BIT

3: Solving Equations and Inequalities

EN1106 - Introductory Mathematics

Level I - Semester 1

UCSC

eLC

3.1 Understands linear equations

- An equation states that two quantities are equal, and will always contain an unknown quantity that we wish to find.

$$5x + 10 = 20$$

In the above equation, unknown quantity is x.

- To solve an equation, we need to find all values of the unknown quantity that can be substituted into the equation so that the left side equals the right side.
- Each such value is called a solution or alternatively a root of the equation.
- Here, the solution is $x = 2$.

3.1 Understands linear equations

- A linear equation is one of the form $ax + b = 0$ where a and b are numbers and the unknown quantity is x.
- The number a is called the coefficient of x. The number b is called the constant term.
- Previous example is an example for a linear equation.
- The coefficient of x is 3 and the constant term is 7
- Note that the unknown quantity occurs only to the first power, that is as x, and not as x^2, x^3, $x^{1/2}$ etc.

3.2 Solving linear equations

- Linear equations are solved by trying to obtain the unknown quantity on its own on the left-hand side; that is, by making the unknown quantity the subject of the equation.

- Example 1: $x + 25 = 75$

  $x + 25 – 25 = 75 – 25$ (Subtract 25 from both sides)

  $x + 0 = 50$

  $x = 50$

3.2 Solving linear equations

- <u>Example 2:</u>

$4x - 7 = 17$

$4x - 7 + 7 = 17 + 7$ (add 7 to both sides to remain only x in LHS)

$4x = 24$

$x = 6$

3.2 Exercises

1. Verify that the given values of x satisfy the given equations:
   (a) $x = 7$ satisfies $3x + 4 = 25$
   (b) $x = -5$ satisfies $2x - 11 = -21$
   (c) $x = -4$ satisfies $-x - 8 = -4$
   (d) $x = \frac{1}{2}$ satisfies $8x + 4 = 8$
   (e) $x = -\frac{1}{3}$ satisfies $27x + 8 = -1$
   (f) $x = 4$ satisfies $3x + 2 = 7x - 14$

2. Solve the following linear equations:
   (a) $3x = 9$ (b) $\dfrac{x}{3} = 9$ (c) $3t + 6 = 0$
   (d) $3x - 13 = 2x + 9$ (e) $3x + 17 = 21$
   (f) $4x - 20 = 3x + 16$
   (g) $5 - 2x = 2 + 3x$
   (h) $\dfrac{x + 3}{2} = 3$ (i) $\dfrac{3x + 2}{2} + 3x = 1$

3. Solve the following equations:
   (a) $5(x + 2) = 13$
   (b) $3(x - 7) = 2(x + 1)$
   (c) $5(1 - 2x) = 2(4 - 2x)$

4. Solve the following equations:
   (a) $3t + 7 = 4t - 2$
   (b) $3v = 17 - 4v$
   (c) $3s + 2 = 14(s - 1)$

Solving simultaneous equations

- Sometimes equations contain more than one unknown quantity. When this happens there are usually two or more equations.

- E.g. $x + 2y = 14$

  $3x + y = 17$ where x and y are unknowns

- Above equations are called **simultaneous equations**
- We must find values of x and y that satisfy both equations at the same time to solve these equations. This can be solved by removing, or eliminating, one of the unknowns.

Solving simultaneous equations

Example 1:

$x + 3y = 14$ ------ (1)

$2x - 3y = -8$ ------ (2)

(1) + (2) (if these two equations are added the unknown y is removed)

$x + 3y + (2x - 3y) = 14 + (-8)$

$3x = 6$, $x = 2$

if $x = 2$ then by substituting x value to equation 1,

$2 + 3y = 14$

$3y = 12$

$y = 4$ (Answer: $x = 2$ and $y = 4$)

Solving simultaneous equations

<u>Example 2:</u>

$5x + 4y = 7$ ------ (1)

$3x - y = 11$ ------ (2)

(1) x 3 + (2) x 5 (multiply the 1st equation by 3 and 2nd equation by 5)

$15x + 12y = 21$ ------- (3)

$15x – 5y = 55$ -------- (4)

(Now, subtract the equation 4 from equation 3)

(3) – (4), $12y – (-5y) = 21 – 55$

$17y = - 34$

$y = -2$ and then by substituting value y to equation 2

$3x = 9$, $x = 3$

Exercises

1. Verify that the given values of x and y satisfy the given simultaneous equations.
   (a) $x = 7$, $y = 1$ satisfy $2x - 3y = 11$, $3x + y = 22$
   (b) $x = -7$, $y = 2$ satisfy $2x + y = -12$, $x - 5y = -17$
   (c) $x = -1$, $y = -1$ satisfy $7x - y = -6$, $x - y = 0$

2. Solve the following pairs of simultaneous equations:
   (a) $3x + y = 1, \quad 2x - y = 2$
   (b) $4x + 5y = 21, \quad 3x + 5y = 17$
   (c) $2x - y = 17, \quad x + 3y = 12$
   (d) $-2x + y = -21, \quad x + 3y = -14$
   (e) $-x + y = -10, \quad 3x + 7y = 20$
   (f) $4x - 2y = 2, \quad 3x - y = 4$

3. Solve the following simultaneous equations:
   (a) $5x + y = 36$, $3x - y = 20$
   (b) $x - 3y = -13$, $4x + 2y = -24$
   (c) $3x + y = 30$, $-5x + 3y = -50$
   (d) $3x - y = -5$, $-7x + 3y = 15$
   (e) $11x + 13y = -24$, $x + y = -2$

Solving Quadratic Equations

A quadratic equation has the form $ax^2 + bx + c = 0$ where a, b and c are

numbers, and **x** represents the unknown we wish to find.

- The number "a" is the coefficient of x^2, b is the coefficient of x, and c is the constant term.
- Sometimes b or c may be zero, although a can never be zero
- **Examples:**

$x^2 + 7x + 2 = 0$, $3x^2 - 2 = 0$, $-2x^2 + 3x = 0$

Solving Quadratic Equations

Solution by Factorization

If we want to solve the following quadratic equation, we need to factorize the left hand side of the equation as follows:

$$(1)\ x^2 + 3x - 10 = (x+5)(x-2) = 0, x = \text{-}5 \text{ or } x = 2$$

$$(2)\ 6x^2 + 5x - 4 = (x+8)(x-3) = 0, \quad x = -8 \text{ or } x = 3$$

$$ax^2 + bx + c = (x+p)(x+q) = 0$$

You need to find values for p and q which should be satisfied the following two equations.

pq=ac ------ (1)   Eg: 24 = 24 x 1, 12 x 2, 8 x 3, 6 x 4,

p+q=b ----- (2)

Solving Quadratic Equations

You can use the following formula to solve the quadratic equations.

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula gives possibly two solutions:

- one solution is obtained by taking the positive square root
- second solution by taking the negative square root

Solving Quadratic Equations

Example:

Solve the equation $x^2 + 9x + 20 = 0$ using the formula

(a = 1, b = 9, c = 20)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-9 \pm \sqrt{9^2 - 4 * 1 * 20}}{2 * 1}$$

$x = \frac{-9+\sqrt{1}}{2}$ or $x = \frac{-9-\sqrt{1}}{2}$

$x = \frac{-8}{2} = -4$ or $x = \frac{-10}{2} = -5$

Solving Quadratic Equations

- If the values of a, b and c are such that $b^2 - 4ac$ is positive, the formula

will produce two solutions known as distinct real roots of the equation.

- If $b^2 - 4ac = 0$ there will be a single root known as a repeated root.
- If the equation is such that $b^2 - 4ac$ is negative, the formula requires us to find the square root of a negative number. In ordinary arithmetic this is impossible and we say that in such a case the quadratic equation does not possess any real roots.

Solving Quadratic Equations

1. Solve the following quadratic equations by factorisation:

(a) $x^2 + x - 2 = 0$ (b) $x^2 - 8x + 15 = 0$ (c) $4x^2 + 6x + 2 = 0$
(d) $x^2 - 6x + 9 = 0$ (e) $x^2 - 81 = 0$ (f) $x^2 + 4x + 3 = 0$
(g) $x^2 + 2x - 3 = 0$ (h) $x^2 + 3x - 4 = 0$ (i) $x^2 + 6x + 5 = 0$
(j) $x^2 - 12x + 35 = 0$ (k) $x^2 + 12x + 35 = 0$ (l) $2x^2 + x - 3 = 0$
(m) $2x^2 - x - 6 = 0$ (n) $2x^2 - 7x - 15 = 0$ (o) $3x^2 - 2x - 1 = 0$
(p) $9x^2 - 12x - 5 = 0$ (q) $7x^2 + x = 0$ (r) $4x^2 + 12x + 9 = 0$

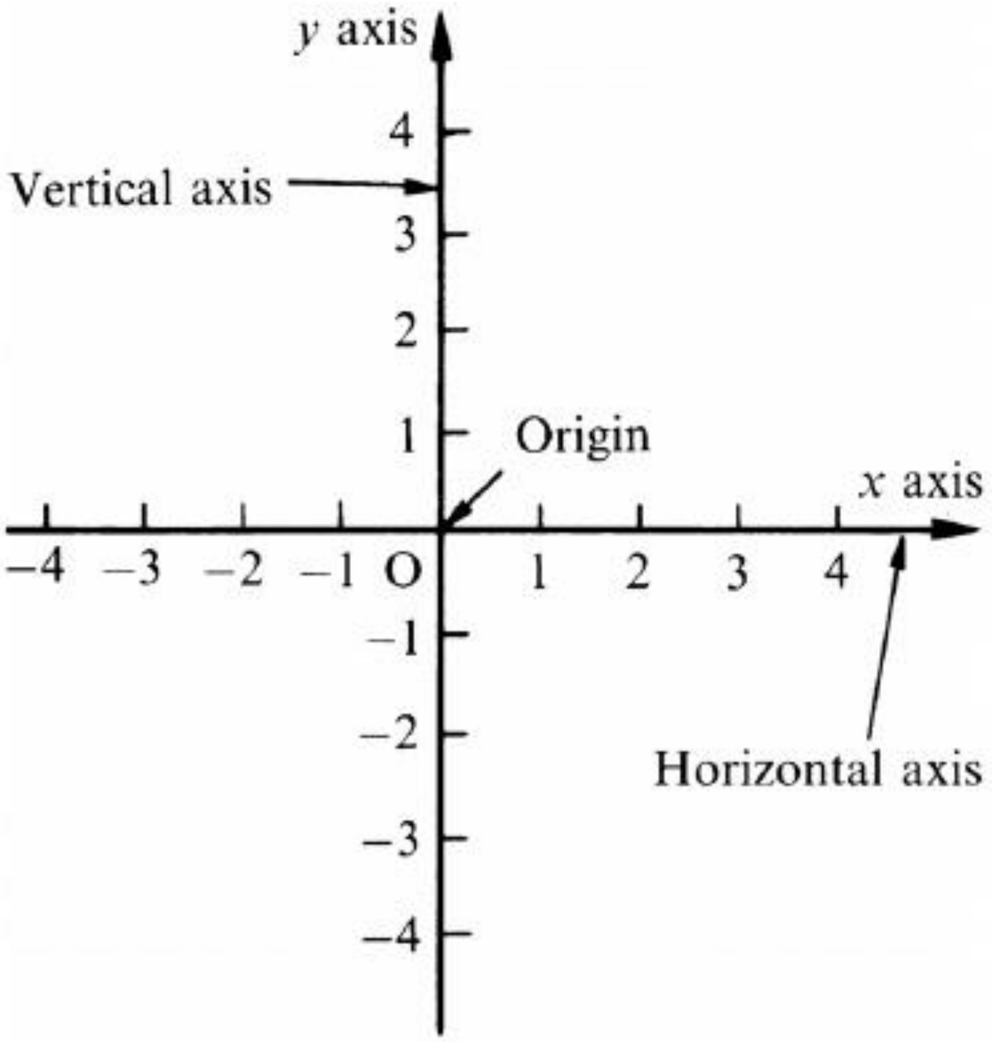
2. Solve the following quadratic equations using the formula:

(a) $3x^2 - 6x - 5 = 0$ (b) $x^2 + 3x - 77 = 0$ (c) $2x^2 - 9x + 2 = 0$
(d) $x^2 + 3x - 4 = 0$ (e) $3x^2 - 3x - 4 = 0$ (f) $4x^2 + x - 1 = 0$
(g) $x^2 - 7x - 3 = 0$ (h) $x^2 + 7x - 3 = 0$ (i) $11x^2 + x + 1 = 0$
(j) $2x^2 - 3x - 7 = 0$

3. Solve the following quadratic equations:

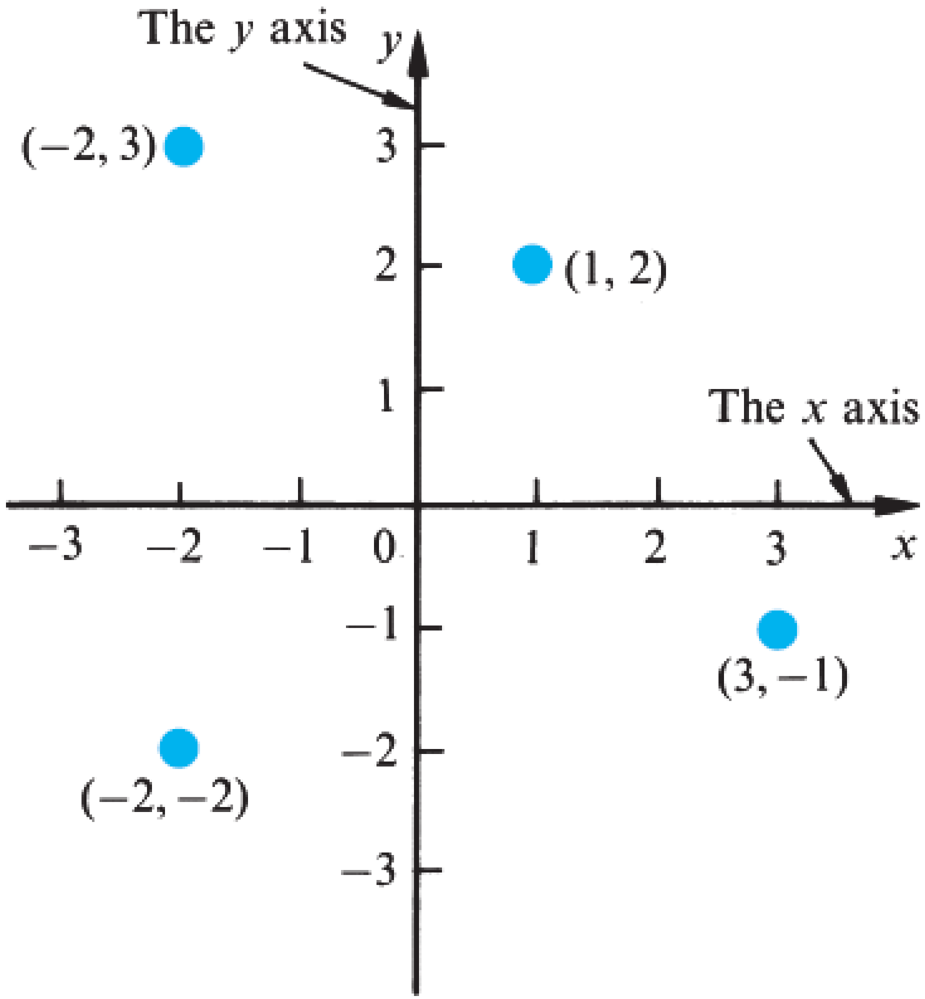
(a) $6x^2 + 13x + 6 = 0$ (b) $3t^2 + 13t + 12 = 0$ (c) $t^2 - 7t + 3 = 0$

X-Y Plane



- Draw the x–y plane and on it mark the points whose coordinates are (1, 2),(3,-1), (-2, 3), (-2,-2).

X-Y Plane



- We usually call the x value the x coordinate and call the y value the y coordinate.
- To refer to a specific point we give both its coordinates in brackets in the form (x; y), always giving the x coordinate first.

3.5 Inequalities and their solutions

- We often need only part of the x axis when plotting graphs.
- For example, we may be interested only in that part of the x axis running from $x = 1$ to $x = 3$ or from $x = -2$ to $x = 7.5$.
- Such parts of the x axis are called intervals.
- The symbols $>$, $\geq$, $<$, $\leq$ are known as inequalities
- We use the symbol $>$ to mean 'greater than'. For example, $6 > 5$ and $-4 > -5$ are both TRUE statements.
- The symbol $\geq$ means 'greater than or equal to', so, for example, $10 \geq 8$ and $8 \geq 8$ are both true.
- The symbol $<$ means 'less than'. We may write $4 < 5$ for example.
- Finally, $\leq$ means 'less than or equal to'. Hence $6 \leq 9$ and $6 \leq 6$ are both TRUE statements.

Manipulation of inequalities

- Adding and subtracting the same quantity from both sides of an inequality leaves the inequality unchanged.
- If $x > y$, then $x + k > y + k$ and $x - k > y – k$, similarly, $x + k < y + k$ and $x – k < y – k$.
- If an inequality is multiplied or divided by a positive quantity then the inequality remains unchanged.
- If $x > y$ and k is positive, $kx > ky$ and $x/k > y/k$.

Manipulation of inequalities

- If the inequality is multiplied or divided by a negative quantity then the inequality is reversed, that is, 'greater than' must be replaced by 'less than' and vice versa.
- If $x > y$ and k is negative, then $kx < ky$ and $x/k < y/k$
- If $x < y$ and k is negative, then $kx > ky$ and $x/k > y/k$

Intervals:

- We often need to represent intervals on the number line. There are three different kinds of interval:
- **Closed interval:** An interval that includes its end-points is called a closed interval. All the numbers from 1 to 3, including both 1 and 3, comprise a closed interval, and this is denoted using square brackets, [1, 3].
- Using set notation:

$$\{x: x \in \mathbb{R},\ 1 \leqslant x \leqslant 3\}$$

Intervals:

- We often need to represent intervals on the number line. There are three different kinds of interval:
- **Open interval:** Any interval that does not include its end-points is called an open interval. For example, all the numbers from 1 to 3, but excluding 1 and 3, comprise an open interval.
- Using set notation:

$$\{x: x \in \mathbb{R},\ 1 < x < 3\}$$

Intervals:

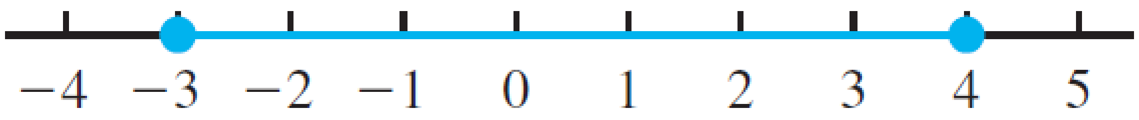
- We often need to represent intervals on the number line. There are three different kinds of interval:
- **Semi-open or semi-closed interval:** An interval may be open at one end and closed at the other. Such an interval is called semi-open or, as some authors say, semi-closed. The interval (1, 3] is a semi-open interval.
- Using set notation:

$$\{x: x \in \mathbb{R},\ 1 < x \leqslant 3\}$$

Example 1

- Describe the interval [-3, 4] using set notation and illustrate it on the x axis.

$$\{x: x \in R, -3 \leq x \leq 4\}$$



- Note: We use colored circle to show that an endpoint is included (i.e. closed) whereas uncolored circle is used to denote an end-point that is not included (i.e. open).

Example 2

- Describe the interval (1, 4) using set notation and illustrate it on the x axis.

$$\{x: x \in R, 1 < x < 4\}$$

- Note: We use colored circle to show that an endpoint is included (i.e. closed) whereas uncolored circle is used to denote an end-point that is not included (i.e. open).

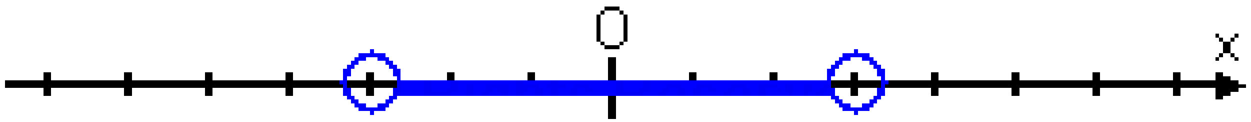
Questions

1. Describe the following intervals using set notation. Draw the intervals on the x axis.
   (a) $[2, 6]$ (b) $(6, 8]$ (c) $(-2, 0)$ (d) $[-3, -1.5)$

2. Which of the following are true?
   (a) $8 > 3$ (b) $-3 < 8$ (c) $-3 \leqslant -3$ (d) $0.5 \geqslant 0.25$ (e) $0 > 9$
   (f) $0 \leqslant 9$ (g) $-7 \geqslant 0$ (h) $-7 < 0$

Absolute-Value Inequalities

- Solve | x | < 3, and graph its solution.
  - This is an inequality. Where the solution to an absolute-value equation is points, the solution to an absolute-value inequality is going to be intervals.
  - In this inequality, they're asking to find all the x-values that are less than three units away from zero in either direction, so the solution is going to be the set of all the points that are less than three units away from zero.



  - Answer = -3 < x < 3

  - Given an inequality in the form | x | < a, the solution will always be of the form –a < x < a.
  - Interval = (-a, a)

Example

- Solve $|-2x + 24| < 8$

$-8 < -2x+24 < 8$

$-8 - 24 < -2x < 8 - 24$

$-32 < -2x < -16$

$-32/2 < -x < -16/2$ (by dividing by 2)

$-16 < -x < -8$

$16 > x > 8$ (by dividing by -1, need to replace the inequality symbol)

Interval = (8, 16)

Plotting the graph of a function

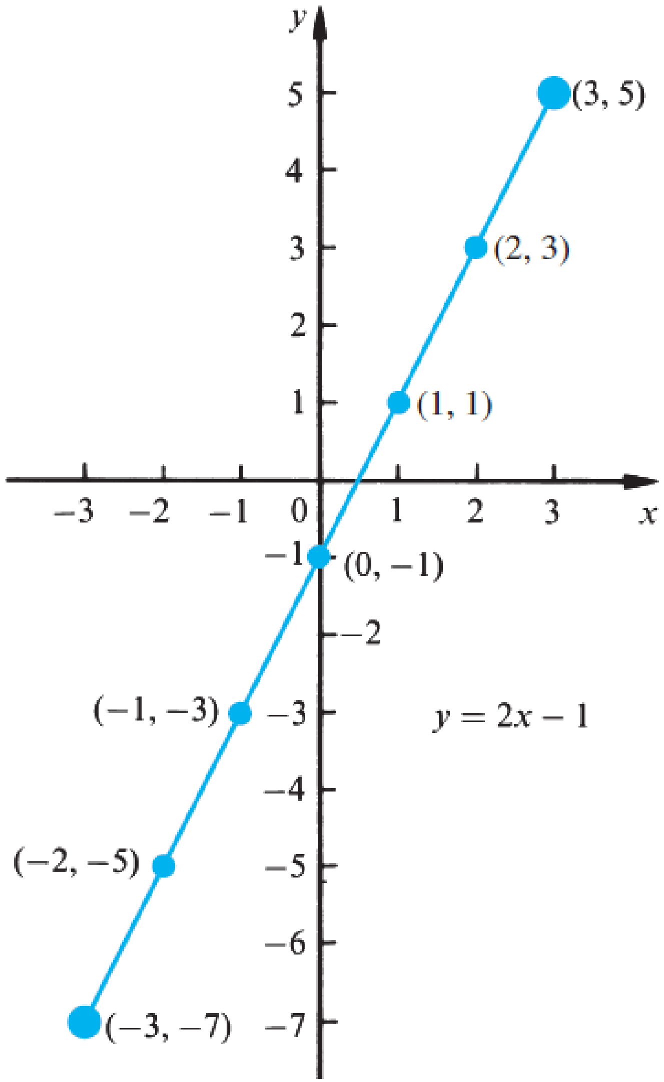
- Plot a graph of y =2x - 1 for -3 ≤ x ≤ 3.

x	-3	-2	-1	0	1	2	3
y	-7	-5	-3	-1	1	3	5

- We first calculate the value of y for several values of x. The independent variable x varies from -3 to 3; the dependent variable y varies from -7 to 5.

Plotting the graph of a function

- $y = 2x - 1$



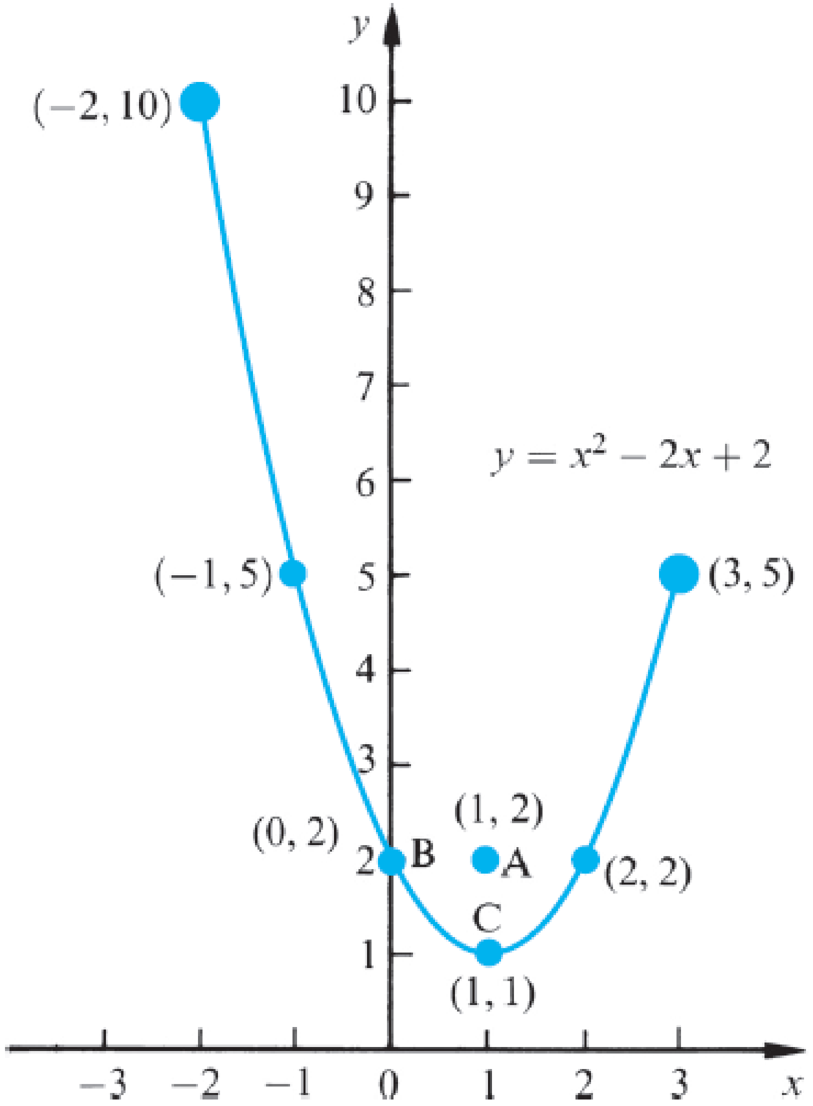
Plotting the graph of a function

- Plot a graph of $y = x^2 - 2x + 2$ for $-2 \le x \le 3$.

x	-2	-1	0	1	2	3
y	10	5	2	1	2	5

Plotting the graph of a function

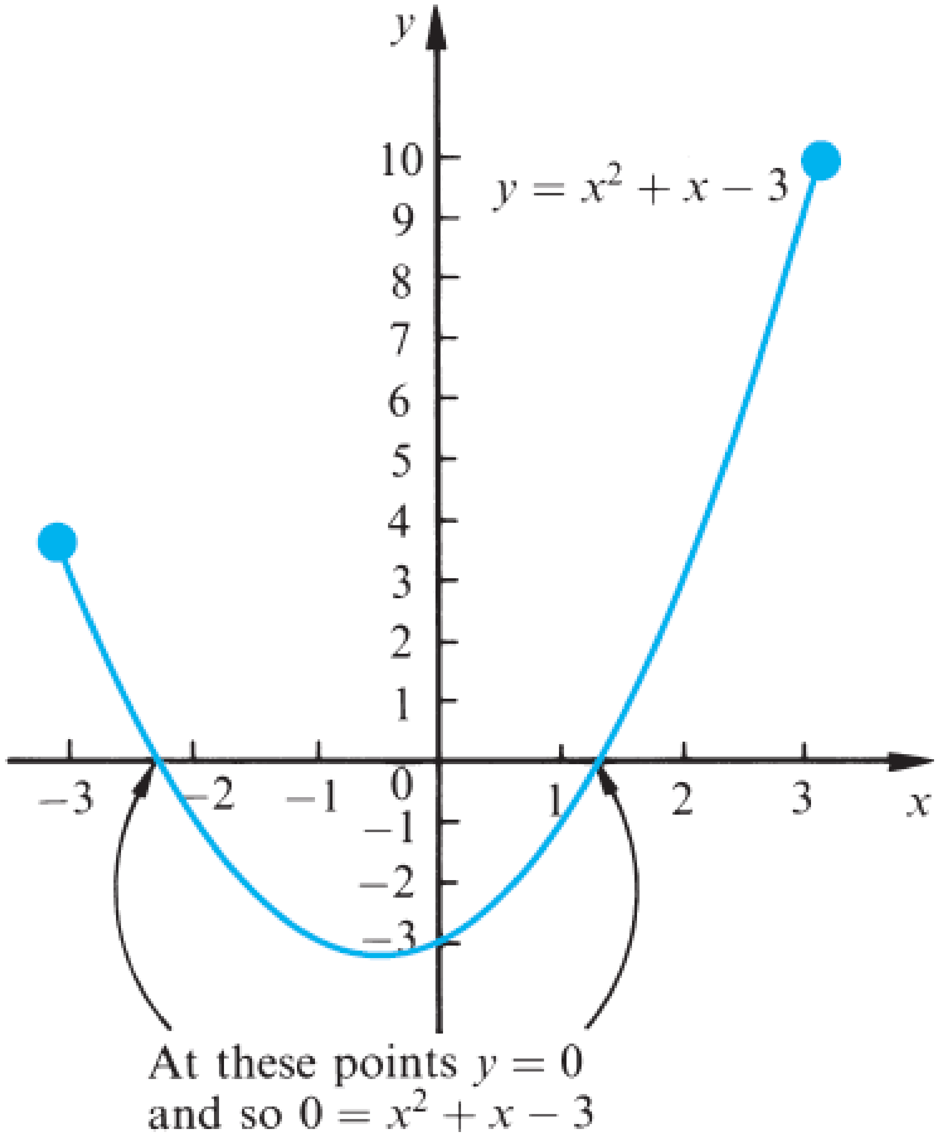
- $y = x^2 - 2x + 2$



Solving equations using graphs

- The following examples illustrate the method of using graphs to solve equations.
- If we want to find all solutions graphically in the interval [-3, 3] of the equation $x^2 + x – 3 = 0$, we first need to draw a graph of $y = x^2 + x – 3$ and then we read from the graph the coordinates of the points at which $y = 0$. Such points must be on the x axis.

Solving equations using graphs



- From the graph the points are (1.3, 0) and (-2.3, 0).
- So the solutions of $x^2 + x - 3 = 0$ are
  - x = 1.3 and x = -2.3

Exercise

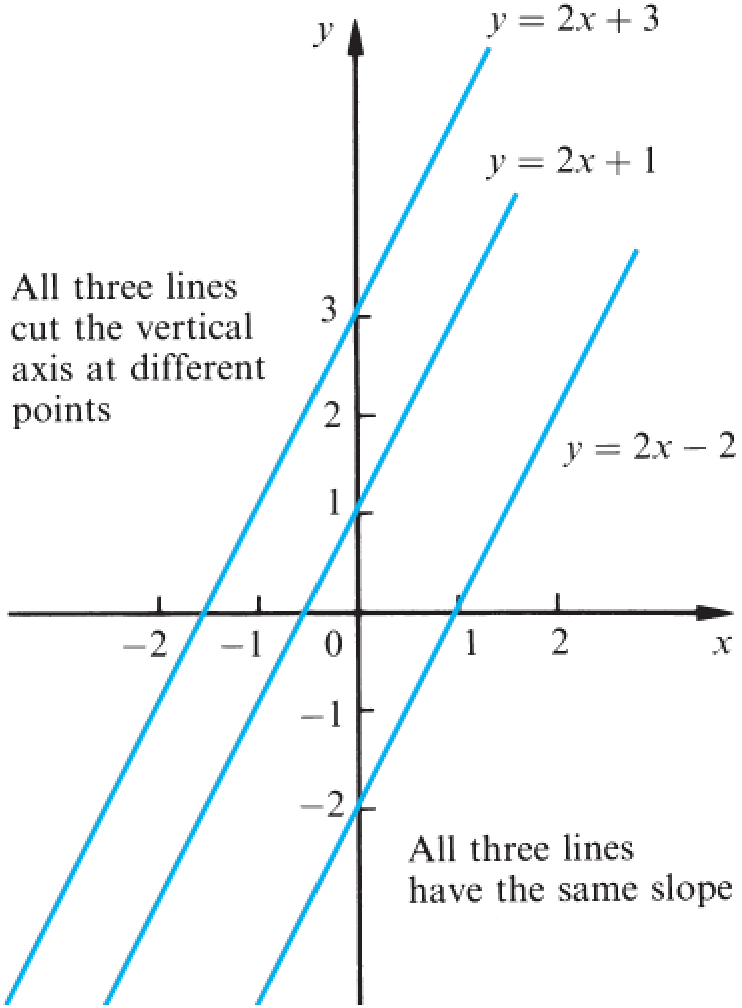
Plot $y = x^2 - x - 1$ for $-3 \leqslant x \leqslant 3$.
Hence solve $x^2 - x - 1 = 0$.

Straight Line Graphs

- Any straight line has an equation of the form $y = mx + c$ where m and c are constants.
- If we consider the $y = 7x + 5$ equation, then it is of the form $y = mx + c$. By comparing those two equations we can identify $m = 7$ and $c = 5$.

Example

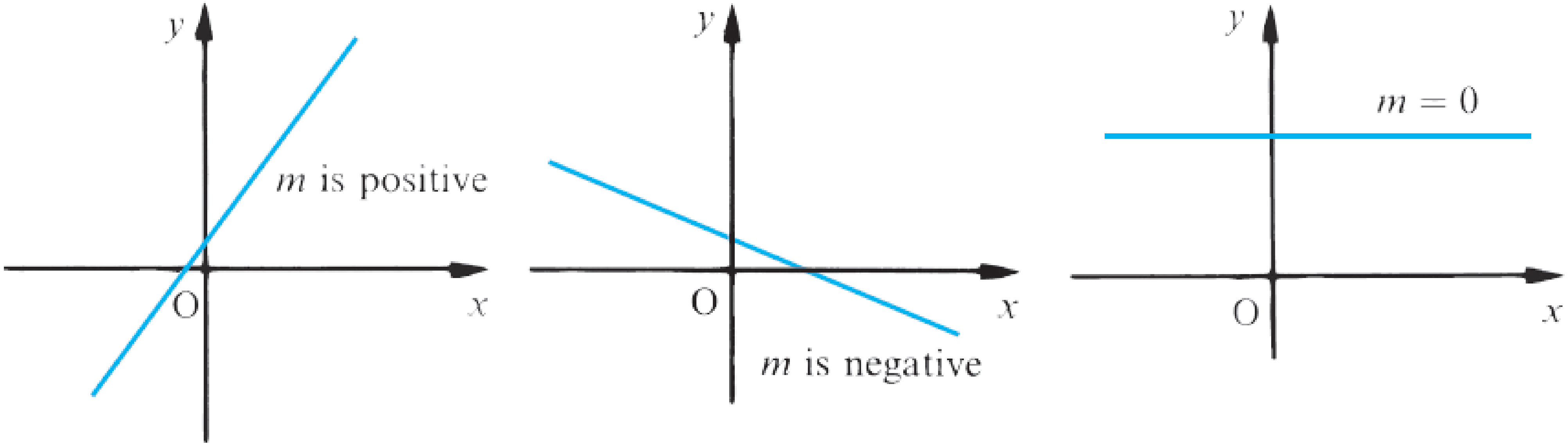
- Plot each of the following graphs: y = 2x + 3, y = 2x + 1 and y = 2x + 2.



- By comparing the general equation y= mx +c and above equations, we can clearly identify that m = 2 in each equation and also according to the diagram all straight lines are parallel to each other.
- When we compare two equations and if those two equations have same m *(m is known as the slope or gradient of the straight line)* then those straight lines are parallel to each other.
- The point where a graph cuts the vertical axis is called the vertical intercept.

Straight Line Graphs

- Straight line graphs with positive, negative and zero gradients



Finding the equation of a straight line

- If the two points on the line are A(x1, y1) and B(x2, y2) then m (gradient) can be obtained from the following method.

$$\text{gradient} = \frac{\text{difference between the } y \text{ coordinates}}{\text{difference between the } x \text{ coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}$$

- Find the gradients of the lines joining A(2, 3) and B(6, 10).

  m = 10 – 3 / 6 – 2 = 7/4

- Equation:

  m = y – y1/x – x1 = y – 3 / x – 2
  7/4 = y-3/x-2 → 7x -14 = 4y -12
  4y = 7x -2 (Equation)

Finding two equations perpendicular to each other

- If two equations are perpendicular to each other then multiplication of two gradient values of those two equations are equals to -1.
- Example:

$y = 2x + 1 \rightarrow$ Gradient (m1) = 2

$y = -0.5x - 5 \rightarrow$ Gradient (m2) = -0.5

If these two equations are perpendicular to each other then m1 x m2 = -1

Let’s check:

$2 \times -0.5 = -1$

Therefore, these two equations are perpendicular to each other.

Exercises

What is the equation of the line perpendicular to the graph of the equation $y+3x-2=0$ and passes through the point $(2,-4)$?