I. RBC with Variable Labor Supply

Table 1 compares volatilities across different parameterizations of η with what is observed in the data. Larger values of η improve the fit of the model. There is greater persistence since more elastic labor supply means shocks affect hours and wages to a greater extent. Consumption and labor supply, however, remain excessively smooth. The classical RBC model implies a greater Frisch elasticity based on the data.

Table 1: Comparing second moments of output, consumption, and labor supply

	Data	$\eta = 0.5$	$\eta = 1$	$\eta = 2$
Consumption	1.27	0.97	1.03	1.09
Output	1.72	1.56	1.67	1.79
Hours	1.59	0.24	0.41	0.60

II. RBC with Variable Capital Utilization

The firm solves the profit maximization problem.

$$\max_{\{N_t, I_t, U_t\}} \mathrm{E}\sum_{t=0}^{\infty}\prod_{s=0}^{t}(1+r_s)^{-1}(Y_t - N_t w_t - I_t)$$

$$Y_t = (U_t K_t)^{\alpha}(Z_t N_t)^{1-\alpha}$$

$$K_{t+1} = (1-\delta(U_t))K_t + I_t$$

We equalize prices of capital and the consumption good as in equilibrium.

1. The first order conditions are given by

$$\text{Labor demand: } w_t = (1-\alpha)Z_t^{1-\alpha}\left(\frac{U_t K_t}{N_t}\right)^{\alpha}$$

$$\text{Shadow value of capital: } q_t = 1$$

$$\text{Euler equation: } q_t = \mathrm{E}(1+r_{t+1})^{-1}\left(\alpha U_{t+1}^{\alpha}\left(\frac{K_{t+1}}{Z_{t+1}N_{t+1}}\right)^{\alpha-1} + q_{t+1}(1-\delta_{t+1})\right)$$

$$\text{Utilization: } \alpha K_t^{\alpha}\left(\frac{U_t}{Z_t N_t}\right)^{\alpha-1} = q_t \delta'(U_t) K_t$$

$$1 + r_{t+1} = \alpha U_{t+1}^{\alpha}\left(\frac{K_{t+1}}{Z_{t+1}N_{t+1}}\right)^{\alpha-1} + 1 - \delta(U_{t+1})$$

Rental rate depends on the marginal product of capital and depreciation which are themselves both dependent on U_t.

2. Utilization satisfies $\alpha K_t^{\alpha}\left(\frac{U_t}{Z_t N_t}\right)^{\alpha-1} = q_t\delta'(U_t)K_t$.

$$\ln\alpha + \alpha\ln K_t + (\alpha-1)(\ln U_t - \ln Z_t - \ln N_t) = \ln q_t + \ln\delta'(U_t) + \ln K_t$$

$$\ln \alpha + (\alpha - 1)(\ln K_t + \ln U_t - \ln Z_t - \ln N_t) = \ln q_t + \ln \delta'(U_t)$$

$$(\alpha - 1)(\check{K}_t + \check{U}_t - \check{Z}_t - \check{N}_t) = \check{q}_t + \frac{\delta''(\bar{U})}{\delta'(\bar{U})}(U_t - \bar{U})$$

$$(\alpha - 1)(\check{K}_t + \check{U}_t - \check{Z}_t - \check{N}_t) = \check{q}_t + \frac{\delta''(\bar{U})\bar{U}}{\delta'(\bar{U})}\check{U}_t = \check{q}_t + \Delta$$

$$\check{U}_t = \frac{\check{q}_t + \Delta}{\alpha - 1}(\check{Z}_t + \check{N}_t - \check{K}_t) = \frac{1}{1+\Delta}(\check{Y}_t - \check{K}_t)$$

The final equality uses $\check{q}_t = 0$ from the first order condition and log linearized production.

3. We can use the previous derivation to reduce $\check{Y}_t$ as a function of technology and inputs.

$$\begin{aligned}
\check{Y}_t &= \alpha(\check{U}_t + \check{K}_t) + (1-\alpha)(\check{Z}_t + \check{N}_t) \\
&= \frac{\alpha}{1+\Delta}(\check{Y}_t - \check{K}_t) + \alpha\check{K}_t + (1-\alpha)(\check{Z}_t + \check{N}_t) \\
\frac{1-\alpha+\Delta}{1+\Delta}\check{Y}_t &= \frac{\alpha\Delta}{1+\Delta}\check{K}_t + (1-\alpha)(\check{Z}_t + \check{N}_t) \\
\check{Y}_t &= \frac{\alpha\Delta}{1-\alpha+\Delta}\check{K}_t + \frac{(1-\alpha)(1+\Delta)}{1-\alpha+\Delta}(\check{Z}_t + \check{N}_t)
\end{aligned}$$

Δ governs the sensitivity of $\check{U}_t$ to the marginal rate of capital. The limiting case where $\Delta \to \infty$ is the standard (linearized) model and $\check{U}_t$ is fixed at full utilization. The limit $\delta \to 0$ represents the case of no utilization so that $\check{Y}_t$ depend solely on technology and labor.

4. The linearized labor demand function is $\check{w}_t = \check{Y}_t - \check{N}_t$. Substituting the expression for $\check{Y}_t$,

$$\check{w}_t = \frac{\alpha\Delta}{1-\alpha+\Delta}\check{K}_t + \frac{(1-\alpha)(1+\Delta)}{1-\alpha+\Delta}(\check{Z}_t + \check{N}_t) - N_t$$

We can obtain an upward sloping demand function if labor exhibits increasing returns to scale.

$$\begin{gathered}
\frac{(1-\alpha)(1+\Delta)}{1-\alpha+\Delta} > 1 \\
(1-\alpha)(1+\Delta) > 1-\alpha+\Delta \\
1-\alpha+\Delta-\Delta\alpha > 1-\alpha+\Delta \\
-\Delta\alpha > 1
\end{gathered}$$

Since $\Delta, \alpha > 0$, indeterminacy is impossible in this model. One way to achieve indeterminacy is to incorporate positive production externalities so that aggregate labor has increasing returns to scale. A model with endogenous capital utilization is more likely to exhibit indeterminacy because it amplifies the importance of labor in production.

III. Macroeconomics of Home Production

The household solves

$$\max_{c_m,c_h,L_h,L_m} (C_m^\rho + C_h^\rho)^{\frac{1}{\rho}} - \left(\frac{1}{\eta}+1\right)^{-1}(L_h+L_m)^{\frac{1}{\eta}+1}$$
$$\text{subject to } C_m \le wL_m, C_h \le L_h$$

The prices of consumption goods are equivalent.

1. The first order conditions are

$$\mathcal{L} = (C_m^\rho + C_h^\rho)^{\frac{1}{\rho}} - \left(\frac{1}{\eta}+1\right)^{-1}(L_h+L_m)^{\frac{1}{\eta}+1} + \lambda(wL_m - C_m) + \xi(L_h - C_h)$$
$$\lambda = (C_m^\rho + C_h^\rho)^{\frac{1-\rho}{\rho}} C_m^{\rho-1}$$
$$\xi = (C_m^\rho + C_h^\rho)^{\frac{1-\rho}{\rho}} C_h^{\rho-1}$$
$$(L_h+L_m)^{\frac{1}{\eta}} = \lambda w = \xi$$

2. $w = \frac{\xi}{\lambda}$. Wage is the ratio of marginal utilities of market and home goods.

3. $\lambda \frac{C_m}{C_h}^{1-\rho} = \xi$. The ratio of marginal utilities depends on the share of consumption.

4. $C_h = L_h = C_m w^{\frac{-1}{1-\rho}}$. The share of consumption depends on the relative wage.

5. Begin with $(L_h+L_m)^{\frac{1}{\eta}} = \lambda w$.

$$\begin{aligned} L_m &= (\lambda w)^\eta - L_h \\ &= (\lambda w)^\eta - C_m w^{\frac{-1}{1-\rho}} \\ &= (\lambda w)^\eta - L_m w^{\frac{-\rho}{1-\rho}} \\ &= (\lambda w)^\eta (1 + w^{\frac{-\rho}{1-\rho}})^{-1} \end{aligned}$$

6. Differentiate the above expression.

$$\frac{\partial L_m}{\partial w} = \frac{(1+W^{\frac{-\rho}{1-\rho}})\lambda^\eta \eta W^{\eta-1} - (\lambda W)^\eta (\frac{-\rho}{1-\rho}) W^{\frac{-\rho}{1-\rho}-1}}{(1+W^{\frac{-\rho}{1-\rho}})^2}$$

$$\begin{aligned} \varepsilon_m = \frac{\partial L_m}{\partial w}\frac{w}{L_m} &= \frac{(1+W^{\frac{-\rho}{1-\rho}})\eta - (\frac{-\rho}{1-\rho})W^{\frac{-\rho}{1-\rho}}}{(1+W^{\frac{-\rho}{1-\rho}})} \\ &= \eta + \left(\frac{\rho}{1-\rho}\right)\left(\frac{W^{\frac{-\rho}{1-\rho}}}{1+W^{\frac{-\rho}{1-\rho}}}\right) \end{aligned}$$

7. Observing labor supply volatilities, the model with home production implies η weakly smaller than in the classical RBC because that model does not account for shifting "employment" in home production. With high elasticity of substitution, $\varepsilon_m \to \infty$ and households will more readily substitute market employment with home production. If goods are complements, ε_m approaches the Frisch elasticity η.

8. Using $\left(C_m^\rho + C_h^\rho\right)^{\frac{1}{\rho}-1} C_m^{\rho-1} = \lambda$ we can substitute the budget constraints and $L_h = L_m W^{\frac{\rho}{\rho-1}}$ to get

$$((WL_m)^\rho + L_h^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1} = \lambda$$

$$((WL_m)^\rho + (W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1} = \lambda$$

Now the previous expression for L_m becomes

$$L_m = \frac{\left[((WL_m)^\rho + (W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

$$L_m = \frac{\left[((W^\rho + W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

$$L_m = \frac{\left[(W^\rho + W^{\frac{\rho^2}{\rho-1}})^{\frac{1}{\rho}-1}L_m^{1-\rho}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

$$L_m = \frac{\left[(W^\rho + W^{\frac{\rho^2}{\rho-1}})^{\frac{1}{\rho}-1}W^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

$$L_m = \left(1+W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-1} W^\eta$$

9. Differentiate the expression.

$$\frac{\partial L_m}{\partial W} = \eta W^{\eta-1}\left(1+W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-1} + W^\eta\left(\eta\left(\frac{1-\rho}{\rho}\right)-1\right)\left(1+W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-2}\left(\frac{\rho}{\rho-1}\right)W^{\frac{1}{\rho-1}}$$

$$\begin{aligned}
\frac{\partial L_m}{\partial W}\frac{W}{L_m} &= \eta + \left(\eta\left(\frac{1-\rho}{\rho}\right)-1\right)\left(\frac{\rho}{\rho-1}\right)\frac{W^{\frac{\rho}{\rho-1}}}{1+W^{\frac{\rho}{\rho-1}}} \\
&= \eta + \left(\frac{\rho}{1-\rho}-\eta\right)\frac{W^{\frac{\rho}{\rho-1}}}{1+W^{\frac{\rho}{\rho-1}}} \\
&= \eta\left(1-\frac{W^{\frac{\rho}{\rho-1}}}{1+W^{\frac{\rho}{\rho-1}}}\right) + \frac{\rho}{1-\rho}\frac{W^{\frac{\rho}{\rho-1}}}{1+W^{\frac{\rho}{\rho-1}}}
\end{aligned}$$

This elasticity has a similar interpretation to the labor elasticity previously derived. The current elasticity is smaller because we control for wealth effects; it only reflects labor responses to relative marginal utilities. In other words, these are the elasticities for the uncompensated and compensated labor supply functions, respectively.

10. Higher elasticity of substitution ($\rho \to 1$) results in higher "total" elasticity which can help explain large labor supply volatilities without an infeasibly high Frisch elasticity.

IV. q-Theory with Variable Capital Utilization

1. The firm solves the profit maximization problem.

$$\max_{\{L_t, I_t, U_t\}} \mathrm{E} \sum_{t=0}^{\infty} \prod_{s=0}^{t} (1+r_s)^{-1} \left[Y_t - L_t w_t - I_t \left(1 + \phi\left(\frac{I_t}{K_t} \right) \right) \right]$$

$$Y_t = Z_t (U_t K_t)^{\alpha} L_t^{1-\alpha}$$

$$K_{t+1} = (1 - \delta(U_t)) K_t + I_t$$

$$Y_t = C_t + I_t$$

We equalize prices of capital and the consumption good as in equilibrium. This is a dynamic problem because of the presence of adjustment costs. Firms face an intertemporal tradeoff when adjusting and utilizing capital stock due to this friction.

2. The Lagrangian and first order conditions are as follows.

$$\mathcal{L} = \mathrm{E} \sum_{t=0}^{\infty} \prod_{s=0}^{t} (1+r_s)^{-1} \left[Z_t (U_t K_t)^{\alpha} L_t^{1-\alpha} - L_t w_t - I_t \left(1 + \phi\left(\frac{I_t}{K_t} \right) \right) \right] + q_t ((1 - \delta(U_t)) K_t + I_t - K_{t+1})$$

$$\text{Labor demand: } w_t = (1-\alpha) \frac{Y_t}{L_t}$$

$$\text{Shadow value of capital: } q_t = 1 + \phi\left(\frac{I_t}{K_t} \right) + \frac{I_t}{K_t} \phi'\left(\frac{I_t}{K_t} \right)$$

$$\text{Euler equation: } q_t = \mathrm{E}(1 + r_{t+1})^{-1} \left(\alpha \frac{Y_{t+1}}{K_{t+1}} + \frac{I_{t+1}^2}{K_{t+1}^2} \phi'\left(\frac{I_t}{K_t} \right) + q_{t+1}(1 - \delta(U_{t+1})) \right)$$

$$\text{Utilization: } \alpha \frac{Y_t}{U_t} = q_t \delta'(U_t) K_t$$

Taken together these define the user cost equivalence.

$$1 + r_{t+1} = \alpha \frac{Y_{t+1}}{K_{t+1}} + \frac{I_{t+1}^2}{K_{t+1}^2} \phi'\left(\frac{I_t}{K_t} \right) + 1 + \delta(U_{t+1})$$

Rental rate depends on U_t through depreciation and the marginal productivity of capital.

3. Now log-linearize the FOC for utilization and solve.

$$\begin{aligned}
\alpha \frac{Y_t}{U_t} &= q_t \delta'(U_t) K_t \\
\ln \alpha + \ln Y_t - \ln U_t &= \ln q_t + \ln \delta'(U_t) + \ln K_t \\
\check{Y}_t - \check{U}_t &= \check{q}_t + \Delta \check{U}_t + \check{K}_t \\
\check{U}_t &= \frac{\check{Y}_t - \check{q}_t - \check{K}_t}{1 + \Delta}
\end{aligned}$$

$\check{q}_t$ affects the tradeoff the firm makes when employing capital. A higher value of capital $\check{q}_t$ makes the cost of utilization, through higher depreciation of capital, more severe.

4. Firms will invest (divest) if q is greater (less) than unity. Hence procyclicality of investment is tightly linked to procyclicality of q. This model predicts that utilization is *countercyclical* since U is inversely related to q. Cyclicality, however, also depends on

5. Consider a positive productivity shock.

V. Fiscal Multipliers in the RBC Model

The log-linearized system is

$$\begin{aligned}
\text{Labor markets: } \check{C}_t + \frac{1}{\eta} \check{L}_t &= \check{Y}_t - \check{L}_t \\
\text{Asset markets: } \mathrm{E}\, \check{C}_{t+1} - \check{C}_t &= \frac{\alpha \frac{\bar{Y}}{\bar{K}}}{\alpha \frac{\bar{Y}}{\bar{K}} + (1-\delta)} (\mathrm{E}\, \check{Y}_{t+1} - \check{K}_{t+1}) \\
\text{Capital law of motion: } \check{K}_{t+1} &= (1-\delta)\check{K}_t + \delta \check{I}_t \\
\text{Production: } \check{Y}_t &= \alpha \check{K}_t + (1-\alpha)\check{L}_t \\
\text{National accounting: } \check{Y}_t &= \frac{\bar{C}}{\bar{Y}} \check{C}_t + \frac{\bar{I}}{\bar{Y}} \check{I}_t + \frac{\bar{G}}{\bar{Y}} \check{G}_t \\
\text{Exogenous fiscal policy: } \check{G}_t &= \rho_g \check{G}_{t-1} + \epsilon_t^g
\end{aligned}$$

1. We can reduce the system to two endogenous variables.

$$\begin{aligned}
\check{K}_{t+1} &= (1-\delta)\check{K}_t + \delta \check{I}_t \\
\check{K}_{t+1} &= (1-\delta)\check{K}_t + \delta \check{I}_t
\end{aligned}$$

Endogenous variables in reduced form are linear in $\check{K}_t$ and $\check{G}_t$.

$$\begin{aligned}
\check{C}_t &= \nu_{CK} \check{K}_t + \nu_{CG} \check{G}_t \\
\check{K}_{t+1} &= \nu_{KK} \check{K}_t + \nu_{KG} \check{G}_t
\end{aligned}$$

2. Fix $Z = 1$ and let $\rho_g = 0.9$.