ECON 210C PROBLEM SET # 3

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1. Variable labor supply in the RBC model

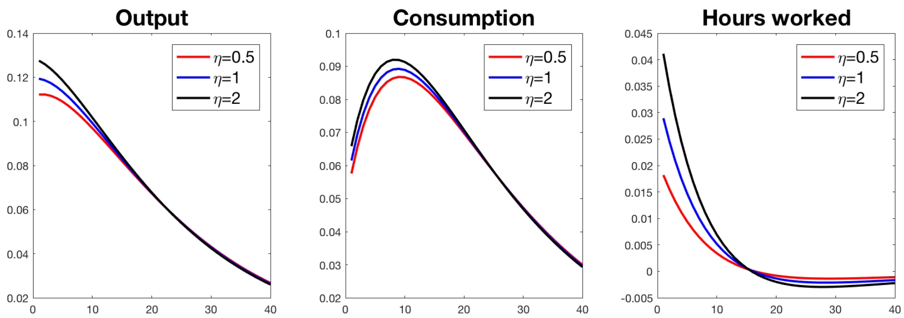

Figure 1. Impulse responses with varying η

	$\eta = 0.5$	$\eta = 1$	$\eta = 2$	Data
$Stdev(Y)$	1.54	1.64	1.74	1.72
$Stdev(C)$	0.97	1.02	1.08	1.27
$Stdev(L)$	0.23	0.37	0.53	1.59

Table 1. Response to a transitory discount factor shock

As one would expect, the fits get better as we calibrate the Frisch elasticity to bigger values. A large Frisch elasticity generates stronger intertemporal substitution of labor suppply, and hence amplifies the effect of shocks. However, even with a large Frisch elasticity, consumption is too smooth, and the volatility of hours generated from the model falls short of the empirical counterpart.

2. Variable capital utilization in an RBC model

(a) The Lagrangian of the firm's profit maximization problem is:

$$\begin{aligned}\mathcal{L} =& \mathbb{E}_t \sum_s \left(\prod_{k=1}^{s} (1+R_{t+k}) \right)^{-1} \\ &\times \left[(U_{t+s}K_{t+s-1})^{\alpha} (Z_{t+s}N_{t+s})^{1-\alpha} - W_{t+s}N_{t+s} - I_{t+s} + q_{t+s}\left(-K_{t+s} + (1-\delta(U_t))K_{t+s-1} + I_{t+s}\right) \right]\end{aligned}$$

The first order conditions are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial N_t} &: \quad W_t = (1-\alpha)\frac{Y_t}{N_t} \\
\frac{\partial \mathcal{L}}{\partial I_t} &: \quad q_t = 1 \\
\frac{\partial \mathcal{L}}{\partial K_t} &: \quad q_t = \mathbb{E}_t \frac{1}{1+R_{t+1}} \left[\alpha \left(U_{t+1}K_t\right)^{\alpha-1} U_{t+1} \left(Z_{t+1}N_{t+1}\right)^{1-\alpha} + q_{t+1}\left(1-\delta(U_{t+1})\right)\right] \\
\frac{\partial \mathcal{L}}{\partial U_t} &: \quad q_t \delta'(U_t) K_{t-1} = \alpha \frac{Y_t}{U_t}
\end{aligned}
$$

Combining the second and the third equations, we get the expression for the rental rate of capital.

$$R_{t+1} = \alpha U_{t+1}^{\alpha} K_t^{\alpha-1} \left(Z_{t+1}N_{t+1}\right)^{1-\alpha} - \delta(U_{t+1})$$

The rental rate depends on U_t because both MPK and depreciation rates depend on U_t.

(b) Log linearized version of $q_t\delta'(U_t)K_{t-1} = \alpha U_t^{\alpha-1} K_{t-1}^{\alpha} \left(Z_t N_t\right)^{1-\alpha}$ is

$$\check{q}_t + \frac{\delta''(\bar{U})\bar{U}}{\delta'(\bar{U})}\check{U}_t + \check{K}_{t-1} = (\alpha-1)\check{U}_t + \alpha \check{K}_{t-1} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right)$$

Using $\check{q}_t = 0$ and $\check{Y}_t = \alpha\left(\check{U}_t + \check{K}_{t-1}\right) + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right)$, we can express $\check{U}_t$ in terms of $\check{Y}_t, \check{K}_t$, and Δ.

$$\check{U}_t = \frac{1}{1+\Delta}\left(\check{Y}_t - \check{K}_{t-1}\right)$$

(c) The production function in a log-linear form is:

$$
\begin{aligned}
\check{Y}_t &= \alpha\left(\check{U}_t + \check{K}_{t-1}\right) + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right) \\
&= \frac{\alpha}{1+\Delta}\left(\check{Y}_t - \check{K}_{t-1}\right) + \alpha \check{K}_{t-1} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right)
\end{aligned}
$$

Isolate $\check{Y}_t$:

$$
\begin{aligned}
\check{Y}_t &= \frac{\Delta\alpha}{1+\Delta-\alpha}\check{K}_{t-1} + \frac{(1+\Delta)(1-\alpha)}{1+\Delta-\alpha}\left(\check{Z}_t + \check{N}_t\right) \\
&= \check{Z}_t + \check{N}_t \quad (\text{when } \Delta = 0) \\
&= \alpha\check{K}_{t-1} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right) \quad (\text{when } \Delta = \infty)
\end{aligned}
$$

(i) $\Delta = 0$ means that the steady state capital utilization rate is zero. Hence no matter how big the capital stock is, it does not contribute to the output. Therefore, deviations of output from its steady state solely depend on technology and labor.

(ii) $\Delta = \infty$ means that steady state capital utilization rate is one. In this case, this model boils down to a model without capital utilization, since 100% of capital stock is always used in production. Therefore, deviations of output from its steady state depend on all three inputs of the production function, with weights corresponding to the inputs same as the Cobb-Douglas coefficients.

(iii) Consider the case when $0 < \Delta < \infty$. In this case, the log linearized production function is written as:

$$\check{Y}_t = \frac{\Delta\alpha}{1+\Delta-\alpha}\check{K_{t-1}} + (1-\alpha)\left(\check{Z}_t + \check{N}_t\right) + \frac{\alpha(1-\alpha)}{1+\Delta-\alpha}\left(\check{Z}_t + \check{N}_t\right)$$

Since capital stock is not fully used in production, the contributions of Z_t and N_t in Y_t is higher than when $\Delta = \infty$.

(d)

3. Homework in macroeconomics

(a) The Lagrangian for the household's maximization problem is:

$$\mathcal{L} = \left(C_m^\rho + C_h^\rho\right)^{\frac{1}{\rho}} - \left(\frac{1}{\eta}+1\right)^{-1}(L_h + L_m)^{\frac{1}{\eta}+1} + \lambda\left(WL_m - C_m\right) + \xi\left(L_h - C_h\right)$$

The first order conditions for the interior solutions are:

$$\left(C_m^\rho + C_h^\rho\right)^{\frac{1}{\rho}-1} C_m^{\rho-1} = \lambda$$
$$\left(C_m^\rho + C_h^\rho\right)^{\frac{1}{\rho}-1} C_h^{\rho-1} = \xi$$
$$(L_h + L_m)^{\frac{1}{\eta}} = \lambda W$$
$$(L_h + L_m)^{\frac{1}{\eta}} = \xi$$

(b) $\xi = \lambda W$

(c) $\xi = \lambda\left(\frac{C_m}{C_h}\right)^{1-\rho}$

(d) Assuming an interior solution, $C_h = L_h = C_m W^{\frac{1}{\rho-1}}$

(e) Combining $L_h = C_h = C_m W^{\frac{1}{\rho-1}}$ and $C_m = WL_m$, we get $L_h = L_m W^{\frac{\rho}{\rho-1}}$. Substituting this into $(L_h + L_m)^{\frac{1}{\eta}} = \lambda W$, we get:

$$L_m\left(1 + W^{\frac{\rho}{\rho-1}}\right) = (\lambda W)^\eta$$

Hence,

$$L_m = \frac{(\lambda W)^\eta}{1 + W^{\frac{\rho}{\rho-1}}}$$

(f) Differentiate L_m with respect to W:

$$\frac{\partial L_h}{\partial W} = \frac{(1+W^{\frac{\rho}{\rho-1}})\lambda^\eta \eta W^{\eta-1} - (\lambda W)^\eta (\frac{\rho}{\rho-1}) W^{\frac{\rho}{\rho-1}-1}}{(1+W^{\frac{\rho}{\rho-1}})^2}$$

Hence the elasticity of L_m with respect to W is:

$$\begin{aligned} \varepsilon_{L_m,W} = \frac{\partial L_m}{\partial W}\cdot\frac{W}{L_m} &= \frac{(1+W^{\frac{\rho}{\rho-1}})\eta - (\frac{\rho}{\rho-1})W^{\frac{\rho}{\rho-1}}}{(1+W^{\frac{\rho}{\rho-1}})} \\ &= \eta + \left(\frac{\rho}{1-\rho}\right)\left(\frac{W^{\frac{\rho}{\rho-1}}}{1+W^{\frac{\rho}{\rho-1}}}\right) \end{aligned}$$

(g) Consider a case where $\rho \to 1$ (C_m and C_h) being perfect substitutes. Then, $\frac{\rho}{1-\rho} \to \infty$, pushing up the Frisch elasticity to infinity. Intuitively, if the household can home-produce everything on the market, there is no reason to supply labor to the market when the wage is lower than the value of the home produced goods.

As ρ gets smaller, the Frisch elasticity also approaches to η. If home produced goods and goods on the market are not substitutable, then the Frisch elasticity is exactly equals to η.

(h)

(i)

(j)

4. q-Theory with Variable Capital Utilization

(a) The Lagrangian of the firm's profit maximization problem is:

$$\begin{aligned} \mathcal{L} =& \mathbb{E}_t \sum_s \left(\prod_{k=1}^{s}(1+r_{t+k})\right)^{-1} \\ &\times \Bigg\{ Z_{t+s}\left(U_{t+s}K_{t+s-1}\right)^\alpha L_{t+s}^{1-\alpha} - W_{t+s}L_{t+s} - I_{t+s}\left[1+\phi\left(\frac{I_{t+s}}{K_{t+s-1}}\right)\right] \\ &\quad + q_{t+s}\left(-K_{t+s} + (1-\delta(U_t))K_{t+s-1} + I_{t+s}\right) \Bigg\} \end{aligned}$$

This problem is truly dynamic because the presence of an adjustment cost links the present and future period investment decisions.

(b)

5. Fiscal multiplier in the RBC model

The log-linearized system of equations is

$$\begin{aligned}
\check{K}_t &= (1-\delta)\check{K}_{t-1} + \delta \check{I}_t \\
\check{C}_t + \frac{1}{\eta}\check{L}_t &= \check{Y}_t - \check{L}_t \\
E_t\check{C}_{t+1} - \check{C}_t &= \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)}(E_t\check{Y}_{t+1} - \check{K}_t) \\
\check{Y}_t &= \alpha\check{K}_{t-1} + (1-\alpha)\check{L}_t \\
\check{Y}_t &= \frac{\bar{C}}{\bar{Y}}\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t + \frac{\bar{G}}{\bar{Y}}\check{G}_t \\
\check{G}_t &= \rho_g\check{G}_{t-1} + \epsilon_t^g
\end{aligned}$$

Guess that the policy functions take the form

$$\begin{aligned}
\check{C}_t &= v_{CK}\check{K}_{t-1} + v_{CG}\check{G}_t \\
\check{K}_t &= v_{KK}\check{K}_{t-1} + v_{KG}\check{G}_t
\end{aligned}$$

Now plug the policy functions into the system of equations and we are left with the log-linearized consumption Euler equation and labor-leisure condition in terms of parameters, coefficients of the policy functions, and state variables:

$$\begin{aligned}
&(v_{CK}v_{KK} - v_{CK})\check{K}_{t-1} + (v_{CK}v_{KG} + v_{CG}\rho_g - v_{CG})\check{G}_t \\
&= \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)}\left[\frac{\bar{C}}{\bar{Y}}v_{CK}v_{KK} + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)v_{KK}\frac{1}{\delta} - v_{KK}\right]\check{K}_{t-1} \\
&\quad + \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)}\left[\frac{\bar{C}}{\bar{Y}}v_{CK}v_{KG} + \frac{\bar{C}}{\bar{Y}}v_{CG}\rho_g + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)v_{KG}\frac{1}{\delta} + \frac{\bar{I}}{\bar{Y}}v_{KG}\rho_g\frac{1}{\delta}\right. \\
&\qquad\qquad \left. + \frac{\bar{G}}{\bar{Y}}\rho_g - v_{KG}\right]\check{G}_t
\end{aligned}$$

$$\begin{aligned}
\left[v_{CK} - \left(\frac{1}{\eta} + 1\right)\frac{\alpha}{1-\alpha}\right]\check{K}_{t-1} + v_{CG}\check{G}_t &= \frac{-\alpha - \frac{1}{\eta}}{1-\alpha}\left[\frac{\bar{C}}{\bar{Y}}v_{CK} + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)\frac{1}{\delta}\right]\check{K}_{t-1} \\
&\quad + \frac{-\alpha - \frac{1}{\eta}}{1-\alpha}\left[\frac{\bar{C}}{\bar{Y}}v_{CG} + \frac{\bar{I}}{\bar{Y}}v_{KG}\frac{1}{\delta} + \frac{\bar{G}}{\bar{Y}}\right]\check{G}_t
\end{aligned}$$

By comparing the coefficients on $\check{K}_{t-1}$ and $\check{G}_t$ in both equations, we obtain 4 equations

$$v_{CK}v_{KK} - v_{CK} = \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)}\left[\frac{\bar{C}}{\bar{Y}}v_{CK}v_{KK} + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)v_{KK}\frac{1}{\delta} - v_{KK}\right]$$

$$
\begin{aligned}
v_{CK}v_{KG} + v_{CG}\rho_g - v_{CG} = \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + (1-\delta)} \Bigg[& \frac{\bar{C}}{\bar{Y}} v_{CK}v_{KG} + \frac{\bar{C}}{\bar{Y}} v_{CG}\rho_g \\
& + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)v_{KG}\frac{1}{\delta} + \frac{\bar{I}}{\bar{Y}} v_{KG}\rho_g \frac{1}{\delta} + \frac{\bar{G}}{\bar{Y}}\rho_g - v_{KG} \Bigg]
\end{aligned}
$$

$$
v_{CK} - \left(\frac{1}{\eta} + 1\right)\frac{\alpha}{1-\alpha} = \frac{-\alpha - \frac{1}{\eta}}{1-\alpha}\left[\frac{\bar{C}}{\bar{Y}} v_{CK} + \frac{\bar{I}}{\bar{Y}}(v_{KK} - 1 + \delta)\frac{1}{\delta}\right]
$$

$$
v_{CG}\check{G}_t = \frac{-\alpha - \frac{1}{\eta}}{1-\alpha}\left[\frac{\bar{C}}{\bar{Y}} v_{CG} + \frac{\bar{I}}{\bar{Y}} v_{KG}\frac{1}{\delta} + \frac{\bar{G}}{\bar{Y}}\right]\check{G}_t
$$

I will now use NLsolve to solve for the 4 unknown coefficients.