I. RBC with Variable Labor Supply

Table 1 compares volatilities across different parameterizations of η with what is observed in the data. Larger values of η improve the fit of the model. There is greater persistence since more elastic labor supply means shocks affect hours and wages to a greater extent. Consumption and labor supply, however, remain excessively smooth. The classical RBC model implies a greater Frisch elasticity based on the data.

Table 1: Comparing second moments of output, consumption, and labor supply

	Data	$\eta = 0.5$	$\eta = 1$	$\eta = 2$
Consumption	1.27	0.97	1.03	1.09
Output	1.72	1.56	1.67	1.79
Hours	1.59	0.24	0.41	0.60

II. RBC with Variable Capital Utilization

The firm solves the profit maximization problem.

$$\max_{\{N_t, I_t, U_t\}} \mathrm{E} \sum_{t=0}^{\infty} \prod_{s=0}^{t} (1 + r_t)^{-1} (Y_t - N_t w_t - I_t)$$
$$Y_t = (U_t K_t)^\alpha (Z_t N_t)^{1-\alpha}$$
$$K_{t+1} = (1 - \delta(U_t))K_t + I_t$$

We equalize prices of capital and the consumption good as in equilibrium.

1. The first order conditions are given by

$$\begin{aligned}
\text{Labor demand: } & w_t = (1-\alpha) Z_t^{1-\alpha} \left(\frac{U_t K_t}{N_t} \right)^\alpha \\
\text{Shadow value of capital: } & q_t = 1 \\
\text{Euler equation: } & \mathrm{E}(1 + r_t)^{-1} \left(\alpha U_{t+1}^\alpha \left(\frac{K_{t+1}}{Z_{t+1} N_{t+1}} \right)^{\alpha - 1} + q_{t+1}(1 - \delta_{t+1}) \right) \\
\text{Utilization: } & \alpha K_t^\alpha \left(\frac{U_t}{Z_t N_t} \right)^{\alpha - 1} = q_t \delta'(U_t) K_t
\end{aligned}$$

$$1 + r_t = \alpha U_{t+1}^\alpha \left(\frac{K_{t+1}}{Z_{t+1} N_{t+1}} \right)^{\alpha - 1} + 1 - \delta(U_t)$$

   Rental rate depends on the marginal product of capital and depreciation which are themselves both dependent on U_t.

2. Utilization satisfies $\alpha K_t^\alpha \left(\frac{U_t}{Z_t N_t}\right)^{\alpha-1} = q_t \delta'(U_t) K_t$.

$$\ln \alpha + \alpha \ln K_t + (\alpha - 1)(\ln U_t - \ln Z_t - \ln N_t) = \ln q_t + \ln \delta'(U_t) + \ln K_t$$

$$\ln \alpha + (\alpha - 1)(\ln K_t + \ln U_t - \ln Z_t - \ln N_t) = \ln q_t + \ln \delta'(U_t)$$

$$(\alpha - 1)(\check{K}_t + \check{U}_t - \check{Z}_t - \check{N}_t) = \check{q}_t + \frac{\delta''(\bar{U})}{\delta'(\bar{U})}(U_t - \bar{U})$$

$$(\alpha - 1)(\check{K}_t + \check{U}_t - \check{Z}_t - \check{N}_t) = \check{q}_t + \frac{\delta''(\bar{U})\bar{U}}{\delta'(\bar{U})}\check{U}_t = \check{q}_t + \Delta$$

$$\check{U}_t = \frac{\check{q}_t + \Delta}{\alpha - 1}(\check{Z}_t + \check{N}_t - \check{K}_t) = \frac{1}{1 + \Delta}(\check{Y}_t - \check{K}_t)$$

The final equality uses $\check{q}_t = 0$ from the first order condition and log linearized production.

3. We can use the previous derivation to reduce $\check{Y}_t$ as a function of technology and inputs.

$$\begin{aligned}
\check{Y}_t &= \alpha(\check{U}_t + \check{K}_t) + (1-\alpha)(\check{Z}_t + \check{N}_t) \\
&= \frac{\alpha}{1+\Delta}(\check{Y}_t - \check{K}_t) + \alpha\check{K}_t + (1-\alpha)(\check{Z}_t + \check{N}_t) \\
\frac{1-\alpha+\Delta}{1+\Delta}\check{Y}_t &= \frac{\alpha\Delta}{1+\Delta}\check{K}_t + (1-\alpha)(\check{Z}_t + \check{N}_t) \\
\check{Y}_t &= \frac{\alpha\Delta}{1-\alpha+\Delta}\check{K}_t + \frac{(1-\alpha)(1+\Delta)}{1-\alpha+\Delta}(\check{Z}_t + \check{N}_t)
\end{aligned}$$

Δ governs the sensitivity of $\check{U}_t$ to the marginal rate of capital. The limiting case where $\Delta \to \infty$ is the standard (linearized) model and $\check{U}_t$ is fixed at full utilization. The limit $\delta \to 0$ represents the case of no utilization so that $\check{Y}_t$ depend solely on technology and labor.

4. The linearized labor demand function is $\check{w}_t = \check{Y}_t - \check{N}_t$. Substituting the expression for $\check{Y}_t$,

$$\check{w}_t = \frac{\alpha\Delta}{1-\alpha+\Delta}\check{K}_t + \frac{(1-\alpha)(1+\Delta)}{1-\alpha+\Delta}(\check{Z}_t + \check{N}_t) - N_t$$

We can obtain an upward sloping demand function if labor exhibits increasing returns to scale.

$$\begin{gathered}
\frac{(1-\alpha)(1+\Delta)}{1-\alpha+\Delta} > 1 \\
(1-\alpha)(1+\Delta) > 1-\alpha+\Delta \\
1-\alpha+\Delta-\Delta\alpha > 1-\alpha+\Delta \\
-\Delta\alpha > 1
\end{gathered}$$

Since $\Delta, \alpha > 0$, indeterminacy is impossible in this model. One way to achieve indeterminacy is to incorporate positive production externalities so that aggregate labor has increasing returns to scale. A model with endogenous capital utilization is more likely to exhibit indeterminacy because it amplifies the importance of labor in production.

III. Macroeconomics of Home Production

The household solves

$$\max_{c_m, c_h, L_h, L_m} (C_m^\rho + C_h^\rho)^{\frac{1}{\rho}} - \left(\frac{1}{\eta} + 1\right)^{-1} (L_h + L_m)^{\frac{1}{\eta}+1}$$
$$\text{subject to } C_m \leq wL_m, C_h \leq L_h$$

The prices of consumption goods are equivalent.

1. The first order conditions are

$$\mathcal{L} = (C_m^\rho + C_h^\rho)^{\frac{1}{\rho}} - \left(\frac{1}{\eta} + 1\right)^{-1} (L_h + L_m)^{\frac{1}{\eta}+1} + \lambda(wL_m - C_m) + \xi(L_h - C_h)$$
$$\lambda = (C_m^\rho + C_h^\rho)^{\frac{1-\rho}{\rho}} C_m^{\rho-1}$$
$$\xi = (C_m^\rho + C_h^\rho)^{\frac{1-\rho}{\rho}} C_h^{\rho-1}$$
$$(L_h + L_m)^{\frac{1}{\eta}} = \lambda w = \xi$$

2. $w = \frac{\xi}{\lambda}$. Wage is the ratio of marginal utilities of market and home goods.

3. $\lambda \frac{C_m}{C_h}^{1-\rho} = \xi$. The ratio of marginal utilities depends on the share of consumption.

4. $C_h = L_h = C_m w^{\frac{-1}{1-\rho}}$. The share of consumption depends on the relative wage.

5. Begin with $(L_h + L_m)^{\frac{1}{\eta}} = \lambda w$.

$$\begin{aligned} L_m &= (\lambda w)^\eta - L_h \\ &= (\lambda w)^\eta - C_m w^{\frac{-1}{1-\rho}} \\ &= (\lambda w)^\eta - L_m w^{\frac{-\rho}{1-\rho}} \\ &= (\lambda w)^\eta (1 + w^{\frac{-\rho}{1-\rho}})^{-1} \end{aligned}$$

6. Differentiate the above expression.

$$\frac{\partial L_m}{\partial w} = \frac{(1 + W^{\frac{-\rho}{1-\rho}})\lambda^\eta \eta W^{\eta-1} - (\lambda W)^\eta (\frac{-\rho}{1-\rho}) W^{\frac{-\rho}{1-\rho}-1}}{(1 + W^{\frac{-\rho}{1-\rho}})^2}$$

$$\varepsilon_m = \frac{\partial L_m}{\partial w}\frac{w}{L_m} = \frac{(1 + W^{\frac{-\rho}{1-\rho}})\eta - (\frac{-\rho}{1-\rho}) W^{\frac{-\rho}{1-\rho}}}{(1 + W^{\frac{-\rho}{1-\rho}})}$$
$$= \eta + \left(\frac{\rho}{1-\rho}\right)\left(\frac{W^{\frac{-\rho}{1-\rho}}}{1 + W^{\frac{-\rho}{1-\rho}}}\right)$$

7. Observing labor supply volatilities, the model with home production implies η weakly smaller than in the classical RBC because that model does not account for shifting "employment" in home production. With high elasticity of substitution, $\varepsilon_m \to \infty$ and households will more readily substitute market employment with home production. If goods are complements, ε_m approaches the Frisch elasticity η.

8. Using $\left(C_m^\rho + C_h^\rho\right)^{\frac{1}{\rho}-1} C_m^{\rho-1} = \lambda$ we can substitute the budget constraints and $L_h = L_m W^{\frac{\rho}{\rho-1}}$ to get

$$((WL_m)^\rho + L_h^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1} = \lambda$$

$$((WL_m)^\rho + (W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1} = \lambda$$

Now the previous expression for L_m becomes

$$L_m = \frac{\left[((WL_m)^\rho + (W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

$$L_m = \frac{\left[((W^\rho + W^{\frac{\rho^2}{\rho-1}})L_m^\rho)^{\frac{1}{\rho}-1}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

$$L_m = \frac{\left[(W^\rho + W^{\frac{\rho^2}{\rho-1}})^{\frac{1}{\rho}-1}L_m^{1-\rho}(WL_m)^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

$$L_m = \frac{\left[(W^\rho + W^{\frac{\rho^2}{\rho-1}})^{\frac{1}{\rho}-1}W^{\rho-1}\right]^\eta W^\eta}{(1+W^{\frac{\rho}{\rho-1}})}$$

$$L_m = \left(1+W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-1} W^\eta$$

9. Differentiate the expression.

$$\frac{\partial L_m}{\partial W} = \eta W^{\eta-1}\left(1+W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-1} + W^\eta\left(\eta\left(\frac{1-\rho}{\rho}\right)-1\right)\left(1+W^{\frac{\rho}{\rho-1}}\right)^{\eta\left(\frac{1-\rho}{\rho}\right)-2}\left(\frac{\rho}{\rho-1}\right)W^{\frac{1}{\rho-1}}$$

$$\begin{aligned}
\frac{\partial L_m}{\partial W}\frac{W}{L_m} &= \eta + \left(\eta\left(\frac{1-\rho}{\rho}\right)-1\right)\left(\frac{\rho}{\rho-1}\right)\frac{W^{\frac{\rho}{\rho-1}}}{1+W^{\frac{\rho}{\rho-1}}} \\
&= \eta + \left(\frac{\rho}{1-\rho}-\eta\right)\frac{W^{\frac{\rho}{\rho-1}}}{1+W^{\frac{\rho}{\rho-1}}} \\
&= \eta\left(1-\frac{W^{\frac{\rho}{\rho-1}}}{1+W^{\frac{\rho}{\rho-1}}}\right) + \frac{\rho}{1-\rho}\frac{W^{\frac{\rho}{\rho-1}}}{1+W^{\frac{\rho}{\rho-1}}}
\end{aligned}$$

This elasticity has a similar interpretation to the labor elasticity previously derived. The current elasticity is smaller because we control for wealth effects; it only reflects labor responses to relative marginal utilities. In other words, these are the elasticities for the uncompensated and compensated labor supply functions, respectively.

10. Higher elasticity of substitution ($\rho \to 1$) results in higher "total" elasticity which can help explain large labor supply volatilities without an infeasibly high Frisch elasticity.

IV. q-Theory with Variable Capital Utilization

The firm solves the profit maximization problem.

$$\max_{\{L_t, I_t, U_t\}} \mathrm{E} \sum_{t=0}^{\infty} \prod_{s=0}^{t} (1 + r_t)^{-1} (Y_t - L_t w_t - I_t)$$

$$Y_t = Z_t (U_t K_t)^{\alpha} L_t^{1-\alpha}$$

$$K_{t+1} = (1 - \delta(U_t)) K_t + I_t$$

We equalize prices of capital and the consumption good as in equilibrium.

V. Fiscal Multipliers in the RBC Model