Econ 210C Problem Set # 3

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April 26, 2018

1 Variable labor supply in the RBC model
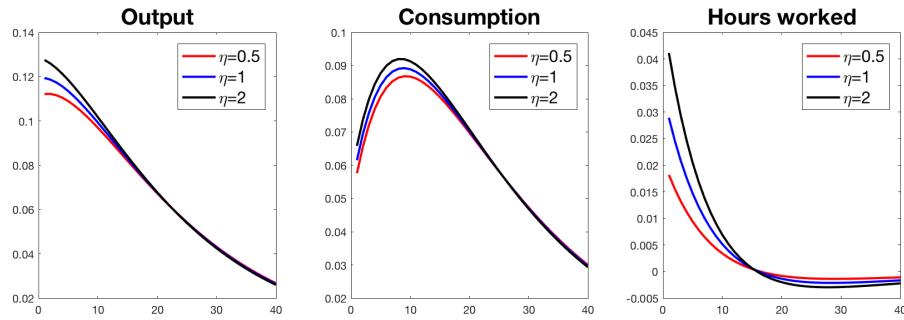

Figure 1: Impulse responses with varying η

	$\eta = 0.5$	$\eta = 1$	$\eta = 2$	Data
$Stdev(Y)$	1.54	1.64	1.74	1.72
$Stdev(C)$	0.97	1.02	1.08	1.27
$Stdev(L)$	0.23	0.37	0.53	1.59

Table 1: Response to a transitory discount factor shock

As one would expect, the fits get better as we calibrate the Frisch elasticity to bigger values. A large Frisch elasticity generates stronger intertemporal substitution of labor suppply, and hence amplifies the effect of shocks. However, even with a large Frisch elasticity, consumption is too smooth, and the volatility of hours generated from the model falls short of the empirical counterpart.

2 Variable capital utilization in an RBC model

(a) The Lagrangian of the firm's profit maximization problem is:

$$\mathcal{L} = \mathbb{E}_t \sum_s \left(\prod_{k=1}^{s} (1 + R_{t+k}) \right)^{-1} \times \left[(U_{t+s} K_{t+s-1})^\alpha (Z_{t+s} N_{t+s})^{1-\alpha} \right.$$
$$\left. - W_{t+s} N_{t+s} - I_{t+s} + q_{t+s} \left(-K_{t+s} + (1 - \delta(U_t)) K_{t+s-1} + I_{t+s} \right) \right]$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial N_t} : \quad W_t = (1 - \alpha) \frac{Y_t}{N_t}$$

$$\frac{\partial \mathcal{L}}{\partial I_t} : \quad q_t = 1$$

$$\frac{\partial \mathcal{L}}{\partial K_t} : \quad q_t = \mathbb{E}_t \frac{1}{1 + R_{t+1}} \left[\alpha (U_{t+1} K_t)^{\alpha-1} U_{t+1} (Z_{t+1} N_{t+1})^{1-\alpha} + q_{t+1} (1 - \delta(U_{t+1})) \right]$$

$$\frac{\partial \mathcal{L}}{\partial U_t} : \quad q_t \delta'(U_t) K_{t-1} = \alpha \frac{Y_t}{U_t}$$

Combining the second and the third equations, we get the expression for the rental rate of capital.

$$R_{t+1} = \alpha U_{t+1}^\alpha K_t^{\alpha-1} (Z_{t+1} N_{t+1})^{1-\alpha} - \delta(U_{t+1})$$

The rental rate depends on U_t because both MPK and depreciation rates depend on U_t.

(b) Log linearized version of $q_t \delta'(U_t) K_{t-1} = \alpha U_t^{\alpha-1} K_{t-1}^\alpha (Z_t N_t)^{1-\alpha}$ is

$$\check{q}_t + \frac{\delta''(\bar{U})\bar{U}}{\delta'(\bar{U})} \check{U}_t + \check{K}_{t-1} = (\alpha - 1)\check{U}_t + \alpha \check{K}_{t-1} + (1 - \alpha)\left(\check{Z}_t + \check{N}_t \right)$$

Using $\check{q}_t = 0$ and $\check{Y}_t = \alpha \left(\check{U}_t + \check{K}_{t-1} \right) + (1 - \alpha)\left(\check{Z}_t + \check{N}_t \right)$, we can express $\check{U}_t$ in terms of $\check{Y}_t$, $\check{K}_t$, and Δ.

$$\check{U}_t = \frac{1}{1 + \Delta} \left(\check{Y}_t - \check{K}_{t-1} \right)$$

(c) The production function in a log-linear form is:

$$\check{Y}_t = \alpha\left(\check{U}_t + \check{K}_{t-1}\right) + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right)$$
$$= \frac{\alpha}{1+\Delta}\left(\check{Y}_t - \check{K}_{t-1}\right) + \alpha\check{K}_{t-1} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right)$$

Isolate $\check{Y}_t$:

$$\check{Y}_t = \frac{\Delta\alpha}{1+\Delta-\alpha}\check{K}_{t-1} + \frac{(1+\Delta)(1-\alpha)}{1+\Delta-\alpha}\left(\check{Z}_t + \check{N}_t\right)$$
$$= \check{Z}_t + \check{N}_t \quad \text{(when } \Delta = 0\text{)}$$
$$= \alpha\check{K}_{t-1} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right) \quad \text{(when } \Delta = \infty\text{)}$$

(i) $\Delta = 0$ means that the steady state capital utilization rate is zero. Hence no matter how big the capital stock is, it does not contribute to the output. Therefore, deviations of output from its steady state solely depend on technology and labor.

(ii) $\Delta = \infty$ means that steady state capital utilization rate is one. In this case, this model boils down to a model without capital utilization, since 100% of capital stock is always used in production. Therefore, deviations of output from its steady state depend on all three inputs of the production function, with weights corresponding to the inputs same as the Cobb-Douglas coefficients.

(iii) Consider the case when $0 < \Delta < \infty$. In this case, the log linearized production function is written as:

$$\check{Y}_t = \frac{\Delta\alpha}{1+\Delta-\alpha}\check{K}_{t-1} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right) + \frac{\alpha(1-\alpha)}{1+\Delta-\alpha}\left(\check{Z}_t + \check{N}_t\right)$$

Since capital stock is not fully used in production, the contributions of Z_t and N_t in Y_t is higher than when $\Delta = \infty$.

(d)

3 Homework in macroeconomics

(a) The Lagrangian for the household's maximization problem is:

$$\mathcal{L} = (C_m^\rho + C_h^\rho)^{\frac{1}{\rho}} - \left(\frac{1}{\eta}+1\right)^{-1}(L_h + L_m)^{\frac{1}{\eta}+1} + \lambda\left(WL_m - C_m\right) + \xi\left(L_h - C_h\right)$$

The first order conditions for the interior solutions are:

$$(C_m^\rho + C_h^\rho)^{\frac{1}{\rho}-1} C_m^{\rho-1} = \lambda$$

$$(C_m^\rho + C_h^\rho)^{\frac{1}{\rho}-1} C_h^{\rho-1} = \xi$$

$$(L_h + L_m)^{\frac{1}{\eta}} = \lambda W$$

$$(L_h + L_m)^{\frac{1}{\eta}} = \xi$$

(b) $\xi = \lambda W$

(c) $\xi = \lambda \left(\frac{C_m}{C_h}\right)^{1-\rho}$

(d) Assuming an interior solution, $C_h = L_h = C_m W^{\frac{1}{\rho-1}}$

(e) Combining $L_h = C_h = C_m W^{\frac{1}{\rho-1}}$ and $C_m = W L_m$, we get $L_h = L_m W^{\frac{\rho}{\rho-1}}$. Substituting this into $(L_h + L_m)^{\frac{1}{\eta}} = \lambda W$, we get:

$$L_m \left(1 + W^{\frac{\rho}{\rho-1}}\right) = (\lambda W)^\eta$$

Hence,

$$L_m = \frac{(\lambda W)^\eta}{1 + W^{\frac{\rho}{\rho-1}}}$$

(f) Differentiate L_m with respect to W:

$$\frac{\partial L_h}{\partial W} = \frac{(1 + W^{\frac{\rho}{\rho-1}})\lambda^\eta \eta W^{\eta-1} - (\lambda W)^\eta (\frac{\rho}{\rho-1}) W^{\frac{\rho}{\rho-1}-1}}{(1 + W^{\frac{\rho}{\rho-1}})^2}$$

Hence the elasticity of L_m with respect to W is:

$$\varepsilon_{L_m,W} = \frac{\partial L_m}{\partial W} \cdot \frac{W}{L_m} = \frac{(1 + W^{\frac{\rho}{\rho-1}})\eta - (\frac{\rho}{\rho-1}) W^{\frac{\rho}{\rho-1}}}{(1 + W^{\frac{\rho}{\rho-1}})}$$

$$= \eta + \left(\frac{\rho}{1-\rho}\right) \left(\frac{W^{\frac{\rho}{\rho-1}}}{1 + W^{\frac{\rho}{\rho-1}}}\right)$$

(g) Consider a case where $\rho \to 1$ (C_m and C_h being perfect substitutes). Then, $\frac{\rho}{1-\rho} \to \infty$, pushing up the Frisch elasticity to infinity. Intuitively, if the household can home-produce everything on the market, there is no reason to supply labor to the market when the wage is lower than the value of the home produced goods.

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As ρ gets smaller, the Frisch elasticity also approaches to η. If home produced goods and goods on the market are not substitutable, then the Frisch elasticity is exactly equals to η.

(h) From the first order condition for market consumption goods, we had

$$(C_m^\rho + C_h^\rho)^{\frac{1}{\rho}-1} C_m^{\rho-1} = \lambda$$

Assuming interior solutions, substitute the budget constraints

$$((WL_m)^\rho + L_h^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1} = \lambda$$

and use the substitution

$$L_h = L_m W^{\frac{\rho}{\rho-1}}$$

to get

$$((WL_m)^\rho + (W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1} = \lambda$$

so now substitute back in to

$$L_m = \frac{(\lambda W)^\eta}{(1 + W^{\frac{\rho}{\rho-1}})}$$

and we have

$$L_m = \frac{\left[((WL_m)^\rho + (W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1 + W^{\frac{\rho}{\rho-1}})}$$

and we can simplify to get

$$L_m = \frac{\left[((W^\rho + W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1 + W^{\frac{\rho}{\rho-1}})}$$

and again to get

$$L_m = \frac{\left[(W^\rho + W^{\frac{\rho^2}{\rho-1}})^{\frac{1}{\rho}-1}L_m^{1-\rho}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1 + W^{\frac{\rho}{\rho-1}})}$$

and the L_m terms on the right side cancel so we have

$$L_m = \frac{\left[(W^\rho + W^{\frac{\rho^2}{\rho-1}})^{\frac{1}{\rho}-1}W^{\rho-1}\right]^\eta W^\eta}{(1 + W^{\frac{\rho}{\rho-1}})}$$

Simplify this a little bit more:

$$L_m = \left(1 + W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-1} W^{\eta}$$

Differentiate L_m w.r.t W

$$\frac{\partial L_m}{\partial W} = \eta W^{\eta-1} \left(1 + W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-1}$$
$$+ W^{\eta} \left(\eta \left(\frac{1-\rho}{\rho}\right) - 1\right)\left(1 + W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-2}\left(\frac{\rho}{\rho-1}\right) W^{\frac{1}{\rho-1}}$$

Hence the elasticity of L_m w.r.t W is

$$\frac{\partial L_m}{\partial W}\frac{W}{L_m} = \eta + \left(\eta\left(\frac{1-\rho}{\rho}\right) - 1\right)\left(\frac{\rho}{\rho-1}\right)\frac{W^{\frac{\rho}{\rho-1}}}{1 + W^{\frac{\rho}{\rho-1}}}$$
$$= \eta + \left(\frac{\rho}{1-\rho} - \eta\right)\frac{W^{\frac{\rho}{\rho-1}}}{1 + W^{\frac{\rho}{\rho-1}}}$$
$$= \eta + \left(\frac{\rho}{1-\rho}\right)\left(\frac{W^{\frac{\rho}{\rho-1}}}{1 + W^{\frac{\rho}{\rho-1}}}\right) - \eta\left(\frac{W^{\frac{\rho}{\rho-1}}}{1 + W^{\frac{\rho}{\rho-1}}}\right)$$

(i) The Frisch elasticity of labor measures the percentage change in hours worked due to the percentage change in wages, holding constant the marginal utility of wealth. This implies that the Frisch elasticity of L_m in our model is expressed as:
$$\varepsilon_{L_m,W}|_{\lambda=\bar{\lambda}} = \frac{\partial L_m}{\partial W} \cdot \frac{W}{L_m}|_{\lambda=\bar{\lambda}}$$

This is what we derived in (h). In other words, the Frisch elasticity is the compensated labor elasticity of wage. However, if calculate the Frisch elasticity of L_m holding the total consumption level C as constant, we have

$$\varepsilon_{L_m,W}|_{C=\bar{C}} = \eta + \left(\frac{\rho}{1-\rho}\right)\left(\frac{W^{\frac{\rho}{\rho-1}}}{1 + W^{\frac{\rho}{\rho-1}}}\right) - \eta\left(\frac{W^{\frac{\rho}{\rho-1}}}{1 + W^{\frac{\rho}{\rho-1}}}\right)$$
$$= \varepsilon_{L_m,W}|_{\lambda=\bar{\lambda}} - \eta\left(\frac{W^{\frac{\rho}{\rho-1}}}{1 + W^{\frac{\rho}{\rho-1}}}\right)$$

The last negative term represents the wealth effect from the rise in W, since we are not holding the wealth effect constant in this case. However, the intuition from (g) still survives.

(j) Even if η itself is small, the model can generate a high Frisch elasticitiy by adjusting the substitutability parameter ρ. If home produced and market produced goods are highly substitutable, households would want to move their labor almost entirely from the home production to the labor market. This induces a high Frisch elasticity regardless of the value of η.

4 q-Theory with Variable Capital Utilization

(a) The Lagrangian of the firm's profit maximization problem is:

$$
\begin{aligned}
\mathcal{L} = \mathbb{E}_t \sum_s & \left(\prod_{k=1}^s (1 + r_{t+k}) \right)^{-1} \\
& \times \left(Z_{t+s} \left(U_{t+s} K_{t+s-1} \right)^\alpha L_{t+s}^{1-\alpha} - W_{t+s} L_{t+s} - I_{t+s} \left[1 + \phi \left(\frac{I_{t+s}}{K_{t+s-1}} \right) \right] \right. \\
& \left. + q_{t+s} \left(-K_{t+s} + (1 - \delta(U_t)) K_{t+s-1} + I_{t+s} \right) \right)
\end{aligned}
$$

This problem is truly dynamic because the presence of an adjustment cost links the present and future period investment decisions.

(b) The first order conditions are:

$$
\frac{\partial \mathcal{L}}{\partial L_t} : \quad w_t = (1 - \alpha) \frac{Y_t}{L_t}
$$

$$
\frac{\partial \mathcal{L}}{\partial L_t} : \quad q_t = 1 + \phi \left(\frac{I_t}{K_t} \right) + \frac{I_t}{K_t} \phi' \left(\frac{I_t}{K_t} \right)
$$

$$
\frac{\partial \mathcal{L}}{\partial K_t} : \quad q_t = (1 + r_{t+1})^{-1} \left(\alpha \frac{Y_{t+1}}{K_{t+1}} + \frac{I_{t+1}^2}{K_{t+1}^2} \phi' \left(\frac{I_t}{K_t} \right) + q_{t+1}(1 - \delta(U_{t+1})) \right)
$$

$$
\frac{\partial \mathcal{L}}{\partial U_t} : \quad \alpha \frac{Y_t}{U_t} = q_t \delta'(U_t) K_t
$$

Combining the second and third equations, we can get an expression for the rental rate of capital:

$$
r_{t+1} = \alpha \frac{Y_{t+1}}{K_{t+1}} + \frac{I_{t+1}^2}{K_{t+1}^2} \phi' \left(\frac{I_t}{K_t} \right) + \delta(U_{t+1})
$$

The rental rate of captial is a function of U because the depreciation rate and the marginal product of capital depend on U. It is also a function of q because -

(c) Log-linearized version of the FOC for utilization is:

$$\check{U}_t = \frac{\check{Y}_t - \check{K}_t - \check{q}_t}{1 + \Delta} = \frac{\check{MPK}_t - \check{q}_t}{1 + \Delta}$$

Hence the sign of the percentage deviation of utilization from its steady state value depends on $\check{MPK}_t - \check{q}_t$. In this model, q_t can be different from 1. Considering that q_t is a Lagrangian multiplier attached to the capital accumulation equation, the interpretation of q_t is *Tobin's marginal q*, which is the ratio of the value of additional unit of capital to its replacement cost. Hence, $\check{MPK}_t - \check{q}_t$ represents the net gain of adding one more unit of capital in the production. When $\check{MPK}_t - \check{q}_t > 0$, it is profitable for the firm to use more units of capital, hence $\check{U}_t > 0$.

(d)

(e)

5 Fiscal multiplier in the RBC model

(a) The log-linearized system of equations is

$$\check{K}_t = (1 - \delta)\check{K}_{t-1} + \delta\check{I}_t$$

$$\check{C}_t + \frac{1}{\eta}\check{L}_t = \check{Y}_t - \check{L}_t$$

$$E_t\check{C}_{t+1} - \check{C}_t = \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1 - \delta)}(E_t\check{Y}_{t+1} - \check{K}_t)$$

$$\check{Y}_t = \alpha\check{K}_{t-1} + (1 - \alpha)\check{L}_t$$

$$\check{Y}_t = \frac{\bar{C}}{\bar{Y}}\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t + \frac{\bar{G}}{\bar{Y}}\check{G}_t$$

$$\check{G}_t = \rho_g\check{G}_{t-1} + \epsilon_t^g$$

Guess that the policy functions take the form

$$\check{C}_t = v_{CK}\check{K}_{t-1} + v_{CG}\check{G}_t$$

$$\check{K}_t = v_{KK}\check{K}_{t-1} + v_{KG}\check{G}_t$$

Now plug the policy functions into the system of equations and we are left with the log-linearized consumption Euler equation and labor-leisure condition in terms of parameters, coefficients of the policy functions, and state

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variables:

$$(v_{CK}v_{KK} - v_{CK})\check{K}_{t-1} + (v_{CK}v_{KG} + v_{CG}\rho_g - v_{CG})\check{G}_t$$

$$= \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)}\left[\frac{\bar{C}}{\bar{Y}}v_{CK}v_{KK} + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)v_{KK}\frac{1}{\delta} - v_{KK}\right]\check{K}_{t-1}$$

$$+ \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)}\left[\frac{\bar{C}}{\bar{Y}}v_{CK}v_{KG} + \frac{\bar{C}}{\bar{Y}}v_{CG}\rho_g + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)v_{KG}\frac{1}{\delta}\right.$$

$$\left. + \frac{\bar{I}}{\bar{Y}}v_{KG}\rho_g\frac{1}{\delta} + \frac{\bar{G}}{\bar{Y}}\rho_g - v_{KG}\right]\check{G}_t$$

$$\left[v_{CK} - \left(\frac{1}{\eta} + 1\right)\frac{\alpha}{1-\alpha}\right]\check{K}_{t-1} + v_{CG}\check{G}_t$$

$$= \frac{-\alpha - \frac{1}{\eta}}{1-\alpha}\left[\frac{\bar{C}}{\bar{Y}}v_{CK} + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)\frac{1}{\delta}\right]\check{K}_{t-1}$$

$$+ \frac{-\alpha - \frac{1}{\eta}}{1-\alpha}\left[\frac{\bar{C}}{\bar{Y}}v_{CG} + \frac{\bar{I}}{\bar{Y}}v_{KG}\frac{1}{\delta} + \frac{\bar{G}}{\bar{Y}}\right]\check{G}_t$$

By comparing the coefficients on $\check{K}_{t-1}$ and $\check{G}_t$ in both equations, we obtain 4 equations

$$v_{CK}v_{KK} - v_{CK} = \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)}\left[\frac{\bar{C}}{\bar{Y}}v_{CK}v_{KK} + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)v_{KK}\frac{1}{\delta} - v_{KK}\right]$$

$$v_{CK}v_{KG} + v_{CG}\rho_g - v_{CG} = \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)}\left[\frac{\bar{C}}{\bar{Y}}v_{CK}v_{KG} + \frac{\bar{C}}{\bar{Y}}v_{CG}\rho_g\right.$$

$$+ \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)v_{KG}\frac{1}{\delta} + \frac{\bar{I}}{\bar{Y}}v_{KG}\rho_g\frac{1}{\delta} + \frac{\bar{G}}{\bar{Y}}\rho_g$$

$$\left. - v_{KG}\right]$$

$$v_{CK} - \left(\frac{1}{\eta} + 1\right)\frac{\alpha}{1-\alpha} = \frac{-\alpha - \frac{1}{\eta}}{1-\alpha}\left[\frac{\bar{C}}{\bar{Y}}v_{CK} + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)\frac{1}{\delta}\right]$$

$$v_{CG}\breve{G}_t = \frac{-\alpha - \frac{1}{\eta}}{1 - \alpha} \left[\frac{\bar{C}}{\bar{Y}} v_{CG} + \frac{\bar{I}}{\bar{Y}} v_{KG} \frac{1}{\delta} + \frac{\bar{G}}{\bar{Y}} \right] \breve{G}_t$$
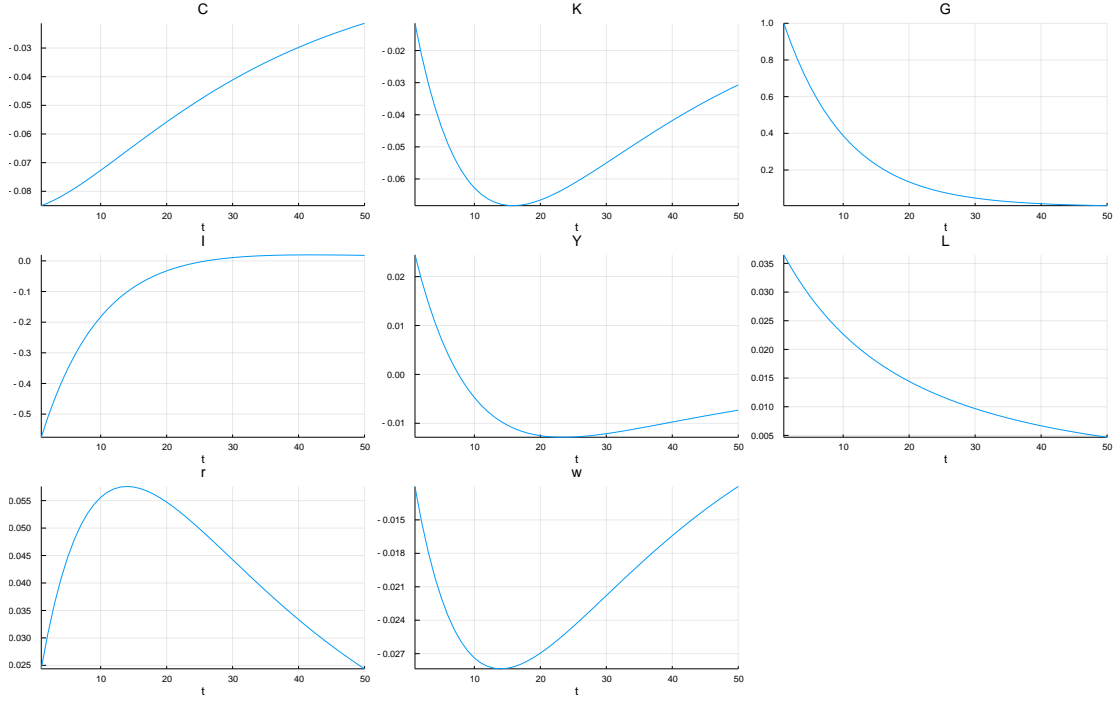

Figure 2: IRFs with $\rho = 0.9$

(b) Increase in government expenditure is only temporary, and the boost in output is transitory. Households compensate for the increased government expenditure by decreasing consumption, decreasing savings, and increasing labor supply. The households desired level of steady-state capital remains unchanged. This causes MPL (and wages) to decline while MPK (and interest rate) to increase.
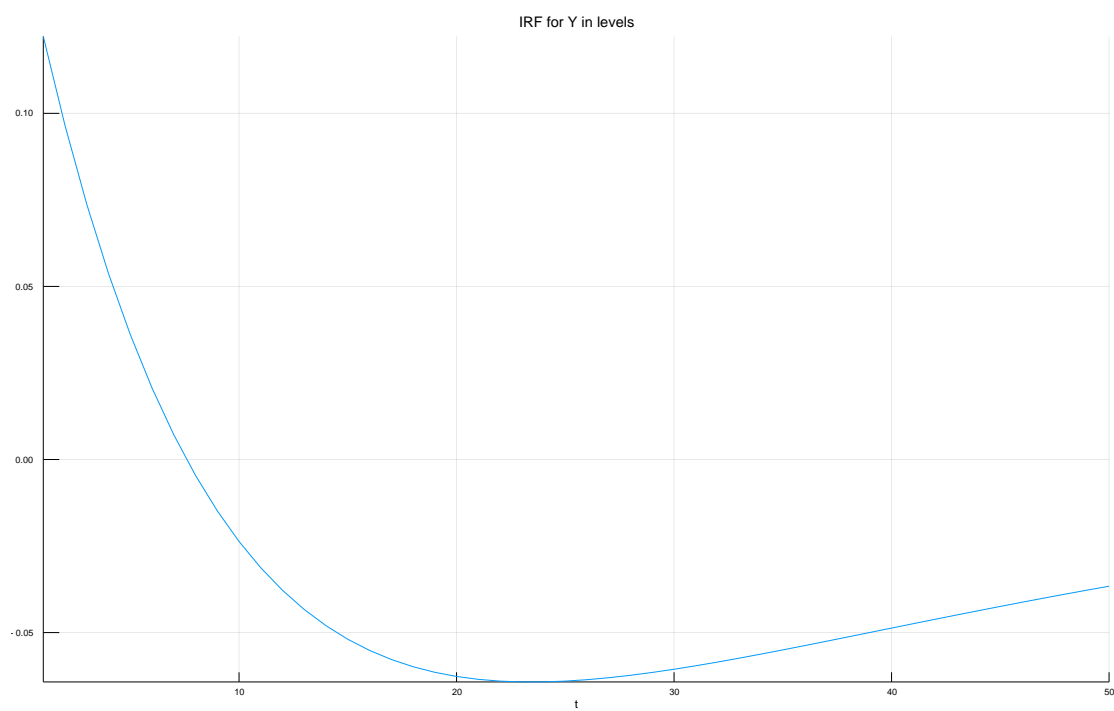
Figure 3: IRF for level of Y with $\rho = 0.9$

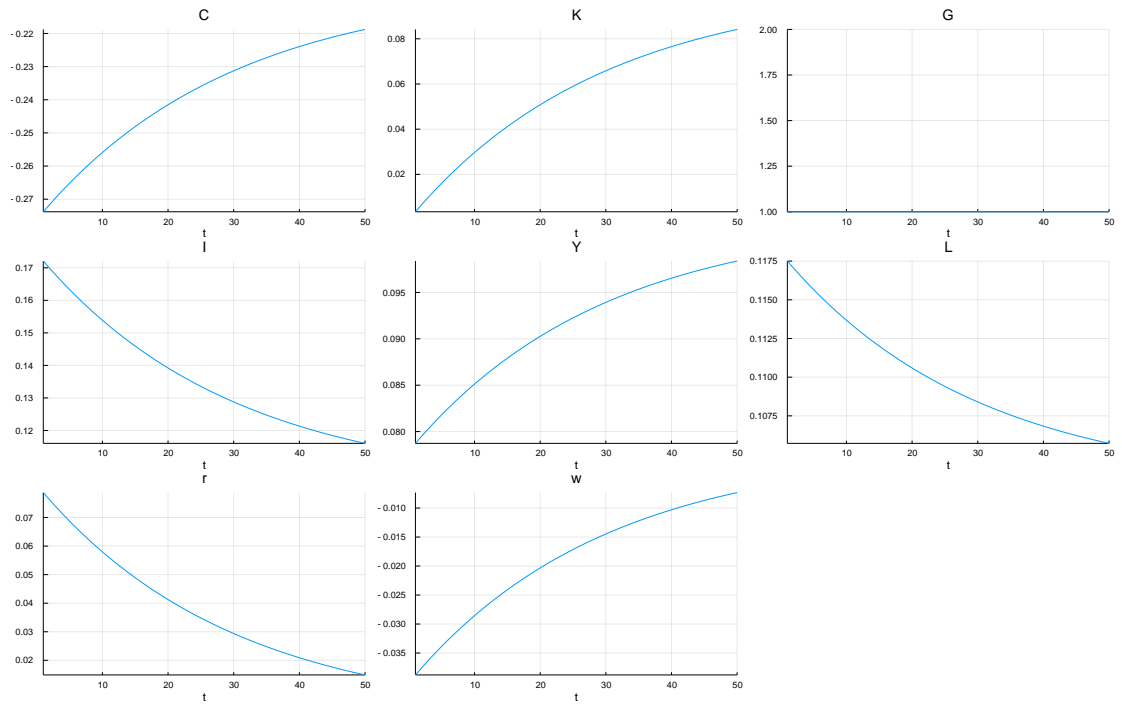(c) The multiplier is about 1.13.
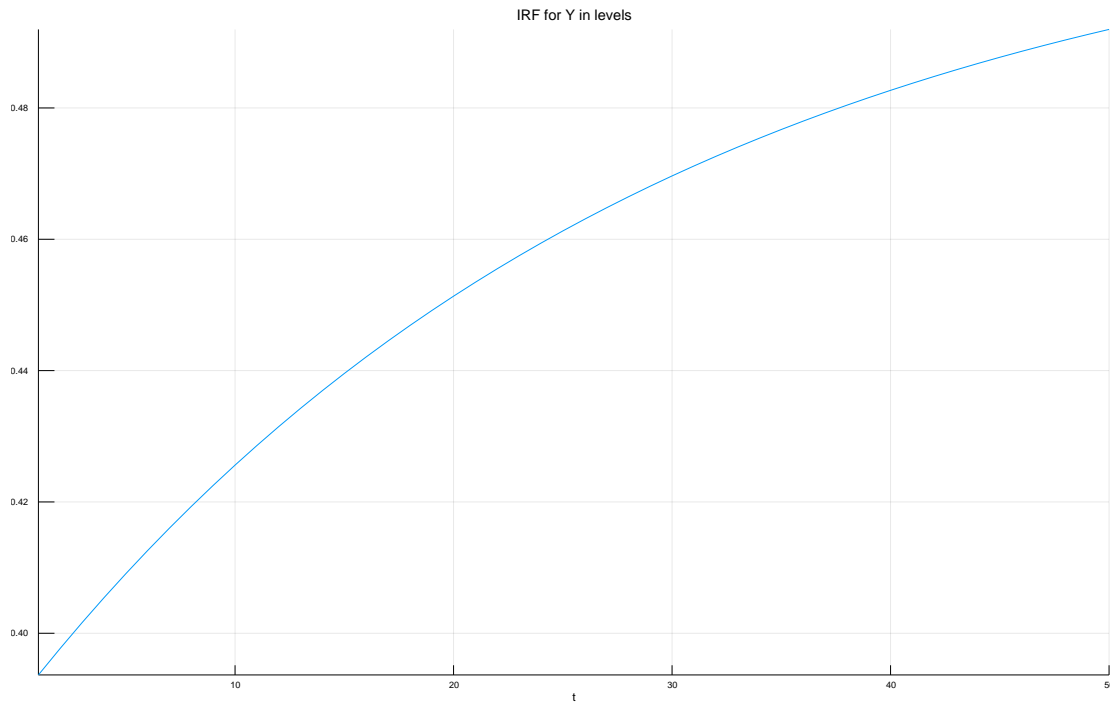
Figure 4: IRFs with $\rho = 1$

Figure 5: IRF for level of Y with $\rho = 1$

(d) Now the increase in government expenditure is permanent, and the boost in output persists forever. Households decrease consumption and increase labor supply by a lot more compared to before because they are increasing their savings in this situation. This is because the households want to attain a higher level of steady-state capital. This sharp increase in labor supply causes wages to decline by more and interest rate to increase more. The level of increase in output is much larger; the multiplier is about 1.4.