Formula Derivation

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MSE(LSB-k) = $\frac{1}{2^k \times 2^k}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}(i-j)^2 = \frac{2^{2k}-1}{6}$

$$\frac{1}{2^{2k}}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}(i-j)^2 = \frac{1}{2^{2k}}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1} i^2 - 2ij + j^2$$

$$= \frac{1}{2^{2k}}\left(\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1} i^2 - \sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1} 2ij + \sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1} j^2\right)$$

$$= \frac{1}{2^{2k}}\left[2 \times 2^k \times \frac{(2^k-1)2^k(2\times(2^k-1)+1)}{6} - 2\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1} ij\right]$$

$$= \frac{1}{2^{2k}}\left[\frac{(2^k-1)(2^k)^2(2^{k+1}-1)}{3} - 2 \times \left(\frac{(2^k-1)2^k}{2}\right)^2\right]$$

$$= \frac{1}{2^{2k}}\left[\frac{(2^k-1)(2^k)^2(2^{k+1}-1)}{3} - \frac{(2^k-1)^2(2^k)^2}{2}\right]$$

$$= \frac{1}{2^{2k}}\left[\frac{2\times(2^k-1)(2^k)^2(2^{k+1}-1)}{6} - \frac{3\times(2^k-1)^2(2^k)^2}{6}\right]$$

$$= \frac{1}{2^{2k}} \times (2^k-1)(2^k)^2\left[\frac{2\times(2^{k+1}-1)}{6} - \frac{3\times(2^k-1)}{6}\right]$$

$$= \frac{2^k-1}{6}[2 \times (2^{k+1}-1) - 3 \times (2^k-1)]$$

$$= \frac{2^k-1}{6}[2^{k+2} - 2 - 3 \times 2^k + 3]$$

$$= \frac{2^k-1}{6}(2^k+1)$$

$$= \frac{2^{2k}-1}{6}$$