# 1: Sets

IT2106 – Mathematics for Computing I

Level I - Semester 2

# Introduction to Sets

- George Cantor (1845-1915), in 1895, was the first to define a set formally.
- Definition - Set

  A set is a unordered collection of zero of more distinct well defined objects.

- The objects that make up a set are called

  **elements** or **members** of the set.

# Specifying Sets

- There are two ways to specify a set

1. If possible, list all the members of the set.

   E.g. A = {a, e, i, o, u}

2. State those properties which characterized the members in the set.

   E.g. B = {x : x is an even integer, x > 0}

   We read this as "B is the set of x such that x is an even integer and x is grater than zero". Note that we can't list all the members in the set B.

   C = {All the students who sat for BIT IT1101 paper in 2003}

   D = {Tall students who are doing BIT} – is not a set because "Tall" is not well defined. But…

   E = {Students who are taller than 6 Feet and who are doing BIT} – is a set.

# Some Properties of Sets

- The order in which the elements are presented in a set is not important.
  - A = {a, e, i, o, u} and
  - B = {e, o, u, a, i} both define the same set.
- The members of a set can be anything.
- In a set the same member does not appear more than once.
  - F = {a, e, i, o, a, u} is incorrect since the element ‘a’ repeats.

# Some Common Sets

- We denote following sets by the following symbols:
  - N = The stet of positive integers = {1, 2, 3, …}
  - Z = The set of integers = {…,-2, -1, 0, 1, 2, …}
  - R = The set of real numbers
  - Q = The set of rational numbers
  - C = The set of complex numbers

# Some Notation

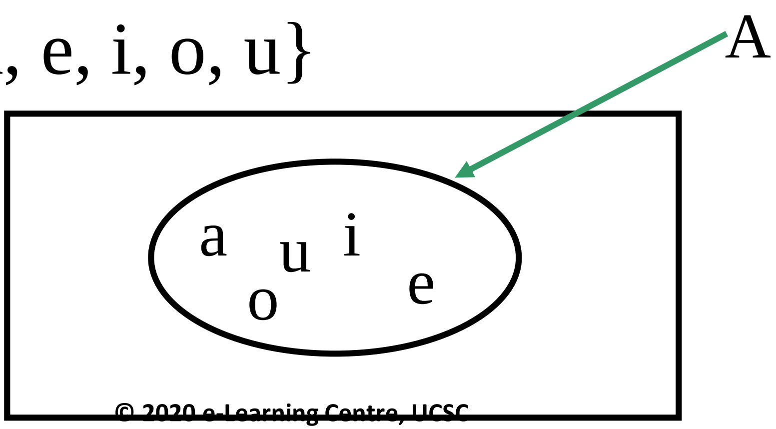
- Consider the set A = {a, e, i, o, u} then
- We write "'a' is a member of 'A'" as:
  - $a \in A$
- We write "'b' is not a member of 'A'" as:
  - $b \notin A$
  - Note: $b \notin A \equiv \neg (b \in A)$

# Universal Set and Empty Set

- The members of all the investigated sets in a particular problem usually belongs to some fixed large set. That set is called the universal set and is usually denoted by ‘U’.
- The set that has no elements is called the empty set and is denoted by Φ or {}.
  – E.g. $\{x \mid x^2 = 4 \text{ and } x \text{ is an odd integer}\} = \Phi$

# Venn Diagrams

- A pictorial way of representing sets.
- The universal set is represented by the interior of a rectangle and the other sets are represented by disks lying within the rectangle.
  - E.g. A = {a, e, i, o, u}



# Equality of two Sets

- A set 'A' is equal to a set 'B' if and only if both sets have the same elements. If sets 'A' and 'B' are equal we write: $A = B$. If sets 'A' and 'B' are not equal we write $A \neq B$.
- In other words we can say:

$$A = B \Leftrightarrow (\forall x, x \in A \Leftrightarrow x \in B)$$

  – E.g.

    $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 1, 3, 5\}$, $C = \{1, 3, 5, 4\}$
    $D = \{x : x \in N \wedge 0 < x < 6\}$, $E = \{1, 10/5, \sqrt{9}, 2^2, 5\}$ then $A = B = D = E$ and $A \neq C$.

# Cardinality of a Set

- The number of elements in a set is called the cardinality of a set.
- Let ‘A’ be any set then its cardinality is denoted by $|A|$
- E.g. $A = \{a, e, i, o, u\}$ then $|A| = 5$.

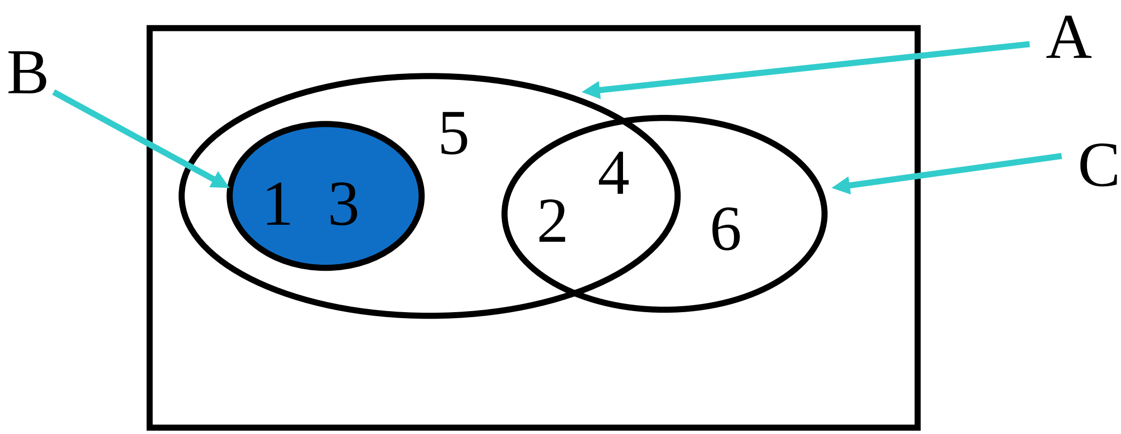
# Subsets

- Set 'A' is called a subset of set 'B' if and only if every element of set 'A' is also an element of set 'B'. We also say that 'A' *is contained in* 'B' or that 'B' *contains* 'A'. It is denoted by $A \subseteq B$ or $B \supseteq A$.
- In other words we can say:

$$(A \subseteq B) \Leftrightarrow (\forall x, x \in A \Rightarrow x \in B)$$

# Subset ctd…

- If ‘A’ is not a subset of ‘B’ then it is denoted by $A \nsubseteq B$ or $B \nsupseteq A$
  - E.g. $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3\}$ and $C = \{2, 4, 6\}$ then $B \subseteq A$ and $C \nsubseteq A$



# Some Properties Regarding Subsets

- For any set 'A', $\Phi \subseteq A \subseteq U$
- For any set 'A', $A \subseteq A$
- $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$
- $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

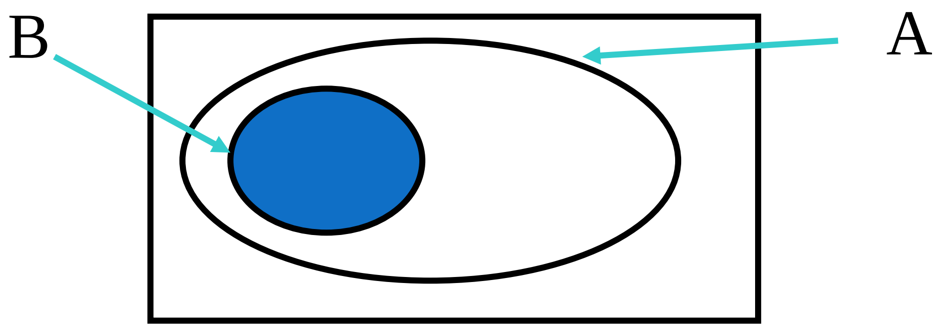
# Proper Subsets

- Notice that when we say $A \subseteq B$ then it is even possible to be $A = B$.
- We say that set 'A' is a proper subset of set 'B' if and only if $A \subseteq B$ and $A \neq B$. We denote it by $A \subset B$ or $B \supset A$.
- In other words we can say:

$$(A \subset B) \Leftrightarrow (\forall x,\ x \in A \Rightarrow x \in B \wedge A \neq B)$$

# Venn Diagram for a Proper Subset

- Note that if A $\subset$ B then the Venn diagram depicting those sets is as follows:




- If A $\subseteq$ B then the disc showing 'B' may overlap with the disc showing 'A' where in this case it is actually A = B

# Power Set

- The set of all subsets of a set ‘S’ is called the power set of ‘S’. It is denoted by P(S) or $2^S$.
- In other words we can say:

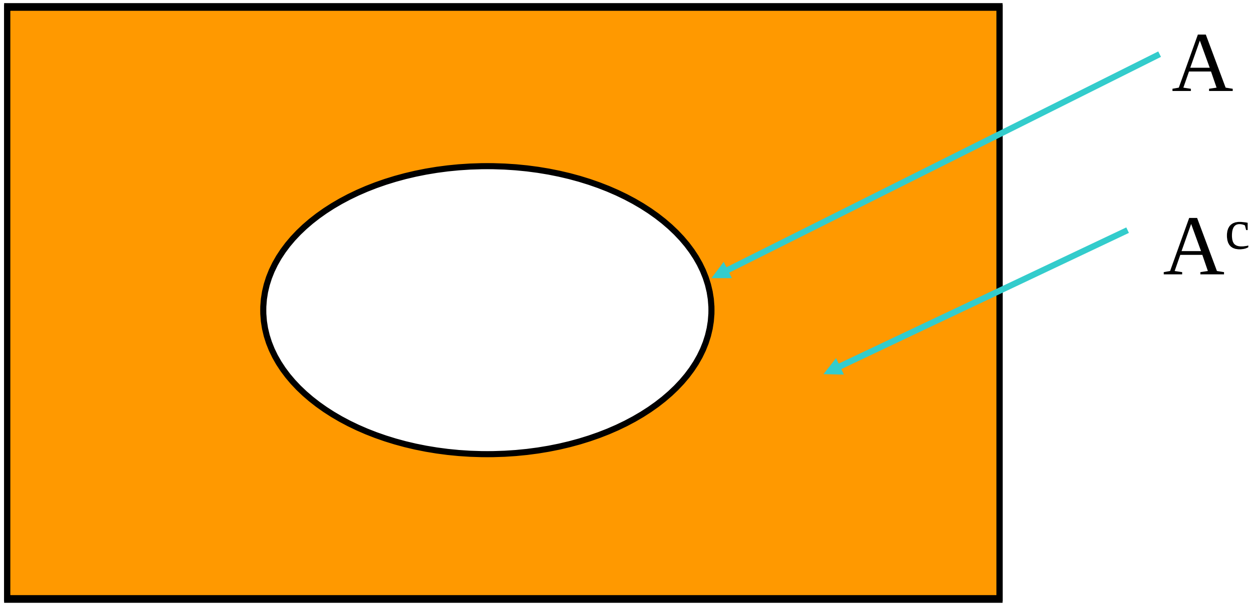
  $P(S) = \{x : x \subseteq S\}$
- E.g. $A = \{1, 2, 3\}$ then

  $P(A) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
- Note that $|P(S)| = 2^{|S|}$.
- E.g. $|P(A)| = 2^{|A|} = 2^3 = 8$.

# Set Operations - Complement

- The (absolute) complement of a set ‘A’ is the set of elements which belong to the universal set but which do not belong to A. This is denoted by $A^c$ or $\bar{A}$ or Á .
- In other words we can say:
- $A^c = \{x : x \in U \wedge x \notin A\}$

# Venn Diagram for the Complement



# Set Operations - ∪nion

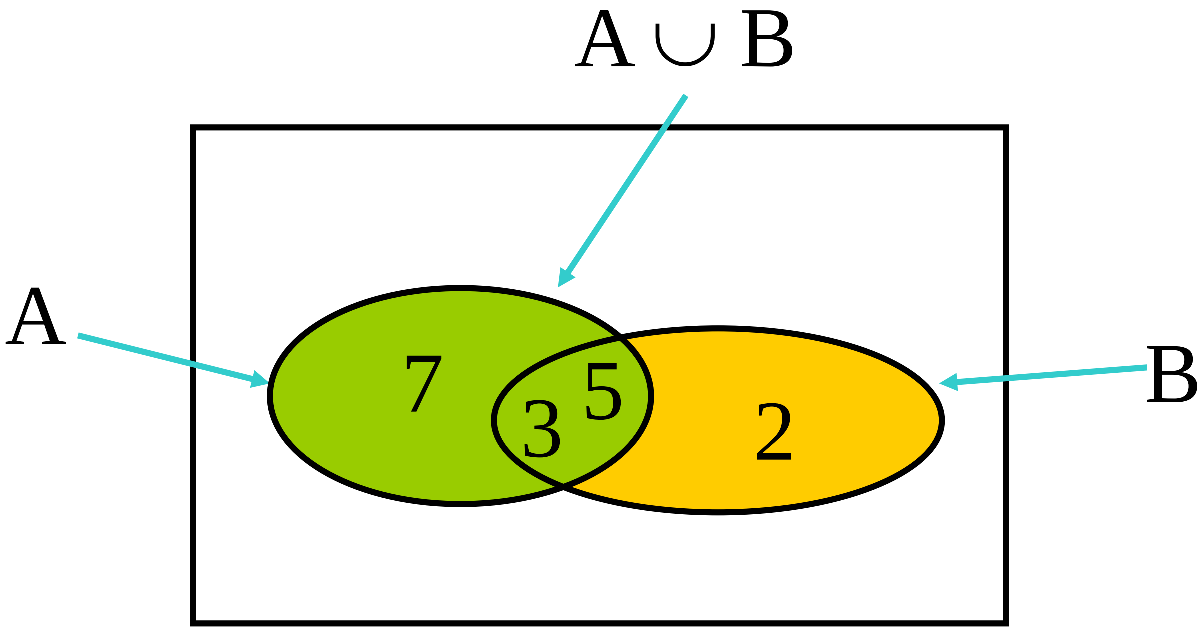
- Union of two sets ‘A’ and ‘B’ is the set of all elements which belong to either ‘A’ or ‘B’ or both. This is denoted by $A \cup B$.
- In other words we can say:

$$A \cup B = \{x : x \in A \vee x \in B\}$$

- E.g. $A = \{3, 5, 7\}$, $B = \{2, 3, 5\}$

$$A \cup B = \{3, 5, 7, 2, 3, 5\} = \{2, 3, 5, 7\}$$

# Venn Diagram Representation for Union



# Set Operations - I∩tersection

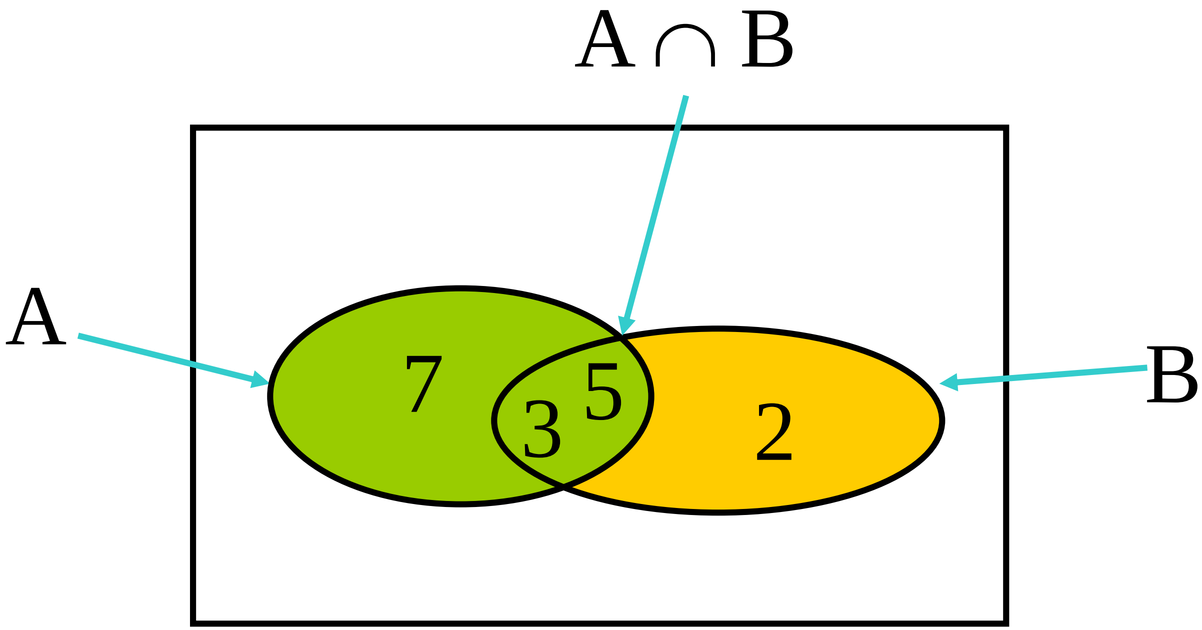
- Intersection of two sets ‘A’ and ‘B’ is the set of all elements which belong to both ‘A’ and ‘B’. This is denoted by $A \cap B$.
- In other words we can say:

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

- E.g. $A = \{3, 5, 7\}$, $B = \{2, 3, 5\}$

$$A \cap B = \{3, 5\}$$

# Venn Diagram Representation for Intersection



# Set Operations - Difference

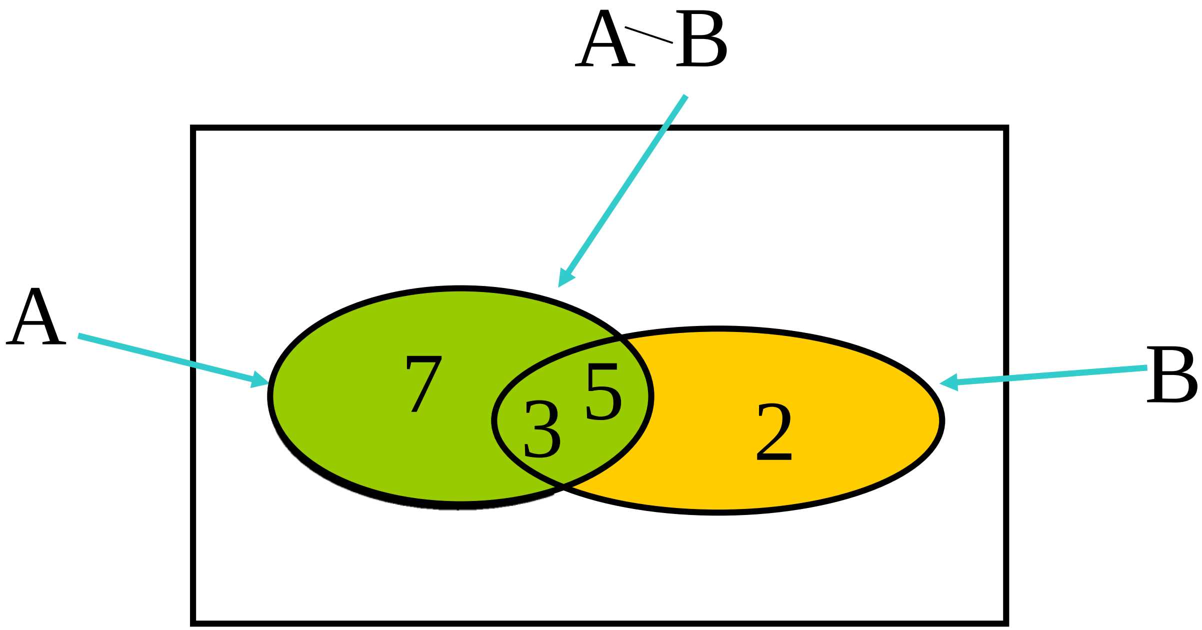
- The difference or the relative complement of a set ‘B’ with respect to a set ‘A’ is the set of elements which belong to ‘A’ but which do not belong to ‘B’. This is denoted by $A \setminus B$.
- In other words we can say:

$$A \setminus B = \{x : x \in A \wedge x \notin B\}$$

- E.g. $A = \{3, 5, 7\}$, $B = \{2, 3, 5\}$

$$A \setminus B = \{\cancel{3}, \cancel{5}, 7\} \setminus \{2, 3, 5\} = \{7\}$$

# Venn Diagram Representation for Difference



# Some Properties

- $A \subseteq A \cup B$ and $B \subseteq A \cup B$
- $A \cap B \subseteq A$ and $A \cap B \subseteq B$
- $|A \cup B| = |A| + |B| - |A \cap B|$
- $A \subseteq B \Rightarrow B^c \subseteq A^c$
- $A \setminus B = A \cap B^c$
- If $A \cap B = \Phi$ then we say 'A' and 'B' are disjoint.

# Algebra of Sets

- Idempotent laws
  - $A \cup A = A$
  - $A \cap A = A$
- Associative laws
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$

# Algebra of Sets ctd…

- Commutative laws
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- Distributive laws
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

# Algebra of Sets ctd…

- Identity laws
  - $A \cup \Phi = A$
  - $A \cap U = A$
  - $A \cup U = U$
  - $A \cap \Phi = \Phi$
- Involution laws
  - $(A^c)^c = A$

# Algebra of Sets ctd…

- Complement laws
  - $A \cup A^c = U$
  - $A \cap A^c = \Phi$
  - $U^c = \Phi$
  - $\Phi^c = U$

# Algebra of Sets ctd…

- De Morgan's laws
  - $(A \cup B)^c = A^c \cap B^c$
  - $(A \cap B)^c = A^c \cup B^c$
- Note: Compare these De Morgan's laws with the De Morgan's laws that you find in logic and see the similarity.

# Proofs

- Basically there are two approaches in proving above mentioned laws and any other set relationship
  - Algebraic method
  - Using Venn diagrams
- For example lets discuss how to prove
  - $(A \cup B)^c = A^c \cap B^c$

# Proofs Using Algebraic Method

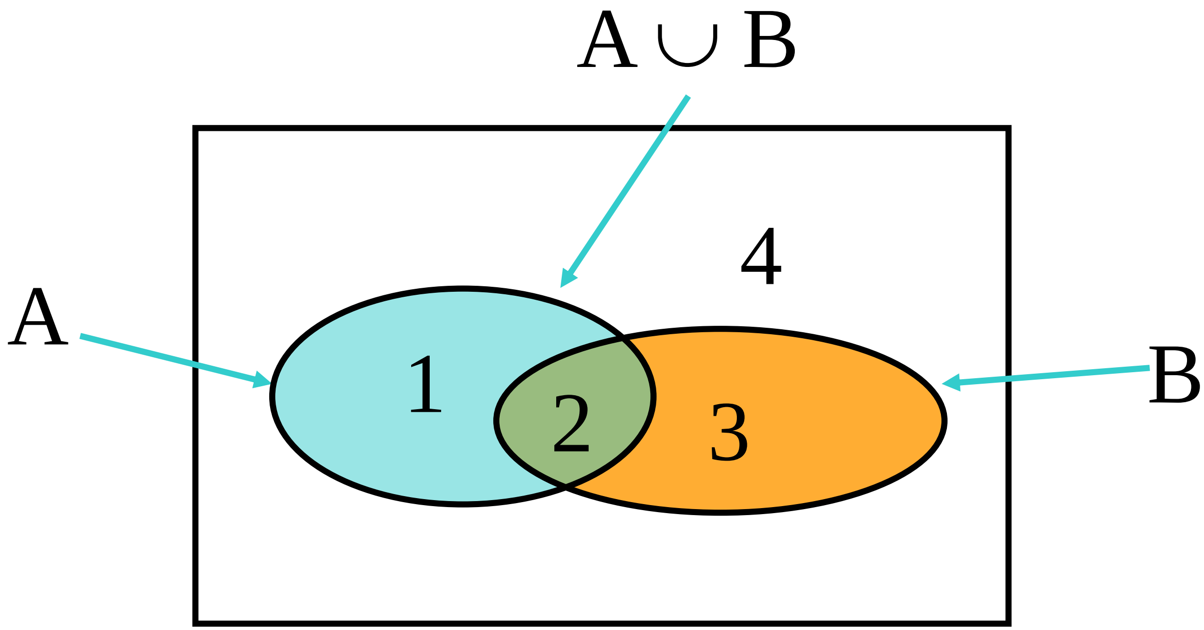
$$
\begin{aligned}
x \in (A \cup B)^c &\Rightarrow x \notin A \cup B \\
&\Rightarrow x \notin A \wedge x \notin B \\
&\Rightarrow x \in A^c \wedge x \in B^c \\
&\Rightarrow x \in A^c \cap B^c \\
&\Rightarrow (A \cup B)^c \subseteq A^c \cap B^c \quad \text{———————} (\alpha)
\end{aligned}
$$

# Proofs Using Algebraic Method ctd…

$$
\begin{aligned}
x \in A^c \cap B^c &\Rightarrow x \in A^c \wedge x \in B^c \\
&\Rightarrow x \notin A \wedge x \notin B \\
&\Rightarrow x \notin A \cup B \\
&\Rightarrow x \in (A \cup B)^c \\
&\Rightarrow A^c \cap B^c \subseteq (A \cup B)^c \quad \text{———————} (\beta)
\end{aligned}
$$

$$
\underline{\underline{(\alpha) \wedge (\beta) \quad \Rightarrow (A \cup B)^c = A^c \cap B^c}}
$$

# Proofs Using Venn Diagrams




- Note that these indicated numbers are not the actual members of each set. They are region numbers.

# Proofs Using Venn Diagrams ctd…

U : 1, 2, 3, 4

A : 1, 2 (i.e. The region for 'A' is 1 and 2)

B : 2, 3

$\therefore$ $A \cup B$ : 1, 2, 3

$\therefore$ $(A \cup B)^c$ : 4 ———————————— ($\alpha$)

# Proofs Using Venn Diagrams ctd…

$A^c$ : 3, 4

$B^c$ : 1, 4

$\therefore A^c \cap B^c$ : 4 ———————————— ($\beta$)

$$(\alpha) \wedge (\beta) \quad \Rightarrow (A \cup B)^c = A^c \cap B^c$$