2 : Linear Programming and Integer Programming

IT5506-Mathematics for Computing II

Level III - Semester 5

Overview

- Linear Programming is a mathematical modeling technique to optimize the limited resources.

- Military, Industry, Agriculture, Transportation, Economics, Health system are few major applications of Linear Programming.

- Highly efficient Computer Codes which are developed by George Dantzigin (in 1940) increase the usefulness of this mathematical modeling technique.

Intended Learning Outcomes

At the end of this lesson, you will be able to;

- Formulate a decision problem either as a linear program, or as a integer program, or a mixed- integer program.
- Solve linear and integer programming 2 variable problems graphically and interpret the solutions.
- Solve 2 and 3 variable linear programming problems using the simplex algorithm, interpret solutions and do sensitivity analysis

List of sub topics

2.1 Introduction to Linear Programming

- 2.1.1 Assumptions
- 2.1.2 Graphical method and Simplex algorithm with standard and general linear programming problems
- 2.1.3 Duality

2.1 Introduction to Linear Programming

2.1.1 Assumptions

- **Divisibility** - Decision variables can have any value (eg:noninteger values) which represents the level of some activity that can have fractional level.

- **Proportionality** - The contribution of each activity to the value of objective function is proportional to the level of the activity. Similarly, the contribution of each activity to the left hand side of each constraint is proportional to the level of the activity.

- **Additivity** - Every function in a linear programming model is the sum of the individual contributions of the respective activities.

- **Certainty** - The value assign to each parameter of the linear programming model is assumed to e known constraint.

General Form of Linear Programming Model

Maximize $f(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$
$$x_1, x_2, \ldots, x_n \geq 0$$

<u>In Matrix form</u>

Maximize $f(x) = c^T x$

Subject to $Ax \leq b$

$x \geq 0$

Hear

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \ c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

where x_i is a decision variable and a_{ji}, b_j, c_i are constants, for each $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$

We can do variations to the objective function like minimizing instead of maximizing and equalities in constrains changing to inequalities.

Formulating Linear Programming Model

STEP 1 - Study the given situation and find the key decision to be taken. Hence, identify the decision variables of the problem.

Eg: $x_1, x_2, x_3, \ldots$

STEP 2 - Identify the objective function of the LP problem and represent it as a function of the decision variables, which is to be maximized or minimized.

STEP 3 - Identify the all the constraints in the problem and express them as linear equations or inequalities which are linear functions of unknown variables.

STEP 4 - Add non-negative constraints.

Terminology

<u>Optimization Problem</u> - A problem in which we are asked to find the best or optimal solution subject to given conditions is called optimization problem

<u>Linear Programming Problem</u> - An optimization problem in which the objective function can be expressed as a linear function of the variables and in which the constraints can be expressed as linear equations or linear inequalities is called a linear programming problem.

<u>Decision Variables</u> - A decision variable is used to represent the level of achievement of a particular course of action. The solution of the linear programming problem will provide the optimum value for each and every decision variable of the model.

Terminology

Objective function coefficient - It is a constant representing the profit per unit or cost per unit of carrying out an activity

Objective Function - It is an expression representing the total profit or cost of carrying out a set of activities at same levels.

Objective function is either maximization type or minimization type. The benefit related objective function will come under maximization type whereas the cost related objective function will come under minimization type.

General form of objective function;

Maximize or Minimize $f(x) = C_1X_1 + C_2X_2 + \cdots + C_nX_n$

Terminology

Technological Coefficient (a_{ij}) - The technological coefficient, a_{ij} is the amount of resource i required for the activity j, where i varies from 1 to m and j varies from 1 to n.

Resource Availability - b_i - Constant i amount of resource i available during the planning period.

Set of constraints - A constraint is a kind of restriction on the total amount of particular resource required to carry out the activity at various level.

Non - Negative Constraints - Each and every decision variable in the LP models non-negative variables.

2.1.2 Graphical Method

Steps of graphical method

1. Formulate the linear programming problem
2. Construct a graph and plot the constraint lines
3. Determine the valid slide of each constraint line
4. Identify feasible solution region
5. Find the optimum points
6. Calculate the coordinates of the optimum points
7. Evaluate the objective function at optimum points to get the requires maximum or minimum value of the objective function

❖ This method is applicable for different types of problems but this gets more complicated when the number of variables and constraints increases.

Example 01

A medical officer gave advice to a family to have a very controlled vitamin C-rich mixed fruit-breakfast which is a good source of dietary fibre as well; in the form of 5 fruit servings per day. They choose apples and bananas as their target fruits, which can be purchased from an online vendor in bulk at a reasonable price.

Banana cost 30 rupees per 6 servings and apples cost 80 rupees per kilogram or 8 servings. Given: 1 banana contains 8.8 mg of Vitamin C and 100-125 g of apples, serving contains 5.2 mg of Vitamin C.

Every person of the family would like to have at least 20 mg of Vitamin C daily but would like to keep the intake under 60 mg.

How much fruit servings would the family have to consume on a daily basis per person to minimize their cost?

Solution

x = number of banana servings taken

y = number of apples servings taken

cost of a banana serving = 30/6 rupees

cost of an apple serving = 80/8 rupees

cost of x banana servings = 5x

cost of y apples servings = 10y

Total cost, C = 5x + 10y

x and y greater than or equal to zero

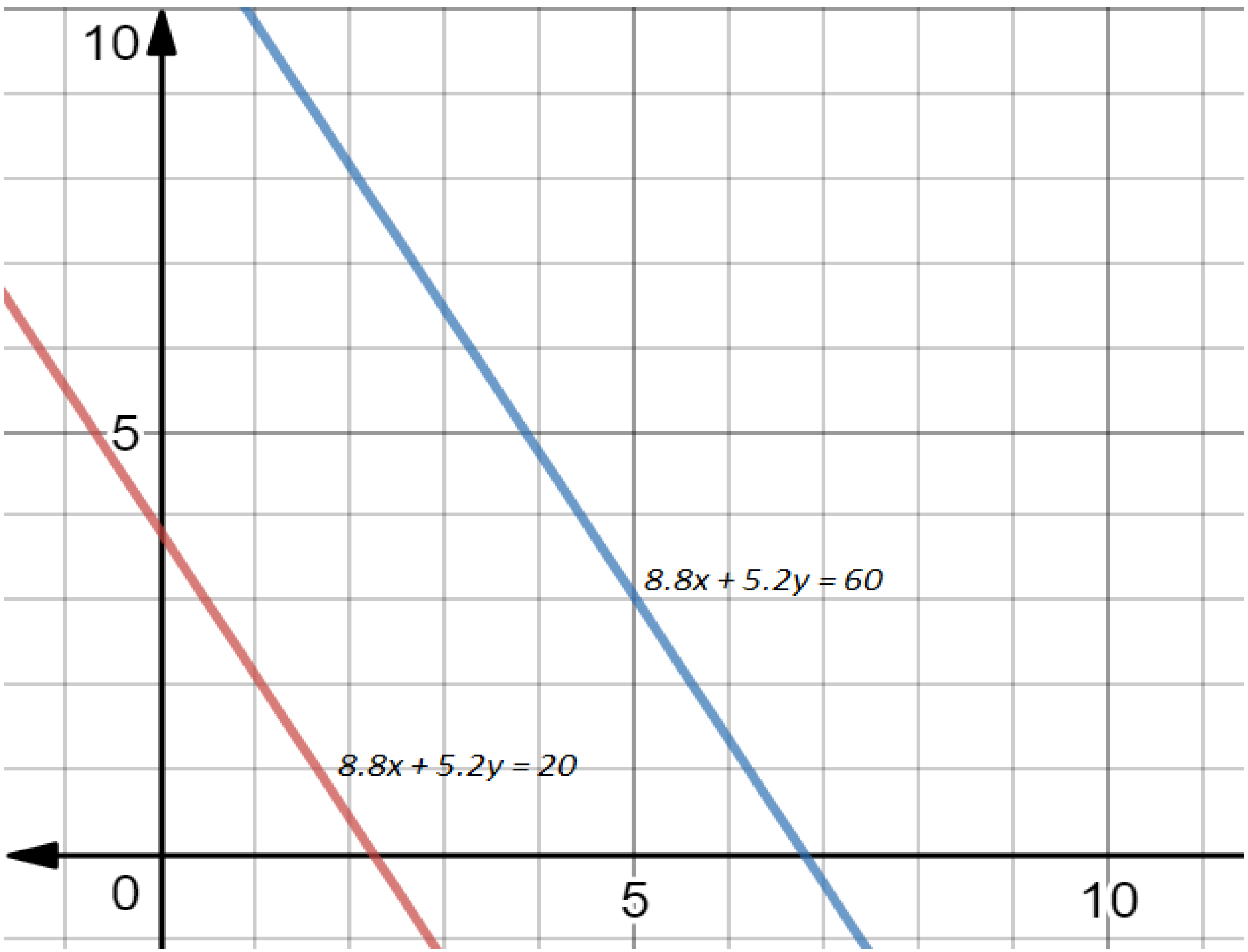
Solution Cont.

Total Vitamin C intake,

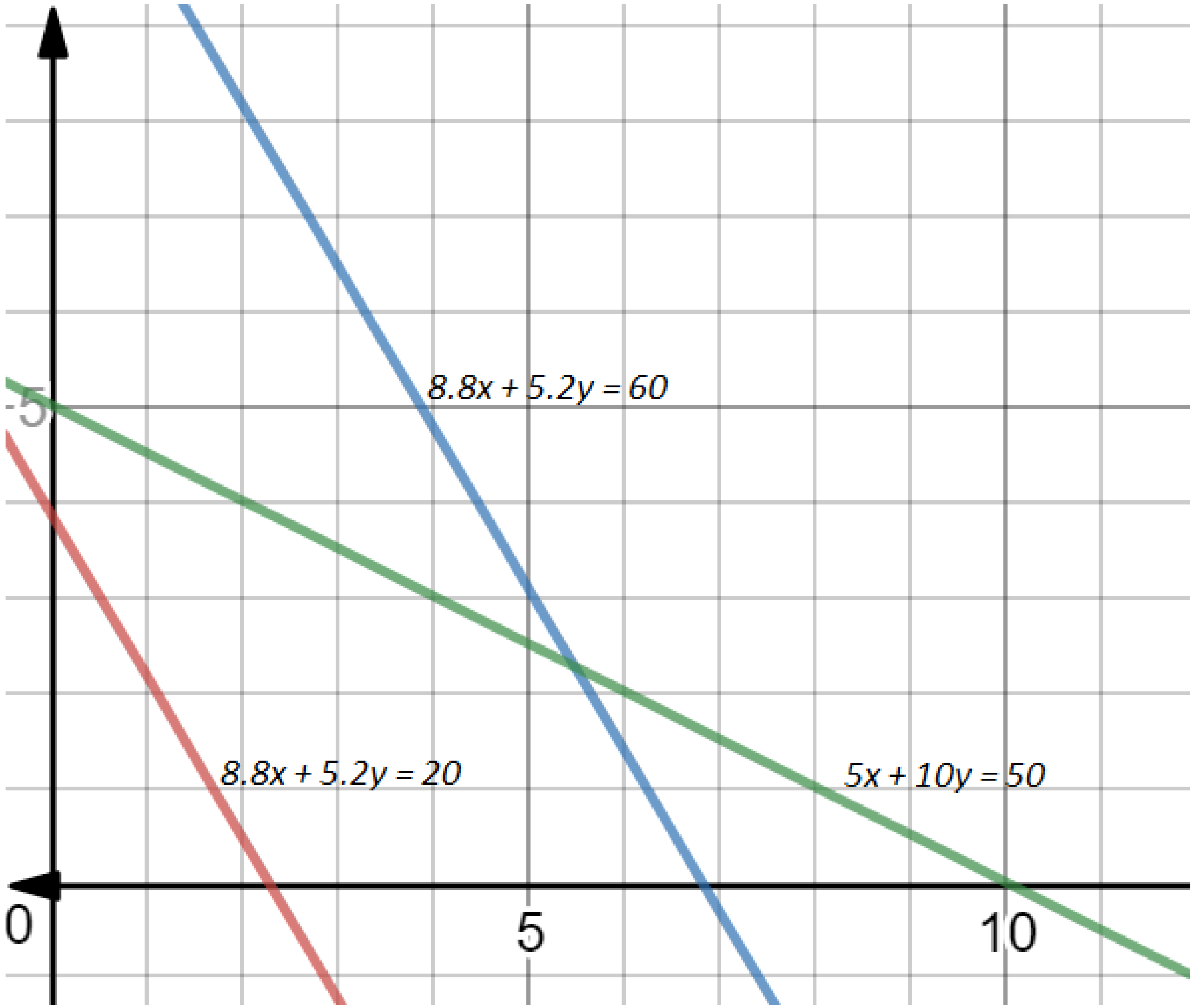
$8.8x + 5.2y \geq 20 \quad (1)$

$8.8x + 5.2y \leq 60 \quad (2)$

Graph with the constraint equations;



Solution Cont.



Solution Cont.

To check for the validity of the equations, put x=0, y=0 in (1). Clearly, it doesn't satisfy the inequality. Therefore, we must choose the side opposite to the origin as our valid region. Similarly, the side towards origin is the valid region for equation 2)

Feasible Region: As per the analysis above, the feasible region for this problem would be the one in between the red and blue lines in the graph! For the direction of the objective function; let us plot $5x+10y = 50$.

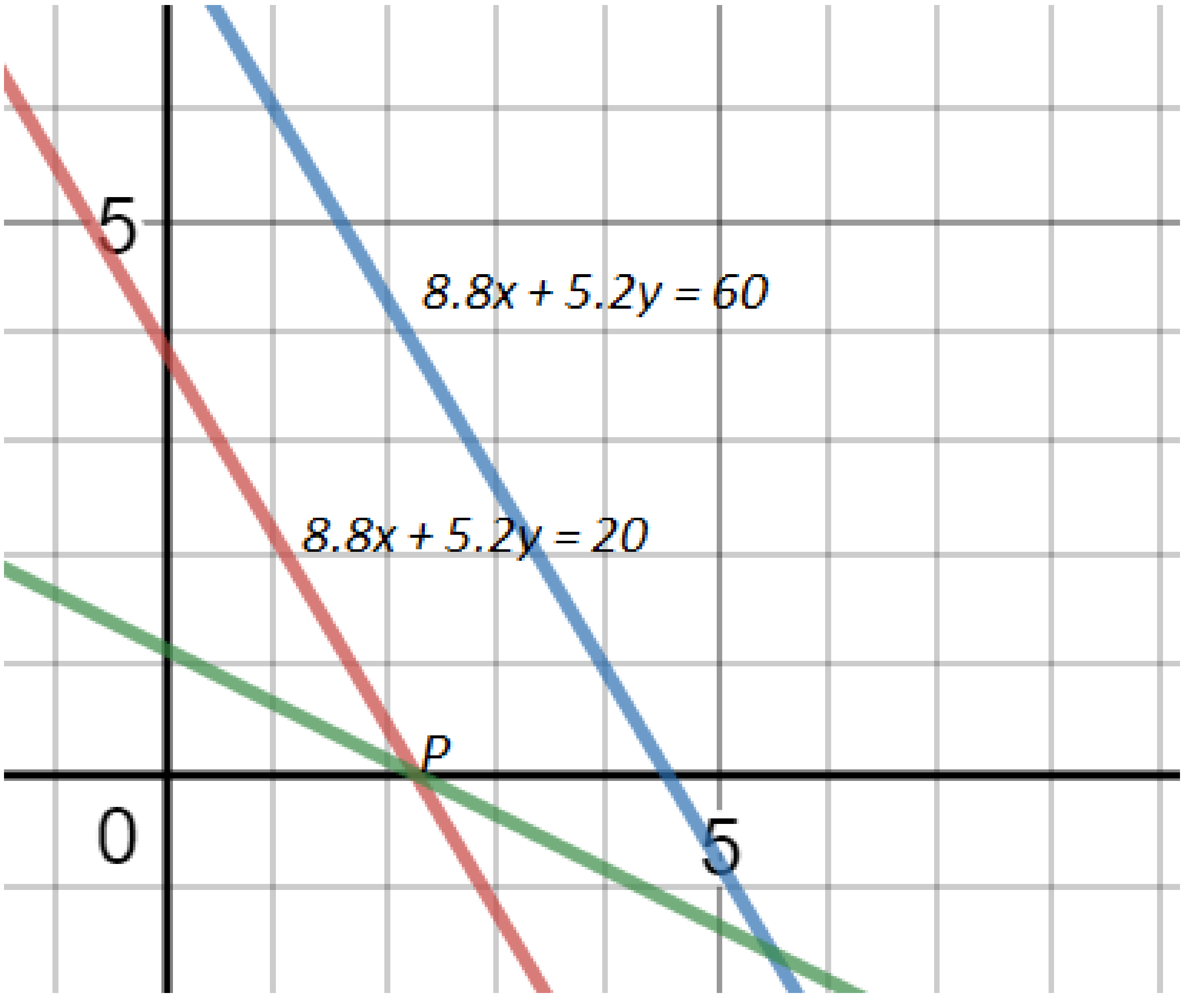
Solution Cont.

Now take a ruler and place it on the straight line of the objective function. Start sliding it from the left end of the graph. What do we want here? We want the minimum value of the cost i.e. the minimum value of the optimum function C. Thus we should slide the ruler in such a way that a point is reached, which:

1) lies in the feasible region
2) is closer to the origin as compared to the other points

This would be our Optimum Point. I’ve marked it as P in the graph. It is the one which you will get at the extreme right side of the feasible region here. I’ve also shown the position in which your ruler needs to be to get this point by the line in green.

Solution Cont.



Solution Cont.

Now we must calculate the coordinates of this point. To do this, just solve the simultaneous pair of linear equations:

$$y = 0$$

$$8.8x + 5.2y = 20$$

We'll get the coordinates of 'P' as (2.27, 0). This implies that the family must consume 2.27 bananas and 0 apples to minimize their cost and function according to their diet plan.

2.1.2 Simplex algorithm with standard and general linear programming problems

Steps

1. Set up the problem
2. Convert the inequalities into equations
3. Construct the initial simplex tableau
4. The most negative entry in the bottom row identifies the pivot column
5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.
6. Perform pivoting to make all other entries in this column zero
7. When there are no more negative entries in the bottom row, we have finished otherwise we should start again from step 4

Example 02

Riz has two part time jobs, job A and job B. He can’t work more than a total of 12 hours a week. He has determined that for every hour works at job A, he wants 2 hours of preparation time and for every hour he works at job B he wants an hour of preparation time. If he makes \$40 an hour at job A and \$30 an hour at job B, how many hours should he work per week at each job to maximize his income?

Solution

x_1 = Number of hours per week Riz should work at job A

x_2 = Number of hours per week Riz should work at job B

Maximize, $z = 40x_1 + 30x_2$

Subject to $x_1 + x_2 \leq 12$

$2x_1 + x_2 \leq 16$

$x_1, x_2 \geq 0$

$$z - 40x_1 - 30x_2 = 0$$

$$x_1 + x_2 + y_1 = 12$$

$$2x_1 + x_2 + y_2 = 16$$

$$x_1, x_2 \geq 0$$

Solution Cont.

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
–40	–30	0	0	1	0

Most negative entry in bottom row identifies the pivot

x_1	x_2	y_1	y_2	Z			
1	1	1	0	0	12	y_1	$12 \div 1 = 12$
$\boxed{2}$	1	0	1	0	16	y_2	$\leftarrow 16 \div 2 = 8$
–40	–30	0	0	1	0	Z	
↑							

x_1	x_2	y_1	y_2	Z	
1	1	1	0	0	12
$\boxed{1}$	1/2	0	1/2	0	8
–40	–30	0	0	1	0

Solution Cont.

x1	x2	y1	y2	Z		
0	1/2	1	–1/2	0	4	y1
$\boxed{1}$	1/2	0	1/2	0	8	x1
0	–10	0	20	1	320	Z

x1	x2	y1	y2	Z			
0	$\boxed{1/2}$	1	–1/2	0	4	y1	← 4 ÷ 1/2 = 8
1	1/2	0	1/2	0	8	x1	8 ÷ 1/2 = 16
0	–10	0	20	1	320	Z	
	↑						

x1	x2	y1	y2	Z	
0	$\boxed{1}$	2	–1	0	8
1	1/2	0	1/2	0	8
0	–10	0	20	1	320

Solution Cont.

x_1	x_2	y_1	y_2	Z		
0	1	2	–1	0	8	x_2
1	0	–1	1	0	4	x_1
0	0	20	10	1	400	Z

$z = 400 - 20y_1 - 10y_2$

Therefore,

$x_1 = 4$ $x_2 = 8$ and $z= 400$

Extra Variables

Slack Variables - Additional variables that are introduced into the linear constraints of linear program to transform them from inequality constraints to equality constraints.

Free Variables - Suppose some component of x is unrestricted in sign, constraint is omitted.

Surplus Variables - To convert " $\geq$ " constraints to standard form, a surplus variable is subtracted on the left hand side of the constraint. That is number of items produced in excess of the requirement. Non-negativity constraints are required for surplus variables

Standard Form of a Linear Program Model

To solve an LPP algebraically, we first put it in the standard form. This means the objective is maximized, all decision variables are non negative and all constraints (other than the non-negativity restrictions) are equations with nonnegative RHS.

Maximize the objective function $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
subject to problem constraints of the form $c_1x_1 + c_2x_2 + \cdots + c_nx_n = b,\ b \geq 0$
with non-negative constraints

$$x_1, x_2, \ldots, x_n \geq 0$$

- If the problem is $min\ Z$, convert it to $max - z$.
  (Minimizing $f(x)$ is equivalent to maximizing $-f(x)$)
- If a constraint is $c_1x_1 + c_2x_2 + \cdots + c_nx_n \leq b$, convert it to an equality constraint by adding a nonnegative slack variable s. The resulting constraint is
  $$c_1x_1 + c_2x_2 + \cdots + c_nx_n + s = b$$
- If a constraint is $c_1x_1 + c_2x_2 + \cdots + c_nx_n \geq b$, convert it to an equality constraint by subtracting a nonnegative surplus variable s. The resulting constraint is
  $$c_1x_1 + c_2x_2 + \cdots + c_nx_n - s = b$$
- If some variable x_i is unrestricted in sign (free), replace it everywhere in the formulation by $u_i - v_i$, where $u_i, v_i \geq 0$.

2.1.3 Duality

Every LPP has another LPP related to it and can be derived from it.

Primal - Original LPP

Dual - Derived LPP

- Initially the **original LPP** should be formulated in its standard form.

  (Standard form : variables of the problem should be non negative and ≥ , ≤ inequality symbols are used in the minimization and maximization case respectively.)

Example for Duality

The following example explains the concept of duality.

Maximize $Z = 50x_1 + 30x_2$

Subject to:

$4x_1 + 3x_2 \leq 100$

$3x_1 + 5x_2 \leq 150$

$X_1, x_2 \geq 0$

The duality can be applied to the above original linear programming problem as:

Minimize

$G = 100y_1 + 150y_2$

Subject to:

$4y_1 + 3y_1 \geq 50$

$3y_1 + 5y_2 \geq 30$

$Y_1, y_2 \geq 0$

Example for Duality

The following observations were made while forming the dual linear programming problem:

1. The primal or original linear programming problem is of the maximization type while the dual problem is of minimization type.
2. The constraint values 100 and 150 of the primal problem have become the coefficient of dual variables y1 and y2 in the objective function of a dual problem and while the coefficient of the variables in the objective function of a primal problem has become the constraint value in the dual problem.
3. The first column in the constraint inequality of primal problem has become the first row in a dual problem and similarly the second column of constraint has become the second row in the dual problem.
4. The directions of inequalities have also changed, i.e. in the dual problem, the sign is the reverse of a primal problem. Such that in the primal problem, the inequality sign was "≤" but in the dual problem, the sign of inequality becomes "≥".

Activity 01

Select the correct constraints for the following scenario.

A company makes two products (X and Y) using two machines (A and B). Each unit of X that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Y that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B.

At the start of the current week there are 30 units of X and 90 units of Y in stock. Available processing time on machine A is forecast to be 40 hours and on machine B is forecast to be 35 hours.

Activity 01 Cont.

The demand for X in the current week is forecast to be 75 units and for Y is forecast to be 95 units. Company policy is to maximise the combined sum of the units of X and the units of Y in stock at the end of the week.

- x be the number of units of X produced in the current week
- y be the number of units of Y produced in the current week

(a) 50x + 24y <= 40(60) machine A time

30x + 33y <= 35(60) machine B time

x >= 75 - 30

(b)50x + 24y <= 40(60) machine A time

30x - 33y <= 35(60) machine B time

x <= 75 - 30

Activity 01 Cont.

(c) $50x + 24y = 40(60)$ machine A time

$-30x + 33y <= 35(60)$ machine B time

$x >= 75 - 30$

(d) $50x + 24y <= 40(60)$ machine A time

$30x + 33y = 35(60)$ machine B time

$x >= 75 - 30$

Activity 02

Select the correct objective function for the following scenario

A company makes two products (X and Y) using two machines (A and B). Each unit of X that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Y that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B.

At the start of the current week there are 30 units of X and 90 units of Y in stock. Available processing time on machine A is forecast to be 40 hours and on machine B is forecast to be 35 hours.

Activity 02 Cont.

The demand for X in the current week is forecast to be 75 units and for Y is forecast to be 95 units. Company policy is to maximise the combined sum of the units of X and the units of Y in stock at the end of the week.

- x be the number of units of X produced in the current week
- y be the number of units of Y produced in the current week

(a) maximise (x+30-75) + (y+90-95) = (x-y-50)
(b) maximise (x+30-75) + (y+90-95) > (x+y-50)
(c) maximise (x+30-75) + (y+90-95) = (x+y-50)
(d) maximise (x+30+75) + (y+90-95) = (x+y-50)

Activity 03

What are the values for x, number of units of X produced in current week and y, number of units of Y produced in the current week?

A company makes two products (X and Y) using two machines (A and B). Each unit of X that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Y that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B.

At the start of the current week there are 30 units of X and 90 units of Y in stock. Available processing time on machine A is forecast to be 40 hours and on machine B is forecast to be 35 hours.

Activity 03 Cont.

The demand for X in the current week is forecast to be 75 units and for Y is forecast to be 95 units. Company policy is to maximise the combined sum of the units of X and the units of Y in stock at the end of the week.

- x be the number of units of X produced in the current week
- y be the number of units of Y produced in the current week

(a) $x = 54, y = 6.25$
(b) $x = 45, y = 6.25$
(c) $x = 45, y = 7$
(d) $x = 45.1, y = 6.25$

Activity 04

Match the correct values.

A company makes two products (X and Y) using two machines (A and B). Each unit of X that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Y that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B.

At the start of the current week there are 30 units of X and 90 units of Y in stock. Available processing time on machine A is forecast to be 40 hours and on machine B is forecast to be 35 hours.

Activity 04 Cont.

The demand for X in the current week is forecast to be 75 units and for Y is forecast to be 95 units. Company policy is to maximise the combined sum of the units of X and the units of Y in stock at the end of the week.

- x be the number of units of X produced in the current week
- y be the number of units of Y produced in the current week

Activity 04 Cont.

x y objective function	8 46 6 6.25 2 45 7 1.25

Activity 05

Select the correct answer

A carpenter makes tables and chairs. Each table can be sold for a profit of £30 and each chair for a profit of £10. The carpenter can afford to spend up to 40 hours per week working and takes six hours to make a table and three hours to make a chair. Customer demand requires that he makes at least three times as many chairs as tables. Tables take up four times as much storage space as chairs and there is room for at most four tables each week.

x_T = number of tables made per week

x_C = number of chairs made per week

Activity 05 Cont.

(a)

$6x_T + 3x_C <= 40$

$(x_C/4) + x_T <= 4$

$x_C >= 3x_T$

maximise $30x_T + 10x_C$

(b)

$6x_T + 3x_C >= 40$

$(x_C/4) + x_T <= 4$

$x_C >= 3x_T$

maximise $30x_T + 10x_C$

(c)

$6x_T + 3x_C <= 40$

$(x_T/4) + x_C <= 4$

$x_C >= 3x_T$

maximise $30x_T + 10x_C$