2: Logic

IT2106 – Mathematics for Computing I

Level I - Semester 2

Definition

Proposition

The basic units of mathematical reasoning are **propositions** (or **statements**)

A **proposition** (or **statement**) is a sentence that is **either true or false**.

Examples

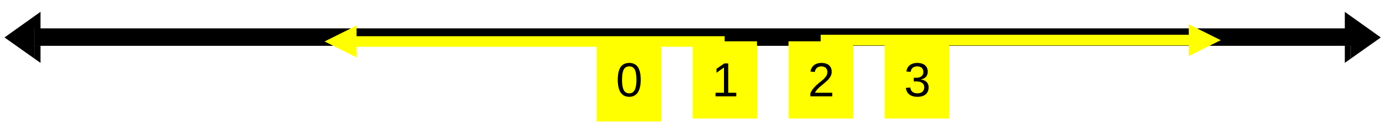
1) 2 is an integer.
2) 2 is not an integer.
3) Is 2 an integer?
4) $X^2 > 10$, Where x is an integer.
5) 5 + 3.

1) is a Proposition
2) is a Proposition
3) is a not a Proposition
4) is a not Proposition
5) is a not Proposition

Examples

The sentence $X^2 > 10$ is not a proposition.

Is the sentence $2 \leq x \leq 1$ a proposition ?



However no matter what number x happens to be the sentence $2 \leq x \leq 1$ is false. Therefore this sentence is a proposition.

In sentences like this it is not important to know the value of the variables involved to determine whether such sentences are true of false.

Definition Truth value of a Proposition

A **proposition** (or **statement**) is a sentence that is **either true or false**.

Each proposition can be assigned with a **truth value**.

The truth value of a proposition is either "**True**" (or **T**) or "**False**" (or **F**) depending on whether the proposition is true or false respectively.

Examples

1) 2 is an integer.
2) 2 is not an integer.
3) Is 2 an integer?
4) $X^2 > 10$
5) $5 + 3$

Can we assign Truth values for all the above sentences ?

- The truth value of proposition 1) is "True" (or T)
- The truth value of proposition 2) is "False" (or F)

Definition Compound Propositions

A proposition is said to be **primitive,** if it cannot be broken down into simple propositions, otherwise the proposition is said to be **composite** (or **compound**).

This means that a composite proposition can always be broken down into a collection of primitive propositions.

Logical Connectives (or Operations)

Logical connectives (operations) allow compound propositions to be built out of simple propositions. Each logical connective is denoted by a symbol.

Operation	Symbol	
not	$\sim$	**Negation**
and	$\wedge$	**Conjunction**
or	$\vee$	**Disjunction**

Examples:

compound propositions

1. $2+2 = 5$ and $5+3 = 8$.
2. Cat is a small animal or Dogs can fly.
3. It is not true that University of Colombo is in Matara.

These examples can be represented by using the symbols for logical connectives as below.

- $2+2 = 5 \wedge 5+3 = 8$.
- Cat is a small animal $\vee$ Dogs can fly.
- ~ (University of Colombo is in Matara).

Definition Proposition Variables

A variable that denotes a proposition is called a propositional variable.

Propositional variables can be replaced by propositions.

In logic the letters p,q,r, ……. Are used to denote propositional variables

Examples :

If p and q denotes two propositional variables

- $\sim p$ denotes the negation of p
- $p \wedge q$ denotes the conjunction of p and q
- $p \vee q$ denotes the disjunction of p and q

The proposition variables in a proposition can be replaced by any proposition.

For example p and q in $p \wedge q$ can be replaced by the two propositions "*A mouse is a small animal*" and "*Nimal is a boy*" resulting the proposition "*A mouse is a small animal* $\wedge$ *Nimal is a boy*"

Truth Tables

The sentences built out of propositions and logical connectives are also propositions. Therefore they must have truth values. The truth values of compound statements built out of the logical connectives are typically defined by using **truth tables**.

Definition of not(~)

p	~p
T	F
F	T

Truth table for ~p

Definition of and ($\wedge$)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table of $p \wedge q$

Definition of or (∨)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth table of $p \vee q$

Examples

Let

p : “It is hot” and

q: “It is sunny”.

Write each of the following sentences symbolically.

a) It is not hot but it is sunny.

b) It is not both hot and sunny.

c) It is neither hot nor sunny.

Examples

Let

p : "It is hot" and

q: "It is sunny".

a) It is not hot **but** it is sunny.

The convention in logic is that the words *but* and *and* mean the same thing.

Solution :

$\sim p \wedge q$

Examples

Let

p : “It is hot” and

q: “It is sunny”.

b) It is not both hot and sunny.

It is not (hot and sunny)

Solution :

$\sim(p \wedge q)$

Examples

Let

p : “It is hot” and

q: “It is sunny”.

c) It is neither hot nor sunny.

It is not hot and It is not sunny.

Solution :

$\sim p \wedge \sim q$

Construction of truth tables

- When constructing a truth table for a compound proposition composed of proposition variables, a column must be allocated for each proposition variable and for the final compound proposition.
- Columns may be allocated to record intermediate compound propositions, which are constructed only to help determination of truth values of the final proposition.

Construction of truth tables

- The table should have enough rows to represent all possible combinations of T and F for all proposition variables.

Number of rows in a Compound Proposition

For example if the compound proposition has two proposition variables there should be 4 (= 2^2) rows in the truth table. If the compound proposition has three proposition variables, there should be 8 (= 2^3) rows. Similarly if the compound proposition has n variables, there should be 2^n rows in the truth table.

Example : Construct the truth table for the compound proposition $\sim(p \wedge \sim q)$

- This proposition has 2 variables, p and q. Therefore the truth table should have $2^2 = 4$ rows.

Example : Construct the truth table for the compound proposition $\sim(p \wedge \sim q)$ …….

- You may allocate two additional columns to record the truth values of $\sim q$ and $(p \wedge \sim q)$ which will help to compute the final truth values easily.

Step 1: Fill in the columns p and q with all possible truth values

p	q	~q	p∧~q	~(p∧~q)
T	T			
T	F			
F	T			
F	F			

Step 2: Fill in the columns ~q. To fill this column only the values of q is required.

p	q	~q	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F		
T	F	T		
F	T	F		
F	F	T		

Step 3: Now fill the column $p \wedge \sim q$ by using the columns for p and $\sim q$.

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	
T	F	T	T	
F	T	F	F	
F	F	T	F	

Step 4: Now fill the final column $\sim(p \wedge \sim q)$ by using the column $p \wedge \sim q$.

p	q	~q	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

It is better to remember the above truth tables in the following way;

(i) $p \wedge q$ is T only in <u>one</u> instance; i.e., when p-T and q-T.

(ii) $p \vee q$ is F only in <u>one</u> instance; i.e., when p-F and q-F.

Examples:

What are the logical meaning of the following

1) $y \leq 5$

2) $5 \leq z \leq 10$

Answers:

1) $y \leq 5$ means $(y < 5)$ v $(y = 5)$

2) $5 \leq z \leq 10$ means $(5 \leq z)$ v $(z \leq 10)$

Tautology

Consider the compound proposition $p \vee \sim p$. From the truth table it is clear that $p \vee \sim p$ is **always T**.

Such compound propositions are called **tautologies**.

p	$p \vee \sim p$
T	T
F	T

Contradictions

Consider the compound proposition $p \wedge \sim p$. From the truth table it is clear that $p \wedge \sim p$ is **always F**.

Such compound propositions are called **contradictions**.

p	$p \wedge \sim p$
T	F
F	F

Conditional statements

A mathematical statement of the form

"**if p then q**"

is called a **conditional** statement. In logic such a statement is denoted by

$$\mathbf{p \to q}$$

Truth table of the conditional statement

$$p \to q$$

$p \to q$ is false only when p is true and q is false.
In all the other cases $p \to q$ is true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional Statements

A mathematical statement of the form

“**p if and only if q**”

is called a **biconditional** statement. In logic such a statement is denoted by

$$\mathbf{p} \leftrightarrow \mathbf{q}$$

Truth table of the Biconditional statement

The biconditional statement $p \leftrightarrow q$ is true if both p and q have the **same truth value** and false if p and q have opposite truth values.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional statements ……

The biconditional “if and only if” sometimes abbreviated as **iff**

A proposition with propositional variables p, q, … can be denoted by **P(p,q,……)**

Example

$P(p,q) : p \wedge q$

This means the proposition $\mathbf{p} \wedge \mathbf{q}$ is denoted by $P(p,q)$

Logical equivalence

Two propositions P(p,q,…) and Q(p,q,..) are said to be **logically equivalent** (or **equivalent** or **equal**), denoted by $\mathbf{P} \equiv \mathbf{Q}$, if they have **identical truth tables**.

Example : Consider the following two propositions.

$$P(p,q) : p \rightarrow q$$

$$Q(p,q) : (\sim q) \rightarrow (\sim p)$$

These two propositions are logically equivalent. This can be verified by constructing truth tables for P(p,q) and Q(p,q).

p	q	$p \rightarrow q$	$(\sim q) \rightarrow (\sim p)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

From this truth table it is obvious that $p \rightarrow q$ and $(\sim q) \rightarrow (\sim p)$ have identical truth tables. Hence these two propositions are logically equivalent.

Some more examples of equivalent propositions

$p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are equivalent.

$p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are equivalent.

$\sim(p \wedge q)$ and $(\sim p) \vee (\sim q)$ are equivalent.

$\sim(p \vee q)$ and $(\sim p) \wedge (\sim q)$ are equivalent.

$p \Rightarrow q$ and $(\sim p) \vee q$ are equivalent.

Exercise: Verify the above by using truth tables.

Algebra of Propositions

Propositions satisfy various laws. These laws are listed below.

1) Idempotent laws

   a) $p \vee p \equiv p$

   b) $p \wedge p \equiv p$

Algebra of Propositions ……

2) Associative laws

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Algebra of Propositions ……

3) Commutative laws

a) $p \vee q \equiv q \vee p$

b) $p \wedge q \equiv q \wedge p$

Algebra of Propositions ……

3) Distributive laws

a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Algebra of Propositions ……

4) Identity laws

a) $p \vee F \equiv p$

b) $p \wedge T \equiv p$

c) $p \vee T \equiv T$

d) $p \wedge F \equiv F$

Algebra of Propositions ……

5) Complement laws

a) $p \vee \sim p \equiv T$

b) $p \wedge \sim p \equiv F$

c) $\sim T \equiv F$

d) $\sim F \equiv T$

Algebra of Propositions ……

6) Involution law

a) $\sim\sim p \equiv p$

Algebra of Propositions ……

5) DeMorgan's laws

a) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

b) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Order of evaluation of logical expressions

Consider the following logical expression.

$$p \vee q \wedge r$$

This expression can be interpreted in two ways as below;

a) $(p \vee q) \wedge r$ b) $p \vee (q \wedge r)$

Question : Are these two expressions logically equivalent ?

Equality of $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$ can be determined by constructing truth tables for these propositions as below.

p	q	r	$p \vee q$	$q \wedge r$	$(p \vee q) \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	F	T	T
F	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	F	T	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

From the truth tables it is clear that $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$ are not logically equivalent. Therefore the meaning of the proposition $p \vee q \wedge r$ is ambiguous.

How can we disambiguate the meaning of such expressions ?

a) Defining precedence for logical connectives

~ has precedence over $\wedge$ which has precedence over $\vee$.

b) By using parenthesis.

Examples:

What is the evaluation of the expression $p \vee$

$\sim q \wedge r$? Solution :

The evaluation order of the expression is

$$p \vee ((\sim q) \wedge r)$$

Arguments

Consider the following:

If prices are high, then wages are high. Prices are high or there are price controls. If there are price controls, then there is no inflation. There is inflation. Therefore, wages are high.

The above is an example of an argument.

Formally, an argument is a **sequence of propositions**. All propositions except the final one are called **premises** (or **assumptions** or **hypotheses**). The final proposition is called the **conclusion**. The symbol $\therefore$ (or $\vdash$), read "therefore", may be placed just before the conclusion.

Consider the following argument

If prices are high, then wages are high. Prices are high or there are price controls. If there are price controls, then there is no inflation. There is inflation. Therefore, wages are high.

In the above argument, we can refer to ‘prices are high’ by p, ‘wages are high’ by q, ‘there are price controls’ by r, and ‘there is inflation’ by s. Then the argument can be represented symbolically as below

$p \to q, p \vee r, r \to \sim s, s$

Therefore q.

An argument can be denoted in any of the following ways.

a) p_1, p_2, p_n

$\therefore\ q$

b) $p_1, p_2, p_n \vdash q$

c) p_1, p_2, p_n

Therefore q

d) p_1, p_2, p_n

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q

In all the above representations $p_1, p_{2,\ldots}$ and p_n denote the premises of the arguments and q denotes the conclusion.

Validity of arguments

An argument is said to be **valid** if the conclusion of the argument is true whenever all premises are true. An argument which is not valid is called a **fallacy**.

How can the validity of an argument be determined ?

Example 1)

Consider the following argument

$p, p \rightarrow q$

Therefore q.

One of the ways to determine the validity of an argument is to construct its truth table and then determine the validity by consulting the truth table.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

From the above truth table it is clear that the conclusion (q) is true whenever all its premises (p, $p \to q$) are true. Therefore the argument is a valid argument.

Example 2)

Consider the following argument

$p \to q, q$

Therefore, p.

Is this argument valid?

Solution : To determine the validity of the argument construct the required truth table.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

From the above truth table it is clear that the both premises ($p \rightarrow q$ and q) are true only in two cases. For this argument to be valid, the conclusion (p) has to be true in both these cases. However the conclusion is not true in both cases. Therefore the argument is not valid. This means the argument is a **fallacy.**

However, when the number of propositions involved in an argument is large, it is not easy to construct the truth table for it. In such cases, the validity of the argument can be determined by analysing the argument.

Consider the following argument

$p \rightarrow q$, $p \vee r$, $r \rightarrow \sim s$, s

Therefore q.

In this argument 4 propositions are involved. If a truth table is to be built for the argument, 16 rows are needed to be constructed and this process takes substantial amount of time.

The validity of the argument can also be determined by analyzing it as below.

Suppose all the premises of the argument are correct.

- Then s is T.
- If s is T, $\sim s$ is F. Therefore $r \rightarrow \sim s$ to be true r has to be F.
- Since r is F, for $p \vee r$ to be true p has to be T.
- If p is true for p $\rightarrow$ q to be true **q has to be T**.

Thus it can be concluded that whenever all premises are T, the conclusion is also T. Therefore the argument is a valid argument.

Now consider the following argument:

If Anil's mark is more than 50, then his mark is more than 55. Anil's mark is more than 50 or his mark is less than 40. If Anil's mark is less than 40, then, Anil does not pass. Anil passes.

Therefore, Anil's mark is more than 55.

In the above argument refer to ‘Anil’s mark is more than 50’ by p, ‘Anil’s mark is more than 55’ by q, ‘Anil’s mark is less than 40’ by r and ‘Anil passes’ by s.

Then the argument is of the form:

$p \rightarrow q$, $p \vee r$, $r \rightarrow \sim s$, s

Therefore q.

We see that this is exactly the same as the earlier argument. Therefore we can conclude that this argument is also valid.

Predicates

- **“x > 10, where x is an integer”**
- **Is this statement proposition?**
- (A **Proposition** is a sentence or statement that can be true or false)

Predicates

- **“x > 10, where x is an integer”**

- This sentence is **not a proposition** as its **truth value cannot be determined**.

- This is due to the presence of a variable x.

Predicates

- **"x > 10, where x is an integer"**

- Once a specific value is assigned to the variable x, then this sentence may become specifically true or false.

  - **If x = 1: False**
  - **If x = 26: True**

Such sentences are called **Predicates**.

Predicate Variables

- A Predicate may contain one or more **variables**.
- We can represent a predicate as a function.
  - For example, **"x > 10, where x is an integer"** be represented as **P(x)**

Example

Predicate Variables

- Consider **$x < y$ where $x, y \in R$**
- This predicate has **two** variables

- Consider **$x > 10$ where $x \in N$**
- This predicate has **one** variable

Definition
Universe of Discourse

- The set of values that a variable (or variables) in a predicate can take is called the **Universe of Discourse** (or the **Domain**) of that variable (or those variable).

Example
Universe of Discourse

- Consider **$x < y$ where $x, y \in R$**

- The **Universe of Discourse or Domain** of both variables x and y is the set of real numbers, R.

Definition

Truth Sets

- Let P(x) be a predicate and A be the Domain of x.

- Then, P(x) could be:
  - true for all values of A,
  - true for some values of A
  - true for no values of A.

- The set of all elements in A for which P(x) is true is called the **truth set** of the predicate P(x).

Example
Truth Sets

- Consider **P(x) : x + 5 > 10, x ∈ N**

- Let x ∈ N
- P(x) is True is and only if
  - x + 5 > 10
  - That is, x > 5
  - That is, x = 6 or 7 or 8 or …

- Truth set is **{6, 7, 8,…}**

The **Truth Set** of a predicate is always a subset of its **Domain**

Example
Truth Sets

- Consider **P(x) : x + y = 4 , x $\in$ N, y $\in$ N**

- By inspection P(x) is true if and only if
  - (x = 1 and y = 3) or
  - (x = 2 and y = 2) or
  - (x = 3 and y =1)

- Truth set is **{(1, 3), (2, 2), (3, 1)}**

(x, y) ordered pairs

Universal Quantifier

- Consider **$x^2 \geq 0$ where $x \in$ R.**

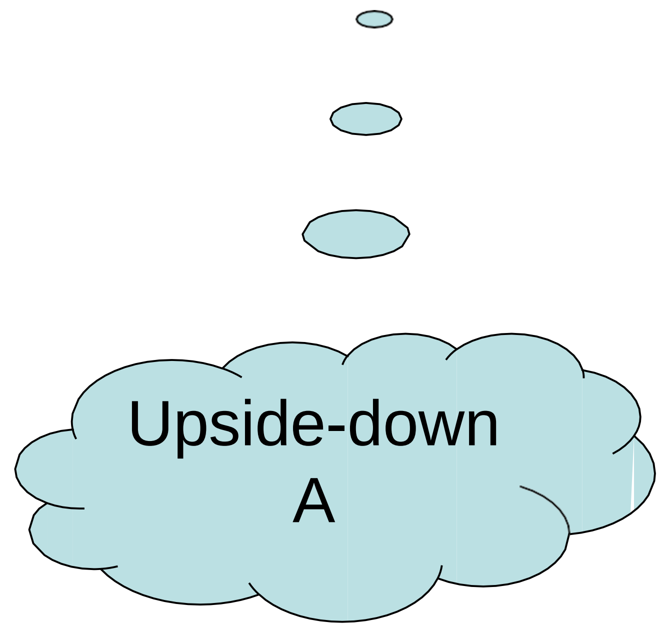
- We know that $x^2 \geq 0$ is **true for any given value of x in R**.

- We can say:
- "For all $x \in$ R, $x^2 \geq 0$ is true"

Definition
Universal Quantifier $\forall$

- We can denote “For all” by the symbol “$\forall$”. The symbol $\forall$ is called the **universal quantifier** and reads as “for any”, “for every” or “ for all”

$\forall$ **$\forall x \in R, x^2 \geq 0$** can be read as:
  - “For any $x \in R, x^2 \geq 0$ is true”
  - “For every $x \in R, x^2 \geq 0$ is true”
  - “For all $x \in R, x^2 \geq 0$ is true”



Example
Universal Quantifier

- Consider P(x) is $x \geq 1$

  $\forall x \in N$, P(x) is True

- $\forall x \in R$, P(x) is False
  Counter example:
  P(0.5) is false because $0.5 < 1$

Example
Universal Quantifier

$\forall x \in N, \forall y \in N, x+y > 1$

"For all $x \in N$, for all $y \in N$, $x + y > 1$"
This is True.

We can extend this to having any number of quantifiers:

$\forall x \in N, \forall y \in N, \forall z \in N, \forall w \in N \ldots$

Existential Quantifier

Consider **$x + 7 = 10$ where $x \in N$**.

Since, $3 + 7 = 10$, this statement is **true for at least one value of x**.

We can say:

“There exists some $x \in N$ such that $x + 7 = 10$”

Definition
Existential Quantifier ∃

- We can denote “There exists” by the symbol “∃”. The symbol ∃ is called the existential quantifier and reads as “There exists”.

$\forall\ \exists$ **$x \in N, x + 7 = 10$** can be read as:

– “There exists $x \in N$ such that $x + 7 = 10$ is true”

Mirror-image of E

Example
Existential Quantifier

- Consider **P(x) is x + 23 = 25.5**

- $\exists x \in R$, P(x) is True
- P(2.5) is True because 2.5 + 23 = 25.5

- $\exists x \in N$, P(x) is False

Example
Existential Quantifier

- **$\exists x \in N, \exists y \in N, x > y$**

- "There exists some x $\in$N and there exists some y $\in$N such that x>y"

- This is True. For example, 2>1,

Quantifier Negation

$\forall$ x, P(x)
"For all x, P(x) is true"

not ($\forall$ x, P(x))
"not (For all x, P(x) is true)"

"There exists some x such that P(x) is false."
$\exists$x, ~P(x)

Quantifier Negation

$\exists$ x, P(x)
“There exists some x such that P(x) is true”

not ($\exists$ x, P(x))
“not (There exists some x such that P(x) is true)”

“For all x, P(x) is false.”
$\forall$x, ~P(x)

Example
Negation

- $\forall x \in N, x > 100$

  $\exists x, x \leq 100$

Quantifiers change
$\forall$ Becomes $\exists$
$\exists$ Becomes $\forall$

$\exists x \in N, x + 2 = 5$

$\forall x \in N, x + 2 \neq 5$

Predicate is negated

Mixed Quantifiers

- "**For all** $x \in N$, **there exists** some $y \in N$ such that $x \leq y$."
- "Given any natural number x, we can find another natural number y, such that x is less than y"
- We can write this as:
  $\forall x \in N, \exists y \in N, x < y$.

Mixed Quantifiers

- $\forall x \in N, \exists y \in N, x < y$

- Let $x \in N$
- Let $y = x+1$.
- Then $y \in N$ and $x < y$.

- Therefore, "$\forall x \in N, \exists y \in N, x < y$" is true.

Mixed Quantifiers

Order

- $\forall x \in N, \exists y \in N, x < y$
- $\exists x \in N, \forall y \in N, x < y$

- Are these both the same?

Mixed Quantifiers
Order

- **$\forall x \in N, \exists y \in N, x < y$**
- "For all $x \in N$, there exists some $y \in N$ such that $x < y$."
  – **This is True**

- **$\exists x \in N, \forall y \in N, x < y$**
- "There exists some $x \in N$ such that for all $y \in N$, $x < y$."

Mixed Quantifiers
Order

- $\exists x \in N, \forall y \in N, x < y$ - We prove this **False** by contradiction.

- Suppose $\exists x \in N, \forall y \in N, x < y$ is true
- Then for some constant $\mathbf{x_0} \in N, \forall y \in N, x_0 < y$
- Since, $1 \in N$, $x_0 < 1$ must be true
- But, all natural numbers (N) are greater or equal to one!
- This is a Contradiction!!!

- Hence, "$\exists x \in N, \forall y \in N, x < y$" must be false

Mixed Quantifiers Order

- $\forall \mathbf{x} \in \mathbf{N}, \exists \mathbf{y} \in \mathbf{N}, \mathbf{x} < \mathbf{y}$
- "For all $x \in N$, there exists some $y \in N$ such that $x < y$."
  – **True**

- $\exists \mathbf{x} \in \mathbf{N}, \forall \mathbf{y} \in \mathbf{N}, \mathbf{x} < \mathbf{y}$
- "There exists some $x \in N$ such that for all $y \in N$, $x < y$."
  - **False**

Mixed Quantifiers
Negation

- $\forall x \in N, \exists y \in N, x < y$
  - not ($\forall x \in N, \exists y \in N, x < y$)
  - $\exists x \in N$, not($\exists y \in N, x < y$)
  - $\exists \mathbf{x} \in \mathbf{N}, \forall \mathbf{y} \in \mathbf{N}, \mathbf{x} \geq \mathbf{y}$

Quantifiers change
$\forall$ Becomes $\exists$
$\exists$ Becomes $\forall$

- $\exists x \in N, \forall y \in N, x < y$
  - $\forall \mathbf{x} \in \mathbf{N}, \exists \mathbf{y} \in \mathbf{N}, \mathbf{x} \geq \mathbf{y}$

Predicate is negated