BIT

1: Numbers and Arithmetic Operations

EN1106 - Introductory Mathematics

Level I - Semester 1

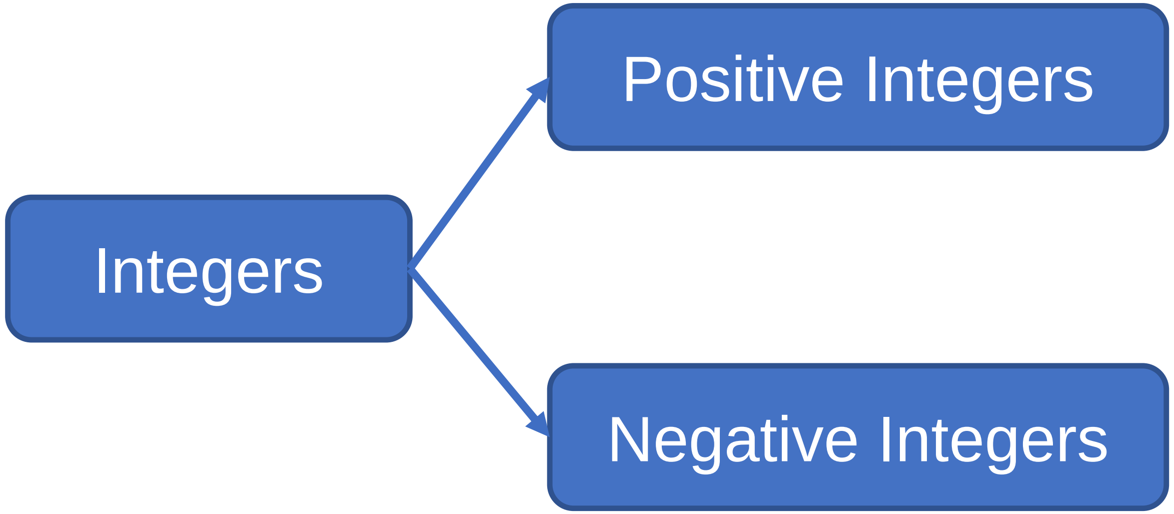
UCSC

Definitions

- Arithmetic is the study of numbers and their manipulation (+, -, x, ÷)
- Sum of two or more numbers: Numbers are added together
- Difference of two numbers: The second number is subtracted from the first number
- Product of two numbers: Multiply the numbers together
- Quotient of two numbers: Divide the first number by the second

Integers

- An integer is a whole number. E.g. 2, 15, -7 , -13



Examples: 1, 2, 3, 4, 5, …..

Examples: -1, -2, -3, -4, -5, …..

NOTE: The number 0 is also an integer but is neither positive nor negative

Examples

- The sum of 7, 6, 3 is:

  $$7 + 6 + 3 = 16$$

- The difference of 8 and 3 is:

  $$8 - 3 = 5$$

- The product of 7 and 2 is:

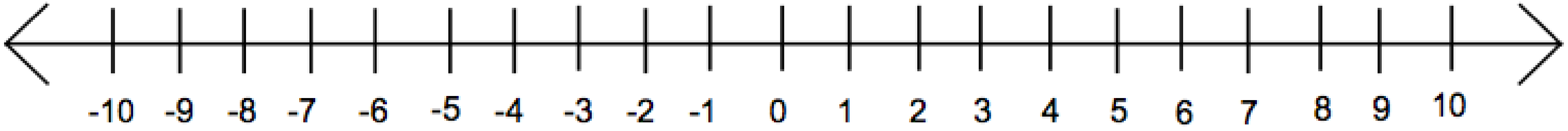
  $$7 \times 2 = 14$$

- The quotient of 20 and 4 is:

  $$(20 \div 4) = 5.$$

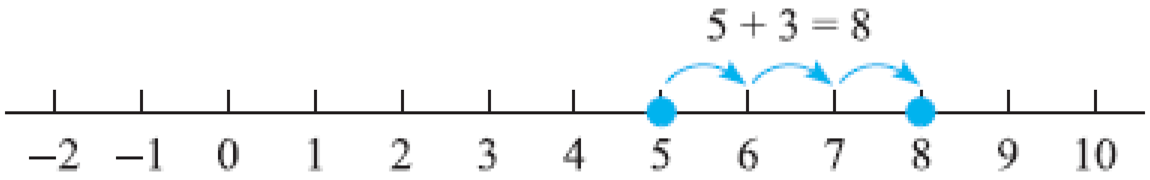
Number Line

- Any number can be represented by a point on the line.
- Positive numbers are on the right-hand side of the line and negative numbers are on the left.



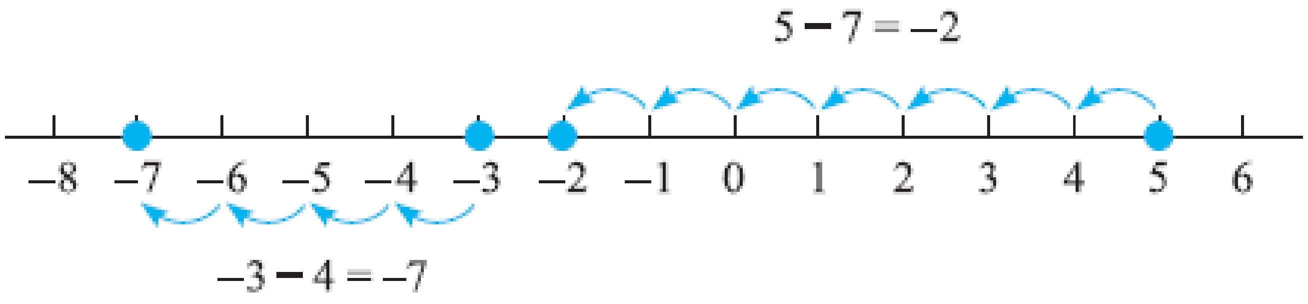
Add positive numbers using number line

- From any given point on the line, we can add a positive number by moving that number of places to the right.
- For example, to find the sum 5 + 3, start at the point 5 and move 3 places to the right, to arrive at 8.



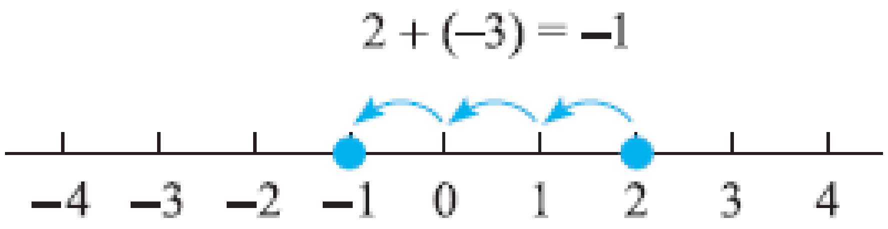
Subtract positive numbers using number line

- To subtract a positive number, we move that number of places to the left.
- For example, to find the difference 5 - 7, start at the point 5 and move 7 places to the left to arrive at -2.



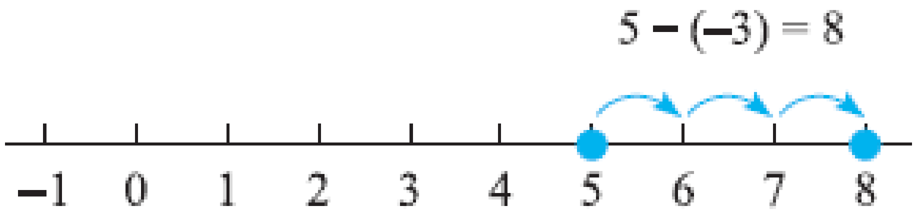
Adding a negative number

- Adding a negative number is equivalent to subtracting a positive number.
- For an example 2 + (-3) = -1



Subtracting a negative number

- Subtracting a negative number is equivalent to adding a positive number.
- For an example 5 – (-3) = 8



Multiplying positive and negative numbers

- Rules to follow
- (positive) x (positive) = positive (E.g.: 7 x 8 = 56)
- (positive) x (negative) = negative (E.g.: 7 x -8 = -56)
- (negative) x (positive) = negative (E.g.: -7 x 8 = -56)
- (negative) x (negative) = positive (E.g.: -7 x -8 = 56)

Dividing positive and negative numbers

- <u>Rules to follow</u>
- (positive) ÷ (positive) = positive (E.g.: 40 ÷ 8 = 5)
- (positive) ÷ (negative) = negative (E.g.: 40 ÷ -8 = -5)
- (negative) ÷ (positive) = negative (E.g.: -40 ÷ 8 = -5)
- (negative) ÷ (negative) = positive (E.g.: -40 ÷ -8 = 5)

Exercises

- Without using a calculator, evaluate each of the following:
  - 18 + (-5)
  - 8 – (-5)
  - -15 + 7
  - -23 -8

The BODMAS Rule

BODMAS stands for

Brackets ( )	First priority
Of $\times$	Second priority
Division $\div$	Second priority
Multiplication $\times$	Second priority
Addition $+$	Third priority
Subtraction $-$	Third priority

Examples

- 2 + 8 x 5
- Using the BODMAS rule we see that multiplication is carried out first. Answer: 2 + 40 = 42

- (3 – 7) x 2
- Using the BODMAS rule we see that the bracketed expression takes priority over all else. Answer: -4 x 2 = -8

Exercises

Exercise 1.2

1. Evaluate the following expressions:
   (a) $6 - 2 \times 2$ (b) $(6 - 2) \times 2$
   (c) $6 \div 2 - 2$ (d) $(6 \div 2) - 2$
   (e) $6 - 2 + 3 \times 2$ (f) $6 - (2 + 3) \times 2$
   (g) $(6 - 2) + 3 \times 2$ (h) $\frac{16}{-2}$ (i) $\frac{-24}{-3}$
   (j) $(-6) \times (-2)$ (k) $(-2)(-3)(-4)$

2. Place brackets in the following expressions to make them correct:
   (a) $6 \times 12 - 3 + 1 = 55$
   (b) $6 \times 12 - 3 + 1 = 68$
   (c) $6 \times 12 - 3 + 1 = 60$
   (d) $5 \times 4 - 3 + 2 = 7$
   (e) $5 \times 4 - 3 + 2 = 15$
   (f) $5 \times 4 - 3 + 2 = -5$

Prime Numbers and Factorization

- Prime Numbers
  - A positive integer, larger than 1, which cannot be expressed as the product of two smaller positive integers.
  - e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23.
  - 8 is not a prime number. 8 = 4 x 2
- Factorise
  - Meaning "writing as a product"
  - e.g. Factorisation of 12 is 3x4=12. Factors of 12 are 4 and 3.
- When a number is written as a product of prime numbers we say the number has been prime factorised.
  - e.g. The prime factorisation of 154 is 2x7x11.

Exercise

1. State which of the following numbers are prime numbers:
   (a) 13 (b) 1000 (c) 2 (d) 29 (e) $\frac{1}{2}$
2. Prime factorise the following numbers:
   (a) 26 (b) 100 (c) 27 (d) 71 (e) 64 (f) 87 (g) 437 (h) 899

Highest Common Factor

- Given two or more numbers, the highest common factor (h.c.f.) is the largest (highest) number that is a factor of all the given numbers.
- Also known as the greatest common divisor (g.c.d).
- e.g.
  - The factors of 12 are 1, 2, 3, 4, 6,and 12. (1 x 12, 2 x 6, 3 x 4)
  - The factors of 18 are 1, 2, 3, 6, 9 and 18. (1 x 18, 2 x 9, 3 x 6)

1, 2, 3 and 6 are common factors of 12 and 18.

The highest common factor of 12 and 18 is 6.

Highest Common Factor

- The factors of 12 are 1, 2, 3, 4, 6,and 12. (1 x 12, 2 x 6, 3 x 4)
- The factors of 18 are 1, 2, 3, 6, 9 and 18. (1 x 18, 2 x 9, 3 x 6)

  1, 2, 3 and 6 are common factors of 12 and 18.

- Highest common factor of 12 and 18 can be obtained directly from their prime factorisation.
- We simply note all the primes common to both factorisations:

  18 = 2 x 3 x 3, 12 = 2 x 2 x 3
- The highest common factor of 12 and 18 is 2 x 3 = 6
- 6 is the highest number that divides exactly into both 12 and 18.

Lowest Common Multiple

- The lowest common multiple (l.c.m.) of a set of numbers is the smallest (lowest) number into which all the given numbers will divide exactly.

- E.g. Given 4 and 6, they both divide exactly into 12, 24, 36, 48, 60 and so on.

  The smallest number into which they both divide is 12.

  Therefore, 12 is the lowest common multiple of 4 and 6.

Lowest Common Multiple

- A more systematic method of finding the l.c.m. involves the use of prime factorisation.
- Find LCM of 15 and 20
  - $15 = \boxed{3} \times 5$
  - $20 = \boxed{2 \times 2} \times \boxed{5}$
- The l.c.m. is the smallest number that contains both of these sets of factors. Consider the number of factors needed (5 is common to both):
  - 3 → 1, 2 → 2, 5 → 1
  - 3 x 2 x 2 x 5 = 60
  - LCM is 60

Lowest Common Multiple

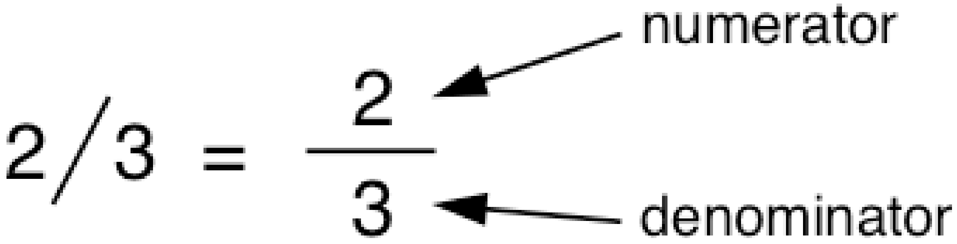
- Find LCM of 20, 24 and 25
  - 20 = 2 x 2 x 5
  - 24 = $\boxed{2 \text{ x } 2 \text{ x } 2}$ x $\boxed{3}$
  - 25 = $\boxed{5 \text{ x } 5}$
- Consider the highest number of 2s, 3s, 5s needed:
  - 2 → 3, 3 → 1, 5 → 2
  - 2 x 2 x 2 x 3 x 5 x 5 = 600
  - LCM is 600

Exercise

1. Calculate the h.c.f. of the following sets of numbers:
   (a) 12, 15, 21 (b) 16, 24, 40 (c) 28, 70, 120, 160 (d) 35, 38, 42 (e) 96, 120, 144
2. Calculate the l.c.m. of the following sets of numbers:
   (a) 5, 6, 8 (b) 20, 30 (c) 7, 9, 12 (d) 100, 150, 235 (e) 96, 120, 144

Fractions

- A fraction is a number of the form p/q, where the letters p and q represent whole numbers or integers.



- A fraction is in its simplest form when there are no factors common to both numerator and denominator.

Proper and Improper Fractions

- Proper fraction
  - Suppose that p and q are both positive numbers. If p is less than q, the fraction is said to be a proper fraction

$\frac{1}{2}$ and $\frac{3}{4}$ are proper fractions

- Improper fraction
  - If p is greater than or equal to q, the fraction is said to be improper

$\frac{11}{8}$, $\frac{7}{4}$ and $\frac{3}{3}$ are all improper fractions

Fractions(2)

- Mixed Fractions

  e.g. $15\frac{1}{4}$

- A fraction is inverted by interchanging its numerator and denominator. This is called, “the reciprocal” of the fraction.

$$\text{reciprocal of } \frac{a}{b} \text{ is } \frac{b}{a} \qquad \frac{4}{5} \text{ is } \frac{5}{4}$$

Expressing a Fraction in Equivalent Forms

- Multiplying or dividing both numerator and denominator of a fraction by the same number produces a fraction having the same value, called an equivalent fraction.
- e.g.

$$\frac{1}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4} \qquad \frac{8}{12} = \frac{8/4}{12/4} = \frac{2}{3}$$

Simplest Form

- A fraction is in its simplest form when there are no factors common to both numerator and denominator
- E.g. simplify 4/6
  - we prime factorise to produce

$$\frac{4}{6} = \frac{2 \times 2}{2 \times 3}$$

  - Dividing both numerator and denominator by 2 leaves $\frac{2}{3}$
  - This is equivalent to cancelling the common factor of 2.

Exercise

2.1 Express $\frac{24}{36}$ in its simplest form.

Solution We seek factors common to both numerator and denominator. To do this we prime factorise 24 and 36:

Prime factorisation has been described in §1.3.

$$24 = 2 \times 2 \times 2 \times 3 \qquad 36 = 2 \times 2 \times 3 \times 3$$

The factors $2 \times 2 \times 3$ are common to both 24 and 36 and so these may be cancelled. Note that only common factors may be cancelled when simplifying a fraction. Hence

Finding the highest common factor (h.c.f.) of two numbers is detailed in §1.4.

$$\frac{24}{36} = \frac{\cancel{2} \times \cancel{2} \times 2 \times \cancel{3}}{\cancel{2} \times \cancel{2} \times \cancel{3} \times 3} = \frac{2}{3}$$

Addition of Fractions

- To add fractions, rewrite each fraction so that they all have the same denominator. This is known as the common denominator.
- The denominator is chosen to be the lowest common multiple of the original denominators. Then the numerators only are added or subtracted as appropriate, and the result is divided by the common denominator.

e.g.(1)

$$\frac{1}{7}+\frac{3}{7}$$

$$\frac{1+3}{7}$$

$$\frac{4}{7}$$

e.g.(2)

$$\frac{1}{3}+\frac{1}{2}$$

$$\frac{2}{6}+\frac{3}{6}$$

$$\frac{5}{6}$$

The denominators are 2 and 3. The l.c.m. of 2 and 3 is 6. Express both fractions with a denominator of 6.

Subtraction of Fractions

- Similar to Addition, rewrite each fraction so that they all have the same denominator and then subtract.
- e.g. $\frac{1}{4} - \frac{5}{9}$

$$\frac{1}{4} = \frac{9}{36} \quad \text{and} \quad \frac{5}{9} = \frac{20}{36}$$

Then

$$\frac{1}{4} - \frac{5}{9} = \frac{9}{36} - \frac{20}{36} = \frac{9-20}{36} = \frac{-11}{36} = -\frac{11}{36}$$

Converting Mixed Fraction into Improper Fraction

- Consider the number $2\frac{3}{4}$.

- This is referred to as a mixed fraction because it contains a whole number part, 2, and a fractional part.
- We can convert this mixed fraction into an improper fraction as follows.
- Recognise that 2 is equivalent to 8/4, and

$$2\frac{3}{4} \text{ is } \frac{8}{4} + \frac{3}{4} = \frac{11}{4}$$

Multiplication of fractions

- The product of two or more fractions is found by multiplying their numerators to form a new numerator, and then multiplying their denominators to form a new denominator.

$$\frac{4}{9} \times \frac{3}{8} = \frac{12}{72}$$

- By converting into Simplest form

$$\frac{12}{72} = \frac{1}{6}$$

Multiplication of fractions

Find $\frac{3}{4}$ of $\frac{5}{9}$.

$\frac{3}{4}$ of $\frac{5}{9}$ is the same as $\frac{3}{4} \times \frac{5}{9}$

By cancelling a factor of 3 from numerator and denominator gives

$\frac{1}{4} \times \frac{5}{3}$, that is $\frac{5}{12}$

Before multiplying, all mixed fractions need to be converted into improper fractions

Division by a Fraction

- To divide one fraction by another fraction, we invert the second fraction and then multiply. When we invert a fraction we interchange the numerator and denominator
- e.g. $$\frac{6}{7} \div \frac{1}{5} = \frac{6}{7} * \frac{5}{1} = \frac{30}{7}$$
- Before multiplying, all mixed fractions need to be converted into improper fractions

Exercises

1. Evaluate

(a) $\frac{3}{4} \div \frac{1}{8}$

(b) $\frac{8}{9} \div \frac{4}{3}$

(c) $\frac{-2}{7} \div \frac{4}{21}$

(d) $\frac{9}{4} \div 1\frac{1}{2}$

(e) $\frac{5}{6} \div \frac{5}{12}$

(f) $\frac{99}{100} \div 1\frac{4}{5}$

(g) $3\frac{1}{4} \div 1\frac{1}{8}$

(h) $\left(2\frac{1}{4} \div \frac{3}{4}\right) \times 2$

(i) $2\frac{1}{4} \div \left(\frac{3}{4} \times 2\right)$

(j) $6\frac{1}{4} \div 2\frac{1}{2} + 5$

(k) $6\frac{1}{4} \div \left(2\frac{1}{2} + 5\right)$

Decimal Numbers

- A decimal point, '.', marks the end of the whole number part, and the numbers that follow it, to the right, form the fractional part.
- A number immediately to the right of the decimal point, that is in the first decimal place, represents tenths.
- Numbers in the second position after the decimal point, or the second decimal place, represent hundredths, and so on.

e.g. 782.124 is $7 \times 100 + 8 \times 10 + 2 \times 1 + \frac{1}{10} + \frac{2}{100} + \frac{4}{1000}$

Decimal Numbers

- A number immediately to the right of the decimal point, that is in the first decimal place, represents tenths, so

$$0.1 = \frac{1}{10}$$

$$0.2 = \frac{2}{10} \quad \text{or} \quad \frac{1}{5}$$

$$0.3 = \frac{3}{10} \quad \text{and so on}$$

- Note that when there are no whole numbers involved it is usual to write a zero in front of the decimal point, thus, .2 would be written 0.2.

Decimal Numbers

- Numbers in the second position after the decimal point, or the second decimal place, represent hundredths, so

$$0.01 = \frac{1}{100}$$

$$0.02 = \frac{2}{100} \quad \text{or} \quad \frac{1}{50}$$

$$0.03 = \frac{3}{100} \quad \text{and so on}$$

Examples

- E.g. $0.25 = 0.2 + 0.05$

$$= \frac{2}{10} + \frac{5}{100}$$

$$= \frac{25}{100}$$

E.g. Express 37.25 as a mixed fraction in its simplest form

$$37.25 = 37 + 0.25$$

$$= 37 + \frac{25}{100}$$

$$= 37 + \frac{1}{4}$$

$$= 37\frac{1}{4}$$

Exercises

1. Express the following decimal numbers as proper fractions in their simplest form:
   (a) 0.7 (b) 0.8 (c) 0.9
2. Express the following decimal numbers as proper fractions in their simplest form:
   (a) 0.55 (b) 0.158 (c) 0.98
   (d) 0.099
3. Express each of the following as a mixed fraction in its simplest form:
   (a) 4.6 (b) 5.2 (c) 8.05 (d) 11.59
   (e) 121.09
4. Write each of the following as a decimal number:
   (a) $\frac{6}{10} + \frac{9}{100} + \frac{7}{1000}$ (b) $\frac{8}{100} + \frac{3}{1000}$
   (c) $\frac{17}{1000} + \frac{5}{10}$

Significant Figures

- Always look at one more digit than the number of significant figures (s.f.) required, and round up or down. (considering digit is greater than or equal to 5 then round up else round down)

e.g. 6.3528 round to,

1st s.f. - 6

2nd s.f. - 6.4

3rd s.f. - 6.35

Examples

Write 36.482 to 3 s.f.

We consider the first four digits, that is 36.48. The final digit is 8 and so we round up 36.48 to 36.5. To 3 s.f. 36.482 is 36.5.

Write 1.0049 to 4 s.f.

To write to 4 s.f. we consider the first five digits, that is 1.0049. The final digit is a 9 and so 1.0049 is rounded up to 1.005.

Write 0.0473 to 1 s.f.

We do not count the initial zeros and consider 0.047. The final digit tells us to round up. Hence to 1 s.f. we have 0.05.

Decimal Places

- e.g. Write 63.4261 to 2 decimal places (d.p.)

Solution:

Consider the number to 3 d.p., that is 63.426.

- If the final digit is 5 or more we round up, otherwise we round down.
- Here the final digit is 6 and therefore, round up 63.426 to 63.43.

Hence 63.4261 to 2 d.p. is 63.43.

Exercises

1. Write to 3 s.f.
   (a) 6962 (b) 70.406 (c) 0.0123
   (d) 0.010991 (e) 45.607 (f) 2345

2. Write 65.999 to
   (a) 4 s.f. (b) 3 s.f. (c) 2 s.f.
   (d) 1 s.f. (e) 2 d.p. (f) 1 d.p.

3. Write 9.99 to
   (a) 1 s.f. (b) 1 d.p.

4. Write 65.4555 to
   (a) 3 d.p. (b) 2 d.p. (c) 1 d.p.
   (d) 5 s.f. (e) 4 s.f. (f) 3 s.f. (g) 2 s.f.
   (h) 1 s.f.