ECON 220C PROBLEM SET # 1

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1. Questions from textbook

1.1. **Romer 5.8.**

(a). We can start with the full Lagrangian, after substituting

$$C_t = K_t + Y_t - K_{t+1} = K_t + AK_t + e_t - K_{t+1} = (1 + A)K_t + e_t - K_{t+1}$$

to get

$$\mathcal{L} = E\left[\sum_{t=0}^{\infty} \frac{u(C_t) + \lambda_t((1 + A)K_t + e_t - K_{t+1})}{(1 + \rho)^t}\right]$$

which gives first order conditions with respect to C_t and K_{t+1} (which are chosen each period) of

$$u'(C_t) = \lambda_t$$

and

$$\lambda_t = \frac{(1 + A)E[\lambda_{t+1}]}{(1 + \rho)}$$

and combining gives

$$u'(C_t) = \frac{(1 + A)E[u'(C_{t+1})]}{(1 + \rho)}$$

but since $A = \rho$, we are left with a more standard

$$u'(C_t) = E[u'(C_{t+1})]$$

and we can substitute for $u'(C_t)$ since we are given the form of the utility function to get

$$u'(C_t) = 1 - 2\theta C_t$$

and we now have

$$1 - 2\theta C_t = E[1 - 2\theta C_{t+1}]$$

and from linearity of expectation we can cancel terms to get

$$C_t = E[C_{t+1}]$$

as our Euler equation.

(b). Substituting the guessed form into the resource constraint
$$C_t = (1+A)K_t + e_t - K_{t+1}$$
gets us
$$\alpha + \beta K_t + \gamma e_t = (1+A)K_t + e_t - K_{t+1}$$
and upon rearranging we have
$$K_{t+1} = (1 + A - \beta)K_t + (1-\gamma)e_t - \alpha$$
as our function for future capital.

(c). We have given that
$$C_t = \alpha + \beta K_t + \gamma e_t$$
so we merely need to find
$$E[C_{t+1}] = E[\alpha + \beta K_{t+1} + \gamma e_{t+1}]$$

By linearity and substituting our earlier result, we have
$$E[C_{t+1}] = \alpha + \beta((1 + A - \beta)K_t + (1-\gamma)e_t - \alpha) + \gamma E[e_{t+1}]$$
Since we are given that ε_t has expectation zero, we can substitute
$$E[e_{t+1}] = \phi e_t$$
(since we are merely applying the law of motion for e in period $t+1$ rather than period t) and so we have
$$E[C_{t+1}] = \alpha + \beta((1 + A - \beta)K_t + (1-\gamma)e_t - \alpha) + \gamma\phi e_t$$
which we can simplify as
$$E[C_{t+1}] = (1-\beta)\alpha + \beta(1 + A - \beta)K_t + (\beta - \gamma\beta + \gamma\phi)e_t$$
and from the Euler equation we have
$$\alpha + \beta K_t + \gamma e_t = (1-\beta)\alpha + \beta(1 + A - \beta)K_t + (\beta - \gamma\beta + \gamma\phi)e_t$$
and so we will have
$$\begin{aligned}\alpha &= (1-\beta)\alpha \\ \beta &= \beta(1 + A - \beta) \\ \gamma &= (\beta - \gamma\beta + \gamma\phi)\end{aligned}$$
which upon solving this system (with 3 equations and 3 unknowns since A and ϕ are given) yield
$$\begin{aligned}\alpha &= 0 \\ \beta &= A \\ \gamma &= \frac{A}{1 + A - \phi}\end{aligned}$$
for the parameters.

(d). First we consider the case of positive ϕ. Consumption rises and stays persistently higher, while capital increases as it approaches a new long-run equilibrium. Output jumps at first, but then declines as it approaches a new long-run level which is still higher than it was.

In the case of negative ϕ, capital and output (and therefore consumption) will also eventually be higher, but they will oscillate around the new long-run level, as ϕ^s will be positive and negative depending on the period.

1.2. **Romer 5.9.**

(a). We can start with the full Lagrangian, after substituting

$$C_t = K_t + Y_t - K_{t+1} = K_t + AK_t + -K_{t+1} = (1 + A)K_t - K_{t+1}$$

to get

$$\mathcal{L} = E\left[\sum_{t=0}^{\infty} \frac{u(C_t) + \lambda_t((1 + A)K_t - K_{t+1})}{(1 + \rho)^t}\right]$$

which gives first order conditions with respect to C_t and K_{t+1} (which are chosen each period) of

$$u'(C_t) = \lambda_t$$

and

$$\lambda_t = \frac{(1 + A)E[\lambda_{t+1}]}{(1 + \rho)}$$

and combining gives

$$u'(C_t) = \frac{(1 + A)E[u'(C_{t+1})]}{(1 + \rho)}$$

but since $A = \rho$, we are left with a more standard

$$u'(C_t) = E[u'(C_{t+1})]$$

and we can substitute for $u'(C_t)$ since we are given the form of the utility function to get

$$1 - 2\theta(C_t + v_t) = 1 - 2\theta(E[C_{t+1}] + E[v_t]) \implies C_t + v_t = E[C_{t+1}]$$

from linearity of expectation and the zero mean shock.

(b). Substituting the guessed form into the resource constraint

$$C_t = (1 + A)K_t - K_{t+1}$$

gets us

$$\alpha + \beta K_t + \gamma v_t = (1 + A)K_t - K_{t+1}$$

and upon rearranging we have

$$K_{t+1} = (1 + A - \beta)K_t - \gamma v_t - \alpha$$

as our function for future capital.

(c). We have given that

$$C_t = \alpha + \beta K_t + \gamma v_t$$

so we merely need to find

$$E[C_{t+1}] = E[\alpha + \beta K_{t+1} + \gamma v_{t+1}]$$

By linearity and substituting our earlier result, we have

$$E[C_{t+1}] = \alpha + \beta((1 + A - \beta)K_t - \gamma v_t - \alpha) + \gamma E[v_{t+1}]$$

Since we are given that v_t has expectation zero, we can substitute to obtain

$$E[C_{t+1}] = \alpha + \beta((1 + A - \beta)K_t - \gamma v_t - \alpha)$$

which we can simplify as

$$E[C_{t+1}] = (1 - \beta)\alpha + \beta(1 + A - \beta)K_t - \gamma\beta v_t$$

and from the Euler equation we have

$$\alpha + \beta K_t + (\gamma + 1)v_t = (1 - \beta)\alpha + \beta(1 + A - \beta)K_t - \gamma\beta v_t$$

and so we will have

$$\begin{aligned} \alpha &= (1 - \beta)\alpha \\ \beta &= \beta(1 + A - \beta) \\ \gamma + 1 &= \gamma\beta \end{aligned}$$

which upon solving this system (with 3 equations and 3 unknowns since A is given) yield

$$\begin{aligned} \alpha &= 0 \\ \beta &= A \\ \gamma &= -\frac{1}{1 + A} \end{aligned}$$

for the parameters.

(d). Solving for consumption gives

$$C_t = AK_t - \frac{1}{1 + A} \times v_t$$

while saving is

$$K_{t+1} = K_t + \frac{1}{1 + A} \times v_t$$

and so a one time positive shock means that the capital stock is higher forever (since there is persistence), and this higher capital stock will mean both output and consumption are higher.

1.3. **Romer 5.11.**

(a). This is the same for every value function: the choice of saving and investment must guarantee that utility in the current period is equal to the discounted expected value of the value function tomorrow, as otherwise the marginal value of allocating resources to today's utility would not be equal to the value of allocating resources for the future, violating the intertemporal first order conditions.

(b). The first order condition for consumption is

$$\frac{1}{C_t} = -\frac{e^{-\rho}\beta_K}{Y_t - C_t}$$

which we can rewrite as

$$\frac{Y_t - C_t}{C_t} = -e^{-\rho}\beta_K$$

and solving for Y we have

$$Y = C_t(1 + e^{-\rho}\beta_K)$$

so consumption is a constant fraction of output.

(c). The first order condition for labor is

$$-\frac{b}{1 - L_t} = -\frac{e^{-\rho}\beta_K}{Y_t - C_t} \times (1-\alpha)K_t^{\alpha}A_t^{1-\alpha}L_t^{-\alpha}$$

which we can rewrite using the definition of Y and the first order condition for consumption as

$$bL_t = (1 - L_t)(1-\alpha)(1 + e^{-\rho}\beta_K)$$

and so we have

$$(1 - L_t) = \frac{b}{(1-\alpha)(1 + e^{-\rho}\beta_K) + b}$$

which shows that the labor supply is constant.

(d). Substitutions give

$$\begin{aligned} V(K_t, A_t) = \alpha \log K_t + &(1-\alpha)\log A_t + (1-\alpha)\log L_t - \log(1 + e^{-\rho}\beta_K) + \\ &b \log\left(\frac{b}{(1-\alpha)(1 + e^{-\rho}\beta_K) + b}\right) + e^{-\rho}\beta_0 + \\ &e^{-\rho}\beta_K[\log e^{-\rho}\beta_k - \log(1 + e^{-\rho}\beta_K) + \alpha \log K_t + (1-\alpha)\log A_t + (1-\alpha)\log L_t] + e^{-\rho}\beta_A\rho_A \log A_t \end{aligned}$$

so we have

$$\beta_K' = \alpha(1 + e^{-\rho}\beta_K)$$

along with

$$\beta_A' = (1-\alpha)(1 + e^{-\rho}\beta_K) + e^{-\rho}\beta_A\rho_A$$

and

$$\beta_0' = (1-\alpha)\log L_t^* - \log(1 + e^{-\rho}\beta_K) + b\log\left(\frac{b}{(1-\alpha)(1 + e^{-\rho}\beta_K) + b}\right) + e^{-\rho}\beta_K[\log e^{-\rho}\beta_k - \log(1 + e^{-\rho}\beta_K) + (1-\alpha)\log L_t^*]$$

(e). We solve the system of equations

$$\beta_K = \alpha(1 + e^{-\rho}\beta_K)$$
$$\beta_A = (1-\alpha)(1 + e^{-\rho}\beta_K) + e^{-\rho}\beta_A\rho_A$$

and so we get that

$$\beta_K = \frac{\alpha}{1-\alpha e^{-\rho}}$$

and

$$\beta_A = \frac{1-\alpha}{(1-\alpha e^{-\rho})(1-\rho_A e^{-\rho})}$$

(f). With this value of β_K, we get

$$\frac{C_t}{Y_t} = \frac{1}{1 + \frac{\alpha e^{-\rho}}{1-\alpha e^{-\rho}}}$$

which we can simplify as

$$\frac{C_t}{Y_t} = 1 - \alpha e^{-\rho}$$

which is the same value.

For labor supply, we have

$$L_t = \frac{1-\alpha}{(1-\alpha) + \frac{b}{1+\frac{\alpha e^{-\rho}}{1-\alpha e^{-\rho}}}}$$

and simplifying we have

$$L_t = \frac{1-\alpha}{(1-\alpha) + b(1-\alpha e^{-\rho})}$$

which is also the same value as in the earlier derivation.

2. Permanent income hypothesis and the "excess smoothness" puzzle

2.1. **Saving responses to shocks.** The Lagrangian is

$$\mathcal{L} = E_t\left\{\sum_{s=0}^{\infty}\beta^s U(C_{t+s}) + \lambda\left[\sum_{k=0}^{\infty}\left(A_t - (1+r)^{-s}(C_{t+k} - Y_{t+k})\right)\right]\right\}$$

The first order conditions for C_t and C_{t+s} are respectively

$$U'(C_t) - \lambda = 0$$
$$E_t\left[\beta^s U'(C_{t+s}) - \lambda(1+r)^{-s}\right] = 0$$

Given we have $\beta = (1+r)^{-1}$, we can combine the FOCs and we get

$$U'(C_t) = E_t\left[U'(C_{t+s})\right]$$

As utility is quadratic, this implies

$$C_t = E_t\left[C_{t+s}\right]$$

Now we use the fact that the budget constrain holds in expectation,

$$A_t = E_t \left[\sum_{s=0}^{\infty} (1+r)^{-s} (C_{t+s} - T_{t+s}) \right]$$

and we plug in $C_t = E_t [C_{t+s}]$ to get

$$C_t = (1-\beta) A_t + (1-\beta) \sum_{s=0}^{\infty} \beta^s E_t [Y_{t+s}]$$

(a) $Y_t = \mu t + \phi Y_{t-1} + \epsilon_t$. Assuming $\epsilon_k = 0$ for $k \neq t$, we can iterate the expression for Y_{t+s} backwards to obtain

$$Y_{t+s} = \sum_{k=0}^{s} \left[\phi^k \mu (t+s-k) \right] + \phi^{s+1} Y_{t-1} + \phi^s \epsilon_t$$

Hence

$$\frac{\partial}{\partial \epsilon_t} E_t [Y_{t+s}] = \phi^s$$

We can then use this to obtain

$$\begin{aligned} \frac{\partial}{\partial \epsilon_t} C_t &= (1-\beta) \sum_{s=0}^{\infty} \beta^s \frac{\partial}{\partial \epsilon_t} E_t [Y_{t+s}] \\ &= (1-\beta) \sum_{s=0}^{\infty} \beta^s \phi^s \\ &= (1-\beta) \frac{1}{1-\beta\phi} \end{aligned}$$

3. Estimation of adjustment costs

3.1. **Optimality conditions.** First we write the Lagrangian

$$\mathcal{L} = E \left\{ \sum_{t=0}^{\infty} R_t \left[(1-\tau) K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - I_t \left(1 + a(I_t/K_t - \delta)\right) \right] + q_t ((1-\delta) K_t + I_t - K_{t+1}) \right\}$$

which gives first order conditions with respect to L_t, I_t, and K_{t+1} (which are chosen each period) of

$$(1-\alpha)(1-\tau) K_t^{\alpha} L_t^{-\alpha} = w_t$$

for L_t,

$$q_t = 1 + a(I_t/K_t - \delta) + a \cdot \frac{I_t}{K_t}$$

for I_t,

$$R_t q_t = E \left[R_{t+1} (\alpha (1-\tau) K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} + a (I_{t+1}/K_{t+1})^2 + (1-\delta) q_{t+1}) \right]$$

for K_{t+1}.

Optimal level of investment requires the capital stock and the value for a, as we can rewrite the investment optimality condition as

$$q_t = 2a\frac{I_t}{K_t} + 1 - \delta a$$

Log-linearizing the first order condition for investment, with $\delta = \frac{\bar{I}}{\bar{K}}$, we have

$$\bar{q} = \delta a + 1 \implies \check{q} = \frac{2\delta a}{1+\delta a} \times (\check{I} - \check{K})$$

Estimating a using this equation would be simply a matter of regressing estimates of q deviations on investment and capital deviations, and then solving for a given that the coefficient would be $frac2\delta a1 + \delta a$. However, the error term would also be picking up transitions to a new steady state in addition to mere measurement error, and since one might expect investment and capital to rise over time, the error term would likely be correlated with growth and would be larger (and serially correlated) in times of more rapid growth.

3.2. **Euler equation.** Substituting the investment optimality condition and the definition of R, we have

$$2a\frac{I_t}{K_t}+1-\delta a = E\left[\frac{1}{1+r_{t+1}} \times (\alpha(1-\tau)K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha} + a(I_{t+1}/K_{t+1})^2 + (1-\delta)(2a\frac{I_{t+1}}{K_{t+1}} + 1 - \delta a))\right]$$

and we can proceed with log linearization.

The steady state value of I/K is just δ, so we have

$$\frac{2\delta a}{1+\delta a}(\check{I}_t - \check{K}_t)$$

for the left hand side.

4. Practice log-linearization

4.1. The original equation is

$$Y_t = C_t + I_t + G_t + NX_t$$

so we take logs to get

$$\log(Y_t) = \log(C_t + I_t + G_t + NX_t)$$

and do the first order Taylor series expansion at the steady state to get

$$\cancel{\log \bar{Y}} + \frac{1}{\bar{Y}}(Y_t - \bar{Y}) = \cancel{\log(\overline{C_t + I_t + G_t + NX_t})} + \frac{1}{\bar{Y}}(C_t - \bar{C}) + \frac{1}{\bar{Y}}(I_t - \bar{I}) + \frac{1}{\bar{Y}}(G_t - \bar{G}) + \frac{1}{\bar{Y}}(NX_t - 0)$$

and we rewrite as

$$\check{y} = \frac{\bar{C}}{\bar{Y}}\frac{(C_t - \bar{C})}{\bar{C}} + \frac{\bar{I}}{\bar{Y}}\frac{(I_t - \bar{I})}{\bar{I}} + \frac{\bar{G}}{\bar{Y}}\frac{(G_t - \bar{G})}{\bar{G}} + \cancel{\frac{\overline{NX}}{\bar{Y}}\frac{(NX_t - \overline{NX})}{\overline{NX}}}$$

which comes out to just

$$\check{Y} = \frac{\bar{C}}{\bar{Y}}\check{C} + \frac{\bar{I}}{\bar{Y}}\check{Y} + \frac{\bar{G}}{\bar{Y}}\check{G}$$

4.2. The original equation is

$$Y_t = (\alpha K_t^\rho + (1-\alpha)(A_t L_t)^\rho)^{\frac{1}{\rho}}$$

so we take logs to get

$$\log(Y_t) = \log((\alpha K_t^\rho + (1-\alpha)(A_t L_t)^\rho)^{\frac{1}{\rho}})$$

and do the first order Taylor series expansion at the steady state to get

$$\check{Y} = \left(\frac{\alpha \bar{K}^{\rho-1}}{\alpha \bar{K}^\rho - (\alpha-1)(\bar{A}\bar{L})^\rho}\right)(K_t - \bar{K}) + \left(\frac{(\alpha-1)(\bar{A}\bar{L})^\rho}{\bar{A}\left((\alpha-1)(\bar{A}\bar{L})^\rho - \alpha\bar{K}^\rho\right)}\right)(A_t - \bar{A}) + \left(\frac{(\alpha-1)(\bar{A}\bar{L})^\rho}{\bar{L}\left((\alpha-1)(\bar{A}\bar{L})^\rho - \alpha\bar{K}^\rho\right)}\right)(L_t - \bar{L})$$

and rewrite as

$$\check{Y} = \left(\frac{\alpha \bar{K}^{\rho}}{\alpha \bar{K}^\rho - (\alpha-1)(\bar{A}\bar{L})^\rho}\right)\check{K} + \left(\frac{(\alpha-1)(\bar{A}\bar{L})^\rho}{\left((\alpha-1)(\bar{A}\bar{L})^\rho - \alpha\bar{K}^\rho\right)}\right)\check{A} + \left(\frac{(\alpha-1)(\bar{A}\bar{L})^\rho}{\left((\alpha-1)(\bar{A}\bar{L})^\rho - \alpha\bar{K}^\rho\right)}\right)\check{L}$$

4.3.

4.4.

4.5. The original equation is

$$\exp(i_t) = \left(\frac{P_t}{P_{t-1}}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y}\exp(\rho i_{t-1})$$

so we take logs to get

$$i_t = \phi_\pi(\log(P_t) - \log(P_{t-1})) + \phi_y(\log(Y_t) - \log(Y_{t-1})) + \rho i_{t-1}$$

and do the first order Taylor series expansion at the steady state to get

$$i_t - \bar{i} = \frac{\phi_\pi}{\bar{P}}(P_t - \bar{P}) - \frac{\phi_\pi}{\bar{P}}(P_{t-1} - \bar{P}) + \frac{\phi_y}{\bar{Y}}(Y_t - \bar{Y}) - \frac{\phi_y}{\bar{Y}}(Y_{t-1} - \bar{Y}) + \rho(i_{t-1} - \bar{i})$$

and we can rewrite as

$$\check{i}_t = \frac{\phi_\pi}{\bar{i}}(\check{P}_t - \check{P}_{t-1}) + \frac{\phi_y}{\bar{i}}(\check{Y}_t - \check{Y}_{t-1}) + \rho\check{i}_{t-1}$$

4.6.

4.7. The original equation is

$$Y_t = K_t^{\alpha_t} L_t^{1-\alpha_t}$$

so we take logs to get

$$\log(Y_t) = \alpha_t \log(K_t) + (1-\alpha_t)\log(L_t)$$

and do the first order Taylor series expansion at the steady state to get

$$\check{Y} = \alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}(K_t - \bar{K}) + (1-\alpha)\bar{K}^\alpha \bar{L}^{-\alpha}(L_t - \bar{L}) + \bar{K}^\alpha \bar{L}^{1-\alpha}(\log(\bar{K}) - \log(\bar{L}))(\alpha_t - \bar{\alpha})$$

and we can rewrite as

$$\check{Y} = \alpha \bar{Y}\check{K} + (1-\alpha)\bar{Y}\check{L} + \bar{\alpha}\bar{Y}(\log(\bar{K}) - \log(\bar{L}))\check{\alpha}$$