I. RBC with Variable Labor Supply

Table 1 compares volatilities across different parameterizations of η with what is observed in the data. Larger values of η improve the fit of the model. There is greater persistence since more elastic labor supply means shocks affect hours and wages to a greater extent. Consumption and labor supply, however, remain excessively smooth. The classical RBC model implies a greater Frisch elasticity based on the data.

Table 1: Comparing second moments of output, consumption, and labor supply

	Data	$\eta = 0.5$	$\eta = 1$	$\eta = 2$
Consumption	1.27	0.97	1.03	1.09
Output	1.72	1.56	1.67	1.79
Hours	1.59	0.24	0.41	0.60

II. RBC with Variable Capital Utilization

The firm solves the profit maximization problem.

$$\max_{\{N_t, I_t, U_t\}} \mathrm{E} \sum_{t=0}^{\infty} \prod_{s=0}^{t} (1+r_t)^{-1} (Y_t - N_t w_t - I_t)$$

$$Y_t = (U_t K_t)^{\alpha} (Z_t N_t)^{1-\alpha}$$

$$K_{t+1} = (1 - \delta(U_t)) K_t + I_t$$

We equalize prices of capital and the consumption good as in equilibrium.

1. The first order conditions are given by

$$\text{Labor demand: } w_t = (1-\alpha) Z_t^{1-\alpha} \left(\frac{U_t K_t}{N_t} \right)^{\alpha}$$

$$\text{Shadow value of capital: } q_t = 1$$

$$\text{Euler equation: } \mathrm{E}(1+r_t)^{-1} \left(\alpha U_{t+1}^{\alpha} \left(\frac{K_{t+1}}{Z_{t+1} N_{t+1}} \right)^{\alpha - 1} + q_{t+1}(1 - \delta_{t+1}) \right)$$

$$\text{Utilization: } \alpha K_t^{\alpha} \left(\frac{U_t}{Z_t N_t} \right)^{\alpha-1} = q_t \delta'(U_t) K_t$$

$$1 + r_t = \alpha U_{t+1}^{\alpha} \left(\frac{K_{t+1}}{Z_{t+1} N_{t+1}} \right)^{\alpha-1} + 1 - \delta(U_t)$$

Rental rate depends on the marginal product of capital and depreciation which are themselves both dependant on U_t.

2. Utilization satisfies $\alpha K_t^{\alpha} \left(\frac{U_t}{Z_t N_t} \right)^{\alpha-1} = q_t \delta'(U_t) K_t$.

$$\ln \alpha + \alpha \ln K_t + (\alpha - 1)(\ln U_t - \ln Z_t - \ln N_t) = \ln q_t + \ln \delta'(U_t) + \ln K_t$$

$$\ln \alpha + (\alpha - 1)(\ln K_t + \ln U_t - \ln Z_t - \ln N_t) = \ln q_t + \ln \delta'(U_t)$$
$$(\alpha - 1)(\check{K}_t + \check{U}_t - \check{Z}_t - \check{N}_t) = \check{q}_t + \frac{\delta''(\bar{U})}{\delta'(\bar{U})}(U_t - \bar{U})$$
$$(\alpha - 1)(\check{K}_t + \check{U}_t - \check{Z}_t - \check{N}_t) = \check{q}_t + \frac{\delta''(\bar{U})\bar{U}}{\delta'(\bar{U})}\check{U}_t = \check{q}_t + \Delta$$
$$\check{U}_t = \frac{\check{q}_t + \Delta}{\alpha - 1}(\check{Z}_t + \check{N}_t - \check{K}_t) = \frac{1}{1+\Delta}(\check{Y}_t - \check{K}_t)$$

The final equality uses $\check{q}_t = 0$ from the first order condition and log linearized production.

3. We can use the previous derivation to reduce $\check{Y}_t$ as a function of technology and inputs.

$$\check{Y}_t = \alpha(\check{U}_t + \check{K}_t) + (1-\alpha)(\check{Z}_t + \check{N}_t)$$
$$= \frac{\alpha}{1+\Delta}(\check{Y}_t - \check{K}_t) + \alpha\check{K}_t + (1-\alpha)(\check{Z}_t + \check{N}_t)$$
$$\frac{1-\alpha+\Delta}{1+\Delta}\check{Y}_t = \frac{\alpha\Delta}{1+\Delta}\check{K}_t + (1-\alpha)(\check{Z}_t + \check{N}_t)$$
$$\check{Y}_t = \frac{\alpha\Delta}{1-\alpha+\Delta}\check{K}_t + \frac{(1-\alpha)(1+\Delta)}{1-\alpha+\Delta}(\check{Z}_t + \check{N}_t)$$

Δ governs the sensitivity of $\check{U}_t$ to the marginal rate of capital. The limiting case where $\Delta \to \infty$ is the standard (linearized) model and $\check{U}_t$ is fixed at full utilization. The limit $\delta \to 0$ represents the case of no utilization so that $\check{Y}_t$ depend solely on technology and labor.

4. The linearized labor demand function is $\check{w}_t = \check{Y}_t - \check{N}_t$. Substituting the expression for $\check{Y}_t$,

$$\check{w}_t = \frac{\alpha\Delta}{1-\alpha+\Delta}\check{K}_t + \frac{(1-\alpha)(1+\Delta)}{1-\alpha+\Delta}(\check{Z}_t + \check{N}_t) - N_t$$

We can obtain an upward sloping demand function if labor exhibits increasing returns to scale.

$$\frac{(1-\alpha)(1+\Delta)}{1-\alpha+\Delta} > 1$$
$$(1-\alpha)(1+\Delta) > 1 - \alpha + \Delta$$
$$1 - \alpha + \Delta - \Delta\alpha > 1 - \alpha + \Delta$$
$$-\Delta\alpha > 1$$

Since $\Delta, \alpha > 0$, indeterminacy is impossible in this model. One way to achieve indeterminacy is to incorporate positive production externalities so that aggregate labor has increasing returns to scale. A model with endogenous capital utilization is more likely to exhibit indeterminacy because it amplifies the importance of labor in production.