1 : Theory of Matrices, Vector spaces and Linear Transformations

IT5506 – Mathematics for Computing II

Level III - Semester 5

Overview

The fundamental problem of linear algebra is to solve m linear equations in n unknowns :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m;$$

where a_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$ and b_i $(1 \leq i \leq m)$ are real numbers.

The **objective** of this course is to find answers for the following questions with the necessary theoretical background in linear algebra:

1. What is the condition (C) that $b = (b_1, b_2, \cdots, b_m)$ should satisfy for solution to exists?
2. When solution does exist, what is the most suitable method to find the solution in a systematic way?
3. When solution does not exist, is it possible to find the best approximate solution for the system? If it possible, what is the most suitable method to find the best approximated solution in a systematic way?

Intended Learning Outcomes

At the end of this lesson, you will be able to;

- solve a system of linear equations using that you studied in O/L Mathematics.
- identify whether a system of linear equations has a solution or not.
- visualize solving a system of linear equations as a geometrical problem.
- visualize solving a system of linear equations as a linear combination of column vectors.

List of sub topics

1.1 Different ways of looking at system of n linear equations in n unknowns (2 hours)

1.1.1 The Geometry of a system of linear Equations

1.1.2 Column Picture of a system of linear equations

1.1.3 Representation of a system of linear equations in matrix form

1.1 Solving System of Linear equations

Solve the following simple system of linear equations (3 unknowns and 3 equations). We have studied how to solve this system by elimination and backward substitution in O/L Mathematics:

$$x_1 + 2x_2 + x_3 = 4 \text{-----} (1)$$

$$2x_1 + 2x_2 + 3x_3 = 7 \text{-----} (2)$$

$$x_1 + x_2 + 4x_3 = 6 \text{-----} (3)$$

Solution:

Step 1: Eliminate x_1 in equations (2) & (3) using equation (1).

$$x_1 + 2x_2 + x_3 = 4 \text{-----} (1')$$

$(1) \times -2 + (2) \Longrightarrow$ $$-2x_2 + x_3 = -1 \text{----} (2')$$

$(1) \times -1 + (3) \Longrightarrow$ $$-x_2 + 3x_3 = 2 \text{-----} (3')$$

This new system of linear equations is equivalent to the given system of linear equations. That is both systems have the some solution.

Solving System of Linear equations

Step 2: Eliminate x_2 in equation (3') using equation (2') to obtain a new system of linear equations that is equivalent to the given system of linear equations.

$$x_1 + 2x_2 + x_3 = 4 -----(1'')$$

$$-2x_2 + x_3 = -1 ----- (2'')$$

$$(2') \times -\frac{1}{2} + (3') \Longrightarrow \qquad \frac{5}{2} x_3 = \frac{5}{2} -----(3'')$$

Step 3: We can find the solution of the given system using back-substitution.

$$(3'') \Longrightarrow \qquad x_3 = 1$$

$$(2'') \Longrightarrow \qquad x_2 = 1$$

$$(1'') \Longrightarrow \qquad x_1 = 1 .$$

This method is known as the **Gaussian Elimination** method and this method is the basis for this course. However, even a simple system of linear equations may not have a solution (singular case) or may have infinitely many solutions.

A System of Linear equations with no solution

Consider the following simple system of linear equations (2 unknowns and 2 equations).

$$x_1 + 2x_2 = 3 ----- -(1)$$

$$4x_1 + 8x_2 = 6 ----- -(2).$$

As usual, we try to solve this system by the **Gaussian Elimination** method.

$$x_1 + 2x_2 = 3 ----- -(1')$$

$$(1) \times -4 + (2) \Longrightarrow \quad 0 = -6 ---- - (2').$$

The equation (2') that we derived from the given system is not true (a contradiction). Therefore this system has no solution. This is an example for a singular case.

A System of Linear equations with infinitely many solutions

Consider the following simple system of linear equations (2 unknowns and 2 equations).

$x_1 + 2x_2 = 3 -----(1)$

$4x_1 + 8x_2 = 12 -----(2)$.

As usual, we try to solve this system by the **Gaussian Elimination** method.

$$x_1 + 2x_2 = 3 -----(1')$$

$$(1) \times -4 + (2) \Longrightarrow \qquad 0 = 0 ----(2').$$

Here we see that the equation (2) is redundant. That is equation (2) can be obtained by multiplying the equation (1) by 4.

Hence, for any values for x_1, x_2 that satisfy equation (1) is a solution for the given system.

Therefore, there are infinitely many solutions for the given system, namely,

$x_2 = t$, and $x_1 = 3 - 2t$, for any real number t.

1.1.1 The Geometry of Linear Equations (Row picture)

Consider the following simple system of linear equations in two unknowns:

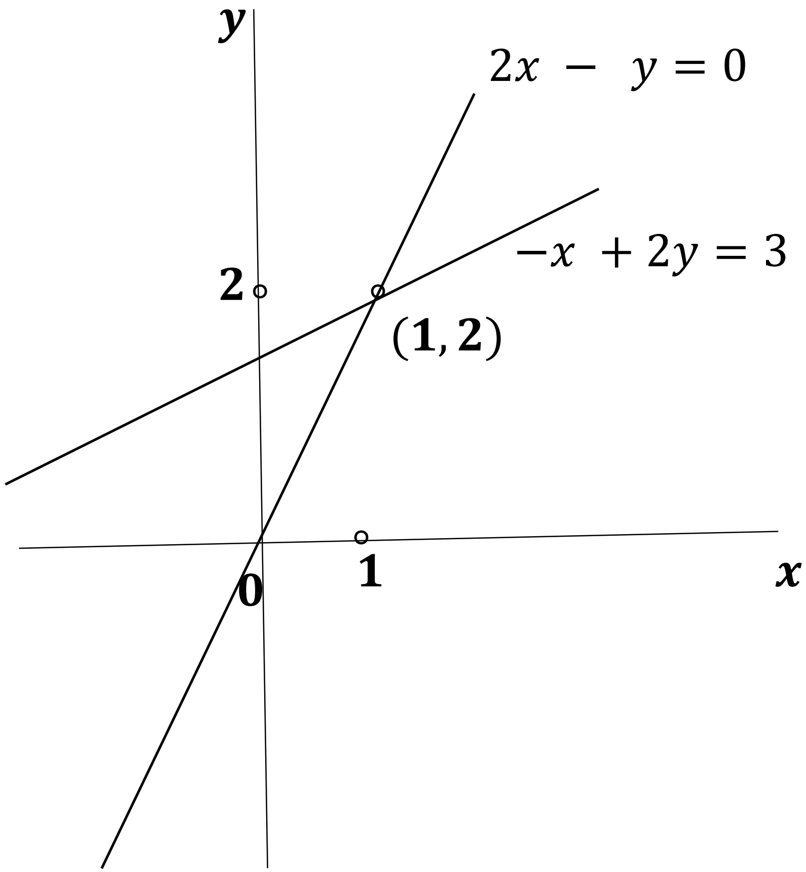
$2x - y = 0$

$-x + 2y = 3.$

These equations can be represented by straight lines in the $x - y$ plane.

The lines $2x - y = 0$ and $-x + 2y = 3$ intersect at the point (1, 2).

Hence $x = 1$, and $y = 2$ is the only solution to the above system of linear equations.



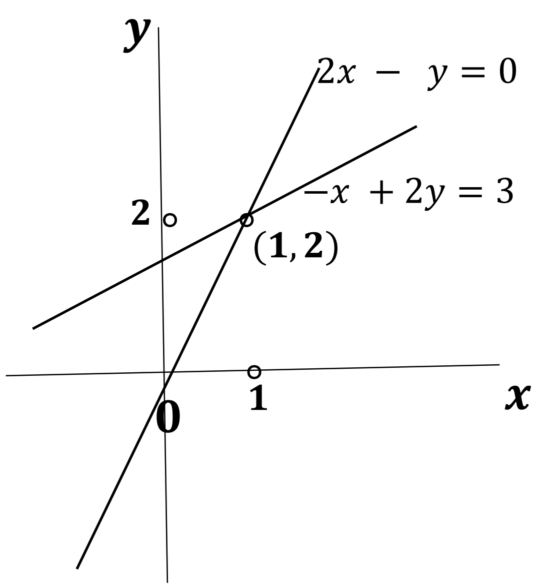
Geometrical interpretation of Different Cases

Consider the following three systems of linear equations in two unknowns:

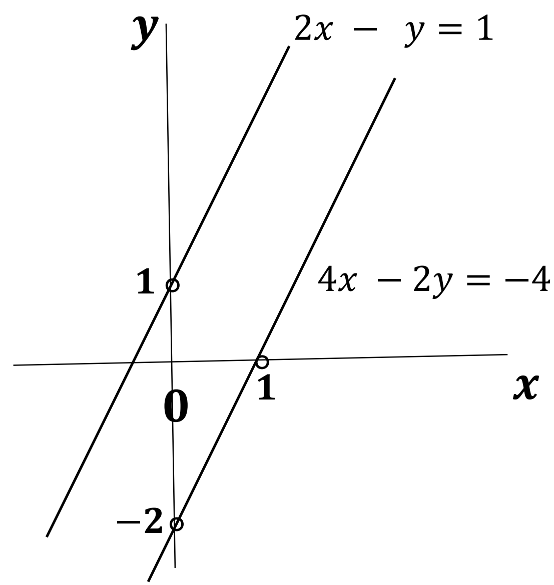
$a.$ $2x - y = 0$
$-x + 2y = 3.$

$b.$ $2x - y = 1$
$4x - 2y = -4.$

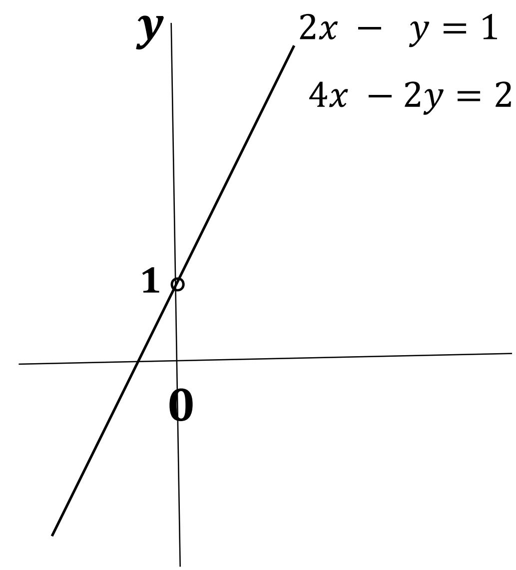
c. $2x - y = 1$
$4x - 2y = -4.$



Unique solution
Two intersecting lines



No solution
Two parallel lines



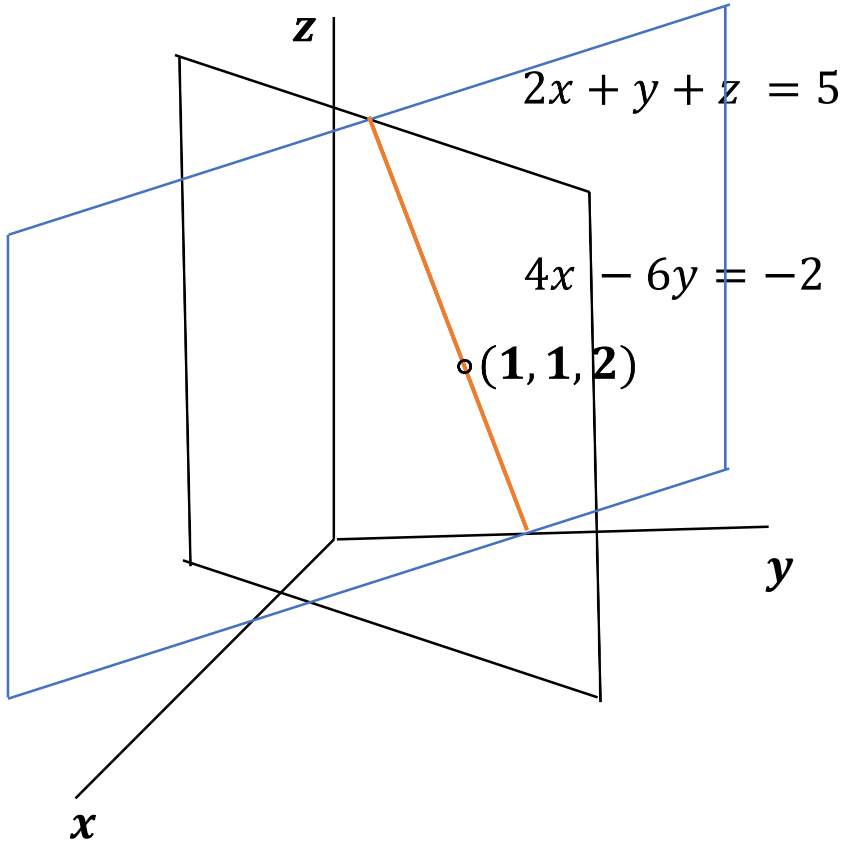
Infinitely many solutions
Two identical lines

Row picture in Three Dimension

Consider the following simple system of linear equations in three unknowns:

$$
\begin{aligned}
2x + y + z &= 5 \\
4x - 6y &= -2 \\
-2x + 7y + 2z &= 9.
\end{aligned}
$$

- Each equation describes a plane in three dimensions.
- The second plane is $4x - 6y = -2$. It is drawn vertically, because z can take any value.
- The figure shows the intersection of the second plane with the first.
- Finally the third plane intersects this line in a point $(1,1,2)$.
- The only solution is $x = 1, y = 1, z = 2$.



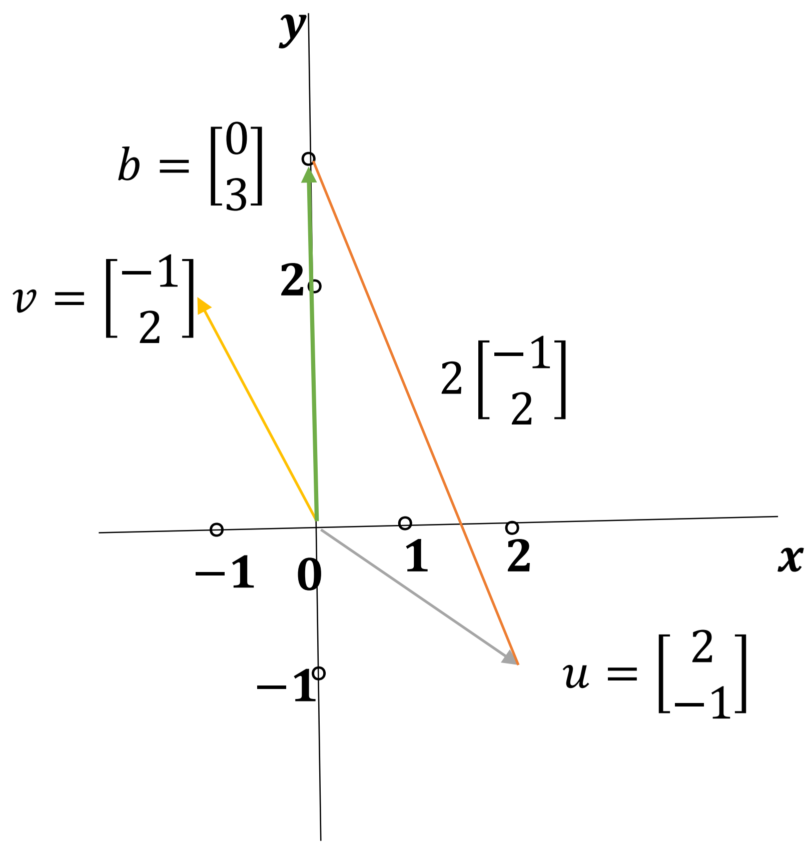
1.1.2 Column Picture

Consider the following simple system of linear equations in two unknowns:

$$2x - y = 0$$
$$-x + 2y = 3.$$

Here we are going to consider the columns of the linear system. The two separate equations are really one vector equation:

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



The problem is to find the combination of the column vectors on the left side that produces the vector on the right side.

Geometrically, we want to find numbers x and y so that $x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ equals $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

As we see from $x = 1$ and $y = 2$, agreeing with the column picture in Figure.

Hence the only solution is $x = 1$, and $y = 2$.

Column Picture

Consider the following simple system of linear equations in three unknowns:

$$\begin{aligned} 2x + \ y + \ z &= 5 \\ 4x - 6y \qquad &= -2 \\ -2x + 7y + 2z &= 9. \end{aligned}$$

This system can be considered as a single vector equation in the following way:

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} z = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

The problem is to find the combination of the column vectors $\boldsymbol{a} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix}$,

and $\boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ on the left side that produces the vector $\boldsymbol{d} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ on the right side.

We would like to find numbers x, y, and z such that $x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} = \boldsymbol{d}$.

It can be easily checked that $x = 1, y = 1$, and $z = 2$ satisfy the above vector equation.

Hence the only solution to the given system is $x = 1$, $y = 1$, and $z = 2$.

Column Picture (Linear independence)

Consider the following simple system of linear equations in two unknowns:

$$x + 2y = b_1$$
$$2x + y = b_2.$$

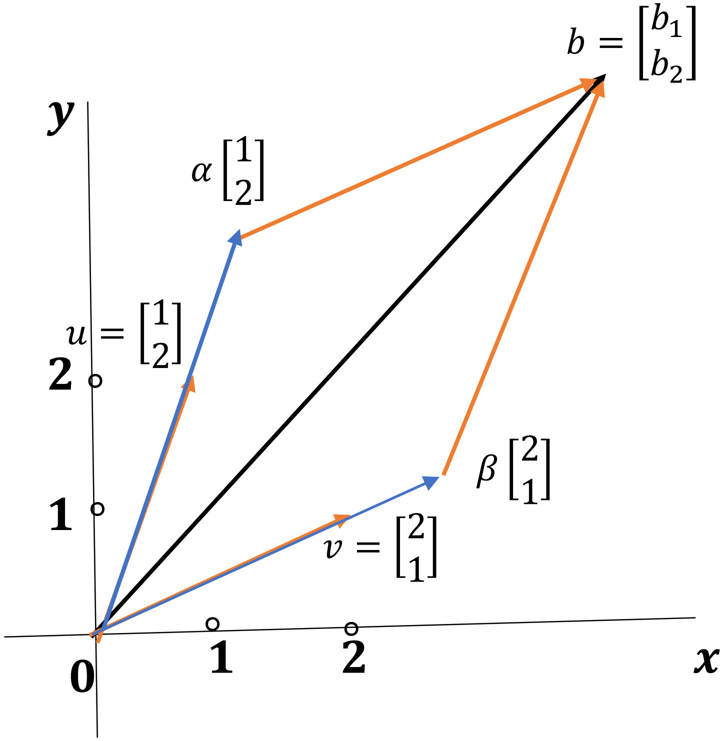
Does this system solvable for any two real numbers b_1 and b_2?

Given system is equivalent to the following single vector equation

$$\begin{bmatrix}1\\2\end{bmatrix}x + \begin{bmatrix}2\\1\end{bmatrix}y = \begin{bmatrix}b_1\\b_2\end{bmatrix}.$$

Since vectors $\mathrm{u} = \begin{bmatrix}1\\2\end{bmatrix}$ and $v = \begin{bmatrix}2\\1\end{bmatrix}$ are not parallel (linear independent), any vector $b = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ in the plane can be uniquely represented by $\alpha u + \beta v = b$, for some real numbers α, β.

Hence, the given system is solvable and the unique solution is $x = \alpha$, and $y = \beta$.



1.1.3 Representation of a Linear System in Matrix Form

Consider the following system of linear equations in 3 unknowns:

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 4 \\
2x_1 + 2x_2 + 3x_3 &= 7 \\
x_1 + x_2 + 4x_3 &= 6.
\end{aligned}
$$

This system can be represented as $A\,x = b$ in the following way:

$$
\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}.
$$

The matrix $\mathrm{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ is called the coefficient matrix of the system, and

the column vector $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is called the vector of unknowns.

Multiplication of a Matrix and a Vector (Ax)

- **Usual way:** We have studied how to multiply two matrices in the O/L Mathematics. For example

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1\times 1 + 2\times 2 + 1\times 3 \\ 2\times 1 + 2\times 2 + 3\times 3 \\ 1\times 1 + 1\times 2 + 4\times 3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 8 \\ 15 \\ 15 \end{bmatrix}_{3\times 1}.$$

- **Other way:** Considering the entries of x as the coefficients of a linear combination of the column vectors of the matrix A:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 15 \end{bmatrix}.$$