ECON 210C PROBLEM SET # 3

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2. Variable capital utilization in an RBC model

(a). Firms choose capital utilization U, capital K, and labor demand N.

The production function that we can use directly (since the output is the numeraire) is

$$Y_t = (U_t K_{t-1})^\alpha (Z_t N_t)^{1-\alpha}$$

and since the firms own capital, they face the constraint

$$K_t = I_t + (1 - \delta(U_t))K_{t-1}$$

but they also have to pay wages $W_t N_t$ and invest I_t so we can set up the Lagrangian

$$\mathcal{L} = E\sum_s (\prod_{k=1}^{s}(1+r_{t+k})^{-1})\Big((U_{t+s}K_{t+s-1})^\alpha (Z_{t+s}N_{t+s})^{1-\alpha} - W_{t+s}N_{t+s} - I_{t+s} + q_{t+s}(-K_{t+s} + I_{t+s} + (1-\delta(U_{t+s}))K_{t+s-1})\Big)$$

so we have first order conditions:

for labor we have

$$W_t = (1-\alpha)(U_t K_{t-1})^\alpha Z_t^{1-\alpha} N_t^{-\alpha}$$

for investment we have

$$q_t = 1$$

for capital at time t we have

$$q_t = E\left[\frac{1}{1+r_{t+1}}\left(\alpha U_{t+1}^\alpha K_t^{\alpha-1}(Z_{t+1}N_{t+1})^{1-\alpha} + q_{t+1}(1-\delta(U_{t+1})\right)\right]$$

and finally we have the condition for utilization

$$\alpha U_t^{\alpha-1} K_{t-1}^\alpha (Z_t N_t)^{1-\alpha} = q_t K_{t-1}\delta'(U_t)$$

Combining the investment and capital optimality conditions yields the expression for the rental rate of capital.

$$R_{t+1} = \alpha U_{t+1}^\alpha K_t^{\alpha-1}\left(Z_{t+1}N_{t+1}\right)^{1-\alpha} - \delta(U_{t+1})$$

The rental rate depends on utilization because the marginal product of capital and its depreciation rate depend on utilization.