ECON 210C PROBLEM SET # 3

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1. Variable labor supply in the RBC model

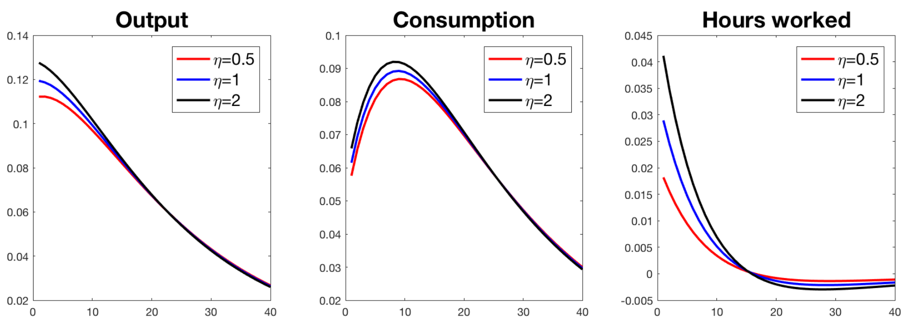

Figure 1. Impulse responses with varying η

	$\eta = 0.5$	$\eta = 1$	$\eta = 2$	Data
σ_Y	1.54	1.64	1.74	1.72
σ_C	0.97	1.02	1.08	1.27
σ_L	0.23	0.37	0.53	1.59

Table 1. Response to a transitory discount factor shock

Larger Frisch elasticity values imply a better fit, as they generate stronger inter-temporal substitution of labor supply, amplifying the effect of shocks. Consumption is still too smooth, and the volatility of hours is too low.

2. Variable capital utilization in an RBC model

(a). Firms choose capital utilization U, capital K, and labor demand N.

The production function that we can use directly (since the output is the numeraire) is

$$Y_t = (U_t K_{t-1})^\alpha (Z_t N_t)^{1-\alpha}$$

and since the firms own capital, they face the constraint

$$K_t = I_t + (1 - \delta(U_t))K_{t-1}$$

but they also have to pay wages $W_t N_t$ and invest I_t so we can set up the Lagrangian

$$\mathcal{L} = E\sum_s (\prod_{k=1}^{s} (1+r_{t+k})^{-1})\Big((U_{t+s}K_{t+s-1})^\alpha (Z_{t+s}N_{t+s})^{1-\alpha} - W_{t+s}N_{t+s} - I_{t+s} + q_{t+s}(-K_{t+s} + I_{t+s} + (1-\delta(U_{t+s}))K_{t+s-1})\Big)$$

so we have first order conditions:

for labor we have

$$W_t = (1-\alpha)(U_t K_{t-1})^\alpha Z_t^{1-\alpha} N_t^{-\alpha}$$

for investment we have

$$q_t = 1$$

for capital at time t we have

$$q_t = E\left[\frac{1}{1 + r_{t+1}} \left(\alpha U_{t+1}^\alpha K_t^{\alpha-1} (Z_{t+1}N_{t+1})^{1-\alpha} + q_{t+1}(1 - \delta(U_{t+1})\right)\right]$$

and finally we have the condition for utilization

$$\alpha U_t^{\alpha-1} K_{t-1}^\alpha (Z_t N_t)^{1-\alpha} = q_t K_{t-1} \delta'(U_t)$$

Combining the investment and capital optimality conditions yields the expression for the rental rate of capital.

$$R_{t+1} = \alpha U_{t+1}^\alpha K_t^{\alpha-1} (Z_{t+1}N_{t+1})^{1-\alpha} - \delta(U_{t+1})$$

The rental rate depends on utilization because the marginal product of capital and its depreciation rate depend on utilization.

(b). The log linearized version of the utilization optimality condition

$$q_t \delta^{'}(U_t) K_{t-1} = \alpha U_t^{\alpha-1} K_{t-1}^\alpha (Z_t N_t)^{1-\alpha}$$

is

$$\check{q}_t + \frac{\delta^{''}(\bar{U})\bar{U}}{\delta^{'}(\bar{U})} \check{U}_t + \check{K}_{t-1} = (\alpha - 1)\check{U}_t + \alpha \check{K_{t-1}} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right)$$

We know $\check{q}_t = 0$ from the investment optimality condition and we also know

$$\check{Y}_t = \alpha\left(\check{U}_t + \check{K_{t-1}}\right) + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right)$$

because the production function is given, so we have

$$\check{U}_t = \frac{1}{1+\Delta}\left(\check{Y}_t - \check{K_{t-1}}\right)$$

(c). The log-linearized production function is

$$\begin{aligned}\check{Y}_t &= \alpha\left(\check{U}_t + \check{K}_{t-1}\right) + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right)\\ &= \frac{\alpha}{1+\Delta}\left(\check{Y}_t - \check{K}_{t-1}\right) + \alpha\check{K}_{t-1} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right)\end{aligned}$$

Isolate $\check{Y}_t$:

$$\begin{aligned}\check{Y}_t &= \frac{\Delta\alpha}{1+\Delta-\alpha}\check{K}_{t-1} + \frac{(1+\Delta)(1-\alpha)}{1+\Delta-\alpha}\left(\check{Z}_t + \check{N}_t\right)\\ &= \check{Z}_t + \check{N}_t \quad \text{(when } \Delta = 0\text{)}\\ &= \alpha\check{K}_{t-1} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right) \quad \text{(when } \Delta = \infty\text{)}\end{aligned}$$

$\Delta = 0$ implies the capital stock is impotent, and output thus only depends on technology and labor. $\Delta = \infty$ implies full utilization, so output depends on all three inputs, with weights equal to the Cobb-Douglas coefficients.

In every other case, we have

$$\check{Y}_t = \frac{\Delta\alpha}{1+\Delta-\alpha}\check{K}_{t-1} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right) + \frac{\alpha(1-\alpha)}{1+\Delta-\alpha}\left(\check{Z}_t + \check{N}_t\right)$$

so Z_t and N_t in Y_t matter relatively more (with capital not being fully utilized) relative to the $\Delta = \infty$ case.

3. Homework in macroeconomics

(a). The Lagrangian for the household's maximization problem is:

$$\mathcal{L} = \left(C_m^\rho + C_h^\rho\right)^{\frac{1}{\rho}} - \left(\frac{1}{\eta}+1\right)^{-1}\left(L_h + L_m\right)^{\frac{1}{\eta}+1} + \lambda\left(WL_m - C_m\right) + \xi\left(L_h - C_h\right)$$

The first order conditions for the interior solutions are:

$$\begin{aligned}\cancel{\frac{1}{\rho}}\left(C_m^\rho + C_h^\rho\right)^{\frac{1}{\rho}-1}\cancel{\rho}C_m^{\rho-1} &= \lambda\\ \cancel{\frac{1}{\rho}}\left(C_m^\rho + C_h^\rho\right)^{\frac{1}{\rho}-1}\cancel{\rho}C_h^{\rho-1} &= \xi\\ \left(L_h + L_m\right)^{\frac{1}{\eta}} &= \lambda W\\ \left(L_h + L_m\right)^{\frac{1}{\eta}} &= \xi\end{aligned}$$

(b). From the two first order conditions for labor, we have

$$\xi = \lambda W$$

(c). From the two first order conditions for consumption, we have

$$\xi = \lambda\left(\frac{C_h}{C_m}\right)^{\rho-1}$$

(d). With the budget constraints binding, we have

$$C_h = L_h$$

and from above we get

$$C_h = C_m W^{\frac{1}{\rho-1}}$$

(e). We now have

$$L_h = C_m W^{\frac{1}{\rho-1}}$$

and we can assume the budget constraint holds for formal markets to make the substitution

$$C_m = W L_m$$

getting us

$$L_h = W L_m W^{\frac{1}{\rho-1}}$$

equivalent to

$$L_h = L_m W^{\frac{\rho}{\rho-1}}$$

and from our first order conditions we have

$$L_h + L_m = (\lambda W)^\eta$$

so we can substitute for L_h to get

$$(\lambda W)^\eta - L_m = L_m W^{\frac{\rho}{\rho-1}}$$

so we have

$$L_m(1 + W^{\frac{\rho}{\rho-1}}) = (\lambda W)^\eta$$

and thus

$$L_m = \frac{(\lambda W)^\eta}{(1 + W^{\frac{\rho}{\rho-1}})}$$

(f). We now have

$$\frac{\partial L_h}{\partial W} = \frac{(1 + W^{\frac{\rho}{\rho-1}})\lambda^\eta \eta W^{\eta-1} - (\lambda W)^\eta (\frac{\rho}{\rho-1}) W^{\frac{\rho}{\rho-1}-1}}{(1 + W^{\frac{\rho}{\rho-1}})^2}$$

with

$$\frac{\partial L_m}{\partial W} \cdot \frac{W}{L_m} = \frac{(1 + W^{\frac{\rho}{\rho-1}})\eta - (\frac{\rho}{\rho-1}) W^{\frac{\rho}{\rho-1}}}{(1 + W^{\frac{\rho}{\rho-1}})}$$

as the elasticity of L_h with respect to W.

(g).

(h). We had

$$\left(C_m^\rho + C_h^\rho\right)^{\frac{1}{\rho}-1} C_m^{\rho-1} = \lambda$$

so substitute the budget constraints

$$((WL_m)^\rho + L_h^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1} = \lambda$$

and use the substitution

$$L_h = L_m W^{\frac{\rho}{\rho-1}}$$

to get

$$((WL_m)^\rho + (W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1} = \lambda$$

so now substitute back in to

$$L_m = \frac{(\lambda W)^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

and we have

$$L_m = \frac{\left[((WL_m)^\rho + (W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

and we can simplify to get

$$L_m = \frac{\left[((W^\rho + W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

and again to get

$$L_m = \frac{\left[(W^\rho + W^{\frac{\rho^2}{\rho-1}})^{\frac{1}{\rho}-1}L_m^{1-\rho}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

and the L_m terms on the right side cancel so we have

$$L_m = \frac{\left[(W^\rho + W^{\frac{\rho^2}{\rho-1}})^{\frac{1}{\rho}-1}W^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

and we can rewrite the numerator to get

$$L_m = \frac{\left[(W^\rho + W^{\frac{\rho^2}{\rho-1}})^{\frac{1}{\rho}-1}W^\rho\right]^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

equivalent to

$$L_m = \frac{\left[(W^\rho(1 + W^{\frac{\rho}{\rho-1}}))^{\frac{1}{\rho}-1}W^\rho\right]^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

and

$$L_m = \frac{\left[(1 + W^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}-1}W^{1-\rho}W^\rho\right]^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

and

$$L_m = \frac{\left[(1+W^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}-1}W\right]^{\eta}}{(1+W^{\frac{\rho}{\rho-1}})}$$

to finally get

$$L_m = \left(1+W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-1} W^{\eta}$$

(i). Differentiating, we have

$$\frac{\partial L_m}{\partial W} = \frac{\rho\left(\frac{\eta(1-\rho)}{\rho}-1\right)W^{\eta+\frac{\rho}{\rho-1}-1}\left(W^{\frac{\rho}{\rho-1}}+1\right)^{\frac{\eta(1-\rho)}{\rho}-2}}{\rho-1} + \eta W^{\eta-1}\left(W^{\frac{\rho}{\rho-1}}+1\right)^{\frac{\eta(1-\rho)}{\rho}-1}$$

and we have the elasticity as

$$\frac{\partial L_m}{\partial W}\cdot\frac{W}{L_m} = \frac{\rho\left(\frac{\eta(1-\rho)}{\rho}-1\right)W^{\frac{\rho}{\rho-1}}\left(W^{\frac{\rho}{\rho-1}}+1\right)^{-1}}{\rho-1} + \eta$$

which simplifies to

$$\frac{\eta\left(W^{\frac{\rho}{\rho-1}}\rho\left(\frac{\eta(1-\rho)}{\rho}-1\right)\right)}{(\rho-1)\left(W^{\frac{\rho}{\rho-1}}+1\right)}$$