

# ECON 210C Homework 1

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# 1. Questions from Romer

(1) Romer 5.8.

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \cdot \left(C_t - \theta C_t^2\right)$$

$$Y_t = C_t + K_{t+1} - C_t$$

$$Y_t = AK_t + e_t, \quad K_{t+1} = K_t + Y_t - C_t$$

Just  $I_t$

$A$  is interest rate.  $A = \rho$ .

$$e_t = \phi e_{t-1} + \varepsilon_t, \quad -1 < \phi < 1$$

$$(a) L = E_0 \max_{C_t, K_{t+1}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \left\{ (C_t - \theta C_t^2) + \gamma_t (A K_t + e_t - C_t - K_{t+1} + C_t) \right\}$$

FOC

$$C_t: 1 - 2\theta C_t = \gamma_t$$

$$K_{t+1}: \gamma_t = E_t \cdot \left(\frac{1}{1+\rho}\right) (1+\rho) \gamma_{t+1}$$

$$\Rightarrow 1 - 2\theta C_t = E_t (1 - 2\theta C_{t+1})$$

$$\therefore C_t = E_t C_{t+1}$$

$$(b) \text{ Guess } C_t = \alpha + \beta K_t + \gamma e_t$$

$$K_{t+1} = K_t + Y_t - \alpha - \beta K_t - \gamma e_t$$

$$= K_t + \rho K_{t+1} + e_t - \alpha - \beta K_t - \gamma e_t$$

$$= (1+\rho-\beta) K_t + (1-\gamma) e_t - \alpha$$

$$\therefore K_{t+1} = (1+\rho-\beta) K_t + (1-\gamma) e_t - \alpha$$

(c) FOC in (a) :  $C_t = E_t C_{t+1}$

Plug in the guess.

$$\begin{aligned} \alpha + \beta K_t + r e_t &= E_t [\alpha + \beta K_{t+1} + r e_{t+1}] \\ &= E_t [\alpha + \beta \{ (1+\rho-\beta) K_t + (1-r) e_t - \alpha \}] \\ &\quad + \gamma (\phi e_t + E_t e_{t+1}) \\ &= \alpha - \beta \alpha + \beta (1+\rho-\beta) K_t + (\beta (1-r) + r \phi) e_t \end{aligned}$$

Method of undetermined coefficients

$$(1) \alpha = \alpha - \beta \alpha \Rightarrow \alpha = \alpha (1 - \beta). \text{ unless } \beta = 0, \alpha = 0.$$

$$(2) \beta = \beta (1 + \rho - \beta) \Rightarrow 1 = 1 + \rho - \beta. \therefore \beta = \rho$$

$$(3) r = \beta (1 - \gamma) + r \phi \Rightarrow r = \rho (1 - \gamma) + r \phi$$

$$\gamma (1 + \rho - \phi) = \rho$$

$$\Rightarrow \gamma = \frac{\rho}{1 + \rho - \phi}$$

$$(d) Y_t = \rho K_t + e_t$$

$$C_t = \rho K_t + \frac{\rho}{1 + \rho - \phi} e_t : \text{only } \frac{\rho}{1 + \rho - \phi} \text{ fraction goes to here}$$

$$K_{t+1} = K_t + \frac{1 - \phi}{1 + \rho - \phi} e_t : \text{the rest are invested. since there's no } \sigma, \text{ it goes to } K_{t+1} 100\%.$$

(2) Romer 5.9.

(a)

$$L = \mathbb{E}_A \max_{C_t, K_{t+1}} \sum_{t=0}^{\infty} \left( \frac{1}{1+\rho} \right)^t \left\{ (C_t - \theta C_t C_t^2 + 2C_t v_t + v_t^2) \right. \\ \left. + \gamma_t (A K_t - C_t - K_{t+1} + l_t) \right\}$$

FOC.

$$C_t: 1 - 2\theta C_t - 2\theta v_t = \gamma_t$$

$$K_{t+1}: \gamma_t = \mathbb{E}_A \cdot \left( \frac{1}{1+\rho} \right) \gamma_{t+1} (A+1) \\ \Rightarrow \gamma_t = \mathbb{E}_A \gamma_{t+1}$$

Combine them.

$$\cancel{1 - 2\theta C_t - 2\theta v_t} = \mathbb{E}_A (\cancel{1 - 2\theta C_{t+1} - 2\theta v_{t+1}}) \\ \Rightarrow C_t + v_t = \mathbb{E}_A C_{t+1} \\ (\text{since } \mathbb{E}_A v_{t+1} = 0)$$

(b) Guess  $C_t = \alpha_t + \beta K_t + \gamma V_t$ .

$$K_{t+1} = K_t + A K_t - C_t \\ = (1+A)K_t - \alpha - \beta K_t - \gamma V_t \\ = (1+A-\beta)K_t - \gamma V_t - \alpha.$$

(c) Plug in the guess

$$\alpha + \beta K_t + \gamma V_t + v_t = \mathbb{E}_A [\alpha + \beta K_{t+1} + \gamma V_{t+1}] \\ = \mathbb{E}_A [\alpha + \beta (1+A-\beta) K_t - \beta \gamma V_t - \beta \alpha] \\ = \alpha - \beta \alpha + \beta (1+A-\beta) K_t - \beta \gamma V_t$$

$$\alpha - \alpha - \beta \alpha \Rightarrow \alpha = 0$$

$$\beta = \beta (1+A-\beta) \Rightarrow \beta = A = \rho$$

$$(1+\gamma) = -\beta \gamma \Rightarrow 1+\gamma+\rho \gamma = 0. \quad \gamma = \frac{-1}{1+\rho}$$

(cd) Plugging in the solutions ...

$$C_t = P_k t - \frac{1}{1+\rho} V_t$$

$$K_{t+1} = K_t + \frac{1}{1+\rho} V_t$$

Because positive  $V_t$  means  
distaste for consumer.

But why  $\frac{1}{1+\rho}$ ? consumption smoothing

### (3) Romer 5. II.

(a) Lifetime maximized value

= maximized value of this period  
+ discounted maximized value from  
the next period.

$$(b) V(K_t, A_t) = \max_{C_t, l_t} \left[ \ln(C_t + b \ln(1-l_t)) + e^{-\rho} \beta_0 + \beta_k \ln(Y_t - C_t) + \beta_A \alpha \ln A_t \right]$$

FOC for  $C_t$ :

$$\frac{1}{C_t} = e^{-\rho} \cdot \beta_k \cdot \frac{1}{Y_t - C_t}$$

$$\text{s.t. } Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

$$C_t/Y_t = e^{\rho} \cdot \frac{1}{\beta_k} \cdot (Y_t - C_t) / Y_t$$

$$= e^{\rho} \cdot \frac{1}{\beta_k} \cdot \left( 1 - \frac{C_t}{Y_t} \right)$$

$$\Rightarrow \frac{C_t}{Y_t} \left( 1 + e^{\rho} \cdot \frac{1}{\beta_k} \right) = e^{\rho} \cdot \frac{1}{\beta_k} \Rightarrow \frac{C_t}{Y_t} = \frac{e^{\rho}}{1 + \frac{e^{\rho}}{\beta_k}} : \text{constant}$$

So consumption - output ratio is constant,  
which is less than 1.

$$(c) \frac{b}{1-l_t} = \beta_k \cdot \frac{(1-\alpha) K_t^\alpha (A_t l_t)^{-\alpha} A_t}{K_t^\alpha (A_t l_t)^{1-\alpha} - C_t}$$

$$= \beta_k \cdot \frac{(1-\alpha) \cdot l_t^{-1} \cdot Y_t}{Y_t - C_t}$$

$$= \beta_k \cdot \frac{(1-\alpha) l_t^{-1}}{1 - \frac{C_t}{Y_t}}$$

Hence  $l_t$  does not depend  
on  $A_t$  or  $K_t$ .

(d) Re-write the Bellman equation.

$$V(K_t, A_t) = \max_{C_t, l_t} \left[ \ln(C_t + b \ln(1-l_t)) + e^{-\rho t} \beta_0 + \beta_k \ln(K_t^\alpha (A_t l_t)^{1-\alpha}) - C_t + \beta_A \rho \ln A_t \right]$$

Results from (b) & (c) are

$$(b) \frac{C_t}{Y_t} = \frac{\frac{e^{\rho}}{\beta_k}}{1 + \frac{e^{\rho}}{\beta_k}} = \frac{e^{\rho}}{\beta_k + e^{\rho}} \Rightarrow C_t = \left( \frac{e^{\rho}}{\beta_k + e^{\rho}} \right) Y_t$$

$$(c) \frac{b}{1-l_t} = \beta_k \cdot \frac{(1-\alpha) l_t}{1 - C_t/Y_t}$$

$$\Rightarrow \frac{l_t}{1-l_t} = \frac{\beta_k}{b} \cdot \frac{(1-\alpha)}{1 + \frac{e^{\rho}}{\beta_k}}$$

$$\Rightarrow b l_t = \beta_k (1-\alpha) (1-l_t) \left( 1 + \frac{e^{\rho}}{\beta_k} \right)$$

$$\Rightarrow l_t = \frac{(1-\alpha)}{(1-\alpha) + \frac{b}{1 + \frac{\beta_k}{e^{\rho}}}}$$

Now we can substitute them for the optimal  $C_t$  and  $l_t$ ...

This is hideous

## 2. P/H and excess - smoothing

2-1.

$$L = \max_{\{C_t\}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t) + \lambda (A_0 - \sum_{t=0}^{\infty} \beta^t (C_t - Y_t))$$

$$\text{FOC: } U'(C_t) = \lambda$$

$$\text{Budget constraint: } A_0 = E_t \sum_{t=0}^{\infty} \beta^t C_t - Y_t$$

Now let's get the impulse response.

No shock at all

(a)

$$\begin{aligned} Y_t &= \mu \cdot t + \phi Y_{t-1} \\ Y_{t+1} &= \mu \cdot (t+1) + \phi Y_t \\ &= \mu(t+1) + \phi(\mu t + \phi Y_{t-1}) \\ &= \mu[t+1 + \phi t] + \phi^2 Y_{t-1} \end{aligned}$$

one-time shock at  $t$

$$\begin{aligned} Y_t &= \mu \cdot t + \phi Y_{t-1} + \epsilon_t \\ Y_{t+1} &= \mu \cdot (t+1) + \phi Y_t \\ &= \mu(t+1) + \phi(\mu t + \phi Y_{t-1} + \epsilon_t) \\ &= \mu[t+1 + \phi t] + \phi^2 Y_{t-1} + \phi \epsilon_t \\ \Rightarrow \frac{\partial Y_{t+s}}{\partial \epsilon_t} &= \phi^s \cdot \epsilon_t \end{aligned}$$

$$(b) \quad Y_t = Y_{t-1}$$

$$Y_{t+1} = Y_t = Y_{t-1}$$

$$Y_t = Y_{t-1} + \epsilon_t$$

$$Y_{t+1} = Y_t = Y_{t-1} + \epsilon_t$$

$$\Rightarrow \frac{\partial Y_{t+s}}{\partial \epsilon_t} = \epsilon_t \quad (\text{shock is permanent})$$

$$(c) \quad \Delta Y_t = \phi \Delta Y_{t-1}$$

$$\Rightarrow Y_t - Y_{t-1} = \phi(Y_{t-1} - Y_{t-2})$$

$$\Delta Y_{t+1} = \phi \Delta Y_t$$

$$Y_{t+1} - Y_t = \phi(Y_t - Y_{t-1})$$

$$Y_t - Y_{t-1} = \phi(Y_{t-1} - Y_{t-2}) + \epsilon_t$$

$$\Rightarrow Y_t = (1+\phi)Y_{t-1} - \phi Y_{t-2} + \epsilon_t$$

$$Y_{t+1} - Y_t = \phi(Y_t - Y_{t-1})$$

$$\Rightarrow Y_{t-1} = (1+\phi)Y_t - \phi Y_{t-1}$$

$$= (1+\phi)((1+\phi)Y_{t-1} - \phi Y_{t-2} + \epsilon_t)$$

$$\Rightarrow \frac{\partial Y_{t+s}}{\partial \epsilon_t} = (1+\phi)^s \epsilon_t$$

shock is explosive.

Now revisit the FOC and the budget constraint.

$$\text{FOC: } U'(c_t) = \lambda \Rightarrow c_1 = c_2 = \dots = c^*$$

$$\text{Budget constraint: } A_0 = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t c(c_t - Y_t)$$

$$\Rightarrow \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t c_t = A_0 + \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t Y_t$$

Quadratic utility implies that consumption follows random walk.  $\mathbb{E}c_{t+1} = c_t$

$$\Rightarrow \frac{1}{1-\beta} \cdot c^* = A_0 + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t Y_t$$

Thus, consumption is a function of lifetime income

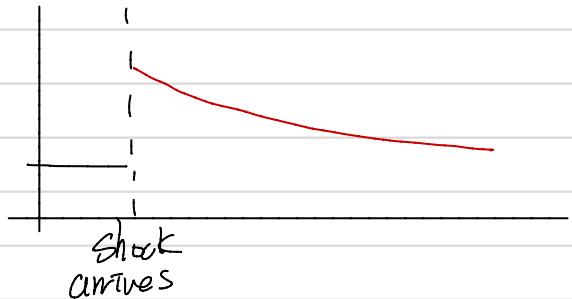
$$c^* = (1-\beta) \cdot A_0 + (1-\beta) \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t Y_t$$

Now we can solve each case analytically.

(a) In this case shocks are transitory. So consumption increases only small amount. precisely, AC is

$$(1-\beta) \sum_{t=0}^{\infty} \phi^t E_t$$

Saving's path is :



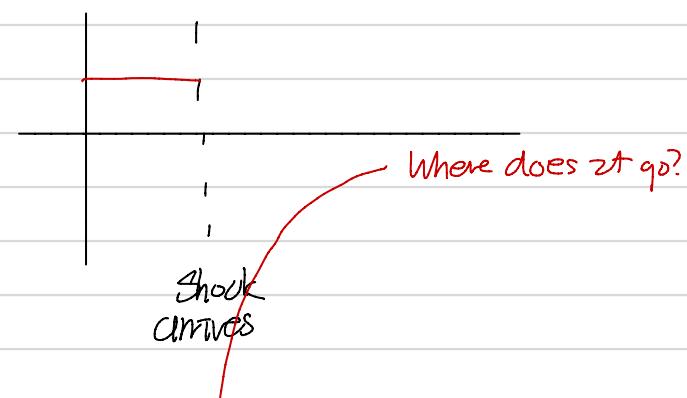
(b) In this case shocks are permanent.

Thus, consumption also jumps by  $E_t$  amount.  
Hence saving does not change

(c) In this case shocks are explosive.

Hence the lifetime income goes to infinity. Therefore consumption goes to infinity as well

Saving's path is :



## 2-2.

Result from (a) implies that as long as shocks are transitory, consumption is always smoother than income.

- tax cut should be ineffective - but not  
    ⇒ Borrowing constraint  
    ⇒ Subsistence level of consumption

## 2-3.

Yes. If income follows a random walk, (b) implies that consumption is no longer smoother than income.

$$3. \quad (1) \quad \mathcal{L} = \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} R_t \left\{ (1-\varepsilon) F(K_t, L_t) - w_t L_t - I_t (1+\alpha \left( \frac{I_t}{K_t} - \delta \right)) + g_t ((1-\delta) k_t + I_t - K_{t+1}) \right\} \right]$$

Choose  $K_{t+1}, L_t, I_t$

$$\frac{\partial \mathcal{L}}{\partial L_t} : w_t = (1-\varepsilon)(1+\alpha) \left( \frac{K_t}{L_t} \right)^{\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} : g_t = \mathbb{E}_t \frac{(1-\delta)g_{t+1} + (1-\varepsilon)\alpha \left( \frac{K_{t+1}}{L_{t+1}} \right)^{\alpha-1} + \alpha \left( \frac{I_t}{K_{t+1}} \right)^2}{1+r_{t+1}}$$

$$\frac{\partial \mathcal{L}}{\partial I_t} : g_t = 1 + \alpha \left( \frac{I_t}{K_t} - \delta \right) + \alpha \cdot \frac{I_t}{K_t}$$

$$\text{Log-linearization. In S.S., } g = 1 + 2\alpha \cdot \frac{I}{K} - \alpha \delta \\ = 1 + 2\alpha \cdot \delta - \alpha \delta = 1 + \alpha \delta$$

$$g_t^v = \frac{2\alpha \delta}{1 + \alpha \delta} (I_t^v - K_t^v)$$

$I_t^v, K_t^v$  are observable, assume that we have an estimate for  $\delta$ .

We can run a regression

$$g_t^v = \beta \cdot (I_t^v - K_t^v) + \epsilon_t$$

and interpret  $\hat{\beta}$  as  $\frac{2\alpha \delta}{1 + \alpha \delta} \Rightarrow \hat{\beta} + \hat{\beta} \hat{\alpha} \delta = 2\hat{\alpha} \delta$

$$\Rightarrow \hat{\beta} = \hat{\alpha} (2\delta - \hat{\beta} \delta) \Rightarrow \hat{\alpha} = \frac{\hat{\beta}}{2\delta - \hat{\beta} \delta}$$

Error term includes unexplained parts of  $g_t^v$ . Value of firm might not be explained solely by investment and capital stock they have.

(2) Euler equation is

$$g_t = E_t \frac{(1-\delta)g_{t+1} + (1-\tau)\alpha\left(\frac{K_{t+1}}{L_{t+1}}\right)^{\alpha-1} + \alpha\left(\frac{I_{t+1}}{K_{t+1}}\right)^2}{1+r_{t+1}}$$

$g_t$  is unobservable. Substitute  $g_t$  using FOC for I

$$\begin{aligned} 1 + \alpha\left(\frac{I_t}{K_t} - \delta\right) + \alpha \cdot \frac{I_t}{K_t} &= (1-\delta)E_t \frac{1}{1+r_{t+1}} \left(1 + \alpha\left(\frac{I_{t+1}}{K_{t+1}} - \delta\right) + \alpha \frac{I_{t+1}}{K_{t+1}}\right) \\ &\quad + E_t \frac{(1-\tau)\alpha\left(\frac{K_{t+1}}{L_{t+1}}\right)^{\alpha-1} + \alpha\left(\frac{I_{t+1}}{K_{t+1}}\right)^2}{1+r_{t+1}} \end{aligned}$$

Log-linearize

$$\text{LHS : } \frac{2\alpha\delta}{1+\alpha\delta} (I_t^V - K_t^V)$$

$$\text{RHS : } \underbrace{\left((1-\delta)g_{t+1} + (1-\tau)\alpha\left(\frac{K_{t+1}}{L_{t+1}}\right)^{\alpha-1} + \alpha\left(\frac{I_{t+1}}{K_{t+1}}\right)^2\right)}_{(1)} - \underbrace{(1+r_{t+1})}_{(2)}$$

In steady state,  $g = \frac{1}{1+r} \cdot [(1-\delta)g + (1-\tau)\alpha\left(\frac{K}{L}\right)^{\alpha-1} + \alpha\left(\frac{I}{K}\right)^2]$

$$\Rightarrow g(1+r) = (1-\delta)g + (1-\tau)\alpha\left(\frac{K}{L}\right)^{\alpha-1} + \alpha\left(\frac{I}{K}\right)^2$$

$$\begin{aligned} (1) : \frac{(1-\delta)g}{g(1+r)} g_{t+1}^V + \frac{(1-\tau)\alpha}{g(1+r)} \left(\frac{K}{L}\right)^{\alpha-1} (\alpha-1)(K_{t+1}^V - L_{t+1}^V) \\ + \frac{2\alpha}{g(1+r)} \left(\frac{I}{K}\right)^2 \cdot (I_{t+1}^V - K_{t+1}^V) \end{aligned}$$

$$\begin{aligned} &= \frac{(1-\delta)}{C(1+r)} g_{t+1}^V + \frac{(1-\tau)\alpha(\alpha-1)}{(1+\alpha\delta)(1+r)} \left(\frac{K}{L}\right)^{\alpha-1} (K_{t+1}^V - L_{t+1}^V) \\ &\quad + \frac{2\alpha\delta^2}{(1+\alpha\delta)(1+r)} (I_{t+1}^V - K_{t+1}^V) \end{aligned}$$

$$(2) : \frac{r}{1+r} r_{t+1}^V$$

Simplifying ...

$$\begin{aligned}
 \frac{2\alpha\delta}{1+\alpha\delta} (\check{x}_t - \check{k}_t) &= \frac{(1-\delta)}{(1+r)} \cdot \frac{2\alpha\delta}{(1+\alpha\delta)} \mathbb{E}_t (\check{x}_{t+1} - \check{k}_{t+1}) \\
 &\quad + \frac{(1-\tau)\alpha(\alpha-1)}{(1+\alpha\delta)(1+r)} \left(\frac{k}{L}\right)^{\alpha-1} \mathbb{E}_t (\check{k}_{t+1} - \check{l}_{t+1}) \\
 &\quad + \frac{2\alpha\delta^2}{(1+\alpha\delta)(1+r)} \mathbb{E}_t (\check{x}_{t+1} - \check{k}_{t+1}) \\
 &\quad - \frac{r}{1+r} \mathbb{E}_t \check{k}_{t+1} \quad // \\
 &= \frac{2\alpha\delta}{(1+r)(1+\alpha\delta)} \mathbb{E}_t (\check{x}_{t+1} - \check{k}_{t+1}) \\
 &\quad + \frac{(1-\tau)\alpha(\alpha-1)}{(1+r)(1+\alpha\delta)} \left(\frac{k}{L}\right)^{\alpha-1} \mathbb{E}_t (\check{k}_{t+1} - \check{l}_{t+1}) \\
 &\quad - \frac{r}{1+r} \mathbb{E}_t \check{k}_{t+1}
 \end{aligned}$$

Using expectation error,  $\mathbb{E}_t x_{t+1} = x_{t+1} - (\check{x}_{t+1} - \mathbb{E}_t \check{x}_{t+1})$   
Denote  $(x_{t+1} - \mathbb{E}_t x_{t+1}) = \tilde{x}_{t+1}$ . Then

$$\mathbb{E}_t x_{t+1} = x_{t+1} - \tilde{x}_{t+1}$$

Also, let  $(1+r)(1+\alpha\delta) = g$

$$\begin{aligned}
 \frac{1}{g} 2\alpha\delta(1+r) (\check{x}_t - \check{k}_t) &= \frac{2\alpha\delta}{g} \mathbb{E}_t (\check{x}_{t+1} - \check{k}_{t+1}) \\
 &\quad + \frac{1}{g} (1-\tau)\alpha(\alpha-1) \left(\frac{k}{L}\right)^{\alpha-1} \mathbb{E}_t (\check{k}_{t+1} - \check{l}_{t+1}) - \frac{r}{1+r} \mathbb{E}_t \check{k}_{t+1}
 \end{aligned}$$

(3)

Rearranging a little bit ...

$$\frac{r}{1+r} \tilde{E}_t[k_{t+1}] = \frac{1}{g} 2\alpha\delta(1+r)(\tilde{I}_t^v - \tilde{K}_t^v) - \frac{2\alpha\delta}{g} \tilde{E}_t(\tilde{I}_{t+1}^v - \tilde{K}_{t+1}^v) \\ - \frac{1}{g} ((-\gamma)\alpha(\alpha+1)) \left( \frac{L}{L} \right)^{\alpha+1} \tilde{E}_t(\tilde{K}_{t+1}^v - \tilde{L}_{t+1}^v)$$

Let's denote ...

$$\frac{r}{1+r} (\tilde{k}_{t+1}^v - \tilde{k}_{t+1}^{\hat{v}}) = A(\tilde{I}_t^v - \tilde{K}_t^v) + B(\tilde{I}_{t+1}^v - \tilde{K}_{t+1}^v - \tilde{I}_{t+1}^{\hat{v}} + \tilde{K}_{t+1}^{\hat{v}}) \\ + C(\tilde{K}_{t+1}^v - \tilde{L}_{t+1}^v - \tilde{K}_{t+1}^{\hat{v}} + \tilde{L}_{t+1}^{\hat{v}})$$

Stack all expectation error into a single error term.

$$\frac{r}{1+r} \tilde{r}_{t+1}^v = A(\tilde{I}_t^v - \tilde{K}_t^v) + B(\tilde{I}_{t+1}^v - \tilde{K}_{t+1}^v) + C(\tilde{K}_{t+1}^v - \tilde{L}_{t+1}^v) \\ + \left( \frac{r}{1+r} \tilde{r}_{t+1}^{\hat{v}} - B(\tilde{I}_{t+1}^{\hat{v}} - \tilde{K}_{t+1}^{\hat{v}}) - C(\tilde{K}_{t+1}^{\hat{v}} - \tilde{L}_{t+1}^{\hat{v}}) \right)$$

$$\therefore \tilde{r}_{t+1}^v = \beta_0(\tilde{I}_t^v - \tilde{K}_t^v) + \beta_1(\tilde{I}_{t+1}^v - \tilde{K}_{t+1}^v) + \beta_2(\tilde{K}_{t+1}^v - \tilde{L}_{t+1}^v) + \varepsilon_{t+1}$$

Unknown parameter:  $\alpha, \delta, \alpha$

so we can back out all of them by  $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ .

(4) Solve for FOC agam w/  $K_t = (1-\delta)K_{t-1} + I_t$

$$L = E_0 \left[ \sum_{t=0}^{\infty} R_t \left\{ (1-\gamma) F(K_t, L_t) - w_t L_t - I_t (1+\alpha \left( \frac{I_t}{K_t} - \delta \right)) + g_t (K_t - (1-\delta)K_{t-1} - I_t) \right\} \right]$$

<FOC>

$$(1) \frac{\partial L}{\partial L_t} : w_t = (1-\gamma)(1+\alpha) \left( \frac{K_t}{L_t} \right)^{\alpha}$$

$$(2) \frac{\partial L}{\partial I_t} : 1 + \alpha \left( \frac{I_t}{K_t} - \delta \right) + \alpha \cdot \frac{I_t}{K_t^2} = g_t$$

$$(3) \frac{\partial L}{\partial K_t} : (1-\gamma)\alpha \left( \frac{K_t}{L_t} \right)^{\alpha-1} + I_t \cdot \alpha \cdot \left( \frac{I_t}{K_t^2} \right) + g_t = \frac{(1-\delta)}{1+r+1} g_{t+1}$$

(1) & (2) are unchanged. Hence,  $\overset{\vee}{g}_t = \frac{2\alpha\delta}{1+\alpha\delta} (\overset{\vee}{I}_t - \overset{\vee}{K}_t)$

Log-linearise (3).

$$\text{In steady state, } \frac{1-\delta}{1+r} \overset{\vee}{g} = \overset{\vee}{g} + (1-\gamma)\alpha \left( \frac{K}{L} \right)^{\alpha-1} + \alpha \left( \frac{I}{K} \right)^2$$

$$\text{RHS: } \overset{\vee}{g}_{t+1} - \frac{r}{1+r} \overset{\vee}{k}_{t+1}$$

$$\text{LHS: Note that } g = 1 + \alpha\delta. \text{ Let } \frac{1}{1+r}(1+\delta)(1+\alpha\delta) = g$$

$$\frac{\overset{\vee}{g}}{(1-\delta)g} \cdot \overset{\vee}{g}_t + \frac{1}{g} (1-\gamma)\alpha \left( \frac{K}{L} \right)^{\alpha-1} (K_t - L_t) + \frac{2\alpha}{g} \left( \frac{I}{K} \right)^2 (I_t - K_t)$$

$$\Rightarrow E \overset{\vee}{g}_{t+1} - \frac{r}{1+r} E \overset{\vee}{k}_{t+1} = \frac{(1+r)}{(1-\delta)} \cdot \frac{2\alpha\delta}{1+\alpha\delta} (I_t - K_t)$$

$$+ \frac{(1+r)(1-\gamma)\alpha(\alpha-1)}{(1-\delta)(1+\alpha\delta)} \left( \frac{K}{L} \right)^{\alpha-1} (K_t - L_t)$$

$$+ \frac{(1+r)2\alpha\delta^2}{(1-\delta)(1+\alpha\delta)} (I_t - K_t)$$

$$\begin{aligned}
 & \frac{2\alpha\delta}{1+\alpha\delta} E_t(I_{t+1}^v - K_{t+1}^v) - \frac{r}{1+r} E_t k_{t+1} \\
 &= \frac{(1+r)(1-\gamma)\alpha(\alpha-1)}{(1-\delta)(1+\alpha\delta)} \left(\frac{K}{L}\right)^{\alpha-1} (K_t^v - L_t^v) \\
 &+ \frac{(1+r)2\alpha\delta(1+\delta)}{(1-\delta)(1+\alpha\delta)} (I_t^v - K_t^v)
 \end{aligned}$$

We can run a similar regression.  
Now error term does not involve  
 $\hat{L}_{t+1}$ .

(5)

The problem when we try to estimate ' $\alpha$ ' as a function of  $g$  is that  $g$  itself is unobservable. So the estimate can be sensitive to how  $g$  is defined, which is related to the specification of the model at hand. This is what the error term in (1) implies.

However, since this specification does not include expectation, the estimate does not suffer from any problems arising from expectation errors.

## 4. Practice log-linearization

$$1. \check{Y} = \frac{C}{F} \check{C} + \frac{I}{F} \check{I} + \frac{G}{F} \check{G}$$

( $\because NX=0$  in steady state)

$$2. \text{ Let } X = (\alpha K^\rho + (1-\alpha)(AL)^\rho)$$

$$Y = X^{\frac{1}{\rho}} \quad \text{in s.s., } \bar{Y} = \bar{X}^{\frac{1}{\rho}}$$

$$\bar{Y}(1+\check{Y}) = \bar{X}^{\frac{1}{\rho}} (1 + \check{X}^{\frac{1}{\rho}}) \Rightarrow \check{Y} = \frac{1}{\rho} \check{X}$$

Now let's get  $\check{X}$

$$X = \alpha K^\rho + (1-\alpha)(AL)^\rho = A + B \quad \begin{aligned} A &= \alpha \bar{K}^\rho \\ B &= (1-\alpha)(\bar{A}\bar{L})^\rho \end{aligned}$$

$$\check{X} = \frac{A}{A+B} (\check{\alpha} \bar{K}^\rho) + \frac{B}{A+B} ((1-\check{\alpha})(\bar{A}\bar{L})^\rho)$$

$$= \frac{A}{A+B} (\rho \check{K}) + \frac{B}{A+B} (\rho (\check{A} + \check{L}))$$

$$\therefore \check{Y} = \frac{1}{\rho} \left[ \frac{\alpha \bar{K}^\rho}{F^\rho} \cdot \rho \check{K} + \frac{(1-\alpha)(\bar{A}\bar{L})^\rho}{F^\rho} \cdot \rho (\check{A} + \check{L}) \right]$$

3.

In steady state

$$k = (1-\delta)k + I - \psi \left( \frac{I}{K} - \delta \right)^2 I$$

If we assume that  $I = \delta K$  in steady state, then log-linearization for this is

$$\check{k}_t = (1-\delta)\check{k}_{t-1} + \delta \check{I}_t$$

If we assume that  $I \neq \delta K$ , then  $\psi \left( \frac{I}{K} - \delta \right)^2 I \neq 0$   
In that case,

$$\begin{aligned} \check{k}_t &= \frac{(1-\delta)K}{K} \cdot \check{k}_{t-1} + \frac{I}{K} \check{I}_t - \frac{I}{K} \psi \left( \frac{I}{K} - \delta \right)^2 \left( \psi \left( \frac{I_t}{K_{t-1}} - \delta \right)^2 I_t \right) \\ &= (1-\delta)\check{k}_{t-1} + \frac{I}{K} \check{I}_t - \frac{I}{K} \psi \left( \frac{I}{K} - \delta \right)^2 \left( \psi \left( \frac{I_t}{K_{t-1}} - \delta \right)^2 I_t \right) \end{aligned}$$

Where

$$\psi \left( \frac{I_t}{K_{t-1}} - \delta \right)^2 I_t = 2 \left( \frac{I_t}{K_{t-1}} - \delta \right) + \check{I}_t$$

Where

$$\left( \frac{I_t}{K_{t-1}} - \delta \right) = \frac{I/k}{\frac{I}{K} - \delta} \cdot \left( \frac{I_t}{K_{t-1}} \right) = \frac{I}{I - \delta K} (\check{I}_t - \check{k}_{t-1})$$

$$\therefore \check{k}_t = (1-\delta)\check{k}_{t-1} + \frac{I}{K} \check{I}_t - \frac{I}{K} \psi \left( \frac{I}{K} - \delta \right)^2 \left( \frac{2I}{I - \delta K} (\check{I}_t - \check{k}_{t-1}) + \check{I}_t \right)$$

4.

In steady state,  $\psi \left( \frac{I_t}{K_{t-1}} - 1 \right)^2 I_t = 0$ .  
Hence the answer is same as #3

5. Take log.

$$\hat{\lambda}_t = \phi_{\pi}(\log p_t - \log p_{t-1}) + \phi_y(\log y_t - \log y_{t-1}) + \rho \hat{\lambda}_{t-1}$$

In steady state,  $\hat{\lambda} = \rho \hat{\lambda} \Rightarrow \hat{\lambda} = 0$

$\hat{\lambda}_t$  is already a small number, with steady state value zero. So it is okay to just state  $\hat{\lambda}_t = \check{\lambda}_t$ . Then next steps are straight forward.

$$\begin{aligned} & \phi_{\pi}(\log p_t - \log p_{t-1}) - \phi_{\pi}(\log p - \log P) \\ &= \phi_{\pi}(\log p_t - \log P) - \phi_{\pi}(\log p_{t-1} - \log P) \end{aligned}$$

$$\therefore \check{\lambda}_t = \phi_{\pi} \check{p}_t + \phi_y \check{y}_t + \rho \check{\lambda}_{t-1}$$

$$6. A \cdot F(L) = \frac{U_L C C, 1-L}{U_C C, 1-L}$$

$$\text{In steady state, } \bar{A} \cdot F(\bar{L}) = \frac{U_L C \bar{C}, 1-\bar{L}}{U_C C \bar{C}, 1-\bar{L}}$$

Take log.

$$\log A + \log F(L) = \log U_L C C, 1-L - \log U_C C, 1-L$$

Taylor approximation

$$\log A \approx \log \bar{A} + \frac{1}{\bar{A}} (A - \bar{A})$$

$$\log F(L) = \log F(\bar{L}) + \frac{F'(\bar{L})}{F(\bar{L})} (L - \bar{L})$$

$$\log U_L C C, 1-L$$

$$= \log U_L C \bar{C}, 1-\bar{L} + \frac{\bar{U}_L}{U_L} (C C - \bar{C} \bar{C}) - \frac{\bar{U}_L}{U_L} (C L - \bar{C} \bar{L})$$

$$\log U_C C, 1-L$$

$$= \log U_C C \bar{C}, 1-\bar{L} + \frac{\bar{U}_C}{U_C} (C C - \bar{C} \bar{C}) - \frac{\bar{U}_C}{U_C} (C L - \bar{C} \bar{L})$$

$$\therefore \bar{A} + \frac{F'(\bar{L})}{F(\bar{L})} \bar{L} = \frac{\bar{U}_L \cdot \bar{C}}{U_L} \bar{C} - \frac{\bar{U}_L \cdot \bar{L}}{U_L} \bar{L}$$

$$+ \frac{\bar{U}_C \bar{C}}{U_C} \bar{C} - \frac{\bar{U}_C \bar{L}}{U_C} \bar{L}$$

$$\Rightarrow \bar{A} + \frac{\bar{L} \cdot F(\bar{L})}{F(\bar{L})} \bar{L} = \bar{U}_L \cdot \bar{C} \cdot \bar{C} \left( \frac{1}{U_L} - \frac{1}{U_C} \right) - \bar{U}_L \cdot \bar{L} \cdot \bar{L} \left( \frac{1}{U_L} - \frac{1}{U_C} \right)$$

7. First take log

$$\log Y_t = \alpha \log K_t + (1-\alpha) \log L_t.$$

Then do 1st order Taylor approximation  
around steady state.

$$\frac{Y_t - \bar{Y}}{\bar{Y}} = (\log K - \log L)(\alpha_t - \alpha) + \frac{\alpha}{K}(K_t - K) + \frac{(1-\alpha)}{L}(L_t - L)$$
$$\therefore \dot{Y}_t = \alpha \dot{K}_t + (1-\alpha) \dot{L}_t + \alpha (\log \frac{K}{L}) \dot{\alpha}_t$$