4: Boolean Algebra

IT2106 – Mathematics for Computing I

Level I - Semester 2

Learning Outcomes

At the end of this lesson students will be able to:

I. use the laws of Boolean algebra as they apply to sets;
II. manipulate Boolean-valued expressions
III. simplify Boolean expressions;
IV. investigate de Morgan's laws
V. apply mathematical knowledge and skills in a problem solving context.

Introduction

The most obvious way to simplify Boolean expressions is to manipulate them in the same way as normal algebraic expressions are manipulated. With regards to logic relations in digital forms, a set of rules for symbolic manipulation is needed in order to solve for the unknowns.

A set of rules formulated by the English mathematician *George Boole* describe certain propositions whose outcome would be either *true* *or* *false*.

Basic Definitions

Let B be a non empty set with two binary operations + and *, a unary operation ‘ ,and two distinct elements 0 and 1.Then B is called *Boolean Algebra* if it holds the basic *axioms* [B_1] to [B_4].

Sometimes we will designate a Boolean algebra by ‹B,+,*,’,0,1› when we want to emphasize its six parts.

Axioms

Where a, b, c are any elements in B:

[B_1] Commutative laws:

(1a) $a + b = b + a$ (1b) $a * b = b * a$

[B_2] Distributive laws:

(2a) $a + (b * c)=(a + b) * (a + c)$ (2b) $a * (b + c) = (a * b) + (a * c)$

[B_3] Identity laws:

(3a) $a + 0 = a$ (3b) $a * 1 = a$

[B_4] Complement laws:

(4a) $a + a' = 1$ (4b) $a * a' = 0$

Boolean Operations

The *complement* is denoted by a '. It is defined by
0' = 1   and   1' = 0.
The *Boolean sum*, denoted by + or by OR,
has the following values:
1 + 1 = 1,   1 + 0 = 1,   0 + 1 = 1,   0 + 0 = 0
The *Boolean product*, denoted by . or by * or by AND,
has the following values:
1 * 1 = 1,   1* 0 = 0,   0 * 1 = 0,   0 * 0 = 0

Precedence

We adopt the usual convention that, unless we are guided by parenthesis, ' has precedence over *, and * has precedence over +.

Example:

$a + b * c$ means $a + (b * c)$ and not $(a + b) * c$

$a * b'$ means $a * (b')$ and not $(a * b)'$

Duality

The *dual* of any statement in a Boolean algebra B is the statement obtained by changing every AND(*) to OR(+), every OR(+) to AND(*) and all 1's to 0's and vice-versa in the original statement.

Example:

The dual of

$(1 + a) * (b + 0) = b$ is $(0 * a) + (b * 1) = b$

Principle of Duality

Theorem 1:

The dual of any theorem in a Boolean algebra is also a theorem.

i.e. if any statement is a consequence of the axioms of a Boolean algebra, then the dual is also a consequence of those axioms since the dual statement can be proven by using the dual of each step of the proof of the original statement.

Basic Theorems

Using the axioms $[B_1]$ through $[B_4]$, we prove the following theorems

Theorem 2: Let a, b, c be any elements in a Boolean Algebra B.

i. Idempotent laws:

(5a) $a + a = a$ (5b) $a * a = a$

ii. Boundedness laws:

(6a) $a + 1 = 1$ (6b) $a * 0 = 0$

iii. Absorption laws:

(7a) $a + (a * b) = a$ (7b) $a * (a + b) = a$

iv. Associative laws:

(8a) $(a + b) + c = a + (b + c)$ (8b) $(a * b) * c = a * (b * c)$

Basic Theorems

Theorem 3: Let a be any element of a Boolean algebra B.

i. (Uniqueness of Complement) If $a + x = 1$ and $a * x = 0$, then $x = a'$.

ii. (Involution law) $(a')' = a$.

iii. (9a) $0' = 1$ (9b) $1' = 0$.

Basic Theorems

Theorem 4 :

DeMorgan's laws :

(10a) $(a + b)' = a' * b'$

(10b) $(a * b)' = a' + b'$