ECON 220B MACROECONOMICS B PROBLEM SET # 4

MINKI KIM

1. Consumer's optimization problem

Consumer/worker's intertemporal constrained optimization problem is written as:

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\theta}}{1-\theta} - \gamma \frac{\varepsilon}{1+\varepsilon} h_t^{\frac{1+\varepsilon}{\varepsilon}} \right) \\ &\text{s.t. } c_t + b_{t+1} \leq w_t(s_t) h_t + (1-\delta) b_t + r_t(s_t) b_t \\ &\qquad b_t \geq 0 \end{aligned}$$

A consumer's disposable income consists of labor income, rate of return from the bond purchased in the previous period, and undepreciated part of the bond. She purchases consumptiong goods and also a bond. Negative $b_t, \forall t$ is interpreted as borrowing. Condition $b_t \geq 0$ implies even an unemployed (or non-working) individual should be able to pay back her debt.

2. Optimal consumption and labor supply

Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\theta}}{1-\theta} - \gamma \frac{\varepsilon}{1+\varepsilon} h_t^{\frac{1+\varepsilon}{\varepsilon}} + \lambda_t(s_t) \left(w_t(s_t) h_t + (1-\delta) b_t + r_t(s_t) b_t - c_t - b_{t+1} \right) + \phi_t(s_t) b_t \right]$$

First order conditions are:

$$\begin{aligned} c_t^{-\theta} &= \lambda_t(s_t) \\ \gamma h_t^{\frac{1}{\varepsilon}} &= \lambda_t(s_t) w_t(s_t) \\ \lambda_t(s_t) &= \beta \lambda_{t+1}(s_{t+1}) \left(1 - \delta + r_t(s_t)\right) + \phi_t(s_t) \end{aligned}$$

Combining the first and third conditions, we get the Euler conditions

$$\begin{cases} c_t^{-\theta} = \beta c_{t+1}^{-\theta} \left(1 - \delta + r_{t+1}(s_{t+1})\right) & \text{Case (1) when } b_t > 0 \\ c_t^{-\theta} = \beta c_{t+1}^{-\theta} \left(1 - \delta + r_{t+1}(s_{t+1})\right) + \beta \phi_{t+1}(s_{t+1}) & \text{Case (2) when } b_t = 0 \end{cases}$$

When the borrowing constraint is not binding(Case (1)), we have usual Euler equation. When borrowing constraint is binding, $\phi_{t+1}(s_{t+1})$, the Lagrange multiplier attached to the borrowing constraint, shows up in the Euler equation. Binding borrowing constraint means that the consumer is borrowing up to his limit to increase current period consumption as much as he can. Therefore, LHS denotes the marginal gain of increased consumption. The first term of RHS denotes the discounted loss in period $t+1$, coming from increased consumption in period t. The second term entails discounted marginal loss coming from forgone bond holdings.

For labor supply elasticity, note that $h_t = \left(\frac{1}{\gamma} c_t^{-\theta} w_t\right)^{\varepsilon}$. It is easy to show that $\epsilon_{h,w} = \frac{\partial h_t}{\partial w_t}\frac{w_t}{h_t} = \varepsilon$.

3. Firm's labor demand

Firm's static maxization problem is written as:

$$\max_{k_t, n_t} A_t(s_t) k_t^{\alpha} n_t^{1-\alpha} - w_t(s_t) n_t - r_t(s_t) k_t$$

From the first order condition for labor we get $w_t(s_t) = (1-\alpha)\frac{y_t(s_t)}{n_t}$

4. Equilibrium employment level

Labor market is cleared when $h_t = n_t$, i.e.

$$h_t^* = \left(\frac{1}{\gamma} c_t^{-\theta} w_t\right)^{\varepsilon} = \left(\frac{1}{\gamma} c_t^{-\theta} (1-\alpha)\frac{y_t}{h_t^*}\right)^{\varepsilon}$$

by isolating h_t^*, we get

$$h_t^* = \left(\frac{1}{\gamma} c_t^{-\theta} (1-\alpha) y_t\right)^{\frac{\varepsilon}{1+\varepsilon}}$$

5. Calibrating labor supply elasticity

Take log to $h_t = \left(\frac{1}{\gamma} c_t^{-\theta}(1-\alpha)y_t\right)^{\frac{\varepsilon}{1+\varepsilon}}$ and calculate its variance.

$$
\begin{aligned}
Var(\log h_t) &= Var\left[\frac{\varepsilon}{1+\varepsilon}\left(-\log\gamma - \theta\log c_t + \log(1-\alpha) + \log y_t\right)\right] \\
&= \left(\frac{\varepsilon}{1+\varepsilon}\right)^2 \left[Var(\log y_t) + \theta^2 Var(\log c_t) - 2\theta Cov(\log y_t, \log c_t)\right] \\
&= \left(\frac{\varepsilon}{1+\varepsilon}\right)^2 \left[Var(\log y_t) + Var(\log c_t) - 2Cov(\log y_t, \log c_t)\right] (\because \theta = 1) \\
&= \left(\frac{\varepsilon}{1+\varepsilon}\right)^2 \Bigg[Var(\log y_t) + Var(\log c_t) \\
&\qquad - 2\left(\sqrt{Var(\log y_t)}\sqrt{Var(\log c_t)}\right)\rho(\log y_t, \log c_t)\Bigg]
\end{aligned}
$$

Normalize $Var(\log y_t) = 1$, Then $Var(\log c_t) = 9/16, Var(\log h_t) = 9/25$. Also, $\rho(\log y_t, \log c_t)$ is given as $3/4$. We get $\varepsilon = 9.7660$ under these conditions. Hence the labor elasticity implied by the model by far exceeds the upper bound of microeconometric estimates.

6. Comparative statics: $\theta = 2$

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\begin{aligned}
Var(\log h_t) &= \left(\frac{\varepsilon}{1+\varepsilon}\right)^2 \left[Var(\log y_t) + 4Var(\log c_t) - 4Cov(\log y_t, \log c_t)\right] \\
&= \left(\frac{\varepsilon}{1+\varepsilon}\right)^2 \Bigg[Var(\log y_t) + 4Var(\log c_t) \\
&\qquad - 4\left(\sqrt{Var(\log y_t)}\sqrt{Var(\log c_t)}\right)\rho(\log y_t, \log c_t)\Bigg]
\end{aligned}
$$

We get $\varepsilon = 1.5$ under the alternative calibration. Suppose that the consumer is temporarily hit by a positive wage income shock. Since higher θ reduces the intertemporal elasticity of substitution (or induces a stronger motive for consumption smoothing), the consumer increases his consumption by less amount than he would do under $\theta = 1$. Because increased consumption is supported by larger labor income from working longer, less Δc_t is translated into less Δh_t, which drives down the required value of ε.

7. Indivisible Labor

Suppose that supply of labor is determined by lotteries. Further suppose that the consumer supplies $\bar{h}$ amount of labor when chosen to work. Assuming that the probability of winning(?) the lottery is ψ_t, consumer's (expected) utility maximization problem is written as follows.

$$\begin{aligned}\max_{c_{1,t},c_{2,t}} \sum_{t=0}^{\infty} \beta^t & \left[\psi_t\left(\frac{c_{1,t}^{1-\theta}}{1-\theta} - \gamma\frac{\varepsilon}{1+\varepsilon}\bar{h}^{\frac{1+\varepsilon}{\varepsilon}}\right) + (1-\psi_t)\frac{c_{2,t}^{1-\theta}}{1-\theta}\right] \\ \text{s.t. } & \psi_t c_{1,t} + (1-\psi_t)c_{2,t} + b_{t+1} \leq \psi_t w_t(s_t)\bar{h} + (1-\delta)b_t + r_t(s_t)b_t \\ & b_t \geq 0\end{aligned}$$

First order conditions for consumptions are:

$$\psi_t c_{1,t}^{-\theta} = \psi_t \lambda_t$$
$$(1-\psi_t)c_{2,t}^{-\theta} = (1-\psi_t)\lambda_t$$

Which implies that $c_{1,t} = c_{2,t}$. We can re-formulate the maximization problem by imposing this result. i.e.

$$\begin{aligned}\max_{c_t,\psi_t} \sum_{t=0}^{\infty} \beta^t & \left(\frac{c_t^{1-\theta}}{1-\theta} - \psi_t\gamma\frac{\varepsilon}{1+\varepsilon}\bar{h}^{\frac{1+\varepsilon}{\varepsilon}}\right) \\ \text{s.t. } & c_t + b_{t+1} \leq \psi_t w_t(s_t)\bar{h} + (1-\delta)b_t + r_t(s_t)b_t \\ & b_t \geq 0\end{aligned}$$

First order conditions for c_t and ψ_t[1] are:

$$c_t^{-\theta} = \lambda_t$$
$$\gamma\frac{\varepsilon}{1+\varepsilon}\bar{h}^{\frac{1+\varepsilon}{\varepsilon}} = \lambda_t w_t(s_t)\bar{h}$$

Since $h_t = \bar{h}$, variance of de-trended hours is zero, which has no relation with the variance of consumption or output. In this case where there is no intensive margin of labor, Frisch elasticity of labor supply is not defined.

[1] Choosing optimal ψ_t can be understood either as consumer solving two-stage maximization problem, or social planner solving optimization problem to maximize agents' utility