1 : Theory of Matrices, Vector spaces and Linear Transformations

IT5506 – Mathematics for Computing II

Level III - Semester 5

Intended Learning Outcomes

At the end of this lesson, you will be able to;

- identify different types of matrices and their basic properties
- perform basic operations on matrices

List of sub topics

1.2 Matrices (2 hours)

1.2.1 Defining various types of matrices

1.2.2 Addition and scalar multiplication of matrices

1.2.3 Different ways of defining (or understanding) matrix multiplication

1.2.4 Special type of matrices and their properties.

1.2.5 Inverse of a square matrix (if it exists) and related results.

1.2.1 Defining various types of matrices

Definition of a Matrix

A matrix is an array of m × n elements arranged in m rows and n columns. Such a matrix A is usually denoted by

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots. & \dots & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n} = [a_{ij}]_{m \times n}$$

Where $a_{11,} a_{12}, \dots\dots\dots a_{mn}$ are called the **elements** of the matrix.

The element a_{ij} is called the ijth entry of the matrix and it appears in the ith row and the jth column of the matrix. A matrix with m rows and n columns is called an **m x n (**read **m by n)** matrix and we say that the matrix is of **order** m x n. We often denote matrices by capital letters.

1.2.1 Defining various types of matrices

Column and row matrices

If a matrix A is such that A consists of just one row, then A is said to be a **row matrix or row Vector.** For example

$$\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}_{1\times 3}$$

is a row matrix with three elements.

Note: In a row matrix, m=1

If a matrix A is such that A consists of just one column, then A is said to be a **column matrix or column Vector**.

$$\begin{pmatrix} -3 \\ 6 \\ 90 \end{pmatrix}_{3\times 1}$$

is a column matrix with three elements.

Note: In a column matrix, n=1

1.2.1 Defining various types of matrices

Square Matrix

If in a matrix A, the number of rows equals the number of columns, then A is said to be a **square matrix**.

If the number of rows in a square matrix is n, then A is called a matrix of order n.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$$ is a square matrix of order 2.

1.2.1 Defining various types of matrices

Diagonal Matrix

Let A = (a_{ij}) be a square matrix of order n. Then A is said to be a **diagonal matrix** if a_{ij} = 0 whenever i $\neq$ j, where i, j $\in$ {1, 2,….. n}. The elements a_{ij} , where i $\in$ {1, 2,….. n} are called diagonal elements.

For example

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3}$$

is a diagonal matrix of order 3.

Note that the diagonal elements of a diagonal matrix may also be zero.

1.2.1 Defining various types of matrices

Null or Zero Matrix

Let A = (a_{ij}) be an m x n matrix. Then A is said to be a **null or zero matrix** if a_{ij} = 0 for all i $\in$ {1, 2,.. m}, j $\in$ {1,2,...n}.

For example

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a null matrix of order 2 x 3.

1.2.1 Defining various types of matrices

Symmetric Matrix

An n x n matrix A = (a_{ij}) is called a **symmetric matrix** if $a_{ij} = a_{ji}$ for all i, j $\in$ {1, 2, ….n}.

For example

$$\begin{pmatrix} 4 & 2 & 0 & 1 \\ 2 & 6 & 5 & -2 \\ 0 & 5 & 0 & 1 \\ 1 & -2 & 1 & 9 \end{pmatrix}_{4\times 4}$$ is symmetric.

Note: a symmetric matrix is a square matrix.

1.2.1 Defining various types of matrices

Skew-symmetric Matrix

An n x n matrix A = (a_{ij}) is called a skew -**symmetric matrix** if a_{ij} = - a_{ji} for all i, j $\in$ {1, 2, ….n}.

For example

$$\begin{pmatrix} 0 & 2 & -4 \\ -2 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}_{3\times 3}$$ is skew-symmetric.

1.2.1 Defining various types of matrices

Upper Triangular Matrix

Let A be an n x n square matrix such that all the entries below the diagonal are zero; i.e. $a_{ij} = 0$ whenever i > j, where i, j $\in$ {1, 2,....n}. Then A is said to be an **upper triangular matrix**.

For example

$$\begin{pmatrix} 0 & 5 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3}$$ is a 3 x 3 upper triangular matrix.

Note that the diagonal elements of an upper triangular matrix need not be zero.

1.2.1 Defining various types of matrices

Lower Triangular Matrix

Let A be an n x n square matrix such that all the entries above the diagonal are zero; i.e. a_{ij} = 0 whenever i < j, where i, j$\in$ {1, 2,....n}. Then A is said to be a **lower triangular matrix**.

For example

$$\begin{pmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}_{3\times 3}$$ is a 3 x 3 lower triangular matrix.

Note that the diagonal elements of a lower triangular matrix need not be zero.

1.2.1 Defining various types of matrices

Identity Matrix

Suppose A is an n x n diagonal matrix such that all the diagonal elements are equal to 1. Then A is said to be an identity **matrix** of order n.

For example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$ is an identity matrix of order 3.

Note that identity matrix is a square matrix.

1.2.1 Defining various types of matrices

Equality of Matrices

Two matrices A = (a_{ij}) and B = (b_{ij}) are said to be **equal** if A and B have the same order, say m x n, and if $a_{ij} = b_{ij}$ for all i $\in$ {1, 2, ..., m}, for all j $\in$ {1, 2, ..., n}.

For example

$$\begin{pmatrix} 3 & 4 & 9 \\ 16 & 25 & 64 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 9 \\ 16 & 25 & 64 \end{pmatrix}$$

1.2.2 Addition and scalar multiplication of matrices

Matrix Addition

Let A = (a_{ij}) and B = (b_{ij}) be two matrices having the same order, say m x n. We define the sum of A and B denoted by A + B to be the matrix C = (c_{ij}) where $c_{ij} = a_{ij} + b_{ij}$ for all i $\in$ {1, 2, ….m} and j $\in$ {1, 2,….n}.

Example.

Let A = $\begin{pmatrix} 1 & 0 & 8 \\ 2 & -1 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 2 & 0 \\ -5 & -4 & 1 \end{pmatrix}$ Then

A + B = $\begin{pmatrix} 4 & 2 & 8 \\ -3 & -5 & 5 \end{pmatrix}$

1.2.2 Addition and scalar multiplication of matrices

Scalar Multiplication of a Matrix

Let A = (a_{ij}) be a m x n matrix. The **product** of the **scalar** k and the matrix A, denoted by k.A (or kA) is the matrix B = (b_{ij}) where b_{ij} = ka_{ij} for all i $\in$ {1, 2,….m} and j $\in${1, 2,….n}.

Example.

Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and k = -2 then kA = $\begin{pmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \\ -14 & -16 & -18 \end{pmatrix}$

Notation: We write

- -A for -1×A and
- A – B for A + (-B)

1.2.2 Addition and scalar multiplication of matrices

Results:

Let A, B and C be three matrices of the same order. Then the following properties hold.

- $A + B = B + A$.
- $A + (B + C) = (A + B) + C$.
- $(k_1 k_2)A = k_1(k_2 A)$.
- $(k_1 + k_2)A = k_1 A + k_2 A$.
- $k(A + B) = kA + kB$.
- $1.A = A$
- $0.A_{m \times n} = 0_{m \times n}$

1.2.2 Addition and scalar multiplication of matrices

Proof of (A+B)=(B+A):

$(A+B)_{ij}$ = ij^{th} entry of (A+B)

= $A_{ij}+B_{ij}$ (definition of matrix addition)

= $B_{ij} + A_{ij}$ (commutativity for numbers)

= ij^{th} entry of (B+A)

= $(B+A)_{ij}$ (definition of matrix addition)

Therefore, (A+B)=(B+A)

1.2.2 Addition and scalar multiplication of matrices

Proof of (A+B)+C=A+(B+C):

$((A+B)+C)_{ij}$ = ij^{th} entry of ((A+B)+C)

= $(A+B)_{ij}+C_{ij}$ (definition of matrix addition)

= $(A_{ij}+B_{ij})+C_{ij}$ (definition of matrix addition)

= $A_{ij}+(B_{ij}+C_{ij})$ (Associativity for numbers)

= $A_{ij}+(B+C)_{ij}$ (definition of matrix addition)

= ij^{th} entry of (A+(B+C))

= $(A+(B+C))_{ij}$ (definition of matrix addition)

Therefore, (A+B)+C=A +(B+C)

1.2.3 Different ways of defining (or understanding) matrix multiplication

Let A = (a_{ij}) be a 1 x n row matrix and B = (b_{ij}) be a n x 1 column matrix.

Then we define the product of the row matrix A and the column matrix B by

$$AB = (a_{11}, a_{12}, \ldots\ldots\ldots a_{1n}) \times \begin{pmatrix} b_{11} \\ b_{21} \\ . \\ . \\ b_{n1} \end{pmatrix} = \left[a_{11}b_{11} + a_{12}b_{21} + \ldots\ldots\ldots + a_{1n}b_{n1}\right]_{1\times 1}$$

1.2.3 Different ways of defining (or understanding) matrix multiplication

Now let A = $(a_{ij})_{m \times p}$ and B = $(b_{ij})_{p \times n}$ be any two matrices of order m x p and p x n respectively. Then the product AB is defined as the matrix C of order m x n whose ij^th entry is obtained by multiplying the i^th row of A by the j^th column of B. That is, if C = $(c_{ij})_{m \times n}$,

$$c_{ij} = (a_{i1}, a_{i2}, \ldots\ldots\ldots a_{ip}) \times \begin{pmatrix} b_{1j} \\ b_{2j} \\ . \\ . \\ b_{pj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots + a_{ip}b_{pj}$$

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Note: The product of two matrices A and B is defined only when the number of columns of A is equal to the number of rows of B.

1.2.3 Defining matrix multiplication

Example:

Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$ and B = $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}_{2\times 3}$ then

$$AB = \begin{pmatrix} 1.1+2.2 & 1.-1+2.0 & 1.0+2.1 \\ 3.1+4.2 & 3.-1+4.0 & 3.0+4.1 \end{pmatrix}_{2\times 3}$$

$$= \begin{pmatrix} 5 & -1 & 2 \\ 11 & -3 & 4 \end{pmatrix}_{2\times 3}$$

Note: Matrix multiplication is not commutative. That is, in general, AB $\neq$ BA.

1.2.3 Defining matrix multiplication

Results:

Let A = (a_{ij}), B = (b_{ij}), and C = (c_{ij}), be three matrices and let k be a constant. Then,

- if A is m x n and B is n x p, then k(AB) = (kA)B = A(kB).
- if B and C are m x n and A is n x p, then (B + C)A = BA + CA.
- if A is m x n and B and C are n x p, then A(B + C) = AB + AC.
- if A is m x n, B is n x p and C is p x q, then A(BC) = (AB)C.
- if A is an n x m matrix, and I is the n x n identity matrix, then IA = A. Also if I is the m x m identity matrix, then AI = A.

1.2.3 Defining matrix multiplication

Proof of (AB)C = A(BC)**:**

$$\left((AB)C\right)_{il} = \sum\nolimits_{k=1}^{p} (AB)_{ik} C_{kl} = \sum_{k=1}^{p} \left(\sum_{j=1}^{n} A_{ij} B_{jk} \right) C_{kl}$$

$$(A(BC))_{il} = \sum\nolimits_{j=1}^{n} A_{ij} (BC)_{jl} = \sum_{j=1}^{n} A_{ij} \left(\sum_{k=1}^{p} B_{jk}\, C_{kl} \right)$$

$$\sum_{k=1}^{p} \left(\sum_{j=1}^{n} A_{ij} B_{jk} \right) C_{kl} = \sum_{k=1}^{p} \sum_{j=1}^{n} (A_{ij} B_{jk} C_{kl}) = \sum_{j=1}^{n} \sum_{k=1}^{p} (A_{ij} B_{jk} C_{kl}) = \sum_{j=1}^{n} A_{ij} \left(\sum_{k=1}^{p} B_{jk}\, C_{kl} \right)$$

$$((AB)C)_{il} = (A(BC))_{il}$$

(AB)C = A(BC)

1.2.4 Special type of matrices and their properties.

Transpose of a Matrix

Let A = (a_{ij}), be an m x n matrix. Then the **transpose of A** denoted by A^T is the n x m matrix (b_{ij}) where $b_{ij} = a_{ij}$ for all $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$.

Example:

Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ the A^T = $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

Results: Let A = (a_{ij}), B = (b_{ij}) be m x n matrices, and let C = (c_{ij}) be an n x p matrix.

- $(A + B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(AC)^T = C^T A^T$

1.2.4 Special type of matrices and their properties.

Orthogonal Matrix

A square matrix A = (a_{ij}) is said to be an **Orthogonal matrix** if

$$\mathbf{AA^T = A^TA = I}$$

For example

$$\begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}$$

is an orthogonal matrix of order 2.

1.2.5 Inverse of a square matrix and related results.

Invertible Matrices

Let A be an n x n square matrix. We say that A is **invertible** if there exists a n x n matrix B such that AB = BA = $I_{n \times n}$ where $I_{n \times n}$ is the n x n identity matrix.

Example:

Let A = $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ then

AB = I and BA = I

We call B the **inverse** of A and denote it by A^{-1}

1.2.5 Inverse of a square matrix (if it exists) and related results.

Properties

1. If B is an inverse of A, then A is also an inverse of B

   If $A^{-1}=B$, then $B^{-1}=A$
2. Inverse of a matrix is unique
3. Every square matrix is not invertible
4. $(A^{-1})^{-1} = A$
5. $(AB)^{-1} = B^{-1}A^{-1}$
6. If A is an invertible diagonal matrix with diagonal elements a_{ij} ,

   then A^{-1} is also a diagonal matrix with diagonal elements $1/a_{ij}$
7. $I^{-1} = I$

1.2.5 Inverse of a square matrix (if it exists) and related results.

Proof: If A and B are non-singular matrixes of order n, then $(AB)^{-1} = B^{-1}A^{-1}$

A and B are non-singular. That is, $|A| \neq 0$ and $|B| \neq 0$.

Therefore, $|AB| \neq 0$

AB is non-singular ---------------------------(1)

$(AB)(B^{-1}A^{-1}) = ((AB)\, B^{-1})\, A^{-1}) = (A(B\, B^{-1}\,)\, A^{-1}) = (AI_n)\, A^{-1}) = I_n$ -------(2)

$(B^{-1}A^{-1})(AB) = ((B^{-1}\, A^{-1})A)\, B) = ((B^{-1}\, (A^{-1}A))B = (B^{-1}\, I_n)\, B) = I_n$ -------(3)

By (1),(2) and (3), $(AB)^{-1} = B^{-1}A^{-1}$