{(1+r)}\cancel{q}} \cdot \check{q}_t + \frac{1}{g}(1-\tau)\alpha\left(\frac{K}{L}\right)^{\alpha-1}(\alpha-1)(\check{K}_t - \check{L}_t) + \frac{2a}{g}\left(\frac{I}{K}\right)^2(\check{I}_t - \check{K}_t)$$

$$\Rightarrow \mathbb{E}_t\check{q}_{t+1} - \frac{r}{1+r}\mathbb{E}_t\check{r}_{t+1} = \frac{(1+r)}{(1-\delta)} \cdot \frac{2a\delta}{1+a\delta}(\check{I}_t - \check{K}_t)$$
$$+ \frac{(1+r)(1-\tau)\alpha(\alpha-1)}{(1-\delta)(1+a\delta)}\left(\frac{K}{L}\right)^{\alpha-1}(\check{K}_t - \check{L}_t)$$
$$+ \frac{(1+r)2a\delta^2}{(1-\delta)(1+a\delta)}(\check{I}_t - \check{K}_t)$$

$$\frac{2a\delta}{1+a\delta} E_t(\check{I}_{t+1} - \check{K}_{t+1}) - \frac{r}{1+r} E_t r_{t+1}$$

$$= \frac{(1+r)(1-\tau)a(a-1)}{(1-\delta)(1+a\delta)} \left(\frac{K}{L}\right)^{a-1} (\check{K}_t - \check{L}_t)$$

$$+ \frac{(1+r)2a\delta(1+\delta)}{(1-\delta)(1+a\delta)} (\check{I}_t - \check{K}_t)$$

We can run a similar regression. Now error term does not involve $\tilde{L}_{t+1}$.

(5) The problem when we try to estimate 'a' as a function of q is that q itself is unobservable. So the estimate can be sensitive to how q is defined, which is related to the specification of the model at hand. This is what the error term in (1) implies.

However, since this specification does not include expectation, the estimate does not suffer from any problems arising from expectation errors.

4. Practice log-linearization

1. $\check{Y} = \frac{\bar{C}}{\bar{Y}}\check{C} + \frac{\bar{I}}{\bar{Y}}\check{I} + \frac{\bar{G}}{\bar{Y}}\check{G}$

(∵ $NX = 0$ in steady state)

2. Let $X = (\alpha K^{\rho} + (1-\alpha)(AL)^{\rho})$

$Y = X^{\frac{1}{\rho}}$ in s.s, $\bar{Y} = \bar{X}^{\frac{1}{\rho}}$

$\bar{Y}(1+\check{Y}) = \bar{X}^{\frac{1}{\rho}}(1+\check{X}^{\frac{1}{\rho}}) \Rightarrow \check{Y} = \frac{1}{\rho}\check{X}$

Now let's get $\check{X}$

$X = \alpha K^{\rho} + (1-\alpha)(AL)^{\rho} = A + B$ $\quad A = \alpha\bar{K}^{\rho}$, $B = (1-\alpha)(\bar{A}\bar{L})^{\rho}$

$$\check{X} = \frac{A}{A+B}(\widecheck{\alpha K^{\rho}}) + \frac{B}{A+B}(\widecheck{(1-\alpha)(AL)^{\rho}})$$

$$= \frac{A}{A+B}(\rho\check{K}) + \frac{B}{A+B}(\rho(\check{A}+\check{L}))$$

$$\therefore \check{Y} = \frac{1}{\rho}\left[\frac{\alpha\bar{K}^{\rho}}{\bar{Y}^{\rho}}\cdot\rho\check{K} + \frac{(1-\alpha)(\bar{A}\bar{L})^{\rho}}{\bar{Y}^{\rho}}\cdot\rho(\check{A}+\check{L})\right]$$

3.

In steady state

$$K = (1-\delta)K + I - \psi\left(\frac{I}{K}-\delta\right)^2 I$$

If we assume that $I = \delta K$ in steady state, then log-linearization for this is

$$\check{K}_t = (1-\delta)\check{K}_{t-1} + \delta \check{I}_t$$

If we assume that $I \neq \delta K$, then $\psi\left(\frac{I}{K}-\delta\right)^2 I \neq 0$

In that case,

$$\check{K}_t = \frac{(1-\delta)K}{K}\cdot \check{K}_{t-1} + \frac{I}{K}\check{I}_t - \frac{I}{K}\psi\left(\frac{I}{K}-\delta\right)^2 \left(\widecheck{\psi\left(\frac{I_t}{K_{t-1}}-\delta\right)^2 I_t}\right)$$

$$= (1-\delta)\check{K}_{t-1} + \frac{I}{K}\check{I}_t - \frac{I}{K}\psi\left(\frac{I}{K}-\delta\right)^2\left(\widecheck{\psi\left(\frac{I_t}{K_{t-1}}-\delta\right)^2 I_t}\right)$$

where

$$\widecheck{\psi\left(\frac{I_t}{K_{t-1}}-\delta\right)^2 I_t} = 2\left(\widecheck{\frac{I_t}{K_{t-1}}-\delta}\right) + \check{I}_t$$

where

$$\left(\widecheck{\frac{I_t}{K_{t-1}}-\delta}\right) = \frac{I/K}{I/K-\delta}\cdot\left(\widecheck{\frac{I_t}{K_{t-1}}}\right) = \frac{I}{I-\delta K}(\check{I}_t - \check{K}_{t-1})$$

$$\therefore \check{K}_t = (1-\delta)\check{K}_{t-1} + \frac{I}{K}\check{I}_t - \frac{I}{K}\psi\left(\frac{I}{K}-\delta\right)^2\cdot\left(\frac{2I}{I-\delta K}(\check{I}_t-\check{K}_{t-1}) + \check{I}_t\right)$$

4.

In steady state, $\phi\left(\frac{I_t}{I_{t-1}}-1\right)^2 I_t = 0$.

Hence the answer is same as #3

5. Take log.

$$\hat{i}_t = \phi_\pi(\log P_t - \log P_{t-1}) + \phi_y(\log y_t - \log y_{t-1}) + \rho\hat{i}_{t-1}$$

In steady state, $\hat{i} = \rho\hat{i} \Rightarrow \hat{i} = 0$

$\hat{i}_t$ is already a small number, with steady state value zero. So it is okay to just state $\hat{i}_t = \check{i}_t$

Then next steps are straight forward.

$$\phi_\pi(\log P_t - \log P_{t-1}) - \phi_\pi(\log P - \log P)$$

$$= \phi_\pi(\log P_t - \log P) - \phi_\pi(\log P_{t-1} - \log P)$$

$$\therefore \check{i}_t = \phi_\pi \check{P}_t + \phi_y \check{y}_t + \rho\check{i}_{t-1}$$

6. $A \cdot F(L) = \dfrac{U_L(C, 1-L)}{U_C(C, 1-L)}$

In steady state, $\bar{A} \cdot F(\bar{L}) = \dfrac{U_L(\bar{C}, 1-\bar{L})}{U_C(\bar{C}, 1-\bar{L})}$

Take log.

$\log A + \log F(L) = \log U_L(C, 1-L) - \log U_C(C, 1-L)$

Taylor approximation

$$\log A \approx \log \bar{A} + \frac{1}{\bar{A}}(A - \bar{A})$$

$$\log F(L) = \log F(\bar{L}) + \frac{F'(\bar{L})}{F(\bar{L})}(L - \bar{L})$$

$\log U_L(C, 1-L)$

$$= \log U_L(\bar{C}, 1-\bar{L}) + \frac{\overline{U_{LC}}}{\overline{U_L}}(C - \bar{C}) - \frac{\overline{U_{CL}}}{\overline{U_L}}(L - \bar{L})$$

$\log U_C(C, 1-L)$

$$= \log U_C(\bar{C}, 1-\bar{L}) + \frac{\overline{U_{CL}}}{\overline{U_C}}(C - \bar{C}) - \frac{\overline{U_{CL}}}{\overline{U_C}}(L - \bar{L})$$

$$\therefore \check{A} + \frac{\bar{L} F'(\bar{L})}{F(\bar{L})} \check{L} = \frac{\overline{U_{LC}} \cdot \bar{C}}{\overline{U_L}} \check{C} - \frac{\overline{U_{CL}} \cdot \bar{L}}{\overline{U_L}} \check{L} + \frac{\overline{U_{LC}}\, \bar{C}}{\overline{U_C}} \cdot \check{C} - \frac{\overline{U_{CL}} \cdot \bar{L}}{\overline{U_C}} \check{L}$$

$$\Rightarrow \check{A} + \frac{\bar{L} \cdot F'(\bar{L})}{F(\bar{L})} \check{L} = \overline{U_{LC}} \cdot \bar{C} \cdot \check{C} \left(\frac{1}{\overline{U_L}} - \frac{1}{\overline{U_C}}\right) - \overline{U_{LC}} \cdot \bar{L} \cdot \check{L} \left(\frac{1}{\overline{U_L}} - \frac{1}{\overline{U_C}}\right)$$

7. First take log

$\log Y_t = \alpha_t \log K_t + (1-\alpha_t) \log L_t$.

Then do 1st order Taylor approximation around steady state.

$$\frac{Y_t - \bar{Y}}{\bar{Y}} = (\log K - \log L)(\alpha_t - \alpha) + \frac{\alpha}{K}(K_t - K) + \frac{(1-\alpha)}{L}(L_t - L)$$

$$\therefore \check{Y}_t = \alpha \check{K}_t + (1-\alpha)\check{L}_t + \alpha\left(\log \frac{K}{L}\right)\check{\alpha}_t$$