Lecture notes for ECON-210C
Business cycles

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[1]These notes build on Yuriy Gorodnichenko's notes for Econ 202B at UC Berkeley. I am grateful for his help and dedication and I aspire to meet the high standards he has set.

Contents

Chapter 1

Basic real business cycle model

We begin our analysis of business cycles by considering an elementary economy perturbed by shocks to productivity. This elementary model, constructed in the spirit of the seminal paper by Kydland and Prescott (1982), is obviously unrealistic. However, this model is a useful benchmark for more sophisticated models which we will consider later. Our objective now is to have an understanding of how variables behave in response to shocks, what basic mechanisms are involved, and how to solve/analyze dynamic models.

In this basic model, we assume that the economy is populated by a representative firm and a representative household and that factor and product markets are competitive and clear at all times.

1.1 Household

Households maximize expected utility subject to budget constraints:

$$\begin{aligned} \max \quad & \mathbb{E}_t \sum_s \beta^s \big(\ln C_{t+s} - \frac{L_{t+s}^{1+1/\eta}}{1+1/\eta} \big) \\ \text{s.t.} \quad & A_{t+s} + C_{t+s} = (1 + R_{t+s}) A_{t+s-1} + W_{t+s} L_{t+s} \end{aligned} \tag{1.1}$$

$$lim_{s \to +\infty} \Big(\prod_{k=1}^{s} (1 + R_{t+k}) \Big)^{-1} A_{t+s} = 0 \tag{1.2}$$

where C_t is consumption, A_t is asset holding, W_t is wages, L_t is labor services supplied by the household, R_t is the return on assets. We assume log utility in consumption to simplify solution and to match the fact that labor supply was insensitive to long-term growth in income. The operator $\mathbb{E}_t$ denotes the household's expectations at time t. We will use rational expectations for most of this course, so $\mathbb{E}_t f(x_{t+1}) = \int f(x_{t+1}) dF(x_{t+1})$. The constraint (1.1) is the budget constraint. The constraint (1.2) is the transversality condition, also known as the "No Ponzi game" condition. It ensures that the debt of the household does not grow exponentially over time.

My timing convention is that $t+s$ dated variables are chosen at $t+s$. Thus, A_{t+s} are asset holdings chosen at time $t+s$ that are then carried into the next period.

The Lagrangian of the problem is:

$$\mathcal{L} = \mathbb{E}_t \sum_s \beta^s \Big(\ln C_{t+s} - \frac{L_{t+s}^{1+1/\eta}}{1+1/\eta} + \lambda_{t+s}(-A_{t+s} - C_{t+s} + (1 + R_{t+s})A_{t+s-1} + W_{t+s}L_{t+s}) \Big)$$

λ_t is the Lagrange multiplier of the budget constraint, and it corresponds to the marginal utility of wealth. The first order conditions (FOCs) are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{1}{C_t} - \lambda_t = 0 \tag{1.3}$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = -L_t^{1/\eta} + \lambda_t W_t = 0 \tag{1.4}$$

$$\frac{\partial \mathcal{L}}{\partial A_t} = -\lambda_t + \mathbb{E}_t \beta (1 + R_{t+1}) \lambda_{t+1} = 0 \tag{1.5}$$

The labor supply equation is derived from equation (1.4):

$$L_t = (\lambda_t W_t)^\eta \tag{1.6}$$

$\frac{\partial \ln L_t}{\partial \ln W_t}\Big|_{\lambda=\text{const.}} = \eta$ is the **Frisch labor supply elasticity**. Using $\lambda_t = \frac{1}{C_t}$ we can eliminate

λ and simplify the FOCs:

$$L_t^{1/\eta} = \frac{W_t}{C_t} \tag{1.7}$$

$$\frac{1}{C_t} = \beta \mathbb{E}_t (1 + R_{t+1}) \frac{1}{C_{t+1}} \tag{1.8}$$

Equation (1.7) is an *intratemporal* condition linking consumption and labor supply. Equation (1.8) is the Euler equation. The Euler equation is an *intertemporal* condition which governs the allocation of resources over time.

1.2 Firm

Households own firms. Firms own capital. The price of the good produced by firms is normalized at 1. Firms maximize the present value of the cash flow $CF_{t+s} = Y_{t+s} - W_{t+s}L_{t+s} - I_{t+s}$

$$\max \quad \mathbb{E}_t \sum_s \left(\prod_{k=1}^{s} (1 + R_{t+k}) \right)^{-1} CF_{t+s} \tag{1.9}$$

subject to capital accumulation constraint

$$K_{t+s} = (1 - \delta) K_{t+s-1} + I_{t+s}. \tag{1.10}$$

where δ is the rate of depreciation of physical capital. The production function is:

$$Y_{t+s} = Z_{t+s} K_{t+s-1}^{\alpha} L_{t+s}^{1-\alpha} \tag{1.11}$$

Note that capital K is dated by time index $t-1$. This timing highlights that capital is a predetermined variable at time t. Recall that time t-dated variable are chosen at time t, so K_t is used in production at time $t+1$.

The Lagrangian of the problem is:

$$\mathcal{L} = \mathbb{E}_t \sum_s \big(\prod_{k=1}^{s}(1+R_{t+k})\big)^{-1}\big(Z_{t+s}K_{t+s-1}^{\alpha}L_{t+s}^{1-\alpha} - W_{t+s}L_{t+s} - I_{t+s} + q_{t+s}(-K_{t+s} + (1-\delta)K_{t+s-1} + I_{t+s})\big)$$

The costate variable q_t is nothing else but Tobin's Q, i.e. the shadow value of capital. The FOCs are:

$$\frac{\partial \mathcal{L}}{\partial L_t} = (1-\alpha)Z_tL_t^{-\alpha}K_{t-1}^{\alpha} - W_t = 0 \tag{1.12}$$

$$\frac{\partial \mathcal{L}}{\partial I_t} = -1 + q_t = 0 \tag{1.13}$$

$$\frac{\partial \mathcal{L}}{\partial K_t} = -q_t + \mathbb{E}_t\frac{1}{(1+R_{t+1})}\big(\alpha Z_{t+1}L_{t+1}^{1-\alpha}K_t^{\alpha-1} + q_{t+1}(1-\delta)\big) = 0 \tag{1.14}$$

To see why q_t captures the value of capital, solve the last equation forward:

$$q_t = \sum_{s=1}^{\infty}\mathbb{E}_t\frac{(1-\delta)^{s-1}}{\prod_{k=1}^{s}(1+R_{t+k})}\alpha Z_{t+s}L_{t+s}^{1-\alpha}K_{t+s-1}^{\alpha-1} \tag{1.15}$$

This is the expected discounted extra output produced from an extra unit of capital K_t.

Note that in this model there is no capital adjustment costs and hence $q_t = 1$ at all times. (The second equation.) This is an arbitrage condition. Because the marginal cost of an extra unit of capital is always 1 unit of output, at the optimum its marginal benefit also has to be equal to 1. Hence we can combine equations (1.13) and (1.14) to get,

$$\alpha Z_{t+1}L_{t+1}^{1-\alpha}K_t^{\alpha-1} = \delta + R_{t+1} \tag{1.16}$$

(dropping the expectations operator for now.)

Note that $MPK_t = \alpha Y_t/K_{t-1} = \alpha Z_tL_t^{1-\alpha}K_{t-1}^{\alpha-1}$ is the **marginal product of capital** (MPK). Hence, equation (1.16) says that the marginal product on capital is equal to the cost of using capital $\delta + R_{t+1}$ which is the *gross* return of capital since it also includes the term due to depreciation of capital (δ).

Equation (1.12) gives the demand for labor:

$$MPL_t = (1-\alpha)Y_t/L_t = (1-\alpha)Z_t L_t^{-\alpha} K_{t-1}^{\alpha} = (1-\alpha)Y_t/L_t = W_t \tag{1.17}$$

MPL_t is the **marginal product of labor** (MPL). The firm hires workers until the MPL equals the market wage. Note that because output has decreasing returns to scale in labor, the MPL (and hence the labor demand curve) is downward sloping in labor.

1.3 Competitive Equilibrium

A competitive equilibrium is a stochastic process for quantities $\{C_t, L_t, L_t^d, L_t^s, A_t, K_t, I_t, Y_t, Z_t\}$ and prices $\{W_t, R_t, q_t\}$ such that households solve their utility-maximization problem (equations (1.7) and (1.8)); firms solve their profit-maximization problem (equations (1.14), (1.12), (1.13) and (1.16)); the technological constraints (1.10) and (1.11) are satisfied; and the capital, labor and goods markets clear, i.e., the following conditions are satisfied:

$$L_t^s = L_t^d = L_t \tag{1.18}$$

$$A_t = K_t \tag{1.19}$$

$$Y_t = C_t + I_t \tag{1.20}$$

Equation (1.18) is the labor market equilibrium: labor supply (L_t^s) equals labor demand (L_t^d). Equation (1.19) is the capital market equilibrium saying that the value of firms is equal to the asset holdings of households. Equation (1.20) is the goods market equilibrium. Since $I_t \equiv K_t - (1-\delta)K_{t-1}$, equilibrium in the goods market can be rewritten: $Y_t = C_t + K_t - (1-\delta)K_{t-1}$. Only two of the three conditions above are necessary according to Walras' Law. Also note that the economy is closed and there is no government spending.

Finally, we assume that technology evolves according to the following exogenous process:

$$Z_t = Z_{t-1}^{\rho} \exp(\epsilon_t), \quad \rho < 1 \tag{1.21}$$

where ϵ_t is an i.i.d. zero-mean random variable. Since our focus is business cycles, we ignore that output/productivity can grow over time.

1.4 Recap

We can now summarize the economy with the following system of equations:

$$
\begin{aligned}
C_t L_t^{1/\eta} &= (1-\alpha) Y_t / L_t = W_t \\
\frac{C_{t+1}}{C_t} &= \beta(1 + R_{t+1}) = \beta(\alpha Y_{t+1}/K_t + (1-\delta)) \\
Y_t &= Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} \\
Y_t &= C_t + I_t \\
K_t &= (1-\delta) K_{t-1} + I_t \\
Z_t &= Z_{t-1}^{\rho} \exp(\epsilon_t)
\end{aligned}
$$

1.5 Steady State

To understand why the system is fluctuating, we first need to find the values the system takes when it is not perturbed with shocks. The **steady state** is the state of the economy when the economy is "still". The steady-state value of variable X_t is denoted with $\bar{X}$. Using the description of the competitive equilibrium given above, we infer that the steady-state is characterized by the following equations:

$$\bar{C}\bar{L}^{1/\eta} = (1-\alpha)\bar{Y}/\bar{L} \tag{1.22}$$

$$\frac{\bar{C}}{\bar{C}} = \beta(\alpha \bar{Y}/\bar{K} + (1-\delta)) \tag{1.23}$$

$$\bar{Y} = \bar{Z}\bar{K}^{\alpha}\bar{L}^{1-\alpha} \tag{1.24}$$

$$\bar{Y} = \bar{C} + \bar{I} \tag{1.25}$$

$$\bar{K} = (1-\delta)\bar{K} + \bar{I} \tag{1.26}$$

$$\bar{Z} = \bar{Z}^{\rho} \tag{1.27}$$

Notice that $\mathbb{E}[\exp \epsilon] = 1$ because there are no shocks in steady state and $\epsilon = 0$ at all times.

We now solve for the steady-state values of the variables. Immediately, from equations (1.24), (1.26) and (1.27) we get:

$$\begin{aligned}
\bar{I} &= \delta \bar{K} \\
\bar{Z} &= 1 \\
\bar{Y} &= \bar{K}^{\alpha} \bar{L}^{1-\alpha}
\end{aligned}$$

Equation (1.23) yields:

$$\begin{aligned}
1 &= \beta(\alpha \bar{Y}/\bar{K} + (1-\delta)) \\
1 &= \beta(\alpha(\bar{K}/\bar{L})^{\alpha-1} + (1-\delta)) \\
A \equiv \bar{K}/\bar{L} &= \Big(\frac{1/\beta - (1-\delta)}{\alpha}\Big)^{\frac{1}{\alpha-1}} \\
\bar{K} &= \Big(\frac{1/\beta - (1-\delta)}{\alpha}\Big)^{\frac{1}{\alpha-1}} \bar{L}
\end{aligned}$$

where A is solely a function of the parameters of the economy. Equation (1.22) leads to:

$$\begin{aligned}
\bar{C} &= (1-\alpha)\bar{L}^{-1/\eta}(\bar{Y}/\bar{L}) \\
\bar{C} &= (1-\alpha)\bar{L}^{-1/\eta}\Big(\frac{\bar{K}}{\bar{L}}\Big)^{\alpha}
\end{aligned}$$

Next, using $\bar{Y} = \bar{C} + \delta \bar{K}$, and plugging in the expressions for $\bar{Y}$, $\bar{C}$, $\bar{K}$ as a function of $\bar{L}$, we find:

$$\Big(\frac{\bar{K}}{\bar{L}}\Big)^{\alpha} \bar{L} = (1-\alpha)\bar{L}^{-1/\eta}\Big(\frac{\bar{K}}{\bar{L}}\Big)^{\alpha} + \delta\Big(\frac{\bar{K}}{\bar{L}}\Big)\bar{L}$$

Finally, $\bar{L}$ is implicitly defined by:

$$A^{\alpha}\bar{L} = (1-\alpha)\bar{L}^{-1/\eta}A^{\alpha} + \delta A\bar{L}$$

This is a non-linear equation in $\bar{L}$ and hence it's hard to have a simple expression for $\bar{L}$. However, once $\bar{L}$ is determined, we can find steady state values for all other variables.

1.6 Log-Linearization

Non-linear systems are hard to analyze. For small perturbations, linear approximations capture the dynamics of the model well and are easy to analyze. We will linearize the system around the steady state and study its properties using the linearized version of the model in the neighborhood of the steady state. It is conventional to work with *percent* deviations from the steady state:

$$\check{X}_t = \frac{X_t - \bar{X}}{\bar{X}} = \frac{dX_t}{\bar{X}} \approx \ln(X_t/\bar{X})$$

The theory and practice of log-linearization will be covered in section. The log-linearized system of equations around the steady state will also be derived in great details in section. Let's work through two simple examples to get a flavor of it. Start with the equation defining investment:

$$\begin{aligned}
K_t &= (1-\delta)K_{t-1} + I_t \\
dK_t &= (1-\delta)dK_{t-1} + dI_t \\
\frac{dK_t}{\bar{K}} &= (1-\delta)\frac{dK_{t-1}}{\bar{K}} + \frac{dI_t}{\bar{K}} \\
\frac{dK_t}{\bar{K}} &= (1-\delta)\frac{dK_{t-1}}{\bar{K}} + \frac{\bar{I}}{\bar{K}}\frac{dI_t}{\bar{I}} \\
\check{K}_t &= (1-\delta)\check{K}_{t-1} + \delta\check{I}_t
\end{aligned}$$

where we used the steady state condition: $\frac{\bar{I}}{\bar{K}} = \delta$.

Consider now the equation describing equilibrium in the goods market:

$$\begin{aligned}
Y_t &= C_t + I_t \\
dY_t &= dC_t + dI_t \\
\frac{dY_t}{\bar{Y}} &= \frac{\bar{C}}{\bar{Y}}\frac{dC_t}{\bar{C}} + \frac{\bar{I}}{\bar{Y}}\frac{dI_t}{\bar{I}} \\
\check{Y}_t &= \frac{\bar{C}}{\bar{Y}}\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t \\
\check{Y}_t &= \left(1 - \frac{\bar{I}}{\bar{Y}}\right)\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t
\end{aligned}$$

Moreover, the ratio $\frac{\bar{I}}{\bar{Y}}$ can be determined using the steady-state equations:

$$\begin{aligned}
\frac{\bar{I}}{\bar{Y}} &= \delta\frac{\bar{K}}{\bar{Y}} \\
1 &= \beta(\alpha\bar{Y}/\bar{K} + (1-\delta))
\end{aligned}$$

Following the same strategy, the remaining equations of the log-linearized system around the steady state can be derived. The log-linearized system is:

$$\check{Y}_t = (1 - \frac{\bar{I}}{\bar{Y}})\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t \tag{1.28}$$

$$\check{K}_t = (1-\delta)\check{K}_{t-1} + \delta\check{I}_t \tag{1.29}$$

$$\check{C}_t + (1/\eta)\check{L}_t = \check{Y}_t - \check{L}_t \tag{1.30}$$

$$\mathbb{E}_t\check{C}_{t+1} - \check{C}_t = \frac{\alpha\bar{Y}/\bar{K}}{\alpha\bar{Y}/\bar{K} + (1-\delta)}\mathbb{E}_t(\check{Y}_{t+1} - \check{K}_t) \tag{1.31}$$

$$\check{Y}_t = \check{Z}_t + \alpha\check{K}_{t-1} + (1-\alpha)\check{L}_t \tag{1.32}$$

$$\check{Z}_t = \rho\check{Z}_{t-1} + \varepsilon_t \tag{1.33}$$

Note that we have put the expectations operators back into the equations. An important property of linear rational expectations models is *certainty equivalence*. This means that even though the future is uncertain ($\varepsilon_t \neq 0$ for $t > 0$), agents behave exactly *as if* they knew with certainty that there were no shocks in the future ($\varepsilon_t = 0$ for $t > 0$). For our purposes it has two convenient properties. First, we can linearize the model with certainty and then simply add expectations operators to any $t+1$-dated variables. Second, when we want to know how the model reacts to a single shock at t_0, we can drop the expectations operators after $t > t_0$.

We will use this system of equations to examine how endogenous variables in this economy respond to shocks to technology and to study other properties of the model.

1.7 Calibration

In the vast majority of cases, general equilibrium models do not have simple analytical solutions. We have to rely on numerical methods to study the properties of these models. To do the numerical analysis, we need to assign parameter values such as capital share, elasticity of labor supply, volatility and persistence of technology shocks, etc. There are several strategies to assign parameter values. The first strategy is to estimate the model using some moments of the data and later use the estimated parameters to evaluate the performance of the model using some other moments of the data. Although this approach is quite intuitive, it is hard to believe that the simple model we consider is a realistic description of the world. Clearly the model is misspecified and hence estimates can be badly biased. This critique is less appealing when we consider more sophisticated models such as Smets and Wouters (2007) and Christiano, Eichenbaum, and Evans (2005) with tens of variables, shocks, and equations.

The second strategy is to calibrate parameter values, that is, assign parameter values that are plausible from the economic standpoint. This strategy is, in some sense, less rigorous but it is also more robust to misspecification. The trick in this strategy is to judiciously pick moments in the data such that we can assign a value to a given parameter without relying on estimation of the whole model. In other words, we can split the model into blocks such that each parameter governs a particular moment. Hence, even if there is a misspecification in one equation (moment) it is less likely to spill over to other parameters/moments. This was the motivation for this strategy in the seminal paper by Kydland and Prescott (1982).

More specifically, Kydland and Prescott (1982) envisioned that the researcher should use estimates from **micro**-level studies to assign parameter values (e.g., labor studies reporting estimates of labor supply elasticity based on micro-level data) and then study the **macroeconomic** properties of the model. This ideal scenario is however not always desirable or feasible. For example, the labor supply elasticity at the macro level has two margins: extensive (= how many people work) and intensive (= how many hours a person works). On the other hand, micro-level studies report typically only the intensive margin and as result can miss an important extensive margin. Hence, in some cases, we have to rely on macroeconomic data to assign parameter values.

Note that we do not want to assign parameter values using a set of moments and then evaluate the calibrated model using the same set of moments. For example, if the model is just-identified, we'll have a perfect fit of the model and it would not be a very informative test of how successful the model is at capturing the properties of the data in general. Kydland and Prescott (1982) suggested the following approach which became the standard practice in macroeconomic analysis of business cycles. The researcher uses the **first** moments of the data (i.e., means, ratios, etc.) to assign parameter values and then uses the **second** or **higher** moments of the data (e.g., variances, autocovariances, covariances, kurtosis, skewness, etc.) to evaluate the model. In general, one can evaluate the performance of the model using moments not employed in calibration. (In the estimation strategy this typically amounts to over-identification tests and out-of-sample forecasts).

Using the first order condition for labor in the firm's optimization problem, we can pin down the value of the elasticity of output with respect to capital α. The labor share in National Income and Product Accounts (NIPA) is relatively fixed over time and equals approximately:

$$1-\alpha = \frac{WL}{Y} \approx 0.66$$

The real rate of return is somewhere between 4% and 6% (return on broad based portfolio in stock market) so an estimate of the quarterly discount factor is:

$$\beta = \Big(\frac{1}{1+\bar{R}}\Big)^{1/4} = \Big(\frac{1}{1.04}\Big)^{1/4} \approx 0.99$$

In the data (NIPA), aggregate investment to capital ratio is about 0.076. Since $\delta = \frac{\bar{I}}{\bar{K}}$, if we assume no growth, the annual depreciation rate is about 0.076. Therefore the quarterly rate is:

$$\delta = (1 - 0.076)^{(1/4)} \approx \frac{1}{4}(0.076) \approx 2\%$$

The labor supply elasticity η is an important parameter, but it is very hard to estimate. η has been estimated using micro-level data. There is a wide range of estimates, but studies suggest that the elasticity of labor supply is low, somewhere between 0 and 0.5. However, for macroeconomic models to generate any interesting dynamics, you need to have a high labor supply elasticity (we'll discuss later why but for now hours or work are as volatile as output and intuitively we need to have a high labor supply elasticity to capture this fact). Also this low elasticity is likely to reflect the intensive margin of the labor supply rather than the intensive + extensive margins of the labor supply.

To compute the volatility and the parameters of technology shocks, we can use a measure of Solow residual and fit an AR(1) autoregressive process . The parameters obtained are usually $\rho \approx 1$ and $\sigma_\epsilon = 0.007$. We will use $\rho = 0.95$.

Let's check the consistency of both the estimates and the model. In the steady state of the model (see equation (1.23)):

$$\beta = \frac{1}{\alpha \frac{Y}{K} + (1-\delta)}$$

We estimated $\delta = 0.076$ and $\alpha = 0.33$. In the data, the capital output ratio is about 3.32. This yields an estimate of β: $\beta \approx \frac{1}{.33\frac{1}{3.33}+(1-0.076)} = 0.976$. So the quarterly discount factor obtained is $\beta^{1/4} = 0.994$, which is consistent with the estimate obtained above using the real rate of return.

1.8 Dynamic Solution

Recall from the previous lecture that our log-linearized model is described by the system of equations (1.28)-(1.33). We will use the **method of undetermined coefficients** to solve the model. To simplify the solution, suppose that labor is supplied inelastically and assume that the labor supply is 1. We can reduce the system to a two-variable analogue (two endogenous variables):

$$\begin{aligned}
\check{K}_t &= (1-\delta)\check{K}_{t-1} + \delta \check{I}_t \\
&= (1-\delta)\check{K}_{t-1} + \delta\frac{\bar{Y}}{\bar{I}}(\check{Y}_t - \frac{\bar{C}}{\bar{Y}}\check{C}_t) \\
&= (1-\delta)\check{K}_{t-1} + \delta\frac{\bar{Y}}{\bar{I}}(\check{Z}_t + \alpha\check{K}_{t-1} - \frac{\bar{C}}{\bar{Y}}\check{C}_t) \\
&= \Big((1-\delta) + \delta\alpha\frac{\bar{Y}}{\bar{I}}\Big)\check{K}_{t-1} + \delta\frac{\bar{Y}}{\bar{I}}\check{Z}_t - \delta\frac{\bar{C}}{\bar{I}}\check{C}_t \\
&= \Big((1-\delta) + 1/\beta - (1-\delta)\Big)\check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Z}_t - \delta\frac{\bar{C}}{\bar{I}}\check{C}_t \\
&= 1/\beta\check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Z}_t - \frac{\bar{C}}{\bar{K}}\check{C}_t
\end{aligned}$$

Next:

$$\begin{aligned}
\mathbb{E}_t\check{C}_{t+1} - \check{C}_t &= \frac{\alpha\bar{Y}/\bar{K}}{\alpha\bar{Y}/\bar{K} + (1-\delta)}\mathbb{E}_t(\check{Y}_{t+1} - \check{K}_t) \\
&= \frac{\alpha\bar{Y}/\bar{K}}{\alpha\bar{Y}/\bar{K} + (1-\delta)}\mathbb{E}_t(\check{Z}_{t+1} + \alpha\check{K}_t - \check{K}_t) \\
&= \frac{\alpha\bar{Y}/\bar{K}}{\alpha\bar{Y}/\bar{K} + (1-\delta)}\mathbb{E}_t(\check{Z}_{t+1} + (\alpha-1)\check{K}_t) \\
&= \frac{1/\beta - (1-\delta)}{1/\beta - (1-\delta) + (1-\delta)}\mathbb{E}_t(\check{Z}_{t+1} + (\alpha-1)\check{K}_t) \\
&= (1-\beta(1-\delta))\mathbb{E}_t(\check{Z}_{t+1} + (\alpha-1)\check{K}_t)
\end{aligned}$$

Thus, we obtain the following system:

$$
\begin{aligned}
\check{K}_t &= 1/\beta \check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Z}_t - \frac{\bar{C}}{\bar{K}}\check{C}_t && (1.34)\\
\mathbb{E}_t\check{C}_{t+1} - \check{C}_t &= (1-\beta(1-\delta))\mathbb{E}_t(\check{Z}_{t+1} + (\alpha-1)\check{K}_t) && (1.35)\\
\check{Z}_t &= \rho\check{Z}_{t-1} + \varepsilon_t && (1.36)
\end{aligned}
$$

In this system, there are two state variables: capital K and level of technology Z. From the optimal control theory, we know that the solution should take the following form:

$$
\begin{aligned}
\check{C}_t &= \nu_{ck}\check{K}_{t-1} + \nu_{cz}\check{Z}_t && (1.37)\\
\check{K}_t &= \nu_{kk}\check{K}_{t-1} + \nu_{kz}\check{Z}_t && (1.38)
\end{aligned}
$$

Plug equations (1.37) and (1.38) into equation (1.34):

$$
\begin{aligned}
\check{K}_t &= 1/\beta \check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Z}_t - \frac{\bar{C}}{\bar{K}}[\nu_{ck}\check{K}_{t-1} + \nu_{cz}\check{Z}_t]\\
&= \Big(1/\beta - \frac{\bar{C}}{\bar{K}}\nu_{ck}\Big)\check{K}_{t-1} + \Big(\frac{\bar{Y}}{\bar{K}} - \frac{\bar{C}}{\bar{K}}\nu_{cz}\Big)\check{Z}_t
\end{aligned}
$$

By equating coefficients in both equations, we get:

$$
\begin{aligned}
\nu_{kk} &= 1/\beta - \frac{\bar{C}}{\bar{K}}\nu_{ck}\\
\nu_{kz} &= \frac{\bar{Y}}{\bar{K}} - \frac{\bar{C}}{\bar{K}}\nu_{cz}
\end{aligned}
$$

which yields the following equations, that we will use repeatedly:

$$
\begin{aligned}
\nu_{ck} &= \frac{\bar{K}}{\bar{C}}[1/\beta - \nu_{kk}] && (1.39)\\
\nu_{cz} &= \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}\nu_{kz} && (1.40)
\end{aligned}
$$

Next we turn to equation (1.35):

$$
\begin{aligned}
0 &= \mathbb{E}_t\Big[\check{C}_t - \check{C}_{t+1} + (1-\beta(1-\delta))(\check{Z}_{t+1} + (\alpha-1)\check{K}_t)\Big] \\
&= \mathbb{E}_t\Big[\nu_{ck}\check{K}_{t-1} + \nu_{cz}\check{Z}_t - \nu_{ck}\check{K}_t - \nu_{cz}\check{Z}_{t+1} \\
&\quad +(1-\beta(1-\delta))\check{Z}_{t+1} + (1-\beta(1-\delta))(\alpha-1)(\nu_{kk}\check{K}_{t-1} + \nu_{kz}\check{Z}_t)\Big] \\
&= \mathbb{E}_t\Big[\nu_{ck}\check{K}_{t-1} + \nu_{cz}\check{Z}_t - \nu_{ck}(\nu_{kk}\check{K}_{t-1} + \nu_{kz}\check{Z}_t) - \nu_{cz}\check{Z}_{t+1} \\
&\quad +(1-\beta(1-\delta))\check{Z}_{t+1} + (1-\beta(1-\delta))(\alpha-1)(\nu_{kk}\check{K}_{t-1} + \nu_{kz}\check{Z}_t)\Big] \\
&= \Big[\nu_{ck} - \nu_{ck}\nu_{kk} + (1-\beta(1-\delta))(\alpha-1)\nu_{kk}\Big]\check{K}_{t-1} + \\
&\quad \Big[\nu_{cz} - \nu_{ck}\nu_{kz} - \nu_{cz}\rho + (1-\beta(1-\delta))\rho + (1-\beta(1-\delta))(\alpha-1)\nu_{kz}\Big]\check{Z}_t
\end{aligned}
$$

This is done for all values of K_{t-1} and Z_t. Hence it must true that:

$$
0 = \nu_{ck} - \nu_{ck}\nu_{kk} + (1-\beta(1-\delta))(\alpha-1)\nu_{kk} \tag{1.41}
$$

$$
0 = \nu_{cz}(1-\rho) - \nu_{ck}\nu_{kz} + (1-\beta(1-\delta))\rho + (1-\beta(1-\delta))(\alpha-1)\nu_{kz} \tag{1.42}
$$

We start with equation (1.41):

$$
\begin{aligned}
0 &= \nu_{ck} - \nu_{ck}\nu_{kk} + (1-\beta(1-\delta))(\alpha-1)\nu_{kk} \\
0 &= \frac{\bar{K}}{\bar{C}}[1/\beta - \nu_{kk}](1-\nu_{kk}) + (1-\beta(1-\delta))(\alpha-1)\nu_{kk} \\
0 &= 1/\beta - (1/\beta+1)\nu_{kk} + \nu_{kk}^2 + \frac{\bar{C}}{\bar{K}}(1-\beta(1-\delta))(\alpha-1)\nu_{kk} \\
0 &= 1/\beta - \Big[1/\beta + 1 - \frac{\bar{C}}{\bar{K}}(1-\beta(1-\delta))(\alpha-1)\Big]\nu_{kk} + \nu_{kk}^2 \\
0 &= 1/\beta - \gamma\nu_{kk} + \nu_{kk}^2
\end{aligned}
$$

where $\gamma \equiv [1/\beta + 1 - \frac{\bar{C}}{\bar{K}}(1-\beta(1-\delta))(\alpha-1)]$. Thus:

$$
\nu_{kk} = \frac{\gamma \pm \sqrt{\gamma^2 - 4/\beta}}{2}
$$

We pick the stable root (which corresponds to the saddle path):

$$\nu_{kk} = \frac{\gamma - \sqrt{\gamma^2 - 4/\beta}}{2}$$

From ν_{kk} we can compute ν_{ck}. Next, we need to find ν_{cz} and ν_{kz}. Recall from equation (1.40) that:

$$\nu_{cz} \quad = \quad \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}\nu_{kz}$$

Using equation (1.42), we have:

$$\begin{aligned}
&\Big[\frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}\nu_{kz}\Big](1-\rho) - \frac{\bar{K}}{\bar{C}}[1/\beta - \nu_{kk}]\nu_{kz} + (1-\beta(1-\delta))\rho + (1-\beta(1-\delta))(\alpha-1)\nu_{kz} = 0 \\
&\frac{\bar{Y}}{\bar{C}}(1-\rho) + (1-\beta(1-\delta))\rho = \Big(\frac{\bar{K}}{\bar{C}}(1-\rho) + \frac{\bar{K}}{\bar{C}}(1/\beta - \nu_{kk}) - (1-\beta(1-\delta))(\alpha-1)\Big)\nu_{kz}
\end{aligned}$$

To conclude:

$$\nu_{kz} = \frac{\frac{\bar{Y}}{\bar{C}}(1-\rho) + (1-\beta(1-\delta))\rho}{\frac{\bar{K}}{\bar{C}}(1-\rho) + \frac{\bar{K}}{\bar{C}}(1/\beta - \nu_{kk}) - (1-\beta(1-\delta))(\alpha-1)}$$

Now we have a solution to the system of rational expectations equations (REE). We can use this solution to simulate the model, generate moments, and compare the model and data moments.

We can use the same technique when we add more shocks to the system. For instance, we will be interested in fiscal shocks:

$$Y = C + I + G \Rightarrow \check{Y}_t = \frac{\bar{C}}{\bar{Y}}\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t + \frac{\bar{G}}{\bar{Y}}\check{G}_t$$

where $\check{G}_t = \phi\check{G}_{t-1} + u_t$, and $u_t \sim$ i.i.d. $(0, \sigma_u^2)$ is a fiscal shock.

1.9 Blanchard and Kahn's Method

A more general approach to solving linear REE models is due to Blanchard and Kahn (1980). The idea is as follows. We can write the REE system as $X_t = AX_{t+1}$ where $X_t = [X_{1t}\ X_{2t}\ X_{3t}\ X_{4t}]$.

- X_1 are control variables
- X_2 are state variables
- X_3 are co-state variables
- X_4 are shock variables

If necessary, we can create dummy variables to write the system in the desired form. For example, consider $X_{t+1} = \alpha X_t + \beta X_{t-1}$. Let $Z_t = X_{t-1}$. Then:

$$\begin{aligned} X_{t+1} &= \alpha X_t + \beta Z_t \\ Z_{t+1} &= X_t \end{aligned}$$

Using $Y_t = [X_t\ Z_t]$, the system can be rewritten as $Y_{t+1} = AY_t$.

Let $A = PVP^{-1}$ be the eigenvalue-eigenvector decomposition of A. We will partition the matrices V into two parts, $V = [V_1, V_2]'$, where V_1 collects all eigenvalues less than 1 and V_2 collects all eigenvalues greater or equal to 1. The associates (inverse) matrix of eigenvectors is given by,

$$P^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Multiply both sides of $Y_t = AY_{t+1}$ by A^{-1} and using certainty equivalence and iterating forward:

$$\begin{bmatrix} X_{1,t+j} \\ X_{2,t+j} \end{bmatrix} = P \begin{bmatrix} V_1^j & 0 \\ 0 & V_2^j \end{bmatrix} \begin{bmatrix} P_{11}X_{1,t} + P_{12}X_{2,t} \\ P_{21}X_{1,t} + P_{22}X_{2,t} \end{bmatrix} \tag{1.43}$$

For general $X \neq 0$, the bottom part of this system will typically be explosive (or at least non-mean-reverting) since V_2 has eigenvalues greater or equal to one. This is where we use the Transversality conditions. Remember, these put an optimal limit on how fast variables can grow in the model. To ensure that our solution remains finite we need that the transversality conditions (TVCs) put n_2 restrictions on V_2, where $n_2 = \#\text{rows}(V_2)$. For instance, the transversality condition (1.2) restricts the growth rate of both assets and consumption since they are linked by the budget constraint!

Finally, note that the TVCs restrict choice variables: For instance, a consumer cannot consume so much and save so little that debt accumulates too fast. Thus another way of stating the condition that we need as many choice ("jump") variables as there are eigenvalues greater or equal to 1. This is known as the Blanchard-Kahn condition. Implicit here is that the TVC restricts these choices.

Our transversality conditions therefore imply that $X_{2,t+j} \to 0$ as $j \to +\infty$ given the initial condition $X \geq 0$. Since $V_2 > I_{n_2}$, the term multiplying V_2 must be zero. We must have:

$$\begin{aligned} 0 &= P_{21}X_{1,t} + P_{22}X_{2,t} \\ X_{2,t} &= -P_{22}^{-1}P_{21}X_{1,t} \end{aligned}$$

Turning to equation (1.43) again:

$$\begin{bmatrix} X_{1,t+j} \\ X_{2,t+j} \end{bmatrix} = P \begin{bmatrix} V_1^j & 0 \\ 0 & V_2^j \end{bmatrix} \begin{bmatrix} P_{11}X_{1,t} - P_{12}P_{22}^{-1}P_{21}X_{1,t} \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
&= P \begin{bmatrix} V_1^j(P_{11}X_{1,t} - P_{12}P_{22}^{-1}P_{21}X_{1,t}) \\ 0 \end{bmatrix} \\
&= P \begin{bmatrix} V_1^j(P_{11} - P_{12}P_{22}^{-1}P_{21}) \\ 0 \end{bmatrix} \begin{bmatrix} X_{1,t} \\ 0 \end{bmatrix}
\end{aligned}
$$

Given initial conditions on $X_{1,t}$ we can now solve the system. This implies that we need $n_1 = \#\text{rows}(V_1)$ initial conditions, or (equivalently) n_1 state variables. Thus, we can also state the Blanchard-Kahn conditions as requiring the number of state variables equal to number of eigenvalues less than 1.

An important result from this solution method is that the system converges at rate $V_1 < I_{n_1}$ back to the steady-state (zero). For instance, in our simply model $V_1 = \frac{\gamma - \sqrt{\gamma^2 - 4/\beta}}{2} < 1$ (see previous section). This implies that given some initial capital stock $\check{K}_{t-1} \neq 0$, the economy will close $1 - V_1$ of the gap, $\check{K}_t - \check{K}_{t-1} = (V_1 - 1)\check{K}_{t-1}$. Thus, the larger the initial distance from the steady-state, the greater the decline in the gap, and the larger (in an absolute sense) is investment. This formed the basis for our assertion that investment is (in an absolute sense) monotonically declining as we converge to the steady-state.

Once we have solved the autonomous system, we can add forcing variables and shocks. (Again we use the certainty-equivalence property.)

$$
X_t = AX_{t+1} + BV_{t+1}
$$

with $\mathbb{E}_t V_{t+1} = 0$. This can be rewritten as:

$$
\begin{aligned}
A^{-1}X_t &= X_{t+1} + A^{-1}BV_{t+1} \\
X_{t+1} &= A^{-1}X_t - A^{-1}BV_{t+1}
\end{aligned}
$$

Chapter 2

Comparative Statics of Equilibrium in RBC

In the previous chapter, we specified, calibrated, log-linearized, and solved the basic real business cycle model. We are almost ready to assess how it fits the economy. However, before we do this, it will be useful to get some intuition about what the model does. Also we need to specify how we are going to assess the model. Finally, we need to make predictions about how the model responds to various shocks so that we can use these predictions to test the model.

Recall the FOCs of the RBC:

$$\frac{\partial \mathcal{L}}{\partial L_t} = -L_t^{1/\eta} + \lambda_t W_t = 0 \tag{2.1}$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{1}{C_t} - \lambda_t = 0 \tag{2.2}$$

$$\frac{\partial \mathcal{L}}{\partial K_t} = -\lambda_t + \beta(1 + R_{t+1})\lambda_{t+1} = 0 \tag{2.3}$$

λ_t is the Lagrange multiplier of the budget constraint. It corresponds to the marginal utility of wealth. It is the shadow price of one unit of capital/wealth. Equation (2.2) is the FOC determining consumption. Equation (2.1) is the FOC determining labor supply. Equation

(2.3) is the Euler equation.

We start with the Euler equation (equation 2.3). Using repeated substitution, we find:

$$\begin{aligned}\lambda_t &= \mathbb{E}_t\beta(1+R_{t+1})\lambda_{t+1} = \mathbb{E}_t\beta(1+R_{t+1})(\beta(1+R_{t+2})\lambda_{t+2}) = \mathbb{E}_t\beta(1+R_{t+1})(\beta(1+R_{t+2})(\beta(1+R_{t+3})\lambda_{t+}\\ &= \lim_{s\to+\infty}\beta^s\mathbb{E}_t\Big(\Big[\prod_{i=1}^{s}(1+R_{t+i})\Big]\ \lambda_{t+s}\Big)\end{aligned}$$

Clearly λ_t is a forward looking variable. Also because λ_t depends on R_{t+i} and λ_{t+s} many periods into the future, it is reasonable to assume that in the short run, the variability of λ due to **temporary** shocks is small. In other words, because λ depends on the future stream of marginal utilities, the variation in λ can mostly depend on the distant future when the economy is close to the steady state provided that the discount factor is close to one. Hence, if shocks in our model are transitory, we can assume that λ_t is relatively stable. Obviously, when shocks are permanent, the steady state will move in response to the shocks and hence λ_t can move significantly too.

This is essentially the permanent income hypothesis. Since consumption is inversely proportional to λ_t, all our statements about λ_t directly map to consumption. Thus, consumption will move relatively little in response to temporary shocks but it can move a lot in response to permanent shock.

Now let's go back to the FOC for labor supply (equation 2.1):

$$L_t = \lambda_t^\eta W_t^\eta \tag{2.4}$$

The labor supply curve L^s is an upward-sloping function of the wage W, as defined in equation (2.4). Movements in λ_t can be interpreted as wealth shocks. Since we can treat λ_t as relatively constant with transitory shocks, it means that L^s is relatively fixed and most of the time, temporary shocks will give us movements along the labor supply curve rather shifts of the labor supply curve.

There is something important about this relationship. Barro and King (1984) used $L_t =$

$\lambda_t^\eta W_t^\eta$ to make an important assessment of how the model is going to fit the data. Note that the FOC for labor can be written more generally as:

$$U'(C_t)W_t = V'(L_t) \tag{2.5}$$

where $V'(L_t)$ is the marginal disutility of labor (in our specification $V(L_t) = \frac{L_t^{1+1/\eta}}{1+1/\eta}$ and hence $V'(L_t) = L_t^{1/\eta}$). Equation (2.5) is the intratemporal condition linking consumption to supply of labor, and it should hold in every point in time. We know that in the business cycle, consumption and labor supply comove. $C \nearrow$ means that $U'(C) \searrow$. $L \nearrow$ means that $V(L) \nearrow$. Therefore W **has to be very procyclical**. This is because consumption and leisure are complementary goods, which means that consumption and labor supply are substitutes. Thus, consumption and leisure will typically move in the same direction (and consumption and labor supply in opposite directions), unless the price of leisure (the real wage) moves a lot.

In contrast, wages in the data are only weakly procyclical. Hence, in this bare-bones formulation of a closed economy, the fit to the data is likely to have problems with matching weak procyclicality of wages and we need to think more carefully about potential ways to alleviate this problem. There are a few caveats however as this result relies on:

1. time separability of utility function;

2. "representative customer" paradigm;

3. free choice of consumption and labor during the cycle.

From firms' optimization, we had:

$$W_t = MPL_t = (1-\alpha)Z_t L_t^{-\alpha} K_{t-1}^{\alpha} = (1-\alpha)Z_t\left(\frac{K_{t-1}}{L_t}\right)^{\alpha} \tag{2.6}$$

Equation (2.6) defines a downward-sloping labor-demand curve L^d. Labor market equilibrium implies that:

$$(1-\alpha)Z_t\Big(\frac{K_{t-1}}{L_t}\Big)^{\alpha} = W_t = \frac{L_t^{1/\eta}}{\lambda_t} = L_t^{1/\eta}C_t \tag{2.7}$$

Equation (2.7) implies that holding K and Z constant, there is a negative relationship between employment and consumption. That is, $C \nearrow$ when $L \searrow$.

Equilibrium in the goods market implies:

$$C + I + G = Y = Z_t L_t\Big(\frac{K_{t-1}}{L_t}\Big)^{\alpha}$$

This condition will help us to sign changes in variables in response to shocks.

Finally, we can observe a useful relationship between wages and interest rate from the first order conditions for inputs:

$$\begin{aligned} W_t &= MPL_t = (1-\alpha)Z_t\Big(\frac{K_{t-1}}{L_t}\Big)^{\alpha} \\ R_t + \delta &= MPK_t = \alpha Z_t\Big(\frac{K_{t-1}}{L_t}\Big)^{(\alpha-1)} \end{aligned}$$

Note that the interest rate and wages are pinned down by two variables: the level of technology Z_t and the ratio of capital to labor. An increase in the capital-to-labor ratio will increases wages and reduces interest. This trade-off is called the factor price possibility frontier (FPPF). Changes in the capital-to-labor ratio correspond to movements along the frontier. Changes in technology shift the frontier. For example, an increase in Z_t allows firms to pay higher wages and interest rates which corresponds to an upward shift of the frontier.

FPPF can provide further insight about the behavior of wages and interest rate in the long run. We know from the Euler equation that $C_{t+1}/C_t = \beta(1+R_{t+1})$. Since consumption is fixed in the steady state, the left hand side of this equation is equal to one in the steady state and therefore the steady-state interest rate is effectively set by the discount factor β. It follows that while shifts of the FPPF due to changes in technology can result in temporary changes of the interest rate, these shifts have no effect the interest rate in the steady state. Furthermore, any changes in technology will be absorbed by wages.

2.1 Derivation of Phase Diagram

Our model is in discrete time which is not very convenient for phase diagrams but it is convenient later when we solve the model and simulate it on a computer. Another issue with the discrete time formulation is that we can have a few timing inconsistencies which do not arise in continuous time. Since the purpose of this exercise is to get intuition rather than get quantitative answers, we will ignore these issues. We will also focus on the linearized version of the model. The advantage of working with the linearized version is that we can locally (that is, in the neighborhood of the steady state) determine the signs of various loci and arms.

Relative to our baseline model, we introduce at this point more explicitly government expenditures G_t. Similar to previous instances of having G_t in the model, we will assume that consumers do not derive the utility from G_t (for example, G_t cannot be a public good such as national defense, radio broadcasting, navigation systems, etc.) and that G_t does not have any productive use (e.g., it does not raise the level of technology via research and development such as the Strategic Defence Initiative or Manhattan Project).

Since the phase diagram will have only two variables, we need to reduce the system of equations to a two-equation system. Using capital accumulation equation and goods market equilibrium, we can eliminate investment from the model and express the law of motion for capital as a function of output, consumption, government expenditures:

$$K_t = (1-\delta)K_{t-1} + I_t \Rightarrow \check{K}_t = (1-\delta)\check{K}_{t-1} + \delta\check{I}_t \tag{2.8}$$

Likewise,

$$
\begin{aligned}
Y_t &= I_t + C_t + G_t \Rightarrow \\
\check{Y}_t &= \frac{\bar{I}}{\bar{Y}}\check{I}_t + \frac{\bar{C}}{\bar{Y}}\check{C}_t + \frac{\bar{G}}{\bar{Y}}\check{G}_t \Rightarrow \\
\check{I}_t &= \frac{\bar{Y}}{\bar{I}}\check{Y}_t - \frac{\bar{C}}{\bar{I}}\check{C}_t - \frac{\bar{G}}{\bar{I}}\check{G}_t \Rightarrow \\
\check{I}_t &= \frac{1}{\delta}\frac{\bar{Y}}{\bar{K}}\check{Y}_t - \frac{1}{\delta}\frac{\bar{C}}{\bar{K}}\check{C}_t - \frac{1}{\delta}\frac{\bar{G}}{\bar{K}}\check{G}_t
\end{aligned} \tag{2.9}
$$

Plugging equation (2.9) into equation (2.8), we obtain:

$$
\begin{aligned}
\check{K}_t &= (1-\delta)\check{K}_{t-1} + \delta\big(\frac{1}{\delta}\frac{\bar{Y}}{\bar{K}}\check{Y}_t - \frac{1}{\delta}\frac{\bar{C}}{\bar{K}}\check{C}_t - \frac{1}{\delta}\frac{\bar{G}}{\bar{K}}\check{G}_t\big) \\
&= (1-\delta)\check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Y}_t - \frac{\bar{C}}{\bar{K}}\check{C}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t
\end{aligned} \tag{2.10}
$$

Now combine equations to get:

$$
\begin{aligned}
\Delta\check{K}_t &\equiv \check{K}_t - \check{K}_{t-1} = \frac{\bar{Y}}{\bar{K}}\check{Y}_t - \frac{\bar{C}}{\bar{K}}\check{C}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t - \delta\check{K}_{t-1} \\
&= \frac{\bar{Y}}{\bar{K}}(\check{Z}_t + (1-\alpha)\check{L}_t + \alpha\check{K}_{t-1}) + \frac{\bar{C}}{\bar{K}}\check{\lambda}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t - \delta\check{K}_{t-1} \\
&= \frac{\bar{Y}}{\bar{K}}\check{Z}_t + \frac{\bar{Y}}{\bar{K}}\frac{(1-\alpha)}{\alpha+1/\eta}\big(\check{\lambda}_t + \check{Z}_t + \alpha\check{K}_{t-1}\big) + \alpha\frac{\bar{Y}}{\bar{K}}\check{K}_{t-1} + \frac{\bar{C}}{\bar{K}}\check{\lambda}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t - \delta\check{K}_{t-1}
\end{aligned}
$$

The last line makes use of the labor market equilibrium. Using FOC for labor supply and the labor market equilibrium condition that wages are equal to the marginal product of labor :

$$
L_t^{1/\eta} = \lambda_t W_t \;\;\Rightarrow\;\; (1/\eta)\check{L}_t = \check{\lambda}_t + \check{W}_t \tag{2.11}
$$

$$
\check{W}_t = M\check{P}L_t \;\;\Rightarrow\;\; \check{W}_t = \check{Y}_t - \check{L}_t = \check{Z}_t + \alpha\check{K}_{t-1} - \alpha\check{L}_t \tag{2.12}
$$

The second line comes from $MPL_t = \alpha\frac{Y_t}{L_t} \Rightarrow M\check{P}L_t = \check{Y}_t - \check{L}_t$ and $\check{Y}_t = \check{Z}_t + \alpha\check{K}_{t-1} + (1-\alpha)\check{L}_t$. To eliminate labor, we do the following steps:

$$
\check{W}_t = \check{Z}_t + \alpha\check{K}_{t-1} - \alpha\check{L}_t \tag{2.13}
$$

$$\begin{aligned}\eta^{-1}\check{L}_t &= \check{\lambda}_t + \check{W}_t && (2.14)\\ \eta^{-1}\check{L}_t &= \check{\lambda}_t + \check{Z}_t + \alpha\check{K}_{t-1} - \alpha\check{L}_t && (2.15)\\ \check{L}_t &= \frac{1}{1/\eta+\alpha}(\check{\lambda}_t + \check{Z}_t + \alpha\check{K}_{t-1}) && (2.16)\end{aligned}$$

Hence, the change in the capital stock is described by

$$\begin{aligned}\Delta\check{K}_t &= \frac{\bar{Y}}{\bar{K}}\Big(1+\frac{1-\alpha}{\alpha+1/\eta}\Big)\check{Z}_t + \Big(\frac{\alpha(1-\alpha)}{\alpha+1/\eta}\frac{\bar{Y}}{\bar{K}} + \alpha\frac{\bar{Y}}{\bar{K}} - \delta\Big)\check{K}_{t-1}\\ &+\Big(\frac{\bar{C}}{\bar{K}} + \frac{\bar{Y}}{\bar{K}}\frac{(1-\alpha)}{\alpha+1/\eta}\Big)\check{\lambda}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t && (2.17)\end{aligned}$$

In the steady state, the change in the capital stock is zero and we can use this fact to find the properties of the locus of points such that the capital stock is not changing. Also note that we have eliminated many endogenous variables and now the change in the capital stock is only a function of past capital stock, marginal utility of wealth λ_t, and exogenous forcing variables Z_t and G_t.

We can do a similar trick with the marginal utility of wealth. Using $\frac{1}{C_t} = \lambda_t \Rightarrow \check{\lambda}_t = -\check{C}_t$, we have from the Euler equation

$$\begin{aligned}\Delta\check{C}_{t+1} &\equiv \check{C}_{t+1} - \check{C}_t = \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}(\check{Y}_{t+1} - \check{K}_t)\\ &= \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}M\check{P}K_{t+1} \Rightarrow\\ \Delta\check{\lambda}_{t+1} &= -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}M\check{P}K_{t+1}\end{aligned}$$

MPK is a function of capital/labor ratio and we need to eliminate labor as before.

$$\begin{aligned}\Delta\check{\lambda}_{t+1} &= -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}\Big(\check{Z}_{t+1} + (\alpha-1)\check{K}_t + (1-\alpha)\check{L}_{t+1}\Big) && (2.18)\\ &= -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}\Big(\check{Z}_{t+1} + (\alpha-1)\check{K}_t + (1-\alpha)\frac{1}{1/\eta+\alpha}(\check{\lambda}_{t+1} + \check{Z}_{t+1} + \alpha\check{K}_t)\Big)\end{aligned}$$

Finally we have:

$$\Delta\check{\lambda}_{t+1} \quad = \quad -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}\Big[(1+\frac{1-\alpha}{1/\eta+\alpha})\check{Z}_{t+1}+(1-\alpha)\Big(\frac{\alpha}{1/\eta+\alpha}-1\Big)\check{K}_t+\frac{1-\alpha}{1/\eta+\alpha}\check{\lambda}_{t+1}\Big]$$

which is the law of motion for the change in the marginal utility of wealth. Note that the marginal utility of wealth does not depend directly on government expenditures (why?). Now by setting $\Delta\check{\lambda}_t = 0$, we have the locus of points where the marginal utility of wealth is not changing. We need to determine the slope of the loci such that capital stock and marginal utility of wealth are not changing. The intersection of these loci will give us the steady state.

Locus $\Delta\check{K}_t = 0$

$$\check{\lambda}_t = -\frac{1}{\left(\frac{\bar{C}}{\bar{K}}+\frac{\bar{Y}}{\bar{K}}\frac{(1-\alpha)}{\alpha+1/\eta}\right)}\Big[\frac{\bar{Y}}{\bar{K}}\Big(1+\frac{1-\alpha}{\alpha+1/\eta}\Big)\check{Z}_t+\Big(\alpha\frac{\bar{Y}}{\bar{K}}(\frac{1-\alpha}{\alpha+1/\eta}+1)-\delta\Big)\check{K}_{t-1}-\frac{\bar{G}}{\bar{K}}\check{G}_t\Big]$$

Because $\alpha\frac{\bar{Y}}{\bar{K}}(\frac{1-\alpha}{\alpha+1/\eta}+1)-\delta > 0$, the locus is **downward sloping**. On this locus, consumption, labor and investment choices are such that the current capital stock can be maintained. It is downward sloping because when less capital and thus less output is available, consumption must fall (and employment will rise) in order to maintain the same capital stock.

Locus $\Delta\check{\lambda}_t = 0$

$$\check{\lambda}_{t+1} \quad = \quad -\frac{1/\eta+\alpha}{1-\alpha}\left\{(1+\frac{1-\alpha}{1/\eta+\alpha})\check{Z}_{t+1}+(1-\alpha)\Big(\frac{\alpha}{1/\eta+\alpha}-1\Big)\check{K}_t\right\}$$

This locus is **upward sloping** because $(1-\alpha)\Big(\frac{\alpha}{1/\eta+\alpha}-1\Big) < 0$ since $1-\alpha > 0$ and $\frac{\alpha}{1/\eta+\alpha} < 1$. On this locus, consumption and labor choices are such that the real interest rate and the MPK are equal to their steady-state values. Recall our Euler equation states that the growth rate of consumption is proportional to the real interest rate, $C_{t+1}/C_t = \beta(1+R_t)$. Thus, for consumption and marginal utility not to change we must have $1 = \beta(1+R_t) = \beta(1+MPK_{t+1}-\delta)$. The locus is upward-sloping because a higher capital stock lowers the

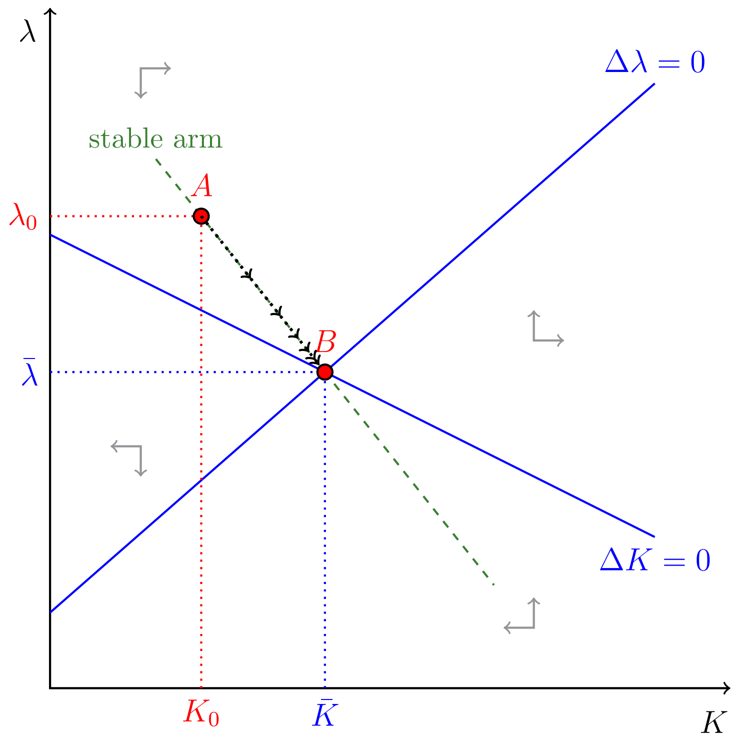


Figure 2.1: Model behavior when capital is below the steady-state at K_0. The economy begins at point A and converges to point B.

MPK and a higher marginal utility of consumption raises labor supply and thus the MPK. On the locus these two forces exactly balance.

2.1.1 Variables in phase diagram

K is given by history (it cannot jump on impact at the time of the shock) while λ is pointed to the future (it can jump on impact at the time of the shock). This means that the phase diagram will have two arms: one stable and one unstable. Here, the stable arm goes from north-west above both $\Delta\check{\lambda}_t = 0$ and $\Delta\check{K}_t = 0$ loci to the steady state and from south-east below both $\Delta\check{\lambda}_t = 0$ and $\Delta\check{K}_t = 0$ loci to the steady state. The unstable arm goes from north-west above $\Delta\check{\lambda}_t = 0$ and below $\Delta\check{K}_t = 0$ loci to the steady state and from south-east below $\Delta\check{\lambda}_t = 0$ and above $\Delta\check{K}_t = 0$ loci to the steady state. This is shown in Figure 2.1.

The stable arm depicts the optimal choice of $\check{\lambda}$ for a given level of capital. All other choices for $\check{\lambda}$ that are either above or below the stable arm produce explosive behavior that eventually converges to the unstable arm. For instance, if $\check{\lambda}$ is too low (consumption is too high), then the economy will converge towards the bottom left, eventually running out of capital and thus any potential for consumption. Above the stable arm the household consumes too little ($\check{\lambda}$ is too high) and keeps accumulating an infinite amount of capital. This violates the transversality condition. This consumer would on average have higher welfare by consuming more today.

Along the stable path the economy gradually converges towards the steady-state. The convergence will be at the rate of the stable eigenvalue in the economy. Based on the arguments in section 1.9, we know that along the stable arm the economy will converge at a rate $\mu_2 < 1$ to the steady-state, where μ_2 is the stable eigenvalue. Thus, in each period the distance to the steady-state gets reduced by a fraction $1 - \mu_2$. In turn, this implies that investment is highest (in an absolute sense) the further away the economy is from the steady state.

2.1.2 Shifts of loci after shocks

Using this linearized system of equations, we can easily get a sense of what should happen with the steady state when there is a permanent shock to technology or government expenditures. An increase in technology always shifts down the $\Delta\check{\lambda} = 0$ and $\Delta\check{K} = 0$ schedules. Therefore $\bar{\lambda}$ unambiguously falls. What happens to $\bar{K}$ depends on which locus shifts more.

When government expenditure G increases, only the locus $\Delta\check{K} = 0$ shifts up. Thus both steady state values of $\bar{\lambda}$ and $\bar{K}$ increase. These shifts are illustrated on Figure 2.2. In what follows, we will examine in detail the responses of the variables at the time of the shock ("impact"), on the transitional path to a new (or the old) steady state ("transition"), and at the time when the system converges to the new (or old) steady state ("steady state").

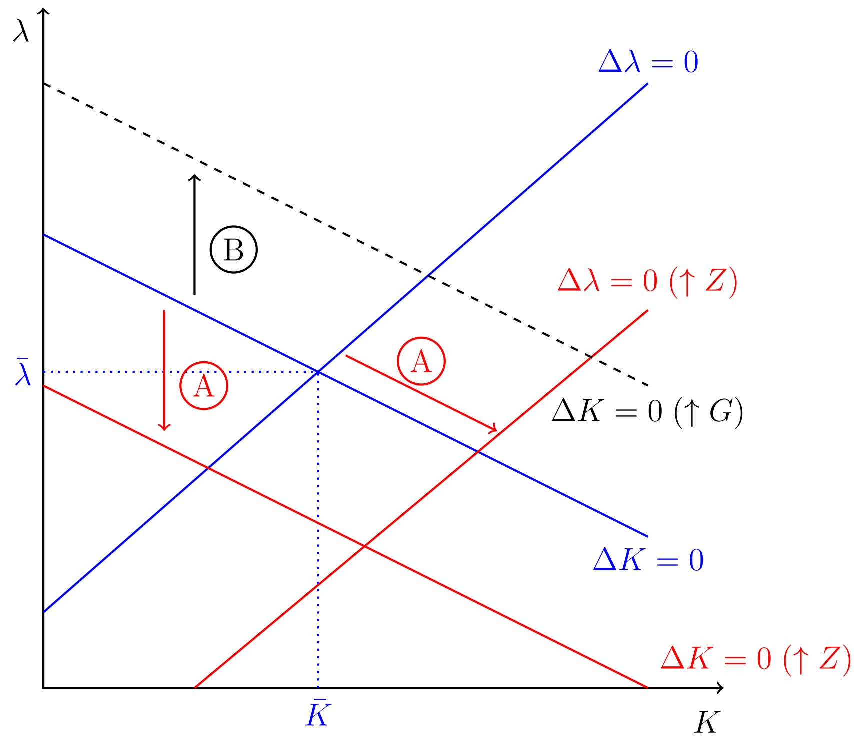


Figure 2.2: Response to technology shocks (A) and government-spending shocks (B).

2.2 Response to a Permanent Government-Spending Shock

We start our analysis of shocks with an easy case: a permanent, unanticipated increase in government spending. We assume that the shock occurs at time t_0. Remember again what this shock entails: the government is effectively throwing away output in this economy and taxing you (lump-sum) to pay for it. Think of wars on foreign territory. It is not investment in roads or schools! That we would have to model very differently. To organize our qualitative predictions about the behavior of the variables, we put results in Table 2.1.

We know from the phase diagram that $\Delta K = 0$ locus shifts up. The new steady state has a higher value of λ and a higher level of capital K (figure 2.3). Since capital is a predetermined variable and it does not move at the time of the shock (hence we put 0 in

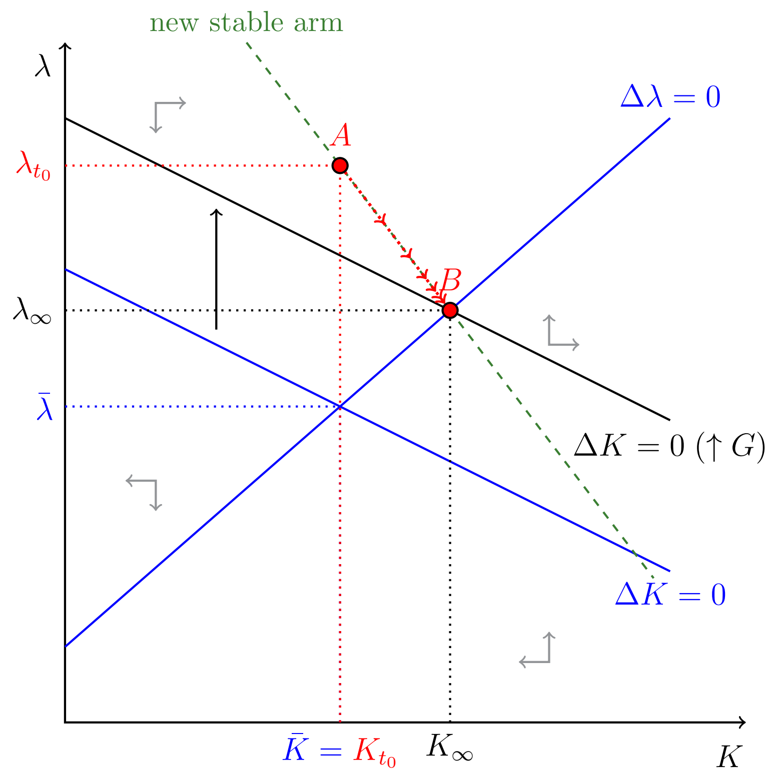


Figure 2.3: Response to permanent government-spending shocks.

the second row of the first column in Table 2.1), at the time of the shock ($t = t_0$) the only variable in the diagram that can react is λ. To put economy on the stable arm (so that the economy converges to the new steady state), λ has to jump up to the new stable arm. We record this movement with $\uparrow$in the first row of the first column of Table 2.1.

From the first order condition for consumption, we know that consumption and λ are inversely related. Therefore, consumption has to fall on impact ($\downarrow$).

The impact effect of a permanent government-spending shock in the labor market is described in Figure 2.4. On impact, K is fixed. There is no change in technology and as a result there is no change in the marginal product of labor: L^d does not move. We also know from the first order condition for labor supply that for a given wage an increase in λ makes the household supply more labor as an additional unit of work is more valuable. Hence, labor supply L^s shifts down. As a result, the labor market clears with $W \searrow$ and $L \nearrow$.

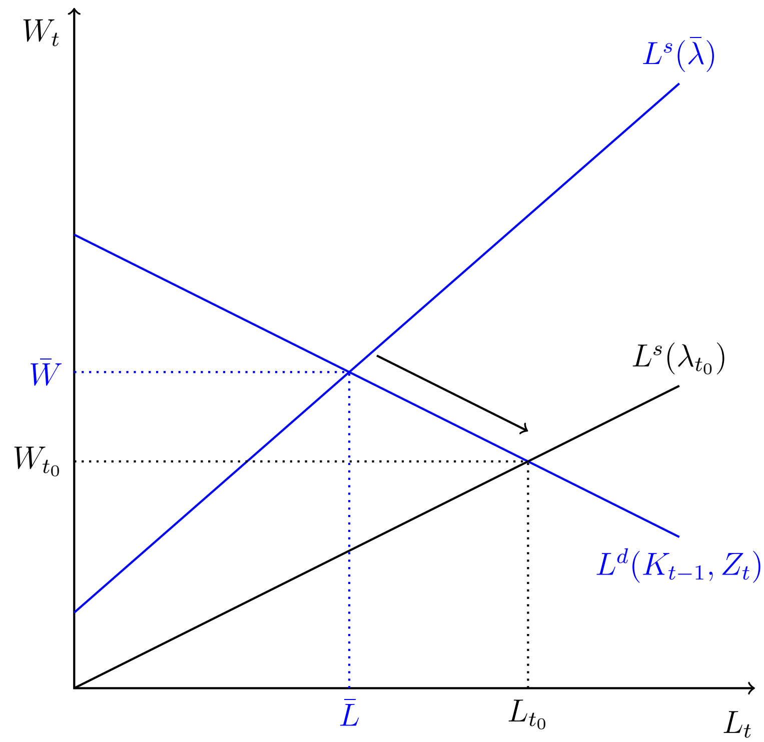


Figure 2.4: Response to permanent government-spending shocks in the labor market in the short run.

What happens with output at $t = t_0$? Using the production function, we find that output has to increase because technology and capital are fixed by labor increases unambiguously. So now we have that consumption falls while output and government spending increase. Unfortunately, this combination of signs means that we can't use the goods market equilibrium ($Y = C + G + I$) to sign investment because we do not know if the change in output is large or small relative to $C + G$. To answer what happens with investment, we have to look elsewhere: the phase diagram. Specifically, we know that while capital is predetermined, investment is not. We also know that the change in investment has to be the largest at the time of the shock since as we move along the stable arm increments in capital are increasingly smaller and therefore investment changes by smaller amounts as we approach the steady state. Therefore, investment has to jump up on impact ($t = t_0$) and then gradually decrease.

Finally, we can sign the reaction of the interest rate using the factor price possibility frontier. With no change in technology, any changes in wages and interest have to occur along the FPPF. Since we know wages fall, it means that the interest rate has to rise. This completes our analysis of how variables react to the shock on impact.

In the next step, we study how variables behave in transition to the new steady state. In this analysis, we again sign changes in variables but these changes are relative to the position of variables right after the shock; that is, relative to values variable took at $t = t_0$. With this "base" for comparison, note that a "decrease" does not mean that a variable is below the value of the variable in the initial steady state if this variable jumps up at $t = t_0$. We record the movements of the variation in the transition stage in the second column of 2.1.

Using the phase diagram, we can easily find that λ falls while capital increases. Given the first order condition for consumption, a fall in λ means a rise in consumption. We have already discussed that investment falls on the transition path. In the labor market, a decrease in λ shift the labor supply curve up while an increase in capital shifts the labor demand up. As a result, wages increase unambiguously. From the FPPF, it follows that the interest rate has to fall (factor prices change along the curve). What happens to output and labor is ambiguous. From the labor market, we know that both labor demand and supply shift up and the movement of labor depends on the relative strength of these shifts. Because the dynamics of labor is ambiguous, we can't use the production function to sign output. Likewise, we cannot use the goods market equilibrium to sign output because consumption increases while investment falls. Thus, we put "?" in the Y and L rows of the second column.

In the final step, we determine how variables behave in the new steady state relative to the initial steady state. From the phase diagram we know that capital is higher, lambda is higher and consequently consumption is lower. From the capital accumulation constraint, we know that in the steady state $\bar{I} = \delta \bar{K}$. Since capital is higher in the new steady state, investment is higher too. The factor price possibility frontier does not move and we know from our previous discussion that the steady-state interest rate is pinned down by the discount factor β. Therefore, nothing happens to the interest rate and wages. However, since capital is

	Impact $t = t_0$	Transition $t \in (t_0, +\infty)$	Steady State $t = +\infty$
λ	$\uparrow$	$\downarrow$	$\uparrow$
K	0	$\uparrow$	$\uparrow$
C	$\downarrow$	$\uparrow$	$\downarrow$
L	$\uparrow$	?	$\uparrow$
Y	$\uparrow$	?	$\uparrow$
I	$\uparrow$	$\downarrow$	$\uparrow$
W	$\downarrow$	$\uparrow$	0
R	$\uparrow$	$\downarrow$	0

Table 2.1: RESPONSE TO A PERMANENT GOVERNMENT-SPENDING SHOCK

higher in the new steady state, we can have the same wages and interest in the new steady state only if labor increased to keep the capital-to-labor ratio constant. Thus, labor is higher in the new steady state. From the production function, it follows that output is higher in the new steady state.

Why is the household behave this way? Remember the government is effectively throwing away resources in this economy and taxing the household for the remainder of time. So the household is worse off as it cannot maintain its original consumption and leisure choices. There are three ways for it to respond and still satisfy the budget constraint (or resource constraint): reduce consumption, increase labor supply, and/or reduce investment (draw on savings). All these margins are costly. In our case, the household decides to use the first two margins. How much of each will depend on parameters: If increasing labor supply is not very costly — the Frisch elasticity is high — then the household will primarily use that margin. Conversely, if raising labor supply is very costly, then the household will lower consumption more.

We also see from the phase diagram that the economy in fact accumulates capital. Why is the household not operating on the third margin? The reason is that the long-run increase in labor supply makes capital much more valuable. For a given level of capital, higher labor usage raises the marginal product of capital and thus the return on saving and investing. Consider what would happen if the household decided to maintain capital at a constant level instead (the argument is more extreme for eating into it): This would raise the real interest rate in the economy permanently, so $\beta(1+\tilde{R}) > 1$, where the $\tilde{X}$ denotes the variable X on this alternative path. Then from the Euler equation, $\tilde{C}_t = [\beta(1+\tilde{R})]^{-1}\tilde{C}_{t+1} = [\beta(1+\tilde{R})]^{-2}\tilde{C}_{t+2} = \lim_{T\to\infty}[\beta(1+\tilde{R})]^{-T}\tilde{C}_{t+T}$. If long-run consumption is finite, then todays consumption is zero since $\lim_{T\to\infty}[\beta(1+\tilde{R})]^{-T} = 0$. The incentives to save are just too strong in this case. This contradicts our premise that todays investment is not increasing: if consumption is zero then income has to be saved for investment. Consider next the case where long-run consumption is infinite. With a constant, finite capital stock (remember we did not invest in this thought experiment), infinite consumption is infeasible unless labor supply is infinite. But that is also not optimal because disutility of labor rises at a faster rate than the utility of consumption. In short, the rise in labor supply raises the marginal product of capital, which provides incentives for this household to accumulate more capital.

Note that incentive to save is particularly strong on impact: At t_0 the marginal product of capital, $MPK_{t+1} = \alpha Z_{t+1}K_t^{\alpha-1}L_{t+1}^{1-\alpha}$, is very high because labor supply has increased a lot and capital is fixed at that instant. From the Euler equation, $\check{C}_{t+1} - \check{C}_t = \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}M\check{P}K_{t+1}$, we know that the higher the interest rate, the greater the increments to consumption. Thus consumption and marginal utility will converge towards the steady-state at a pace commensurate to the real interest rate. As the real interest rate declines this pace slows. From the phase diagram it then follows that the investment rate is also slowing as capital and consumption gradually approach the steady-state. Intuitively, investment is most valuable when the marginal product of capital is highest, which occurs at t_0, so the investment will be highest then. As capital accumulates, the real interest rate declines and so do the incentives to save and invest. Thus, the consumer reduces her investments during (t_1,∞) relative to t_0.

These qualitative predictions about the dynamics of the variables lead us to an important conclusion. In this basic RBC model, consumption and output move in opposite direction in response to permanent government spending shocks. This pattern contradicts a key empirical regularity that consumption and output comove over the business cycle. As a result, one may conclude that shocks similar to government spending could not be a key driving force of business cycle (if there were, we should have observed a negative correlation between output and consumption).

2.3 Response to a Transitory Government-Spending Shock

In the previous section we found that permanent government spending shocks appear to lead to results inconsistent with the comovement of macroeconomic variables observed in the data. In this section, we explore if unanticipated, *transitory* shocks to government spending may match data better. We assume that G increases at time t_0 and stays high until t_1 when it returns to the initial level. All agents in the economy know that government spending returns to the initial level at time t_1. So the change in government spending at time t_1 is perfectly anticipated. Similar to the case of permanent government spending shocks, we will summarize the dynamics of the variables in Table 2.2.

Since the shock is temporary, we know that the economy reverts to the initial steady state and therefore there is no change in the position of the loci $\Delta\check{\lambda} = 0$ and $\Delta\check{K} = 0$ in the long run. As a result, we can immediately put "0" in the last column of Table 2.2. However, for $t \in [t_0, t_1)$ the economy is governed by $\Delta\check{K} = 0$ locus shifted up. We know from the previous case that λ jumps up at the time of a permanent, unanticipated increase in government spending to put the economy on the stable arm of the phase diagram. In the case of a transitory shock, λ also jumps up but not as much as in the case of a permanent shock. Intuitively, the economy has to be on the stable arm only at time t_1 when the shock dissipates and between t_0 and t_1 the economy can be on an explosive path. However, this explosive path has to be such that precisely at time t_1 the economy hits the stable arm. This explosive path will be below the stable arm of the economy with a permanent increase in government spending and hence an increase in λ for a temporary shock is smaller. For now

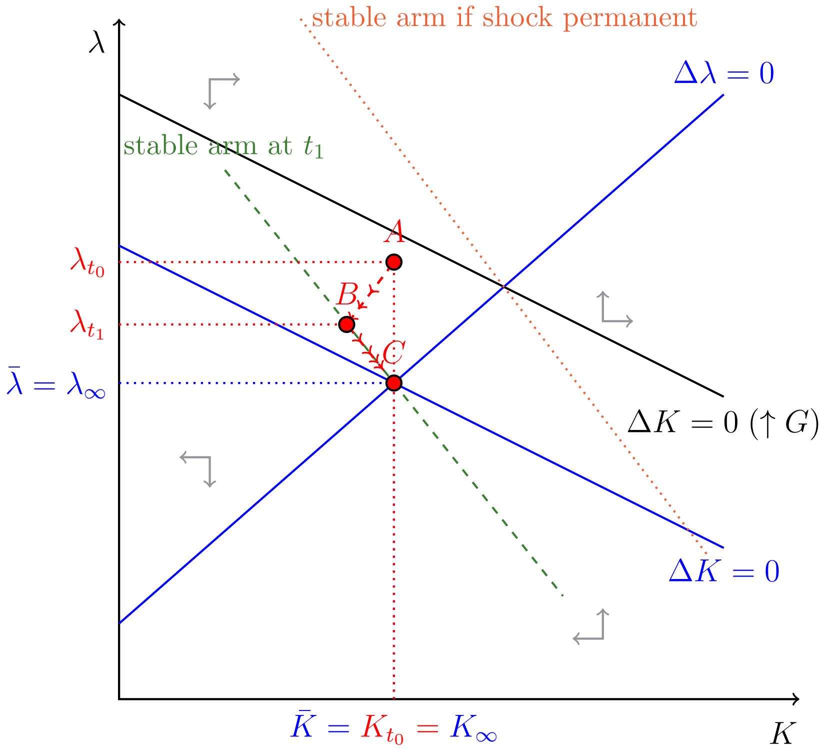


Figure 2.5: Response to transitory government-spending shocks.

we consider the case where λ jumps to a value *below* the new $\Delta K = 0$ locus. As we argue below, this will be the case when the shock to government spending is relatively transitory. (Remember we showed earlier that transitory shocks will affect marginal utility relatively less, because it is forward looking.) Note that similar to the case of a permanent shock, λ is falling for any $t \in (t_0, +\infty)$ and so the qualitative dynamics for λ is similar for permanent and transitory shocks.

In contrast, the dynamics of capital is different. In the case of a permanent shock to government spending, the economy accumulates more capital to support increased spending while in the case of a transitory shock, capital falls for $t \in (t_0, t_1)$ and rises back to the initial level of capital for $t \in (t_1, +\infty)$. Thus, the economy "eats" capital (this is the only form of saving in the economy) during the time of increased government spending. Note that at time t_1 there is no jump in the stock of capital because it is a predetermined variable. We can use the dynamics of capital to infer the dynamics of investment since investment is effectively a time derivative of capital. On impact ($t = t_0$) investment jumps down since capital starts to decrease. What happens on the explosive path may depend on parameter values (how persistent the shock is), but in this case investment will keep declining: over (t_0, t_1) consumption rises while output falls and G_t is constant. Then the only way the goods market equilibrium ($Y_t = C_t + I_t + G_t$) can hold is if investment declines further over (t_0, t_1). That is, over (t_0, t_1), the economy eats increasingly more of its capital stock.

At time $t = t_1$, there is a change in the direction of change for capital: it starts to rise. Hence, investment has to jump up ($\uparrow$in column 3). When the economy is on the stable arm, accumulation of capital slows down as the economy approaches the steady state and hence investment should fall relative to it's position at time t_1.

Now we can turn to labor markets. On impact, labor supply curve shift to the right while labor demand curve does not move (neither capital nor technology change). Therefore, employment increases while wages fall. Since employment increases, we can use the production function to predict that output increases at t_0.

Between t_0 and t_1, λ is falling and hence the labor supply curve is shifting up. At the same time, capital is generally falling over the same period and hence the labor demand curve shifts down. This combination of shifts means that labor unambiguously decreases relative to its level at $t = t_0$ but we cannot sign the effect on wages from the labor market alone. Let us check if using FPPF can help us to sign the change in wages. First, both capital and labor decrease and thus we do not know what happens with capital-to-labor ratio. So this avenue to use FPPF is not fruitful. Second, we know from the Euler equation that the change in consumption is proportional to the interest rate. We know from the phase diagram that consumption is rising monotonically between t_0 and t_1. Hence, the interest rate has to be above steady state. But to sign the change in the interest rate we need to know what happens with the change of changes (i.e, acceleration, second time derivative) in consumption. Because the economy is on an explosive path, the acceleration depends potentially on many factors and we cannot sign it. Thus, using FPPF can't help and we have ambiguous predictions about the direction of change for the interest rate and wages (we put "?" for both of these variables in the second column).

At $t = t_1$, λ continues to fall which pushes labor supply further but given no discontinuity in λ we will record movements in λ and consumption as "fixed" to highlight that none of these variables jumps at this time. Capital is fixed (this is the turning point for capital) and hence labor demand curve is fixed. Since both the supply and demand labor curves are fixed, there is not change in wages, employment, output (from the production function), and interest rate (from the FPPF). Note that at $t = t_1$, the government spending shock dissipates and thus from the goods market equilibrium one may expect a decrease in output too. However, a fall in government spending is "compensated" by a jump up in investment so that we do not observe a change in output.

When the economy reaches the stable arm and stays on it ($t \in (t_1, +\infty)$), consumption keeps rising as λ is falling. In the labor market, labor supply curve keeps shifting up (λ is falling) and labor demand curve starts shifting up as the economy starts to accumulate capital. Consequently, wages rise unambiguously in this period but we cannot sign the change in employment over this period because the outcome depends on the relative strength of the shifts. From the FPPF, a rise in wages entails a decrease in the interest rate. An ambiguous dynamics of labor means that we cannot use the production function to sign the change in output. We can try to use the goods market equilibrium to sign output. After t_1, consumption is rising but investment is falling and so this does not help to sign output either. Output's dynamics is ambiguous over $t \in (t_1, +\infty)$.

We note that on some dimensions the model behaves very similarly following a permanent shock. In particular, consumption falls, labor supply rises, and output rises. Again the consumer responds to being poorer by reducing consumption and raising labor supply. But the behavior of investment and capital are very different: while investment and capital rose following a permanent shock, they declined following a temporary shock. Why is this the case? We noted earlier that a permanent shock raises the value of capital, because it caused a permanent rise in labor supply. But with a temporary shock, there is no longer a long-run benefit from raising the capital stock, because in the long-run labor supply returns to normal. Thus, the returns to investment rise less when the shock is temporary than when it is permanent all else equal. We therefore know that investment must rise less than in the permanent case.

But in our example investment not only rose less; in fact, it declined. Note again from the Euler equation that today's consumption response is given by,

$$\check{C}_t = -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}M\check{P}K_{t+1} + \check{C}_{t+1} = -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}\sum_{s=0}^{\infty} M\check{P}K_{t+1+s} + \underbrace{\lim_{s\to\infty}\check{C}_{t+s}}_{=0 \text{ for temporary shock}} .$$

If the marginal product of capital (the interest rate) does not rise very much, then consumption will fall relatively little. From the resource constraint, $Y_t = C_t + I_t + G_t$, the household then uses investment to smooth out the government spending shock. Intuitively, for a transitory shock that goes away quickly, it is not optimal for the household to give up consumption and accumulate long-lasting capital. That capital will not be valuable for very long! It can instead achieve a higher consumption path by eating into the capital stock while the shock lasts and then rebuilding it once the shock is over. (You can also check that our earlier proof by contradiction when the shock is permanent no longer works when the shock is temporary.)

This argument also suggests that for a shock that is temporary but very persistent, it may still make sense for the consumer to accumulate some capital (although not as much as with a permanent shock). The more persistent the shock, the longer is labor supply elevated, and the longer is the marginal product high. In fact that is exactly what will happen when the shock is so persistent that λ_{t_0} is located above the new $\Delta K = 0$ locus. This is what happens in figure 2.6. The main changes will be to the behavior of investment on impact and the transition for investment and capital. Note that as the shock becomes increasingly more persistent the behavior of the economy will approach that of the permanent shock. Note though, that the consumer would never optimally chose to be above the stable arm for the permanent shock! (Recall that temporary shocks have smaller effects on marginal utility than temporary shocks.)

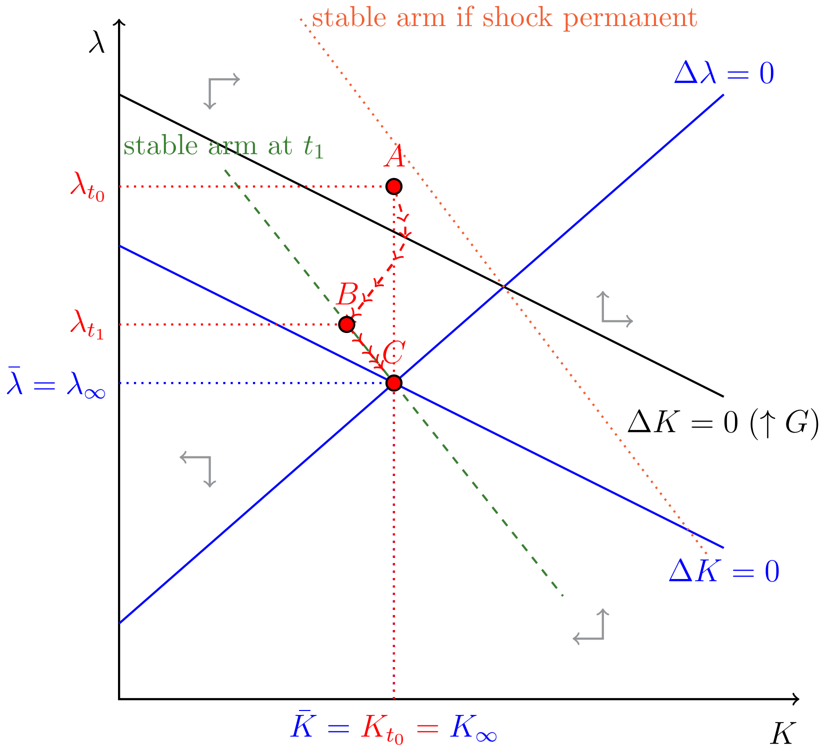


Figure 2.6: Response to a transitory but very persistent government-spending shocks.

However, even in this case where government spending shocks are very persistent, we still observe opposing movements in consumption and output. So our conclusion regarding business cycle facts are unchanged.

To see why investment must switch sign on an optimal path, note that this consumer desires smooth consumption and smooth labor supply. If those variables follow "continuous" paths (as they would at the optimum), then predetermined capital also implies that output, $Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}$, is continuous. From the resource constraint in the economy, $Y_t = C_t + I_t + G_t$, the only variable that can then exhibit a discontinuity at t_1, is investment. It exactly compensates for the decline in G_{t_1} at that instant.

In summary, using temporary spending shocks as a source of business cycles is also problematic. Consumption and now investment move in directions opposite to where employment and output move which is inconsistent with the data. As a result, temporary government spending shocks cannot be the key source of business cycles in this class of models. This

	Impact $t = t_0$	Transition I $t \in (t_0, t_1)$	Inflection $t = t_1$	Transition II $t \in (t_1, +\infty)$	Steady State $t = +\infty$
λ	$\uparrow$	$\downarrow$	0	$\downarrow$	0
K	0	$\downarrow$	0	$\uparrow$	0
C	$\downarrow$	$\uparrow$	0	$\uparrow$	0
L	$\uparrow$	$\downarrow$	0	?	0
Y	$\uparrow$	$\downarrow$	0	?	0
I	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	0
W	$\downarrow$	?	0	$\uparrow$	0
R	$\uparrow$	?	0	$\downarrow$	0

Table 2.2: RESPONSE TO A TRANSITORY GOVERNMENT-SPENDING SHOCK

analysis also highlights the importance of capital for smoothing out and propagating shocks. Households use capital (their only form of saving) to smooth a temporary shock.

2.4 Response to a Permanent Technology Shock

Now we turn to another source of fluctuations in the model: technology shocks. To understand the reaction of the variables to these shocks, we will again use the phase diagram, labor market equilibrium, and FPPF. We start with a permanent, unanticipated increase in technology at time t_0. Since two loci shift in response to the shock, there are a few possibilities for where the new steady state is located. We focus here on the case in which $\bar{\lambda}$ **is lower** and $\bar{K}$ **is higher** after the shock, and in which the new stable arm in the phase diagram is **below the original steady state**. We draw the phase diagram in figure 2.7 collect the responses of key macroeconomic variables in Table 2.3.

Given our assumption about the position of the new steady state relative to the initial steady state, we know that, on impact ($t = t_0$), λ has to jump down to put the economy on the stable arm. There is no change in capital at the time of the shock because capital is predetermined but investment jumps up as the economy starts to accumulate more capital. A decrease in λ means an increase in consumption (first order condition for consumption).

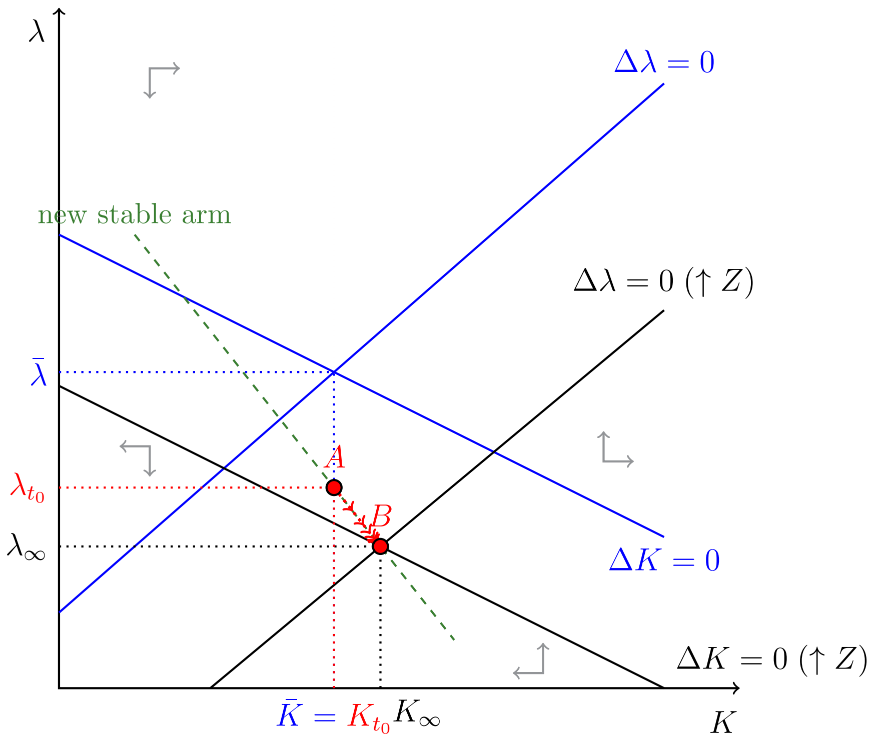


Figure 2.7: Response to permanent technology shock.

Since capital is higher in the new steady state, investment has to jump up on impact.

In the labor market, labor demand shifts up (technology raises MPL) and labor supply shifts to the left (workers fill richer as λ goes down), see Figure 2.8. As a result, wages increase unambiguously but we can't sign the change in employment. Because of this ambiguity for employment, we can't use the production function to sign the change in output. However, we can use the goods market equilibrium to do so. We know that both consumption and investment increase at $t = t_0$ and therefore output increases on impact. We have not signed the change of the interest rate yet but we will do it later when we understand the dynamics of other variables.

On transition path, λ falls, K increases, and consumption rises (recall that we do signs of changes relative to the position of the variables at t_0). Because capital increases at decreasing rate, we know that investment has to fall relative to its level at $t = t_0$. Falling λ means that labor supply curve keeps shifting up. Increasing capital stock means that labor demand keeps shifting up to. Hence wages rise but again we can't sign changes in employment. Because investment is falling and consumption is rising we cannot use the goods market equilibrium to sign the change in output. Likewise, using the production function is not helpful because labor dynamics is ambiguous. Hence, in general we can't sign the dynamics of output during this period but in plausible parameterizations output increases while the economy is in transition.

In the steady state, capital is higher and λ is lower (by assumptions we made earlier) than in the initial steady state. Hence, consumption is higher and investment is higher (recall that $\bar{I} = \delta\bar{K}$). Wages are higher because labor demand and supply are higher. Since both consumption and investment are higher, we know that output is higher in the new steady state too.

Now we turn to the interest rate. An increase in wages does not mean that the interest rate has to fall because the FPPF shifts up so that both higher wages and interest rate can be sustained. However, we can use "backward induction" to sign the change in the interest rate at the time of the shock. We know that the steady-state interest rate is fixed by the discount factor and hence even if the FPPF shifts we have the same interest rate in steady state. The economy converges to the new steady state monotonically after the shock and jumps at t_0. We also know that wages jump up at t_0 and continue to rise afterwards. This means that the interest rate must approach its steady state level from from above. Therefore, the interest rate jumps up on impact and then falls while in transition to the new steady state.

In contrast to spending shocks, technology shocks generate comovement of macroeconomic variables similar to what we see in the data. Thus, at least in the context of this basic model, we can make this model consistent with the data only if technology shocks is the primary force of fluctuations.

	Impact $t = t_0$	Transition $t \in (t_0, +\infty)$	Steady State $t = +\infty$
λ	$\downarrow$	$\downarrow$	$\downarrow$
K	0	$\uparrow$	$\uparrow$
C	$\uparrow$	$\uparrow$	$\uparrow$
L	?	?	?
Y	$\uparrow$	$\uparrow$(generally)	$\uparrow$
I	$\uparrow$	$\downarrow$	$\uparrow$
W	$\uparrow$	$\uparrow$	$\uparrow$
R	$\uparrow$	$\downarrow$	0

Table 2.3: RESPONSE TO A PERMANENT TECHNOLOGY SHOCK

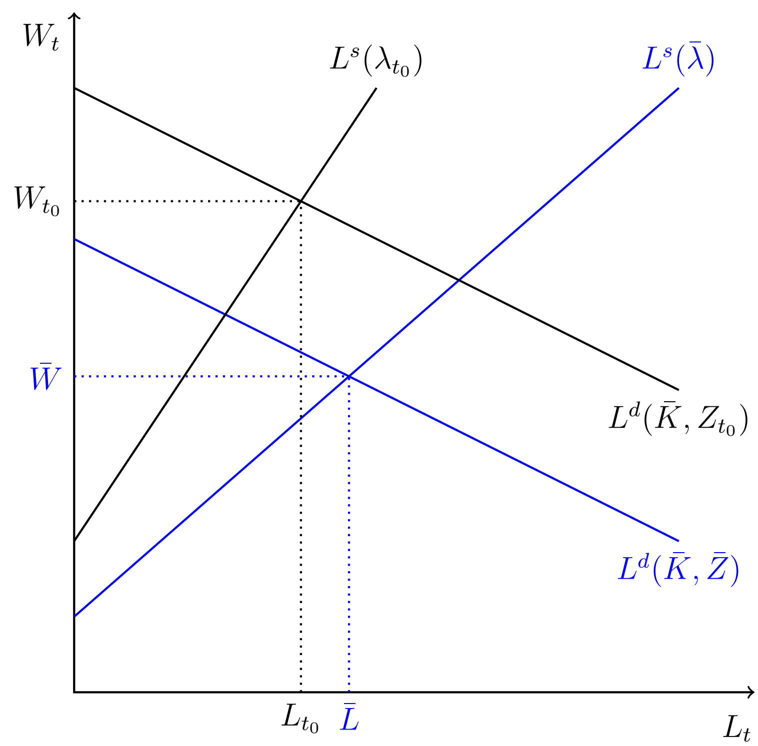


Figure 2.8: Response to permanent technology shocks in the labor market in the short run.

2.5 Response to a Transitory Technology Shock

In our analysis of temporary vs. permanent shocks to government spending, we saw important differences in the dynamics of variables. In this section, we explore how temporary shocks for technology affect variables in our basic model. Similar to the case of temporary

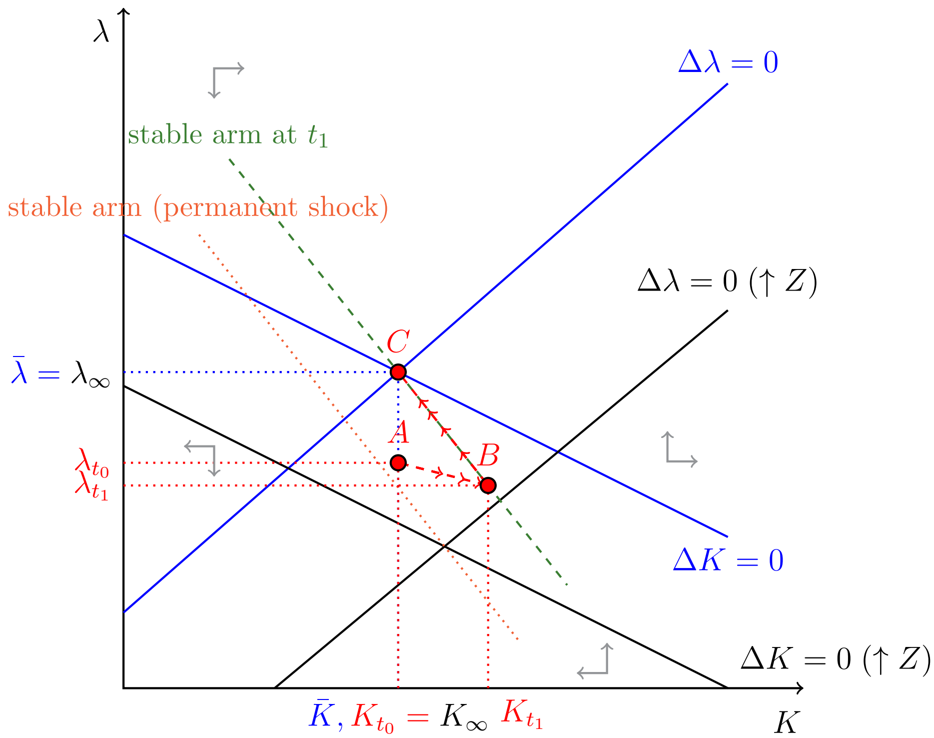


Figure 2.9: Response to transitory technology shock.

government spending shocks, we assume that there is an unanticipated, positive shock to technology at time t_0 which lasts until t_1 and then technology reverts to its initial level. The reversal to the initial level at t_1 is anticipated by all agents in the economy.

To make this analysis comparable to the case of a permanent technology shock we have considered above, we assume that the size of the shock and shifts in the loci are such that if the shock stayed permanent it would have resulted in the new steady state with a lower λ and a higher K as in figure 2.9. So for $t \in [t_0, t_1)$ the economy is governed by the same phase diagram we had for the permanent technology shock, and for $t \in [t_1, +)$ the economy is governed by the initial phase diagram. The effects of a transitory technology shock are summarized in Table 2.4. Because the shock is temporary, there is no change in the steady state and hence we put zeros in the last column of the table.

On impact ($t = t_0$), we know (given our assumptions) that λ falls. But it does not fall as much as in the case of a permanent shock because the economy has to be on the stable arm of the initial phase diagram at time t_1 and therefore the economy has to be on an explosive path between t_0 and t_1 governed by the new phase diagram. Since λ falls, consumption rises. Capital is a predetermined variables and so it does not move at the time of the shock. However, investment jumps up as the economy starts to accumulate capital. Since investment and consumption increase, we know that output increases on impact.

At the time of the shock, the labor supply curve shifts up because households feel richer (λ falls). At the same time, labor demand curve shifts up too because technology improves. Hence, wages rise unambiguously but we can't sign the change in employment. Even though wages unambiguously rise, we cannot use the FPPF because it shifts out and so higher wages *and* interest rate can be sustained. We will sign the change in the interest rate later when we understand the full dynamics of other variables.

On transition path $t \in (t_0, t_1)$, λ will (generally) fall and hence consumption will continue to rise (remember these statements are relative to the values variables take at time $t = t_0$, that is, at the time of the shock). While capital stock increases during this period, the economy is on an explosive path and hence we do not know if investment falls or rises relative to its value at $t = t_0$ (that is, accumulation of capital can accelerate or decelerate). Falling λ means that the labor supply curve will continue to shift up and continued accumulation of capital means that the labor demand will continue to shift up. Therefore, we observe rising wages but again we do not know what happens with employment as the reaction depends on the relative strength of shifts in the supply and demand curves. Since labor dynamics is ambiguous and we can't sign changes in investment, using either production function or goods market equilibrium can't help us to sign changes in output.

At $t = t_1$, technology reverts to its original level. At this time, there is not jump in λ (and hence consumption) or capital (but they both have infliction points at this time). Other variables, however, jump. Labor demand curve shift down because technology falls while there is no movement in the labor supply curve (λ is fixed). Hence, both wages and employment fall at $t = t_1$. Employment and technology fall, hence production function suggests that output has to decrease at this time too. The economy starts to "spend" capital and therefore investment has to go down.

In the second phase of the transition ($t \in (t_1, +\infty)$), λ is rising back to the initial steady state level and hence consumption falls. Capital is falling towards its initial level. Since economy is on the stable arm, we know that accumulation of capital decelerates and therefore investment has to be rising (i.e., it is negative but it becomes less negative). Decreasing capital shock and rising λ shift labor demand and supply curves down and therefore wages decrease unambiguously and, again, we can't say anything about what happens to employment. Since investment is rising and consumption is falling, we can't sign output from the goods market equilibrium. We also cannot use the production function because the dynamics of employment is ambiguous.

Now we will try to sign the dynamics of the interest rate. At time t_0 we know that consumption will be growing in the future. We can use therefore use Euler equation, $C_{t+1}/C_t = \beta(1 + R_{t+1})$, to infer that interest rates must have risen at t_0. At the time of the shock the FPPF shifts out and for $t \in [t_0, t_1)$ it is governed by this new FPPF. During this period, wages are rising and hence the interest rate has to be falling relative to where it was at $t = t_0$ (keep in mind that given the shift, it does not mean it's lower relative to the initial steady-state level of the interest rate). At time $t = t_1$, wages fall but it does not necessarily mean that interest rate falls. To sign the change at t_1, we again use the Euler equation: $C_{t+1}/C_t = \beta(1 + R_{t+1})$. There is no discontinuity in consumption at t_1, but we know that consumption changes from rising to falling and therefore C_{t+1}/C_t changes from being positive (generally) to being negative. This means that interest rate has to go down at t_1. For $t \in (t_1, +\infty)$, wages fall and thus as the economy moves along the FPPF, interest rate has to rise. Since the economy converges monotonically the the new steady state, we know that the interest rate approaches the steady state level from below. Thus, we can sign all dynamics of the interest rate.

In summary, the dynamics of the economy in response to a temporary shock is similar to the reaction in response to a permanent shock. These dynamics also highlights the importance of capital in smoothing consumption over time as the representative household

	Impact $t = t_0$	Transition I $t \in (t_0, t_1)$	Inflection $t = t_1$	Transition II $t \in (t_1, +\infty)$	Steady State $t = +\infty$
λ	$\downarrow$	$\downarrow$	0	$\uparrow$	0
K	0	$\uparrow$	0	$\downarrow$	0
C	$\uparrow$	$\uparrow$	0	$\downarrow$	0
L	?	?	$\downarrow$	?	0
Y	$\uparrow$	?	$\downarrow$	?	0
I	$\uparrow$	?	$\downarrow$	$\uparrow$	0
W	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	0
R	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$	0

Table 2.4: RESPONSE TO A TRANSITORY TECHNOLOGY SHOCK

uses capital to "save" temporarily high productivity and therefore spread it over longer period of time.

2.6 Recap

These qualitative predictions gives us a sense about what shocks should be an important source of business cycles. Table 2.5 summarized qualitative predictions about correlations of variables in the data and in the model conditional on types of shocks. Clearly, government spending shocks have trouble reproducing some key correlations observed in the data. For example, $\rho(C, Y)$ is negative conditional on this shock, but this correlation is positive in the data. Technology shocks appear to have a better match but these also have issues. For example, $\rho(L, Y)$ is potentially ambiguous and its sign depends on parameters of the model (e.g., the Frisch elasticity of labor supply is very important here). In addition, technology shocks lead to highly procyclical wages which is inconsistent with only weak cyclicality of wages in the data.

	$\rho(Y,L)$	$\rho(Y,C)$	$\rho(Y,I)$	$\rho(Y,W)$
Z shocks	?	+	+	+
G shocks	+	-	+ (permanent shock) - (transitory shock)	-
data	+	+	+	+, ≈ 0

Table 2.5: Actual and Predicted Correlations

Chapter 3

Quantitative Validity of the Real Business Cycle Model

We are now ready to assess the empirical success of the theoretical model we developed so far. We will evaluate our models along many dimensions. Specifically, we will look at the following properties of the data and check whether the model can reproduce them:

- volatility
- correlations
- amplification of shocks
- propagation of shocks
  - internal dynamics
  - built-in persistence (serial correlations of exogenous forcing variables)

Ideally, we would like to have a model which does not rely heavily on the persistence and volatility of exogenous forcing variables. Instead we want to have a model which can generate ***endogenously*** persistence and volatility.

Mechanisms that increase endogenous volatility are often called **amplification mechanisms**. To get a sense of endogenous amplification, consider a simple model with inelastic labor supply where output is fully reinvested in capital and every period capital is fully depreciated (i.e., $K_t = Y_t$; in the Solow growth model this would correspond to the saving rate equal to one). The linearized production function is $\check{Y}_t = \check{Z}_t + \alpha\check{K}_{t-1}$. Note that an immediate effect of Δ change in $\check{Z}$ on output is Δ since given capital stock fixed in the short run we have $\check{Y}_t = \check{Z}_t$. Hence, a one percent increase in Z_t increases output Y_t by one percent. This change in output is determined exogenously.

Over time more capital is accumulated. In the long run, $\check{Y}_{LR} = \Delta + \alpha\check{Y}_{LR}$ and hence the long run response to a change in technology is $\check{Y}_{LR} = \frac{1}{1-\alpha}\Delta$. In other words, 1% increase in technology $\check{Z}$ raises output by $\frac{1}{1-\alpha}$%. Here, capital accumulation is the amplification mechanism which can generate an extra kick to output beyond the immediate effect. We can say that $\frac{1}{1-\alpha} - 1 > 0$ is the **endogenous amplification**.

Mechanisms that increase endogenous persistence are called **propagation** mechanisms. In our simple model with full depreciation and complete reinvestment, capital accumulation also works as a propagation mechanisms because the system reaches equilibrium gradually and adjustment is not instantaneous. If the model relies too much on properties of exogenous variables, then this model is not very satisfactory since we end up explaining the data with postulated exogenous variables.

In summary, our desiderata in modeling business cycles are strong amplification and propagation in the model.

3.1 Canonical Real Business Cycle Model

Cooley and Prescott (1995) develop and calibrate a standard ("bare-bones") model in business cycle literature. They focus only on technology shocks as the driving force of the business cycles. This class of models is often called "***real business cycle***" (RBC) models to emphasize that shocks in the model are real (e.g., technology shock) and not nominal (e.g., monetary shock). The model in Cooley and Prescott (1995) is similar to what we developed and analyzed before and their calibration is similar to ours. Hence, we will take results of their model simulations to assess the fit of the model to the data.

The numerical results from their analysis are presented in Table 3.1. The moments of the data (variances and correlations) are computed after removing the trend component in the macroeconomic series (Cooley and Prescott use the Hodrick-Prescott filter which is a way to remove the trending component in a flexible way). On the other hand, we simulate the calibrated model, HP-filter the simulated series, compute the moments of the filtered series, and then compare the moments of these series with the moments in the data. Broadly, the match is not perfect, but it is not terrible either. Some prominent facts about the business cycle are reproduced by the model. For instance: $\sigma_I > \sigma_Y > \sigma_C$. This finding could be somewhat surprising given how simple our model is and this is why we do not use any formal metric such as an over-identifying restrictions test to falsify the model. The objective here is to check whether the model can "roughly" reproduce salient features of business cycles.

Summary of findings:

1. overall the match is not terrible;

2. level of volatility is ok;

3. dynamic properties of the model are off;

4. a lot of variability is driven by "exogenous" shocks (forcing variable);

5. too much correlation with output and productivity.

While the model has a number of limitations, it also has a number of desirable properties. First, the model studies business cycles in general equilibrium and hence we take into account equilibrium responses of variables to technology shocks. Second, the framework is a Walrasian economy with no frictions and all markets clear. This is important because

Variable	Data st.dev.	Model st.dev.
Output	1.72	1.35
Consumption	Total: 1.27	0.329
	Non-durable: 0.86	
Investment	Total: 8.24	5.95
	Fixed: 5.34	
Hours	1.59	0.769
Employment	1.14	
Serial corr. in output, $\rho(Y_t, Y_{t-1})$	0.85	0.698
$\rho(Y_t, C_t)$	0.83	0.84
$\rho(Y_t, I_t)$	0.91	0.992
$\rho(Y_t, L_t)$	Hours: 0.86	0.986
	Employment: 0.85	
$\rho(Y_t, Z_t)$	0.77 (Solow resid.)	0.98
$\rho(Y_t, W_t)$	Hourly rate: 0.68	0.978
	$\frac{Compensation}{hour}$: 0.03	

Table 3.1: NUMERICAL ANALYSIS IN COOLEY AND PRESCOTT (1995)

we don't need to rely on ad hoc mechanisms of how and why the markets are not clearing. Third, the model is very parsimonious. We have only a handful of parameters. We can use over-identification tests (the model is hugely over-identified) to check the validity of the model. Fourth, we can use the same model to explain growth and fluctuations around growth trend. Fifth, we try to explain macro facts based on solid/rigorous micro-foundations where all economic agents are optimizing and therefore we can study changes in policies without being subject to the Lucas critique.

In summary, with one model we can address many things and at the same time we put enormous discipline on the model. An important point about RBC models is that fluctuations are fully optimal and, hence, there is no need for policy in these models. In other words, recessions are not terribly bad in the sense that they are not reflecting disruptions to how markets operate. If taken literally, it means that enormous unemployment during the Great Depression was a great vacation since the unemployed ***optimally*** chose not to work and enjoy leisure!

3.2 Critique of the RBC Model

1. When we look at the simulated data we see that the ***quantitative*** match is far from perfect:

   - consumption is too smooth
   - wages are too procyclical

   To be fair, there is only one shock in the model. By introducing more shocks, we can improve the fit. If we introduce shocks to government expenditures G, we can reduce procyclicality of wages. The intuition is that:

   - $Z \nearrow \Rightarrow Y \nearrow, W \nearrow \Rightarrow$ wages are very procyclical.
   - $G \nearrow \Rightarrow Y \nearrow, W \searrow \Rightarrow$ wages are countercyclical.

   Thus if you have the right mix of shocks, you can improve the fit (see e.g. Christiano and Eichenbaum (1992)). There are constraints to this kind of approach. It is true that one can keep adding shocks but it does not mean that you necessary improve the fit. For instance:

   - $Z \nearrow \Rightarrow Y \nearrow, C \nearrow$ which fits the data.
   - $G \nearrow \Rightarrow Y \nearrow, C \searrow \Rightarrow C$ is countercyclical which we do not observe in the data.

   Thus if we want to keep emphasizing shocks to G, we can get deteriorated results on consumption and improved results on wages.

2. Technology shocks can be controversial:

   - What is a "decline" in technology? How is it possible to have a technological regress? We measure technology using the Solow residual:

   $$\check{Z}_t = \check{Y}_t - (1-\alpha)\check{L}_t - \alpha\check{K}_{t-1}$$

α is estimated with $1-\alpha = \frac{wL}{Y}$. It looks like the TFP fell a lot during the Great Depression in the US. But technological regress is unlikely to be the source of the Depression. At least, we are not aware of any immediately apparent technological problem leading to the Great Depression.[1] More generally, we cannot identify large movements in technology (as we see in TFP) with particular events and discoveries.

- Another problem with using the Solow residual in RBC models as a structural shock is that Z_t does not seem "exogenous". Intuitively, we want to have an exogenous shock that is not correlated with other shocks if we want to identify the effects of this shock. Otherwise, when we look at the response of the system to a shock, we don't know if we see the effect of a particular force "technology" or if we see an effect of technology being correlated with something else. Hence, we often require that $\rho(Z, G) = 0$. Indeed, there is no particular reason to believe that shocks to government expenditure cause higher levels of technology (at least in the short run). However, when we look at the Solow residual in the data we see that it is correlated with non-technology shocks:

  - Hall (1988): $\rho(Z, \text{ demand shifters}) \neq 0$
  - Evans (1992): $\rho(Z, \text{ monetary policy shocks}) \neq 0$

  This is a problem because if technology shocks are endogenous, we can have a good fit of the model but there is no causal effect of technology shocks on fluctuations. We could not say that technology drives business cycles.

- RBC models typically have only real shocks:

  - technology
  - government expenditure
  - taxes

[1] Lucas (1994) notes: "Imagine trying to rewrite the Great Contraction chapter of A Monetary History with shocks of this kind playing the role Friedman and Schwartz assign to monetary contractions. What technological or psychological events could have induced such behavior in a large, diversified economy? How could such events have gone unremarked at the time, and remain invisible even to hindsight?"

- preferences

  However, nominal/monetary shocks apparently can be very important (e.g., Friedman and Schwartz (1963)). Omitting these shocks can lead to erroneous conclusions.

3. There is a concern about the plausibility of assumed parameters in the model. For instance, for RBC models to work, one should have a high labor supply elasticity, and also a high intertemporal labor supply elasticity. However, micro-level evidence suggest that these elasticities are pretty small. When we work with aggregate model, we have a combination of two things: i) micro-level elasticities; ii) aggregation. Aggregation can be very important. One striking example is due to Houtthakker (REStud, 1955). He shows that if every firm has Leontieff-type production function and distribution of firms is Pareto then aggregate production function is Cobb-Douglas. Hence the distinction between intensive and extensive margins of adjustment could be very important.

4. RBC models lack strong propagation mechanisms (Cogley and Nason (1995)). Basically to match persistence we see in the data, we have to rely on very persistent shocks. But if we have to stick very persistent shocks into the model, what is the value of the model in explaining persistence of Y, C, I? We get the conclusion by construction. Furthermore, Rotemberg and Woodford (1996) find that the model fails to reproduce predictable movements (and comovements) of variables we observed in the data. In other words, the model has to excessively rely on the volatility of exogenous shocks to match the level of volatility of variables in the data.

3.2.1 Cogley and Nason (1995)

Cogley and Nason (1995) note that in the data output growth rate is serially correlated. That is, if output growth rate is high today, it is likely to be high in next periods. To document this fact, Cogley and Nason use a variety of techniques:

- autocorrelation function (ACF) (in the data, they found that at lags 1 and 2, the autocorrelation is positive and significant).

- spectrum and impulse response (here the idea is that you can approximate series with sine and cosine functions, and since these functions are periodic we can look at the contribution of various cycles to total variation).

Then Cogley and Nason write down a basic RBC model and note that the model has four broad mechanisms of propagating shocks.

1. built-in persistence of shocks: $Z_t = \rho Z_{t-1} + \epsilon_t$ where ρ governs the persistence of Z_t.

2. capital accumulation: when technology is high, I save my output in the form of capital and then enjoy higher output in the future by using my capital in production and having smaller investment for some time.

3. intertemporal substitution: I can have large labor supply in the time of the shock and then spread below-average labor supply over subsequent periods.

4. adjustment costs, gestation lags, etc.: I get a shock today but some of its effect on K, L, or other variables are going to realize a few periods later.

The model analyzed by Cogley and Nason has two shocks:

- technology shock Z_t: random walk (RW)

- government spending G_t: AR(1) process

Then they calibrate and simulate the model (simple model with no adjustment costs). They obtain no serial correlation in the growth rate of output in the model (see Figure 3.1).

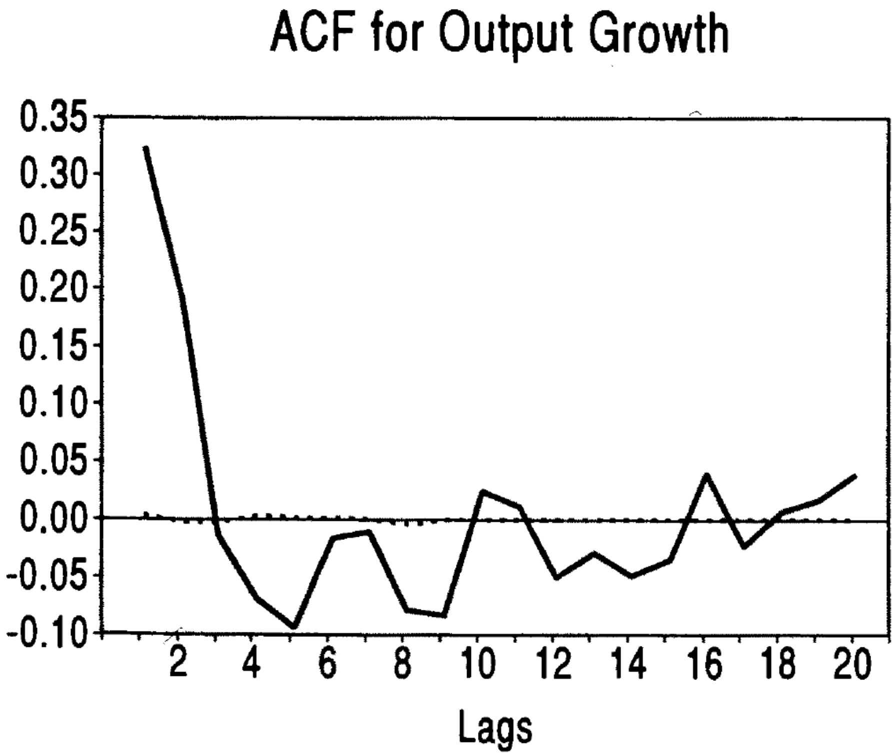


Figure 3.1: Autocorrelation function for output growth rate

Notes: This figure is taken from Cogley and Nason (1995), Figure 3. Solid lines show sample moments, and dotted lines show moments that are generated by an RBC model.

Another metric they use is empirical impulse responses. They identify the shocks from the data:

$$\begin{bmatrix} \Delta Y_t \\ L_t \end{bmatrix} = \begin{bmatrix} a_{11}(L) & a_{12}(L) \\ a_{21}(L) & a_{22}(L) \end{bmatrix} \begin{bmatrix} \epsilon_t^S \\ \epsilon_t^D \end{bmatrix} \tag{3.1}$$

$\epsilon_t^S \sim Z_t$ and $\epsilon_t^D \sim G_t$. The identifying assumption is that only Z_t has permanent effect on Y_t.[2]

Then Cogley and Nason suppose that $\Delta Z_t = \rho \Delta Z_{t-1} + \epsilon_t$ (ARIMA(1,1,0) process). Using this process they can match the serial correlation in the growth rate of output but they get counterfactual predictions in other dimensions:

[2]Obviously the question is what happens if you have a long lasting demand shock, e.g. Cold War.

- too much serial correlation in TFP
- fail to match other facts: response of growth of output to a transitory shock in technology. They cannot generate the hump-shaped response observed in the data.

If you add capital-adjustment:

$$Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \frac{\alpha_K}{2}\left(\frac{\Delta K_t}{K_{t-1}}\right)^2$$

or gestation lags (invest now, get capital installed and ready to work in S periods), the response of growth output is not satisfying either: one can get some persistence in growth output, but not a hump-shaped response. Labor-adjustment models also have problems.

The bottom line in Cogley and Nason (1995) is that you need to build a lot of persistence to shocks because the model does not have an ability to predict very persistent responses to serially uncorrelated shocks.

3.2.2 Rotemberg and Woodford (1996)

Rotemberg and Woodford (1996) have a simple but powerful idea: in the data shocks to technology lead to large ***predictable*** movements and ***co***movements in the series. But in the basic RBC model, ***predictable*** movements are small. Furthermore, ***predictable*** comovements can in fact be of the wrong sign.

To identify technology shocks, they also use VAR with three variables growth are of output Δy_t, consumption to output ratio $c_t - y_t$, and hours h_t:

$$\begin{bmatrix} \Delta y_t \\ c_t - y_t \\ h_t \end{bmatrix} = u_t = A u_{t-1} + \epsilon_t \tag{3.2}$$

Note that variables in this system are stationary. We want to know the predictable movements in the variables. To construct these predictable moments, Rotemberg and Woodford use impulse response functions (IRFs), which is a path of a variable in response to a shock. In the empirical model considered by Rotemberg and Woodford, the impulse response for the growth rate of output is:

1. $e_1 I \epsilon_t$

2. $e_1 A \epsilon_t$

3. $e_1 A^2 \epsilon_t$

   $\vdots$

(k) $e_1 A^k \epsilon_t$

where $e_1 = [1\ 0\ 0]$ is the selection vector. Note that this response is for the first difference of output y. The cumulative response of the growth rates of output (which gives us the ***level*** response of output y) is:

$$\begin{aligned}
\Delta\hat{y}_t + \Delta\hat{y}_{t+1} + \ldots + \Delta\hat{y}_{t+k} &= \hat{y}_t - \hat{y}_{t-1} + \hat{y}_{t+1} - \hat{y}_t + \ldots + \hat{y}_{t+k} - \hat{y}_{t+k-1} \\
&= \hat{y}_{t+k} - \hat{y}_{t-1} = \hat{y}_{t+k} - y_{t-1} \\
&= e_1(I + A + A^2 + \ldots + A^k)\epsilon_t
\end{aligned}$$

where hats indicate that we look at ***predictable*** movements. We can also re-write this expression as follows:

$$\begin{aligned}
\Delta\hat{y}_t^k &\equiv \hat{y}_{t+k} - \hat{y}_t \\
&= e_1(A + A^2 + \ldots + A^k)\epsilon_t \\
\Delta\hat{y}_t^k &= B_y^k \epsilon_t
\end{aligned}$$

Here $\Delta\hat{y}_t^k$ provides an estimate (prediction) of how output is going to move between period t and $t+k$. That is, standing in time t, we can expect that on average output is going to change by $\Delta\hat{y}_t^k$ in k periods from t.

We can also consider the response of the model in the very long run by sending k to infinity:

$$\begin{aligned} \Delta \hat{y}_t^{+\infty} &= \lim_{k \to +\infty} \hat{y}_{t+k} - \hat{y}_t \\ &= e_1 (I - A)^{-1} A \epsilon_t \end{aligned}$$

Using these predictable movements in output, we can assess how much variation in output is predictable at a given horizon k. Specifically, the predictable variation in output at horizon k due to a given shock ϵ_t is given by

$$\text{cov}(\Delta \hat{y}_t^k, \Delta \hat{y}_t^k) = \text{cov}(B_y^k \epsilon_t, B_y^k \epsilon_t) = B_y^k \Omega_\epsilon (B_y^k)^T$$

Likewise we can compute predictable ***co***movements across variables. For example, the predictable covariance between output and consumption is given by:

$$\text{cov}(\Delta \hat{y}_t^k, \Delta \hat{c}_t^k) = \text{cov}(B_y^k \epsilon_t, B_c^k \epsilon_t) = B_y^k \Omega_\epsilon B_c^k$$

We can then compare $\text{cov}(\Delta \hat{y}_t^k, \Delta \hat{y}_t^k)$, which are predictable movements, with $\text{cov}(\Delta y_t^k, \Delta y_t^k)$, which are total movements. Given estimates of A in the data and A implied by the model, we can look at horizon $k = 12$ quarters and get:

- $\text{cov}(\Delta \hat{y}_t^k, \Delta \hat{y}_t^k) = 0.0322$ in the data
- $\text{cov}(\Delta \hat{y}_t^k, \Delta \hat{y}_t^k) = 0.0023$ in the model
- $\text{cov}(\Delta y_t^k, \Delta y_t^k) = 0.0445$ in the data
- $\text{cov}(\Delta y_t^k, \Delta y_t^k) = 0.0212$ in the model

Importantly, predictable movements are a big factor of total output movements in the data. In the model, predictable movements are only a tiny fraction of variation in output.

One may use the following example to better understand this result. Suppose we are interested in variable X_t which follows an AR(1) process $X_t = \rho X_{t-1} + \epsilon_t$ with $\epsilon_t \sim iid(0, \sigma_\epsilon^2)$. The variance of the variable is $\sigma_\epsilon^2/(1-\rho^2)$. Thus a certain level of volatility for the variable can be achieved by increasing σ_ϵ^2 or by increasing ρ. However, these two approaches have sharply different implications for predictability of X_t. In the extreme case of $\rho = 0$, X_t is not predictable at all. In contrast with ρ close to one, predictability is high. The Rotemberg-Woodford result may be interpreted as suggesting that RBC models match volatility via σ_ϵ^2 which contrasts with the data were volatility is high via ρ.

To assess the importance of predictable comovements, we can regress one predictable movement on another. For example, if we regress the change in consumption at horizon k $\Delta\hat{c}_t^k$ on the change in output at horizon k $\Delta\hat{y}_t^k$, we get:

- 0.2-0.5 in the data
- 2.5 in the model

Likewise, regress the change in hours at horizon k $\Delta\hat{h}_t^k$ on $\Delta\hat{y}_t^k$ and get:

- 1 in the data
- -1 in the model

There are two striking differences. First, hours in the model are falling when you expect high output while in the data high hours predict high output. Second, the response of consumption in the model is too high relative to what we see in the data (10 times bigger). This means that the model predicts that the variability of predictable consumption movements should be larger than the variability of predictable output movements whereas, in practice, it is not. The model makes this counterfactual prediction because the high interest-rate changes accompanying a shortfall of capital from its steady state induce a substantial postponement of consumption. The model's high consumption coefficient implies that periods of high expected-output growth should also be periods in which the ratio of consumption to income is low. In other words, in the data $C/Y \nearrow$ means that Y has to grow in the future. In the model, for me to anticipate high output in the future, I should consume less now, i.e. C/Y is low. Intuitively, I need to postpone consumption until good times. Intertemporal substitution explains why hours h have a negative coefficient. I need to work hard now because the interest rate is high.

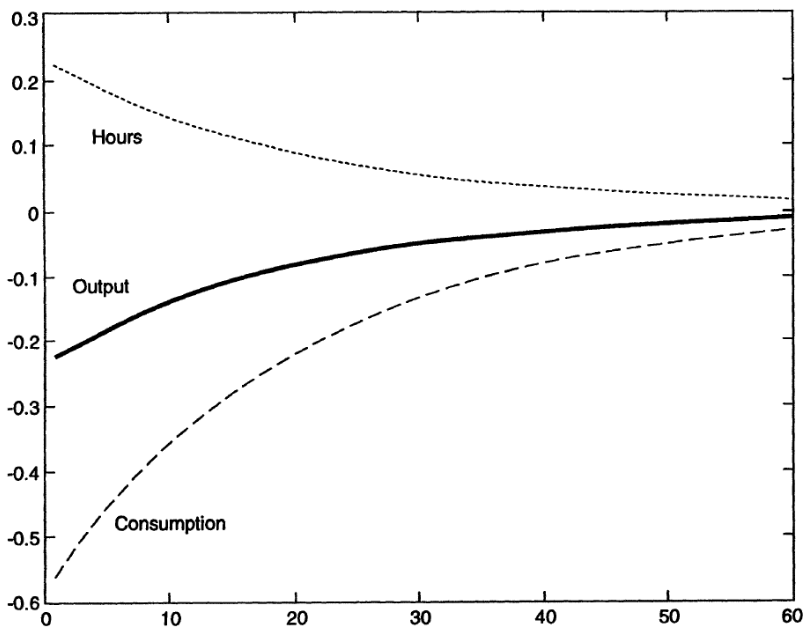


Figure 3.2: Evolution of output, hours, and consumption when capital starts one percent below its steady state; Rotemberg and Woodford (1996), Figure 2.

The most fundamental failing of the model can be explained simply as follows. The model generates predictable movements only as a result of departures of the current capital stock from its steady-state level. When the current capital stock is relatively low, its marginal product is relatively high so rates of return are high as well. With relatively strong intertemporal substitution, this leads individuals to enjoy both less current consumption and less current leisure than in the steady state. So, consumption is expected to rise while hours of work are expected to fall. More generally, consumption and hours are expected to move in opposite directions. This means that the regression coefficient of consumption on output and the regression coefficient of hours on output should have opposite signs. This intuition is illustrated by IRFs in a basic RBC model (see Figure 3.2). Instead, the data suggests that both of these coefficients are positive.

Why do we have a failure? Rotemberg and Woodford suggest that the model is not sufficiently rich and we miss some margins (e.g., variable factor utilization) and we need more than one shock.

Chapter 4

Extensions of the basic RBC model

In the previous lecture, we investigated the properties of the baseline real business cycle (RBC) model. Overall, the conclusion was that to a first-order approximation, the properties of the model follow from the properties of the forcing variables:

1. The model does relatively little to amplify shocks. In other words, we have to rely on very volatile exogenous shocks to match the volatility of endogenous variables.

2. The model has very weak internal propagation mechanisms. In other words, we have to rely on persistent exogenous shocks to match the persistence of endogenous variables.

3. Wages in the model are too procyclical.

We need to find mechanisms that can fix these important issues. Although there are many suggestions in the literature, we are going to focus in this lecture only on a few specific mechanisms: externalities, variable capital utilization, countercyclical markups and structure of labor markets. Each of these mechanisms are theoretically and empirically appealing. We will see that these mechanism often generate flat labor demand curves which help to reduce the volatility of wages, increase volatility of employment and more generally improve amplification and propagation properties of business cycle models.

4.1 Capital Utilization

4.1.1 Empirical regularity

We have examined cyclical properties of many variables but we did not investigate how capital utilization varies over business cycle. Capital utilization measures how intensively machines and structures are employed in the production process. There are several measures of capital utilization in the literature. Note that in the literature ***capacity*** utilization is often confused with ***capital*** utilization. The former is a measure of how intensively all resources are used while the latter refers only to capital input. In practice, however, capacity and capital utilization rates are highly correlated. Since at this point we are interested in qualitative conclusions only, we can focus only on qualitative facts about cyclical properties of variable capital utilization. Although the properties of variable capital utilization can vary from industry to industry, we are going to ignore these differences and examine only aggregate measures of capital utilization.[1] These facts are:

- capital utilization is highly procyclical
- capital utilization is a least as volatile as output at the business cycle frequencies

Note that in the basic RBC model we assume that capital utilization is the same which is sharply at odds with the data.

4.1.2 A RBC model with capital utilization

Consider the following model. The central planner solves[2]:

$$\max \qquad \mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t \left\{ \ln C_t - \frac{L_t^{1+1/\eta}}{1+1/\eta} \right\}$$

[1] Typically three types of industries are distinguished: conveyer (e.g., auto assembly line), workstation (e.g., watch factory), continuous (e.g., oil refinery or steel mill). Capital utilization is least variable for the continuous process industries since in these industries capital is used 24/7.

[2] You can show that this central planner problem leads to the same first order conditions as in the decentralized economy.

$$\text{s.t.} \qquad Y_t = Z_t(u_t K_{t-1})^{\alpha} L_t^{1-\alpha} \tag{4.1}$$

$$Y_t = C_t + I_t \tag{4.2}$$

$$K_t = (1-\delta_t)K_{t-1} + I_t \tag{4.3}$$

$$\delta_t = \frac{1}{\theta} u_t^{\theta} \tag{4.4}$$

u_t is the capital utilization rate. Here, the cost of using capital more intensively comes from faster depreciation of capital. We assume that $\theta > \alpha$ (otherwise utilization is always full, i.e. $u = 1$).

To understand the role of capital utilization in amplifying and propagating business cycles, consider the FOC for utilization:

$$\alpha \frac{Y_t}{u_t} = u_t^{\theta-1} K_{t-1}.$$

The LHS ($\alpha \frac{Y_t}{u_t}$) is the marginal product from more intensive utilization of capital. The RHS ($u_t^{\alpha-1} K_{t-1}$) is the marginal cost of using capital more intensively. This FOC yields:

$$u_t = \left(\alpha \frac{Y_t}{K_{t-1}}\right)^{1/\theta}. \tag{4.5}$$

Thus the optimal utilization is determined by the marginal product of capital. Here, this condition is saying that capital should be used more intensively during booms when MPK is high. From this FOC we can immediately say that u will be procyclical since capital cannot change very much over the cycle.

Now, substitute condition 4.5 into the production function to get:

$$Y_t = A Z_t^{\frac{\theta}{\theta-\alpha}} K_{t-1}^{\frac{\alpha(\theta-1)}{\theta-\alpha}} L_t^{\frac{(1-\alpha)\theta}{\theta-\alpha}} \tag{4.6}$$

Note that $\frac{\theta}{\theta-\alpha} > 1$, so utilization directly amplifies response to Z_t. Given that the production function has constant return to scale (CRS; i.e. $\alpha + (1-\alpha) = 1$), capital utilization has no effect on return to scale:

$$\frac{\alpha(\theta-1)}{\theta-\alpha}+\frac{(1-\alpha)\theta}{\theta-\alpha}=1 \tag{4.7}$$

Now suppose that $\alpha = 0.3$ and $\theta = 1.4$. The effective capital share is:

$$\frac{\alpha(\theta-1)}{\theta-\alpha}=0.3\times\frac{1.4-1}{1.4-0.3}\approx 0.1$$

This value is much smaller than the usual estimate of 0.3. The effective elasticity of output with respect to labor is ≈ 0.9, which we often see in the data. Hence, when technology $Z \nearrow$:

- direct effect from production function
- "elasticity effect" resulting from the positive effect of capital utilization on labor:
  - this amplifies technology shocks as it effectively increases the responsiveness of output to shocks (capital is fixed in the short run).
  - output becomes more elastic with respect to labor

Intuitively, utilization creates intratemporal substitution away from leisure and toward consumption. Thus it generates procyclical effects on labor and consumption. When $Z \nearrow$, MPL $\nearrow$, and $L \nearrow$ and because $Y \nearrow$ so much, you can increase C and I at the same time. There is no need to sacrifice consumption today to allow for capital accumulation.

There is no need to have very high intertemporal elasticity of substitution to generate persistence and amplification (i.e. no need to let labor respond intrinsically strongly to shocks to productivity). Furthermore, variable capital utilization makes labor demand curve more elastic. The reason why it may be important is that when we have demand shocks which affect marginal utility of wealth and hence shift labor supply (e.g., government expenditures shock), we can have large moments in employment and relatively small movements in wages.

4.1.3 Results

Burnside and Eichenbaum (1996) estimate and calibrate a model similar to what is described above. They show that:

1. Once we take utilization into account, the volatility of technology shocks falls by 30%. Not surprising since $TFP = \check{Y}_t - \alpha \check{K}_{t-1} - (1-\alpha)\check{L}_t = Z_t + \alpha u_t$.

2. Variable utilization helps to match the volatility of endogenous variables in the data. This is good news: with smaller shocks we can do the job, we don't need to rely on implausibly large fluctuations in technology.

3. The impulse response functions (IRFs) to shocks are more persistent than in the baseline RBC model.

4. The model with variable utilization can match serial correlation in output without relying on counterfactual serial correlation in technology changes.

In summary, endogenously varying capital utilization can serve as a useful amplification and propagation mechanism.

4.2 Externalities

Another useful mechanism for improving RBCs is externalities. There are many ways to introduce externalities. We will work with a specific case that can be supported by a competitive equilibrium. This formulation is due to Baxter and King (1991) who introduce externalities via increasing return to scale (IRS) in production at the aggregate level and constant returns to scale (CRS) in production at the firm level.

4.2.1 Assumptions

Firms

Continuum of firms indexed by i:

$$Y_{it} = Z_t E_t K_{i,t-1}^{\alpha} L_{it}^{1-\alpha}$$

where Z_t is technology and E_t is externality. Firms treat Z_t and E_t as exogenous. Assume that $E_t = Y_t^{1-1/\gamma}$ where:

$$Y_t = \int Y_{it} di \tag{4.8}$$

and $\gamma \geq 1$. Having a continuum of firms ensures that no firm has an effect on aggregate outcomes.

Firms maximize the net present value (NPV) of profits and equalize the cost of inputs with the marginal return of inputs:

$$\begin{aligned}\text{rental rate of capital: } R_t &= \alpha \frac{Y_{it}}{K_{i,t-1}} - \delta \\ \text{wages: } W_t &= (1-\alpha)\frac{Y_{it}}{L_{it}}\end{aligned}$$

With a constant return-to-scale technology (CRS):

$$(R_t + \delta)K_{i,t-1} + W_t L_{it} = Y_{it}$$

In a symmetric equilibrium, firms are identical: $\forall i,\ K_{it} = K_t,\ Y_{it} = Y_t,\ L_{it} = L_t$. The aggregate output is:

$$\begin{aligned}Y_t &= \int Y_{it} di \\ &= \int Z_t E_t K_{i,t-1}^{\alpha} L_{it}^{1-\alpha} di\end{aligned}$$

$$
\begin{aligned}
&= Z_t E_t K_{t-1}^{\alpha} L_t^{1-\alpha} \\
&= Z_t Y_t^{1-1/\gamma} K_{t-1}^{\alpha} L_t^{1-\alpha} \\
&= (Z_t K_{t-1}^{\alpha} L_t^{1-\alpha})^{\gamma}
\end{aligned}
$$

At the aggregate level, there are increasing return-to-scale (IRS) in production. The marginal products are:

$$
\begin{aligned}
MPK_t &= \gamma\alpha \frac{Y_t}{K_{t-1}} \\
MPL_t &= \gamma(1-\alpha)\frac{Y_t}{L_t}
\end{aligned}
$$

To finish the description of the production side, we need the definition of investment and the resource constraint:

$$
\begin{aligned}
K_t &= (1-\delta)K_{t-1} + I_t \\
Y_t &= C_t + I_t + G_t
\end{aligned}
$$

Households

Households maximize the following utility:

$$
\mathbb{E}\sum \beta^t \left\{\ln(C_t - \Delta_t) + \ln(1 - L_t)\right\}
$$

where Δ_t is a preference shock. Δ_t is introduced to generate acyclical real wages. The budget constraint is:

$$
A_{t+1} + C_t = (1 + r_t)A_t + W_t L_t + \Pi_t
$$

A_t is the quantity of assets held in period t, $r_t = R_t - \delta$ is the interest rate, and Π_t is firms' profit, which is redistributed to households. The FOCs are:

$$\begin{aligned}\frac{1}{C_t - \Delta_t} &= \lambda_t \Rightarrow C_t = \frac{1}{\lambda_t} + \Delta_t \\ \frac{1}{1 - L_t} &= \lambda_t W_t\end{aligned}$$

Aggregate production function

The log-linearized production function is given by:

$$\check{Y}_t = \gamma(\alpha \check{K}_{t-1} + (1-\alpha)\check{L}_t) + \gamma \check{Z}_t$$

Having IRS has a direct effect on output since Z_t is multiplied by $\gamma > 1$. When $\gamma > 1$, labor demand L^d is flatter, and MPL is more sensitive to Z_t. The social MPL is flatter than individual MPLs, as showed on Figure 4.1.

Thus:

$$\begin{aligned}G \nearrow &\Rightarrow Y \nearrow \\ Y \nearrow &\Rightarrow E \nearrow \Rightarrow L^d \nearrow\end{aligned}$$

If γ is really large, the social labor demand could even be upward-sloping., i.e. $\gamma(1-\alpha) > 1$. Then demand-side shocks could increase both wages and output. This is very important because before we discounted demand shocks as they qualitatively generated countercyclical movements in wages. Let's make an even bolder assumption and suppose that the social L^d is steeper than L^s.

Suppose that there is a shock to preferences due to "animal spirits": $\Delta_t \searrow \Rightarrow \lambda_t \searrow$. Now shocks in expectations are driving business cycles. Consumers feel wealthy because they may think that output is higher. They consume more and indeed output becomes higher. Shocks

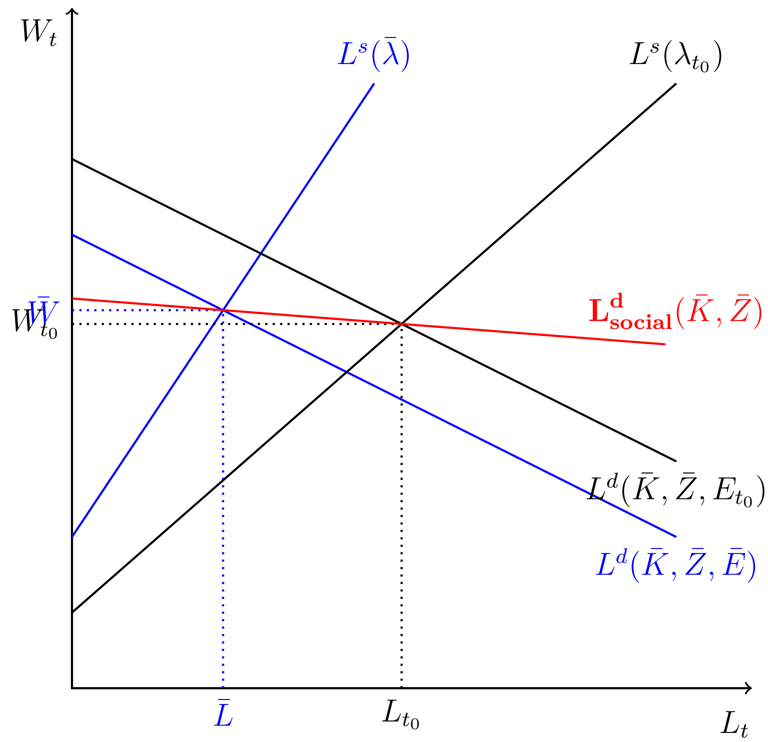


Figure 4.1: Social labor demand.

to expectations become self-fulfilling.

4.2.2 Recap

In our analysis of the basic RBC model, we found that the basic RBC model lacks strong internal amplification and propagation mechanisms. We considered two extensions that improve the performance of the RBC model:

- capital utilization
- increasing return-to-scale (IRS) in production

These two mechanisms enhance the properties of the model as they generate more endogenous amplification and propagation. These modifications in the model were done in competitive equilibrium. Now we extend our framework to models where firms (and maybe

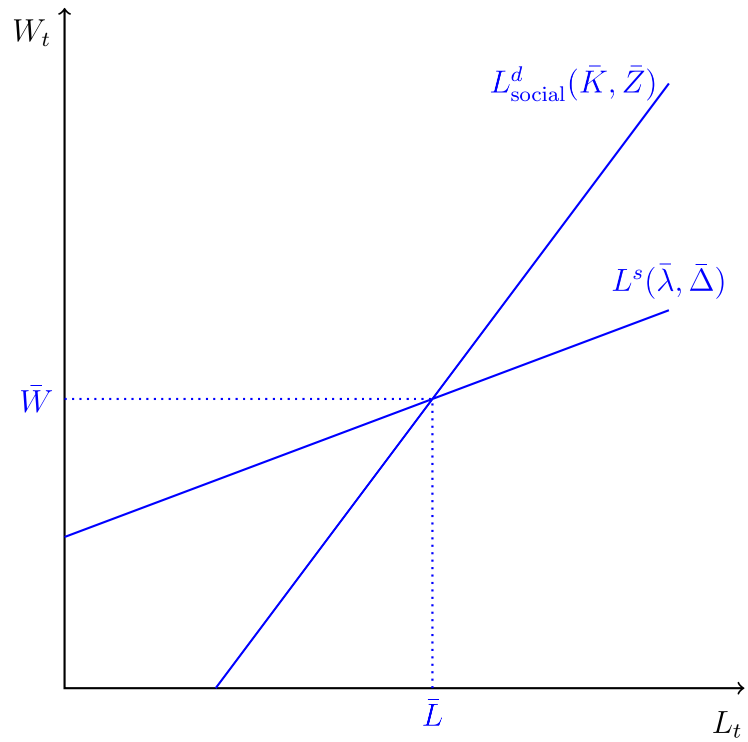


Figure 4.2: Multiple equilibria with social labor demand.

workers) have market power. This extension will have immediate ramifications for the properties of the basic RBC model and it will also help us later when we will analyze models where prices are not entirely flexible. Table 4.1 puts our approach into perspective.

4.3 Imperfect Competition

4.3.1 Producer's behavior

We will assume that firms maximize profits, hire labor and rent capital from households.

$$\begin{aligned} \max \quad & P(Y)Y - RK_i - WL_i \\ \text{s.t.} \quad & Y_i = F(K_i, L_i) \quad (\eta) \end{aligned}$$

		Money neutral?	
		Yes	No
Competitive markets?	Yes	RBC ⓐ	
	No	Externalities ⓑ Imperfect competition	Sticky price models ⓒ

Table 4.1: Our approach: ⓐ→ⓑ→ⓒ

where $P(Y)$ is the function describing the demand curve and Lagrange multiplier η is interpreted as marginal cost of producing one extra unit of Y. The FOCs are:

$$\begin{aligned} P(Y) + P'(Y)Y &= \eta \\ R &= \eta F'_K \\ W &= \eta F'_L \end{aligned}$$

This yields:

$$\eta = P(1 + P'(Y)Y/P) = P(1 + 1/\epsilon^D) = \frac{P}{\mu}$$

$\mu = P/\eta = P/MC$ is the markup over marginal cost. Importantly, the markup summarizes a great deal of information about the elasticity of demand and it can serve as a sufficient statistic for demand conditions and the intensity of competition. [3]

Note that profit is:

$$\begin{aligned} \Pi &= P(Y)Y - RK - WL \\ &= P(Y)Y - \eta F'_K K - \eta F'_L L = P(Y)Y - \eta Y\Big(\frac{F'_K K}{Y} + \frac{F'_L L}{Y}\Big) = P(Y)Y - \eta Y\gamma \\ &= P(Y)Y\Big(1 - \frac{\gamma MC}{P}\Big) \\ &= P(Y)Y\Big(1 - \frac{\gamma}{\mu}\Big) \end{aligned}$$

[3] We can get the same result if we relax the assumption of profit maximization and impose a weaker condition that firms minimize costs.

The firm with increasing returns to scale in production $\gamma > 1$ can make a positive profit only if $\mu > 1$. We can rewrite this condition and find that the share of ***economic*** profits in total revenue is intimately related to returns to scale in production and markup:

$$s_\pi = \Pi/[P(Y)Y] = \left(1 - \frac{\gamma}{\mu}\right)$$

Hence, we can use the profit share as a useful check for estimates of returns to scale and markup.

4.3.2 Demand side

We will assume that consumers have a love for variety. They want to consume a bit of every good. The idea here is that the marginal utility of consuming any given good may be quickly declining in the consumption of this good, but you can attenuate this problem by consuming more goods. Although there are many formulations of the love for variety, we will consider only the standard one which is due to Dixit and Stiglitz (1977). We assume that households maximize the following utility function

$$\begin{aligned} \sum \beta^t U(C_t, L_t) &= \sum \beta^t (\ln(C_t) + \ln(1 - L_t)) \\ C_t &= \left[\int_0^1 C_{it}^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

Here households consume C_t which is a bundle or composite of various goods. To remove the strategic effects of consuming any particular good on the bundle C_t, we assume a continuum of goods. Parameter σ gives the elasticity of substitution across goods and also the slope of the demand curve. In other words $\sigma = |\epsilon^D|$. In this setup, the optimal markup is given by:

$$\begin{aligned} \eta &= P(1 + 1/\epsilon^D) \\ \mu &= P/\eta = \frac{1}{1 + 1/\epsilon^D} \\ \mu &= \frac{\sigma}{\sigma - 1} \end{aligned}$$

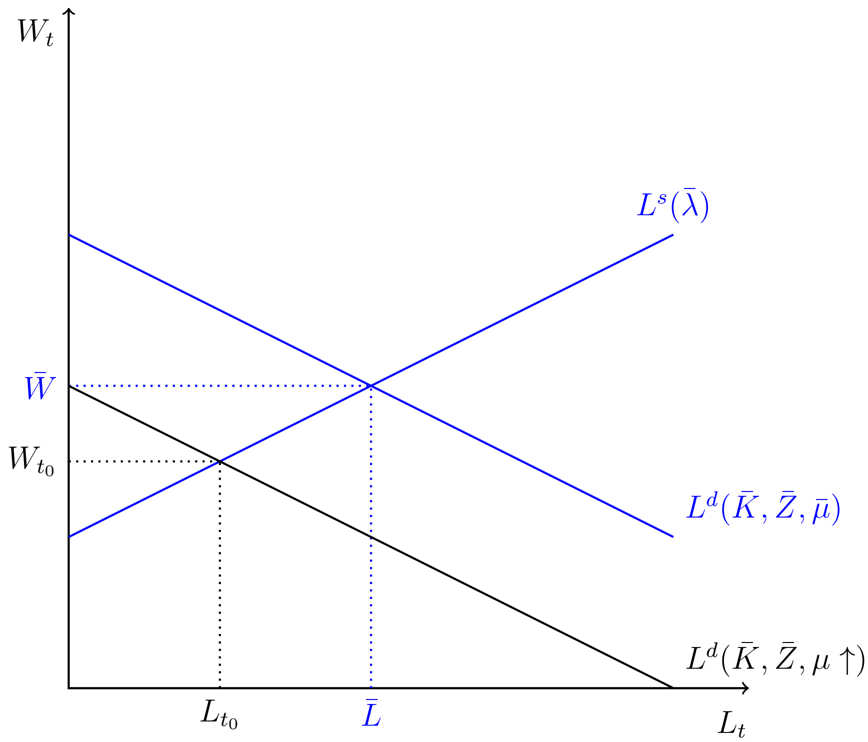


Figure 4.3: Response of labor demand to increase in markup μ.

Note that in this formulation markup does not depend on any aggregate variable. However, suppose markup can vary. What would be the effects of changes in markup on wages and employment? To answer this question consider the FOC for labor demand:

$$
\begin{aligned}
W &= \eta F'_L \\
&= P(1 + 1/\epsilon^D)F'_L \\
W/P &= (1 + 1/\epsilon^D)F'_L \\
&= \frac{F'_L}{\mu} \\
\ln(W/P) &= -\ln(\mu) + \alpha \ln(K/L) + \ln(Z)
\end{aligned}
$$

Hence a variable markup would work as another shift variable which is illustrated in Figure 4.3.

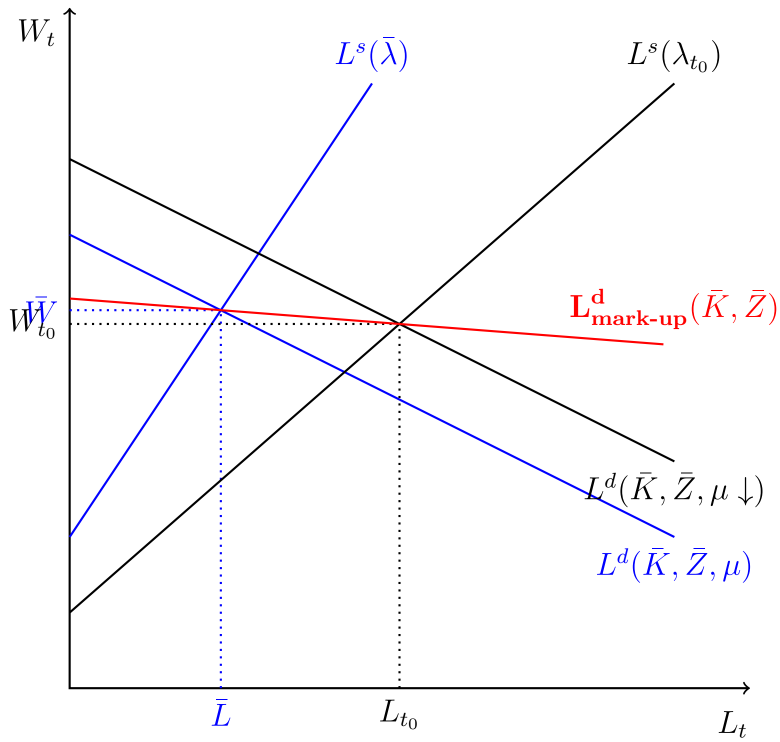


Figure 4.4: Labor demand with countercyclical mark-ups.

4.3.3 Countercyclical markups

Now suppose that the markup is countercyclical, i.e. $\mu'(Y) < 0$. With countercyclical markups, we can generate flatter labor demand curve, as shown in Figure 4.4.

1. $G \uparrow \Rightarrow \lambda \downarrow \Rightarrow L \uparrow$

2. $L \uparrow \Rightarrow Y \uparrow \Rightarrow \mu \downarrow \Rightarrow L^d \uparrow$

Hence with countercyclical markups, labor demand curve get flatter and we can achieve results similar to the results we have in models with externalities and variable capital utilization. We will study models of variable markups in the next section.

4.4 Models of Markup

Recall that markup is given $\mu = (1 + 1/\epsilon^D)^{-1}$ where ϵ^D is the elasticity of demand. If ϵ^D is countercyclical then μ is countercyclical. However, we have to be careful here. If output is non-stationary and grows over time, we cannot simply have $\mu'(Y) < 0$ because we would converge to $\mu = 1$ as output keeps growing. We need to formulate the relationship as $\mu = \mu(\check{Y}_t)$, where $\check{Y}_t$ is deviation from steady-state or balanced growth path.

4.4.1 Cyclical Composition of Demand

Let $s_t^I = \frac{I_t}{Y_t}$ be the share of investment in output. Suppose consumers use $C_i \in [0,1]$ for $C_t = \left[\int_0^1 C_{it}^{\frac{\sigma_C - 1}{\sigma_C}} di\right]^{\frac{\sigma_C}{\sigma_C - 1}}$. Firms use the same variety of goods to construct composite investment:

$$I_t = \left[\int_0^1 I_{it}^{\frac{\sigma_I - 1}{\sigma_I}} di\right]^{\frac{\sigma_I}{\sigma_I - 1}}$$

Now suppose that $\sigma_I > \sigma_C$, i.e. the elasticity of substitution in the investment sector is larger than the elasticity of substitution in the consumption sector. Then the aggregate markup is the average of markups for firms and consumers:

$$\mu_t \simeq s_t^I \frac{\sigma_I}{\sigma_I - 1} + s_t^C \frac{\sigma_C}{\sigma_C - 1}$$

Since s_t^I is procyclical, μ_t is countercyclical if $\sigma_I > \sigma_C$. See Gali (1994) for more details.

4.4.2 Sticky prices

If prices are inflexible (or sticky), one can also generate countercyclical markups. Recall that $\mu = \frac{\bar{P}}{MC(Y)}$ and $MC'(Y) > 0$. Thus $\mu'(Y) < 0$. Most prices do not change frequency and so this explanation is particularly appealing on empirical grounds. We will cover models with inflexible prices in more details later in the class.

4.5 Evidence on Returns to Scale

Since qualitative behavior of business cycle models can depend on aggregate returns to scale in production, it is useful to examine empirical evidence on this matter. However, before we look at the empirical estimates of the returns to scale, having some theory on what is meant by returns to scale is helpful. Consider the following production function:

$$Y = F(K_t, L_t) - \Phi = K^{\alpha} L^{\beta} - \Phi$$

where Φ is the fixed cost of running a business and does not depend on produced quantity $F(K_t, L_t)$ as long as it is positive. It turns out that returns to scale depend on α, β and Φ. Consider these combinations:

(a) CRS: $\alpha + \beta = 1$, $\Phi = 0$

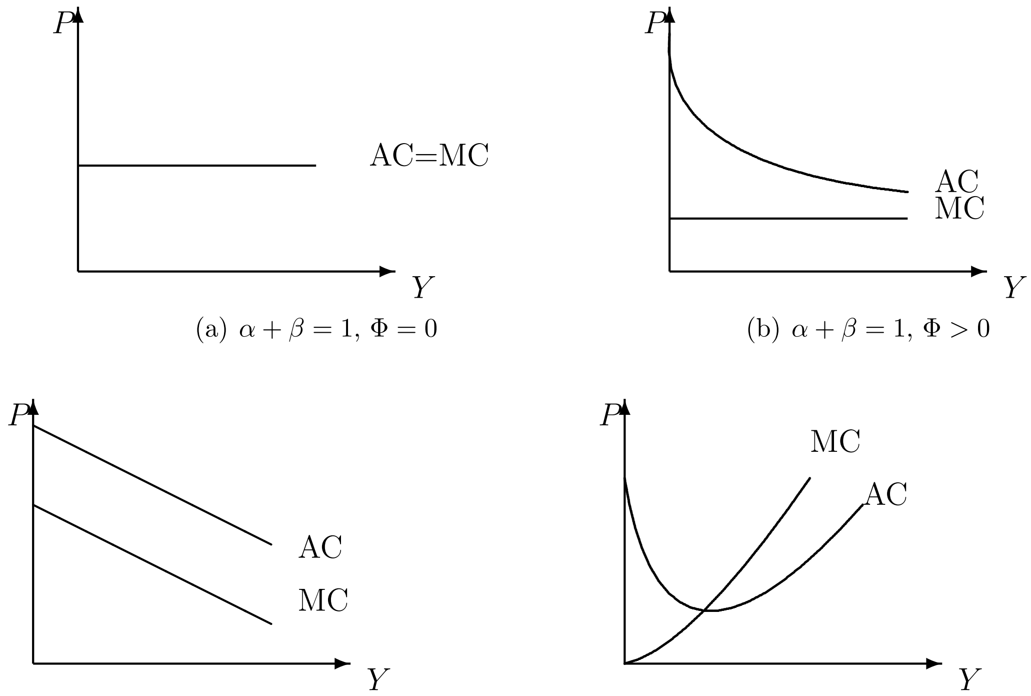
(b) $\alpha + \beta = 1$, $\Phi > 0$. $\gamma = \frac{AC}{MC} > 1$. Hence, Φ causes $s_{\pi} \Rightarrow \mu \Rightarrow \gamma$. So γ can be an endogenous parameter here because the MC is "normal", L^d is going to have a normal slope.

(c) $\alpha + \beta > 1$, $\Phi = 0$. L^d is going to be flatter.

(d) $\alpha + \beta < 1$, $\Phi > 0$. IRS is possible but L^d is going to be steeper because $MC \uparrow$. If $MC \downarrow$, production, inventory is procyclical (evidence from Bils and Klenow (AER 2000)).

We can link returns to scale γ to markup μ and the share of profits in sales s_{π}:

$$\begin{aligned}
\gamma &= \frac{F'_K K}{Y} + \frac{F'_L L}{Y} = \frac{(R/\eta)K + (W/\eta)L}{Y} \\
&= \frac{(RK + WL)/Y}{\eta} \\
\gamma &= \frac{AC}{MC} \\
\mu &= \frac{P}{MC} = \frac{P}{AC}\frac{AC}{MC} = \frac{P}{AC}\gamma = \frac{PY}{PY - \Pi}\gamma
\end{aligned}$$



(a) $\alpha+\beta=1$, $\Phi=0$

(b) $\alpha+\beta=1$, $\Phi>0$

(c) $\alpha+\beta>1$, $\Phi=0$

(d) $\alpha+\beta<1$, $\Phi>0$

$$= \quad \gamma\frac{1}{1-s_\pi}$$

If $s_\pi \approx 0$, then $\gamma \approx \mu$. There is evidence that $s_\pi \approx 0$:

- $Q \approx 1$ in the stock market
- Differences between producer and consumer prices are small.

One can use firm level data to estimate returns to scale at the micro level. Much of the previous literature argued that at the firm level production function is best characterized as CRS. In short, the evidence amount to finding CRS when one regresses appropriately deflated sales on measures of inputs such as employment (or hours), capital, materials, electricity, etc. Gorodnichenko (2007) argues that this interpretation of CRS applies for the revenue function, not production functions. Since CRS in revenue weakly implies IRS in production, the previous evidence suggest locally increasing returns to scale.

While the evidence from firm level data is certainly useful, we know that micro-level elasticities could provide a distorted picture of macro-level elasticities (recall our discussion of labor supply elasticity) and so ideally we would like to have estimates based on aggregate data. In a series of papers, Robert Hall (Hall (1988), Hall (1990)) uses demand shifters as instruments to estimate returns to scale: world oil prices, government defense spending, changes in monetary policy. These variables are unlikely to be correlated with technology (the error term in the regression) and hence are valid instruments. Hall finds that $\mu \in [2, 4]$ which is a very high estimate of markups. Since $s_\pi \approx 0$, it must be that γ is very high.

Hall does not make a distinction about whether returns to scale are internal or external. However, the source of increasing returns to scale is important. To establish the source, Caballero and Lyons (1992) estimate the following regression

$$y_i = \mu_i x + \beta_i \tilde{y} + \tilde{z}$$

where x are inputs and $\tilde{y}$ is total industry output. β is a measure of external returns to scale (RTS), which can be directly related to externality in the Baxter-King model. They estimate $\hat{\beta} \approx 0.3 - 0.4$. Hence, the implied returns to scale at the aggregate level is about $1.3 - 1.4$.

Much of the evidence on increasing returns to scale was undermined by works of Susanto Basu and John Fernald (e.g. Basu and Fernald (1997)). They argue that previous estimates of increasing returns to scale based on aggregaget data are likely to suffer from measurement errors and misspecification and one has to correct these issues to obtain correct estimates. For exaple, Basu and Fernald correct for inputs (RHS variable in the regression) for variable utilization rates by using directly observed utilization (Shapiro (1993), Shapiro (1986)) or using proxies for utilization such as consumption of electricity (Burnside and Eichenbaum (1996)) or hours of work per employees (Basu and Kimball (1997)). They find that $\hat{\mu} = 1.2 \Rightarrow \hat{\beta} \approx 0$. Thus, $\hat{\mu} = 1.2$ is consistent with $P > MC$ but it does not imply implausibly large markups. In summary, while increasing returns to scale is a potentially power amplification and propagation mechanism in theoretical models, it appears that the evidence to support it is weak.

4.6 Labor Markets

One of the problems in the basic RBC model was that wages were two procyclical while employment was not sufficiently procyclical. In the end we typically need L^s being flat and L^d being flat. We solved this problem partially by making labor demand very elastic when we considered countercyclical markups, increasing returns to scale and variable capital utilization. We can also consider several modifications that make aggregate labor supply more elastic even when labor supply for each worker is highly inelastic.[4]

- **Shirking model**: Shapiro and Stiglitz (1984) model of efficiency wages.
- **Gift exchange model**: Akerlof and Yellen (1990).
- **Effort model**: Solow (1979).

To get a flavor of how these models operate, consider the Solow model. Let $L^* = eL$ where L^* is effective labor and e is effort, which depends on wage w: $e = e(w)$, $e'(w) > 0$. The firm maximizes:

$$\begin{aligned} \max_{w} \quad & PF(e(w)L) - wL \\ \text{FOC:} \quad & PF'(e(w)L)e(w) = w \quad \text{(w.r.t. } L) \\ & PF'(e(w)L)e'(w)L = L \quad \text{(w.r.t. } w) \end{aligned}$$

Taking the ratio of the two FOCs yields:

$$\frac{e(w)}{e'(w)} = w \Rightarrow \epsilon_w := \frac{e'(w)w}{e(w)} = 1$$

[4]See Romer's textbook for more details.

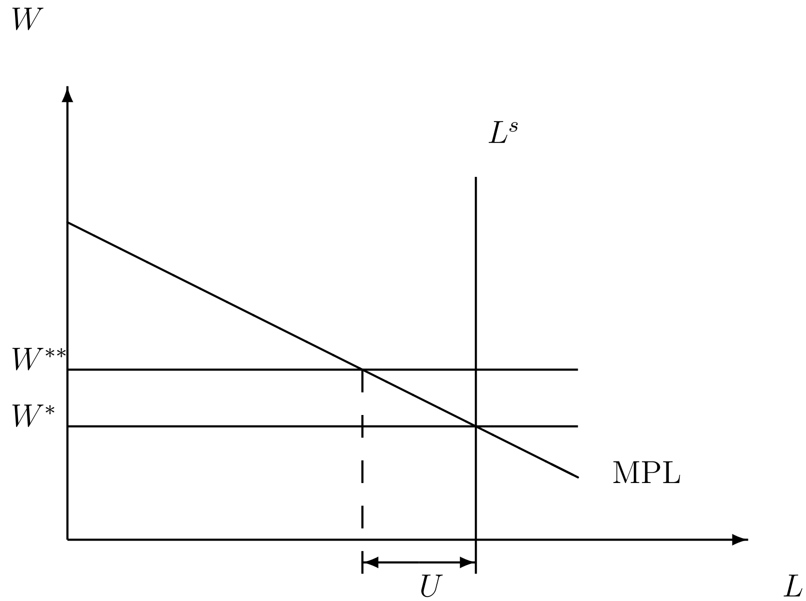


Figure 4.5: Unemployment in efficiency wage model.

So the optimality condition for the wage is:

$$\frac{e'(w^*)w^*}{e(w^*)} = 1$$

Note that the wage is not tied to the reservation wage of workers. In fact, the wage in this model does not necessarily fluctuate with the aggregate conditions as long as effort function is insensitive to these fluctuations. This makes wages very stable.

In general, models of efficiency wages (Shapiro-Stiglitz, Akerlof-Yellen, Solow) lead to some from of unemployment because either marginal product is decoupled from wages or because effective labor supply faced by firms is decoupled from the labor supply of workers. In these models, it is typical that fluctuations in demand can lead to fluctuations in unemployment (see Figure 5).

One can also make labor supply more elastic by introducing a home production sector. The idea here is that if one has an alternative of working at home then his or her supply of labor in formal markets is going to be more elastic. This is similar in spirit to why competition makes elasticities of demand larger.

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