1 : Theory of Matrices, Vector spaces and Linear Transformations

IT5506 – Mathematics for Computing II

Level III - Semester 5

Overview

- Vector spaces play a role in many branches of Mathematics and Computer Science.

- In various theoretical problems, we have set X, whose elements are called vectors and these elements can be added and multiplied by scalars (constants) in a natural way the result being an element of X. Such concrete situations suggest the concept of Vector Space.

- Scalars are from a field $\mathcal{F}$. For most of the mathematical applications the field $\mathcal{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. In general the field can be either a finite field $\mathbb{Z}_p$, where p is a prime number or an infinite field ($\mathbb{R}$ or $\mathbb{C}$).

Intended Learning Outcomes

At the end of this lesson, you will be able to;

- develop the concept of a vector space through axioms.
- Use the vector space axioms to determine if a set and its operations constitute a vector space
- determine whether a set of vectors is a subspace of a given vector space.
- determine whether two sets of vectors span the same subspace.
- determine if a set is spanning a vector space and/or is linearly independent.
- find a basis of a given vector space.
- determine the dimension of a vector space.
- extend a linearly independent set of vectors to a basis.
- shrink a spanning set of vectors to a basis.

List of sub topics

1.6 Vector spaces (6 hours)

- 1.6.1 Axiomatic definition of a Field and a vector space with suitable examples
- 1.6.2 Subspaces of a vector space with examples and identifying all possible subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$
- 1.6.3 Linear combination and linear span, fundamental subspaces associated with a matrix.
- 1.6.5 Linear independence and dependence, the rank of a matrix.
- 1.6.4 Basis and dimension of a vector space.
- 1.6.6 Finite dimensional vector space and constructing basis for it.

1.6.1 Vector Spaces and Fields

Definition: A **field** is a set F equipped with two algebraic operations $+$ and $\cdot$ called scalar addition and scalar multiplication, respectively and two special elements 0 and 1, such that for every $a, b, c \in \mathrm{F}$ the following conditions are satisfied:

F.1 $a + b \in \mathcal{F}$ and $a \cdot b \in \mathrm{F}$ (F is closed under addition and multiplication)

F.2 $a + b = b + a$ and $a \cdot b = b \cdot a$ (commutativity of addition and multiplication),

F.3 $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity of addition and multiplication),

F.4 $0 + a = a$ and $1 \cdot a = a$ (additive and multiplicative identities)

F.5 $\exists -a \in \mathrm{F}$ such that $a + (-a) = 0$ (additive inverse), and

if $b \neq 0$, then $\exists\, b^{-1} \in \mathrm{F}$ such that $b \cdot b^{-1} = 1$ (multiplicative inverse),

F.6 $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity).

Examples of Fields

1. The set $\mathbb{R}$ of real numbers, with the usual operations $+$ and $\cdot$, and the usual 0 and 1 is a field.

2. The set $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$ of complex numbers, with the usual $+, \cdot, 0, 1$ is also a field.

3. The set $\mathbb{Q} = \left\{\frac{m}{n} \ : m, n \in \mathbb{Z} \ \ and \ n \neq 0\right\}$ of rational numbers, with the usual $+, \cdot, 0, 1$ is a field.

4. The set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with $\oplus, \ and \ \odot$ defined respectively for every $a, b \in \mathbb{Z}_n$ as $a \oplus b = \ (a + b) \ mod \ n$, and $a \odot b = (a \cdot b) mod \ n$ is a finite field, where n is a prime number.

Note: Students should be able to verify that the above sets satisfy all the axioms of a field with respect to the addition and multiplication.

Vector Space

Definition: A **Vector space** $(V, \oplus, \odot, \mathrm{F})$ over the field F is a nonempty set V together with two algebraic operations called vector addition and multiplication of vectors by scalars which satisfy the following conditions:

A. Vector addition ($\oplus$):

A.1 $x \oplus y \in V, \quad \forall x, y \in V$

A.2 $x \oplus y = y \oplus x, \ \forall\, x, y \in V$

A.3 $x \oplus (y \oplus z) = (x \oplus y) \oplus z, \ \forall\, x, y, z \in V$

A.4 $\exists\ 0_v \in V$ (zero vector) such that $x \oplus 0_v = x, \ \forall x \in V$

A.5 $\forall x \in V, \ \exists\ -x \in V$ (additive inverse of x) such that $x \oplus (-x) = 0_v$.

B. Multiplication by scalars:

B.1 $\propto \odot\, x \in V, \quad \forall \propto \in \mathrm{F}, \ \forall\, x \in V$

B.2 $\alpha \odot (\beta \odot x) = (\alpha\beta) \odot x, \ \forall\, \alpha, \beta \in \mathrm{F}, \ \forall\, x \in V$

B.3 $\exists\ 1 \in F$ (unit element of F) such that $1 \odot x = x, \forall x \in V$.

B.4 $\alpha \odot (x \oplus y) = \alpha \odot x \oplus \alpha \odot y, \ \forall \alpha \in \mathrm{F}, \ x, y \in V$

B.5 $(\alpha + \beta) \odot x = \alpha \odot x \oplus \beta \odot x, \ \forall \alpha, \beta \in \mathrm{F}, \ \forall x \in V$.

Remarks

- The elements of F are called scalars, and that of V are called vectors.

- From the definition of Vector space, vector addition is a function (binary operation) $V \times V \longrightarrow V$, whereas multiplication by scalars is a function $\mathrm{F} \times V \longrightarrow V$.

- X is called a **real** vector space if the field $\mathrm{F} = \mathbb{R}$ and is called a **complex** vector space if the field $\mathrm{F} = \mathbb{C}$.

Some Basic Consequences of the Vector Space Axioms:

Theorem 1.6.1.1: Let V be a vector space over the field $\boldsymbol{F}$.

1. Zero vector of V (0_V) is unique.
2. Additive inverse of each element of V is unique.
3. Scalar identity 1 of the field $\boldsymbol{F}$ is unique.

Proof of (1): Suppose that 0_V and $0_V'$ are two zero vectors of V. Then $x \oplus 0_V = x$ and $x \oplus 0_V' = x$, for all $x \in V$.
Therefore,

$$\begin{aligned} 0_V' &= 0_V' \oplus 0_V && (\text{as } 0_V \text{ is a zero vector}) \\ &= 0_V \oplus 0_V' && (\because \text{By } A.2) \\ &= 0_V && (\text{as } 0_V' \text{ is a zero vector}). \end{aligned}$$

Hence, $0_V = 0_V'$, showing that the zero vector is unique.

Remaining parts of the theorem are left as exercise.

Some Basic Consequences of the Vector Space Axioms:

Theorem 1.6.1.2: Let V be a vector space over the field $\boldsymbol{F}$. Then

1. $0 \odot x = 0_V, \ \forall x \in V$, where 0_V is the zero vector of V.
2. $\alpha \odot 0_V = 0_V, \ \forall \boldsymbol{\alpha} \in \boldsymbol{F}$
3. $(-\alpha) \odot x = -(\alpha \odot x), \ \forall x \in X, \forall \boldsymbol{\alpha} \in \boldsymbol{F}$.
4. If $x \oplus y = x$, then $y = 0_V$.
5. If $\alpha \odot x = 0_V$, then $\alpha = 0$ or $x = 0_V$.

Proof of (4): Suppose that $x \oplus y = x$. for some $x, y \in V$.
By axiom A.5, $\exists \ -x \in V$ such that $x \oplus (-x) = 0_v$. That is;

$$
\begin{aligned}
0_v = x \oplus (-x) &= (-x) \oplus x && (by \quad A.2) \\
&= (-x) \oplus (x + y) && (\because \ x \oplus y = x) \\
&= \big((-x) \oplus x\big) + y && (By \quad \mathrm{A.3}) \\
&= 0_v \oplus y && \big(\because \ \big((-x) \oplus x\big) = 0_v\big) \\
&= y && (by \quad A.4).
\end{aligned}
$$

Remaining parts of the theorem are left as exercise.

Examples of Vector Spaces

Example 1.6.1.1: The set $\mathbb{R}$ of real numbers, with the usual addition and multiplication (i.e., $\oplus \equiv +$ and $\odot \equiv \cdot$) forms a vector space over $\mathbb{R}$.

Example 1.6.1.2: Consider the set $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ of complex numbers, where $i = \sqrt{-1}$.

1. For any $x_1 + iy_1,\ x_2 + iy_2 \in \mathbb{C}$, and $\alpha \in \mathbb{R}$ define vector addition and scalar multiplication respectively by

   $(x_1 + iy_1) \oplus (x_2 + iy_2) = (x_1 + x_1) + i(y_1 + y_2)$ and

   $\propto \odot (x_1 + iy_1) = (\alpha x_1) + i(\alpha y_1)$.

   Then $\mathbb{C}$ forms a **Real** vector space.

2. For any $x_1 + iy_1,\ x_2 + iy_2 \in \mathbb{C}$, and $\alpha + i\beta \in \mathbb{C}$ define $\oplus$ and $\odot$ by

   $(x_1 + iy_1) \oplus (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$ and

   $(\alpha + i\beta) \odot (x_1 + iy_1) = (\alpha x_1 - \beta y_1) + i(\alpha y_1 + \beta x_1)$.

   Then $\mathbb{C}$ forms a **Complex** vector space.

Note: Students should verify that these satisfy all the axioms of a vector space.

Examples of Vector Spaces

Example 1.6.1.3: The **Euclidean space** $\mathbb{R}^{\boldsymbol{n}} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, x_2, \cdots, x_n \in \mathbb{R} \right\}$ is a real vector space(over the field $\mathbb{R}$) with respect to the two algebraic operations defined in the usual fashion as for any $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}, \text{and} \qquad \alpha \odot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}.$$

Note: Students should be able to verify that $\mathbb{R}^{\boldsymbol{n}}$ satisfy all the axioms of a vector space.

Examples of Vector Spaces

Example 1.6.1.4: The **unitary space** $\mathbb{C}^{\boldsymbol{n}} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, x_2, \cdots, x_n \in \mathbb{C} \right\}$ is a complex

vector

space (over the field $\mathbb{C}$) with respect to the two algebraic operations defined in the

usual fashion as for any $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}, \text{and} \qquad \alpha \odot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}.$$

Note: Students should be able to verify that $\mathbb{C}^{\boldsymbol{n}}$ satisfy all the axioms of a vector space.

Examples of Vector Spaces

Example 1.6.1.5: Let $M_{m,n}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \mid a_{ij} \in \mathbb{R} \right\}$ be the set of all

$\mathrm{m} \times n$ real matrices. Then $M_{m,n}(\mathbb{R})$ is a **real vector space** with respect to the two algebraic operations defined in the usual fashion as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \oplus \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix},$$

and for any $\alpha \in \mathbb{R}$, $\alpha \odot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha\, a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}.$

Note: Students should be able to verify that $M_{m,n}(\mathbb{R})$satisfy all the axioms of a vector space.

Examples of Vector Spaces

Example 1.6.1.6: Fix a positive integer n, Consider the set, $P_n(\mathbb{R})$, of all polynomials of degree $\leq n$ with coefficients from $\mathbb{R}$ in the indeterminate x.

That is; $P_n(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_0, a_1, a_2, \cdots, a_n \in \mathbb{R}\}$.

Let $f(x), g(x) \in P_n(\mathbb{R})$. Then $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, and

$g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$ for some $a_i, b_i \in \mathbb{R}$, $0 \leq i \leq n$.

It can be easily verified that $P_n(\mathbb{R})$ is a real vector space with the addition and scalar multiplication defined by:

$f(x) \oplus g(x) = (a_0 + b_0) + (a_1 + b_1)\,x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$, and

$\alpha \odot f(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \cdots + \alpha a_n x^n$, for $\alpha \in R$.

Examples of Vector Spaces

Example 1.6.1.7: Consider the set, $P(\mathbb{R})$, of all polynomials with coefficients from $\mathbb{R}$ in the indeterminate x.

That is; $P(\mathbb{R}) = \{a_0 + a_1x + \ a_2x^2 + \cdots + a_nx^n \,|n \in \mathbb{N},\ \text{and } a_0, a_1, a_2, \cdots, a_n \in \mathbb{R}\}$.

Let $f(x), g(x) \in P(\mathbb{R})$. Observe that a polynomial of the form $a_0 + a_1x + \cdots + a_mx^m$ can be written as $a_0 + a_1x + \cdots + a_mx^m + 0\,x^{m+1} + \ \cdots + 0\,x^n$ for any $n > m$.

Hence, we can assume $f(x) = a_0 + a_1x + \cdots + a_px^p$, and $g(x) = b_0 + b_1x + \cdots + b_px^p$ for some $a_i, b_i \in \mathbb{R},\ 0 \le i \le p$, for some large positive integer p.

We now define the vector addition and scalar multiplication as follows:

$f(x) \oplus g(x) = \ (a_0 + b_0) + (a_1 + b_1)\,x + (a_2 + b_2)x^2 + \cdots + \left(a_p + b_p\right)x^p$, and

$\alpha \odot f(x) = \alpha a_0 + \alpha a_1x + \alpha a_2x^2 + \cdots + \alpha a_px^p$, for $\alpha \in R$.

Note: Students should be able to verify that $P(\mathbb{R})$ satisfy all the axioms of a vector space.

Activity

1. Consider the set $S = \{(x_1, y_1): \ x_1, y_1 \in \mathbb{R}\}$ with the following non-standard operations of addition and scalar multiplication:
   $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 - 1, y_1 + y_2 - 1)$, and
   $\alpha \odot (x_1, y_1) = (\alpha x_1, \alpha y_1)$.
   Show that is a vector space with these operations. Hint: the zero vector is not (0, 0), but (1, 1).

2. Show that the sets defined in examples 1.6.1.1 through 1.6.1.7 are vector spaces.

1.6.2 Sub Space of a Vector Space

Definition: A subspace of a vector space $(V, \oplus, \odot, \mathrm{F})$ is a nonempty subset Y of V such that $(Y, \oplus, \odot, \mathrm{F})$ is also a vector space.

Lemma 1.6.2.1 (Sub Space Test): Let V be a vector space over the field F, and let $W \subseteq V$ be a subset of V . Then W is a subspace of V if and only if

1. $\mathbf{0}_{\boldsymbol{V}} \in W$ (the zero vector of V must be in W),
2. if $x, y \in W$, then $x + y \in W$ (i.e., W is closed under vector addition), and
3. if $x \in W$ and $c \in F$, then $cx \in W$ (i.e., W is closed under scalar multiplication).

Proof of this Lemma is left as an exercise.

Examples of Subspaces:

Example 1.6.2.1: Let $\mathrm{V} = (V, \oplus, \mathrm{F}, \odot)$ be a vector space. Then

a. $S = \{0_v\}$, where 0_v is the zero vector of V,

b. $S = V$

are vector subspaces of V. These are called trivial subspaces of V.

Example 1.6.2.2: The subspaces of $\mathbb{R}^2$:

1. $\mathbb{R}^2$ itself is a sub space of $\mathbb{R}^2$ (Trivial sub space).
2. Any line through origin. $X = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : a, b \in \mathbb{R},\ ax + by = 0 \right\}$ is a subspace of $\mathbb{R}^2$.
3. The zero vector alone, $0_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^2$(Trivial sub space).

Note: Students are advised to verify these examples using **Lemma 1.6.2.1**.

Examples of Subspaces:

Example 1.6.2.3: The subspaces of $\mathbb{R}^3$:

1. $\mathbb{R}^3$ itself is a sub space of $\mathbb{R}^3$ (Trivial sub space).

2. Any plane through origin, $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : a, b, c \in \mathbb{R},\ ax + by + cz = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

3. Any line through origin.
   $L = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : a, b, c \in \mathbb{R},\ x = at, y = bt, z = ct, \text{where } t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$

4. The zero vector alone, $0_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^3$(Trivial sub space).

Note: Students are advised to verify these using **Lemma 1.6.2.1**.

Intersection of Two Sub Spaces

Theorem 1.6.2.2: Let W_1, W_2 be two sub spaces of a vector space V over the field F. Then $W_1 \cap W_2$ is a sub space of V.

Proof: Suppose that W_1, W_2 be two sub spaces of a vector space V. Since $0_V \in W_1$ and $0_V \in W_2$, $0_V \in W_1 \cap W_2$, where 0_V is the zero vector of V. Hence, $W_1 \cap W_2$ is non-empty.

Let $x, y \in W_1 \cap W_2$. Then $x, y \in W_1$, and $x, y \in W_2$.

Since W_1, W_2 are subspaces of V, we have $x + y \in W_1$, and $x + y \in W_2$. Hence, $x + y \in W_1 \cap W_2$.

Let $x \in W_1 \cap W_2$ and $\alpha \in \mathrm{F}$. Then $x \in W_1$, and $x \in W_2$. Since W_1, W_2 are subspaces of V, we have $\alpha x \in W_1$, and $\alpha x \in W_2$. Hence, $\alpha x \in W_1 \cap W_2$.

Therefore, by **Lemma 1.6.2.1**, $W_1 \cap W_2$ is a subset of V.

Union of Two Subspaces

Exercise: Let W_1, W_2 be two sub spaces of a vector space V over the field F.
Is $W_1 \cup W_2$ is a subset of V? Justify your answer.

Solution: $W_1 \cup W_2$ is not necessarily a subspace of V.

For example, consider the two subspaces W_1, W_2 of $\mathbb{R}^2$, where $W_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$, and $W_2 = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : y \in \mathbb{R} \right\}$.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W_1$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W_2$, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in both W_1 and W_2.

Hence, $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$.

Activity

Determine whether or not the following subsets of the vector space $\mathbb{R}^4$ are subspaces of $\mathbb{R}^4$:

a. $\{[a \quad b \quad c \quad d]^T : a + b = c + d\}$;

b. $\{[a \quad b \quad c \quad d]^T : a + b = 1\}$;

c. $\{[a \quad b \quad c \quad d]^T : a^2 + b^2 = 0\}$;

d. $\{[a \quad b \quad c \quad d]^T : a^2 + b^2 = 1\}$.

1.6.3 Linear Combinations and Linear Span

Definition: Let V be a vector space over the field F. A **linear combination** of the vectors $u_1, u_2, \cdots, u_n$ in V is a sum of the form $a_1u_1 + a_2u_2 + \cdots + a_nu_n$, where $a_1, a_2, \cdots, a_n \in F$. We call $a_1, a_2, \cdots, a_n$ the coefficients of the linear combination.

Definition: Let V be a vector space over the field F. Let S be a non-empty subset of V. The set of all linear combinations of vectors in S is called as the **linear span** of S and is written as **Span (S)** $or \langle \boldsymbol{S} \rangle$. Using set notation, we can write

Span (S) $= \{a_1u_1 + a_2u_2 + \cdots + a_nu_n \mid n \in \mathbb{N}, u_1, \cdots, u_n \in V \;\&\; a_1, \cdots, a_n \in F\}$.

Remarks:

- If S is a finite set (say) $S = \{u_1, u_2, \cdots, u_n\}$, then the Span ($S$) is the set of all linear combinations of vectors $u_1, u_2, \cdots, u_n$. That is;

  Span (S) $=\{a_1u_1 + a_2u_2 + \cdots + a_nu_n \mid a_1, a_2, \cdots, a_n \in F\} = \langle \boldsymbol{S} \rangle$.

- A vector space V is said to be **finitely generated** or **finite dimensional** if there exists a finite set $S = \{u_1, u_2, \cdots, u_n\}$ of V such that $V = \langle S \rangle$. In this course, we are going to concentrate mainly on **finite dimensional** spaces, specially $\mathbb{R}^n$.

Linear Combinations and Linear Span

Theorem 1.6.3.1: Let $S = \{u_1, u_2, \cdots, u_n\}$ be a non-empty subset of a vector space V over the field F.

1. The set $\langle \boldsymbol{S} \rangle$ is a subspace of V which contains S.
2. If W is any subspace of V containing S, then $\langle \boldsymbol{S} \rangle \subseteq W$.

(That is; $\langle \boldsymbol{S} \rangle$ is the smallest subspace of V which contains S.)

Proof of (1): Since $0_V = 0u_1 + 0u_2 + \cdots + 0u_n$, $0_V \in \langle \boldsymbol{S} \rangle$. Hence, $\langle \boldsymbol{S} \rangle$ is non-empty.
Let $u, v \in \langle \boldsymbol{S} \rangle$ and $\alpha \in F$. Then $u = a_1u_1 + a_2u_2 + \cdots + a_nu_n$ and $v = b_1u_1 + b_2u_2 + \cdots + b_nu_n$, where $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \in F$.

Now, $u + v = (a_1+b_1)u_1 + (a_2+b_2)u_2 + \cdots + (a_n+b_n)u_n \in \langle \boldsymbol{S} \rangle$**,** and
$\alpha u = \alpha(a_1u_1 + a_2u_2 + \cdots + a_nu_n) = (\alpha a_1)u_1 + (\alpha a_2)u_2 + \cdots + (\alpha a_n)u_n \in \langle \boldsymbol{S} \rangle$.
Therefore, $\langle \boldsymbol{S} \rangle$ is a subspace of V.

Proof of (2): Suppose that W is a subspace of V and $S \subseteq W$. Let $u \in \langle S \rangle$. Then there exists $a_1, a_2, \cdots, a_n \in F$ such that $u = a_1u_1 + a_2u_2 + \cdots + a_nu_n$.
Since $u_1, u_2, \cdots, u_n \in S \subseteq W$ and W is a subspace of V, $u \in W$.
Hence, $\langle S \rangle \subseteq W$.

Example

Example 1.6.3.1:

Let $V = \mathbb{R}^{\mathbf{3}}$, and $W = \left\{\begin{bmatrix}\alpha+\beta\\ \alpha-\beta\\ \alpha\end{bmatrix} : \ \alpha,\beta \in \mathbb{R}\right\}$. Then

$$\begin{bmatrix}\alpha+\beta\\ \alpha-\beta\\ \alpha\end{bmatrix} = \begin{bmatrix}\alpha\\ \alpha\\ \alpha\end{bmatrix} + \begin{bmatrix}\beta\\ -\beta\\ 0\end{bmatrix} = \alpha\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} + \beta\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}.$$

Therefore, $W = \left\{\begin{bmatrix}\alpha+\beta\\ \alpha-\beta\\ \alpha\end{bmatrix} : \ \alpha,\beta \in \mathbb{R}\right\} = \left\{\alpha\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} + \beta\begin{bmatrix}1\\ -1\\ 0\end{bmatrix} : \ \alpha,\beta \in \mathbb{R}\right\}$ is a linear span of $u_1 = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$, and $u_2 = \begin{bmatrix}1\\ -1\\ 0\end{bmatrix}$.

That is; W is a subspace of V and $W = \langle\{u_1, u_2\}\rangle$.

Fundamental Question in Linear Algebra

Question: Given $v \in V$ and $u_1, u_2, \cdots, u_n \in V$, how does one determine whether v is a linear combination of the vectors $u_1, u_2, \cdots, u_n$?

Answer: This reduces to solving a system of linear equations.

Example: Is $\mathrm{u} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, a linear combination of $u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$?

This is equivalent to, are there any real numbers α_1, α_2 such that

$\alpha_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$?

This is equivalent to, is the following system of linear equations solvable?

$$\alpha_1 + \alpha_2 = 1$$

$$-\alpha_1 + 2\alpha_2 = -4.$$

This system is solvable and the solution is $\alpha_1 = 2$, and $\alpha_2 = -1$.

Hence u is a linear combination of u_1, u_2 and $u = 2u_1 + (-1)u_2$.

Example

Example 1.6.3.2: Write the polynomial $p(x) = 7x^2 + 4x - 3$ a linear combination of polynomials $q_1(x) = x^2, q_2(x) = (x + 1)^2$, and $q_3(x) = (x + 2)^2$ in the vector space $P(\mathbb{R})$?

Solution: We must find coefficients a, b, c such that $p(x) = aq_1(x) + bq_2(x) + cq_3(x)$, or equivalently,

$$
\begin{aligned}
7x^2 + 4x - 3 &= ax^2 + b(x^2 + 2x + 1) + c(x^2 + 4x + 4) \\
&= (a + b + c)x^2 + (2b + 4c)x + (b + 4c).
\end{aligned}
$$

Since two polynomials are equal if and only if each corresponding coefficient is equal, this yields a system of three equations in three variables

$$
\begin{aligned}
a + \ b + \ c &= 7 \\
2b + 4\mathrm{c} &= 4 \\
b + 4c &= -3.
\end{aligned}
$$

We can easily solve this system of equations and find that the unique solution is $a = {}^5/_2\,, b = 7$, and $c = {}^{-5}/_2\,$.

Therefore, $p(x) = \frac{5}{2}\, q_1(x) + 7\, q_2(x) - \frac{5}{2}\, q_3(x)$.

Exercises

- Are vectors $\mathrm{u} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$, $\mathrm{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathrm{w} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$ a linear combination of $u_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix},\ and\ u_4 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$?

- Describe the span of the vectors $\mathrm{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\mathrm{w} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$.

- Let $\mathrm{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathrm{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $\mathrm{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Show that Span$\{u, v, w\}$ = Span$\{u, v\}$.

- Suppose $\{u_1, u_2, \cdots, u_n\}$ is a set of vectors in a vector space V. Show that the zero vector of V is in the Span $\{u_1, u_2, \cdots, u_n\}$.

Redundant vectors

Definition: Let $u_1, u_2, \cdots, u_n$ be a finite sequence of n vectors in a vector space V. We say that the vector u_j $(1 \leq j \leq n)$ is **redundant** if it can be written as a linear combination of earlier vectors in the sequence. That is; if $u_j = a_1u_1 + a_2u_2 + \cdots + a_{j-1}u_{j-1}$ for some scalars $a_1, a_2, \cdots, a_{j-1}$.

Example 1.6.3.3: Find the redundant vectors in the following sequence of vectors in $\mathbb{R}^4$ and write each redundant vector as a linear combination of previous non-redundant vectors:

$$u_1 = \begin{bmatrix}1\\2\\2\\3\end{bmatrix}, u_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, u_3 = \begin{bmatrix}2\\3\\3\\4\end{bmatrix}, u_4 = \begin{bmatrix}0\\1\\2\\3\end{bmatrix} and\ \ u_5 = \begin{bmatrix}3\\3\\2\\2\end{bmatrix}.$$

The casting-out algorithm

Finding the redundant vectors in a sequence of large number of vectors is a very difficult job. The casting-out algorithm is a systematic way to find the redundant vectors among the vectors and write each redundant vector as a linear combination of previous non-redundant vectors.

Suppose that $u_1, u_2, \cdots, u_k$ is a given list of vectors in $\mathbb{R}^n$.

Find set of indices j such that u_j is redundant, and a set of coefficients for writing each redundant vector as a linear combination of previous vectors.

Algorithm:

1. Write the vectors $u_1, u_2, \cdots, u_k$ as the columns of an $n \times k$ – matrix A.
2. Using elementary row operations, reduce matrix A to its reduced echelon form.
3. Every non-pivot column, if any, corresponds to a redundant vector.
4. If u_j is a redundant vector, then the entries in the j^{th} column of the reduced echelon form are coefficients for writing u_j as a linear combination of previous non-redundant vectors.

Application of the casting-out algorithm

Solution of Example 1.6.3.3:

- Reduce the matrix A whose columns are $u_1, u_2, u_3, u_4,$ and u_5 to reduced echelon form using elementary row operations:

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 2 & 1 & 3 & 1 & 3 \\ 2 & 1 & 3 & 2 & 2 \\ 3 & 1 & 4 & 3 & 2 \end{bmatrix} \xrightarrow[\text{opearations}]{\text{elementary row}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- The non-pivot columns are columns 3, and 5, and therefore, the vectors u_3, and u_5 are redundant. The non-redundant vectors are u_1, u_2, and u_4.

- The entries in the fifth column of reduced echelon form are 1, 2, and −1. Note that this means that the fifth column of A can be written as 1 times the first column of A plus 2 times the second column of A plus (−1) times the fourth column. The same coefficients can be used to write u_5 as a linear combination of previous non-redundant columns, namely $u_5 = u_1 + 2u_2 - u_4$.

- By considering 1st , 2nd , and 3rd columns, one can easily verify that $u_3 = u_1 + u_2$.

Fundamental Subspaces Associated with a Matrix

Row Space, Column Space, and Null Space of a Matrix:

Definition: Let $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ be an $m \times n$ real matrix. Let $r_1, r_2, \cdots, r_m$ be the row vectors of A, and $c_1, c_2, \cdots, c_n$ be the column vectors of A. Then

- The **row space** of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$ spanned by the row vectors of A and is denoted by $Row\ (A)$.

$$Row(A) = \langle \{r_1, r_2, \cdots, r_m\} \rangle.$$

- The **column space** of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^m$ spanned by the column vectors of A and is denoted by $\mathrm{C}ol(A)$.

$$\mathrm{C}ol(A) = \langle \{c_1, c_2, \cdots, c_n\} \rangle.$$

- The Null space (or Kernel) of $\boldsymbol{A}$ consists of all vectors $x \in \mathbb{R}^n$ such that $Ax = \mathbf{0}$. It is denoted by $Null(A)$.

$$\mathrm{N}ull(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\} \subseteq \mathbb{R}^n.$$

Proposition 1.6.3.2: The Null space ($Null(A)$) is a subspace of $\mathbb{R}^n$.

Hint: Use Lemma 1.6.2.1 (Sub Space Test) to prove this.

1.6.5 Linear Dependence, and Linear Independence

Definition: Let $S = \{u_1, u_2, \cdots, u_n\}$ be a finite set of n vectors in a vector space V. We say that S is **linearly independent** if the equation $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0_V$ has only the trivial solution $a_i = 0$ for all $i \in \{1, 2, \cdots, n\}$.
An infinite set S of vectors is called linearly independent if every finite subset of S is linearly independent.
A set S of vectors is called **linearly dependent** if it is not linearly independent.

Note: From the definition it is clear that $S = \{u_1, u_2, \cdots, u_n\}$ is linearly dependent if there exists scalers $a_1, a_2, \cdots, a_n$ not all are zero such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0_V$.

Example

Example 1.6.5.1: Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

$$u_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, u_2 = \begin{bmatrix}1\\3\\1\end{bmatrix}, and \;\; u_3 = \begin{bmatrix}0\\4\\2\end{bmatrix}.$$

Solution: we must check whether the equation $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \mathbf{0}_3$ has a non-trivial solution. By doing the elementary row operations to the augmented matrix of this system, we get

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 0 \\ 1 & 3 & 4 & \vdots & 0 \\ 0 & 1 & 2 & \vdots & 0 \end{bmatrix} \xrightarrow[\text{operations}]{\text{elementary row}} \begin{bmatrix} 1 & 1 & 0 & \vdots & 0 \\ 0 & 2 & 4 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Since column 3 is not a pivot column, α_3 is a free variable. Therefore, the system has a non-trivial solution, and the vectors are **linearly dependent**.

By setting $\alpha_3 = 1$, and back substitution, we get $(2, -2, 1)$ is a solution. In other words, $2u_1 - 2u_2 + u_3 = \mathbf{0}_3$ or equivalently, $u_3 = -2u_1 + 2u_2$.

Example

Example 1.6.5.2: Determine whether the following vectors are linearly independent in $\mathbb{R}^4$:

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, and \;\; u_4 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix}.$$

Solution: We must check whether the equation

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} + \alpha_4 \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \equiv \qquad \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

has a non-trivial solution. If it does, the vectors are **linearly dependent**. On the other hand, if there is only the trivial solution, the vectors are linearly **independent**.

Solution of Example 1.6.5.2

By doing the elementary row operations to the augmented matrix of this system, we get

$$\begin{bmatrix} 1 & 0 & 1 & 2 & \vdots & 0 \\ 1 & 1 & 2 & 3 & \vdots & 0 \\ 2 & 1 & 3 & 3 & \vdots & 0 \\ 0 & 1 & 2 & 1 & \vdots & 0 \end{bmatrix} \xRightarrow[\text{opearations}]{\text{elementary row}} \begin{bmatrix} 1 & 0 & 1 & 2 & \vdots & 0 \\ 0 & 1 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 2 & \vdots & 0 \end{bmatrix}.$$

Since every column is a pivot column, there are no free variables; the system of equations has a unique solution, which is $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, i.e., the trivial solution. Therefore, the vectors u_1, u_2, u_3, u_4 are **linearly independent**.

Properties of Linear Independence

Theorem 1.6.5.1: Let V be a vector space over the field $\mathbb{F}$.

1. If a sequence $u_1, u_2, \cdots, u_n$ of n vectors in any V is linearly independent, then so is any reordering of the sequence.
2. For any $v \in V$ the set $\{0_V, v\}$ be is linearly dependent.
3. Let y be any non-zero vector in V. Then $\{y\}$ is linearly independent.
4. Let Y be any linearly dependent subset of V. If W is a subset of V containing Y, then W is linearly dependent.
5. Let X, Y be two subsets of V such that $X \subseteq Y$. If Y is linearly independent, then X is linearly independent.
6. Assume $u_1, u_2, \cdots, u_n$ are linearly independent vectors in V. Then every vector $v \in Span\,\{u_1, u_2, \cdots, u_n\}$ can be written as a linear combination of $u_1, u_2, \cdots, u_n$ in a unique way.

Linear dependence and redundant vectors

Theorem 1.6.5.2: Let V be a vector space, and let $u_1, u_2, \cdots, u_n$ be a finite sequence of vectors in V. If $u_1, u_2, \cdots, u_n$ are linearly dependent, then at least one of the vectors can be written as a linear combination of earlier vectors in the sequence: $u_j = a_1 u_1 + a_2 u_2 + \cdots + a_{j-1} u_{j-1}$ for some scalars $a_1, a_2, \cdots, a_{j-1}$ for some j.

Proof: Suppose that the vectors $u_1, u_2, \cdots, u_n$ are linearly dependent. Then the equation $b_1 u_1 + b_2 u_2 + \cdots + b_n u_n$ has a non-trivial solution for some k.

In other words, there exist scalars $b_1, b_2, \cdots, b_n$, not all equal to zero, such that $b_1 u_1 + b_2 u_2 + \cdots + b_n u_n = 0_V$. Let j be the largest index such that $b_j \neq 0$.

Then $b_1 u_1 + b_2 u_2 + \cdots + b_j u_j = 0_V$.

Dividing by b_j and solving for u_j, we have

$$u_j = -\frac{b_1}{b_j}\, u_1 - \frac{b_2}{b_j}\, u_2 - \cdots - \frac{b_{j-1}}{b_j}\, u_{j-1}.$$

Hence, u_j can be written as a linear combination of earlier vectors as claimed.

Removing redundant vectors

Theorem 1.6.5.3: Let $u_1, u_2, \cdots, u_n$ be a sequence of vectors, and suppose that $u_{j_1}, u_{j_2}, \cdots, u_{j_m}$ is the subsequence of vectors that is obtained by removing all of the redundant vectors. Then $u_{j_1}, u_{j_2}, \cdots, u_{j_m}$ are linearly independent and

$$Span\left\{u_{j_1}, u_{j_2}, \cdots, u_{j_m}\right\} = Span\,\{u_1, u_2, \cdots, u_n\}.$$

Proof: Remove the redundant vectors one by one, from right to left. Each time a redundant vector is removed, the span does not change. Moreover, the resulting sequence of vectors $u_{j_1}, u_{j_2}, \cdots, u_{j_m}$ is linearly independent, because if any of these vectors were a linear combination of earlier ones, then it would have been redundant in the original sequence of vectors, and would have therefore been removed.

Example

Example 1.6.5.3: Find a subset of $\{u_1, u_2, u_3, u_4\}$ in $\mathbb{R}^4$ that is linearly independent and has the same span as $\{u_1, u_2, u_3, u_4\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -6 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, and \;\; u_4 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 5 \end{bmatrix}.$$

Solution: By doing the elementary row operations to the matrix formed by the column vectors of u_1, u_2, u_3, u_4, we get

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ -2 & 4 & 2 & 2 \\ 3 & -6 & 1 & 5 \end{bmatrix} \xrightarrow[\text{opearations}]{\text{elementary row}} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the redundant vectors are u_2 and u_4. We remove them and are left with u_1 and u_3. Therefore, by Theorem 1.6.5.3 $\{u_1, u_3\}$ is linearly independent and $Span\{u_1, u_3\} = Span\{u_1, u_2, u_3, u_4\}$.

Adding a vector to a linearly independent set

Theorem 1.6.5.4: Let V be a vector space and let $\{u_1, u_2, \cdots, u_n\}$ be a linearly independent set of vectors in V. Suppose that $v \notin Span\ (\{u_1, u_2, \cdots, u_n\})$. Then $\{u_1, u_2, \cdots, u_n, v\}$ is also a linearly independent set.

Proof: Suppose that the set $\{u_1, u_2, \cdots, u_n, v\}$ is linearly dependent. Then by Theorem 1.6.5.2, one of the vectors can be written as a linear combination of earlier vectors. This vector cannot be one of the u_i because $u_1, u_2, \cdots, u_n$ are linearly independent. It also cannot be v, because $v \notin Span\ (\{u_1, u_2, \cdots, u_n\})$.

Therefore, our assumption cannot be true, and the set $\{u_1, u_2, \cdots, u_n, v\}$ is linearly independent.

Activity

Use the method of Theorem 1.6.5.3 to determine whether the following vectors in $\mathbb{R}^3$ are linearly independent. If they are linearly dependent, find a non-trivial linear combination of the vectors that is equal to 0.

$$u_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, u_3 = \begin{bmatrix} -3 \\ -4 \\ -2 \end{bmatrix} and \ u_4 = \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}.$$

1.6.4 Basis and Dimension of a Vector Space

Definition: A non-empty subset $\mathcal{B}$ of a vector space V over the field $\mathcal{F}$ is called a basis of V if

a. $\mathcal{B}$ is a linearly independent set, and
b. Linear span of $\mathcal{B}$ is V, , i.e., every vector in V can be expressed as a linear combination of the elements of $\mathcal{B}$.

A vector in $\mathcal{B}$ is called a basis vector.

Remark: By convention, the linear span of an empty set is $\{0\}$. Hence, the empty set is a basis of the vector space $\{0\}$.

Theorem 1.6.4.1: Every vector space has a basis.

Proof of this theorem is beyond the scope of this course.

However, we can show that every subspace of $\mathbb{R}^n$ has a basis (see theorem 1.6.6.3).

Example for Basis:

Example 1.6.4.1: A basis for the **Euclidean space** $\mathbb{R}^{\mathbf{3}} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} \mid x_1, x_2, x_{\mathbf{3}} \in \mathbb{R} \right\}$ is $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This is known as the **standard basis** for $\mathbb{R}^{\mathbf{3}}$.

Clearly $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_{\mathbf{3}}\}$ is a linearly independent set in $\mathbb{R}^{\mathbf{3}}$. Hence, to show $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_{\mathbf{3}}\}$ is a basis for $\mathbb{R}^{\mathbf{3}}$, it is enough to show that any $x = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$ can be expressed as a linear combination of $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_{\mathbf{3}}$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_{\mathbf{3}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_{\mathbf{2}} + x_3 \boldsymbol{e}_{\mathbf{3}}.$$

Exercise: Check that the vectors $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^{\mathbf{3}}$.

We have to show that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_{\mathbf{3}}\}$ is linearly independent and $\mathbb{R}^{\mathbf{3}} = Span\{u_1, u_2, u_3\}$.
This is one of a non-standard basis for $\mathbb{R}^{\mathbf{3}}$.

Example for Basis:

Example 1.6.4.2: Consider $M_{2,3}(\mathbb{R}) = \left\{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \mid a_{ij} \in \mathbb{R}\right\}$ be the set of all

2×3 real matrices. This forms a real vector space with respect to the matrix addition and scalar multiplication. Find a basis for it.

Solution:

One can easily show that $e_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $e_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $e_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

$e_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $e_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and $e_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ forms a basis for $M_{2,3}(\mathbb{R})$

by showing that $\{e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}\}$ is linearly independent and $M_{2,3}(\mathbb{R})$ is the span of $\{e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}\}$.

This is known as the standard basis for $M_{2,3}(\mathbb{R})$.

Example for Basis:

Example 1.6.4.3: Consider the vector space $P_2(\mathbb{R})$ of polynomials of degree at most 2 with coefficients in a field $\mathbb{R}$. Then the following sets form a basis for $P_2(\mathbb{R})$.

1. $\{1, x, x^2\}$
2. $\{x^2, (x+1)^2, (x+2)^2\}$
3. $\{1, x-1, (x-1)^2\}$.

Solution: It is easy to verify that each set of vectors is linearly independent and spanning $P_2(\mathbb{R})$.

Note: Unlike $\mathbb{R}^n$, a vector space like $P_2(\mathbb{R})$ does not necessarily have a "standard" basis. One basis might be useful for one application, and another basis for a different application.

Example for Basis:

Example 1.6.4.4: Consider the vector space $P(\mathbb{R})$, the set of all polynomials with coefficients from $\mathbb{R}$ in the indeterminate x with respect to the usual polynomial addition and scalar multiplication. Find a basis for it.

Find a basis for it.

Solution: Let $\mathcal{B} = \{p_0, p_1, \cdots, p_n, \cdots\}$, where $p_n = x^n, \forall i = 0, 1, 2, \cdots$.

1. $\mathcal{B}$ is a linearly independent set in $P(\mathbb{R})$. (Every finite subset of $\mathcal{B}$ is a linearly independent)
2. Let p $\in P(\mathbb{R})$. Then there exists $k \in \mathbb{N}$ such that $p = a_0 + a_1\, x + \cdots + a_k\, x^k$, where $a_0, a_2, \cdots, a_k \in \mathbb{R}$. That is $p = a_0 p_0 + a_1\, p_1 + \cdots + a_k p_k$. Hence p is in the linear span of $\mathcal{B}$.

Therefore, $\mathcal{B}$ is a basis for $P(\mathbb{R})$.

Remark: This basis has infinite number of vectors as the degree of the polynomial can be any positive integer.

Criterion for Basis

Theorem 1.6.4.2: Let V be a vector space over the field F and let $B = \{u_1, u_2, \cdots, u_n\} \subseteq V$. B is a basis for V if and only if every $v \in V$ can be written uniquely in the form $v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$, where $a_1, a_2, \cdots, a_n \in \mathrm{F}$.

Proof: Suppose that $B = \{u_1, u_2, \cdots, u_n\}$ is a basis of V. Let $v \in V$. Since $V = Span\ (B)$,

$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$, where $a_1, a_2, \cdots, a_n \in \mathrm{F}$.

Suppose that v also can be written as $v = b_1 u_1 + b_2 u_2 + \cdots + b_n u_n$ for some scalars $b_1, b_2, \cdots, b_n$. Then

$(a_1 - b_1) u_1 + (a_2 - b_2) u_2 + \cdots + (a_\mathrm{n} - b_n) u_n = 0_V$, where 0_V is the zero vector of V.

This implies that $a_\mathrm{i} = b_i$ for every $i \in \{1,2, \cdots, n\}$ because $\{u_1, u_2, \cdots, u_n\}$ is linearly independent.

Conversely, suppose that every $v \in V$ can be written uniquely in the form $v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$, where $a_1, a_2, \cdots, a_n \in \mathrm{F}$. From this it is clear that $u_1, u_2, \cdots, u_n$ span V. To show that $B = \{u_1, u_2, \cdots, u_n\}$ is linearly independent, suppose that $a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0_V$, for some scalars $a_1, a_2, \cdots, a_n$.

Since $a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0_V = 0u_1 + 0u_2 + \cdots + 0u_n$, our assumption implies that $a_\mathrm{i} = 0$ for every $i \in \{1,2, \cdots, n\}$. Thus $B = \{u_1, u_2, \cdots, u_n\}$ is linearly independent and hence is a basis for V.

Exchange Lemma

Lemma 1.6.4.3: Suppose $u_1, u_2, \cdots, u_r$ are linearly independent vectors of a subspace spanned by $\{v_1, v_2, \cdots, v_s\}$. Then $r \leq s$.

Proof: Since each $u_j \in Span\{v_1, v_2, \cdots, v_s\}$, there exist scalars a_{ij} $(1 \leq i \leq s)$ such that $u_j = a_{1\mathrm{j}}v_1 + \alpha_{2\mathrm{j}}v_2 + \cdots + a_{sj}v_s$ for $1 \leq j \leq r$.

Let $A = [a_{ij}]$ $(1 \leq i \leq s$, $1 \leq j \leq r)$.

Now suppose that $r > s$. Then the system of linear equations given by $Ax = \mathbf{0}$ has a non-trivial solution x. That is; there exists $x \neq 0$ such that $Ax = \mathbf{0}$. In other words, $a_{\mathrm{i}1}x_1 + \alpha_{\mathrm{i}2}x_2 + \cdots + a_{ir}x_r = 0$ for all $i = 1, 2, \cdots, s$.

Therefore,

$$\begin{aligned}
x_1u_1 + \cdots + x_ru_r &= x_1(a_{11}v_1 + \cdots + a_{s1}v_s) + \cdots + x_r(a_{1r}v_1 + \cdots + a_{sr}v_s) \\
&= (a_{11}x_1 + \cdots + a_{1r}x_r)v_1 + \cdots + (a_{\mathrm{s}1}x_1 + \cdots + a_{sr}x_r)v_s \\
&= 0\,v_1 + \cdots + 0\,v_s \\
&= \mathbf{0}.
\end{aligned}$$

This contradicts the assumption that $u_1, u_2, \cdots, u_r$ are linearly independent. Since we assumed $r > s$ and obtained a contradiction, it follows that $r \leq s$.

Dimension of a Vector Space

Theorem 1.6.4.5: Let V be a vector space over the field F and let B_1 and B_2 be bases of V. Then either B_1 and B_2 are both finite and have the same number of elements, or else B_1 and B_2 are both infinite.

Proof: We first show that B_1 and B_2 are either both finite or both infinite. Assume one of them, say B_1, is finite and contains s vectors. Since B_1 is spanning and B_2 is linearly independent, it follows from the Exchange Lemma 1.6.4.3 that B_2 cannot contain more than s vectors, and in particular, B_2 must be finite. So the sets are either both finite or both infinite.

If they are both finite, say B_1 has s elements and B_2 has r elements, then by the Exchange Lemma 1.6.4.3, we have $s \leq r$ and $r \leq s$, hence $r = s$.

Definition: Let V be a vector space over the field F. If V has a basis consisting of n vectors, we say that V has dimension n, and we write $\dim(V) = n$. In this case we also say that V is a **finite-dimensional vector space**.

If V has an infinite basis, we say that V is infinite-dimensional, and we write $\dim(V) = \infty$.

Examples

1. What is the dimension of $\mathbb{R}^{\mathbf{3}}$? The standard basis of $\mathbb{R}^{\mathbf{3}}$ is $\{e_1, e_2, e_3\}$, where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, $\dim\left(\mathbb{R}^{\mathbf{3}}\right) = 3$.
2. The vector space $P_2(\mathbb{R})$ of polynomials of degree at most 2 with coefficients in a field $\mathbb{R}$ has a basis $\{1, x, x^2\}$. Hence, $\dim\left(P_2(\mathbb{R})\right) = 3$.
3. The vector space $M_{m,n}(\mathbb{R})$, the set of all $\mathrm{m} \times n$ real matrices, has dimension mn. A possible basis consists of all the matrices that contain a single 1 and zeros everywhere else.
4. $\dim(\mathbb{R}^{\boldsymbol{n}}) = n$.
5. $\dim(\mathbb{C}^{\boldsymbol{n}}) = n$.
6. Since a basis for the vector space $P(\mathbb{R})$, the set of all polynomials with coefficients from $\mathbb{R}$, is infinite, $P(\mathbb{R})$ is infinite-dimensional, and $\dim(W) = \infty$.

1.6.6 Finite Dimensional vector spaces and Its Basis

Consequence of the Exchange Lemma 1.6.4.3:

Corollary 1.6.6.1(Size of a linearly independent or spanning set of vectors):

Let V be a finite dimensional vector space with $\dim(V) = n$. Then

a. Every linearly independent set of vectors in W has at most n vectors.

b. Every spanning set of vectors in W has at least n vectors.

Proof: Since $\dim(V) = n$, it has some basis consisting of n vectors $v_1, v_2, \cdots, v_n$.

a. Suppose $u_1, u_2, \cdots, u_m$ are linearly independent vectors in V. Since $u_1, u_2, \cdots, u_m$ are linearly independent in V and $v_1, v_2, \cdots, v_n$ are spanning V, the Exchange Lemma 1.6.4.3 implies that $m \leq n$.

b. Suppose the vectors $u_1, u_2, \cdots, u_s$ span V. Since $v_1, v_2, \cdots, v_n$ are linearly independent and $u_1, u_2, \cdots, u_s$ are spanning W, the Exchange Lemma 1.6.4.3 implies that $n \leq s$.

Basis test for n vectors in n-dimensional space

Theorem 1.6.6.2: Let W be a n −dimensional subspace of a vector space V. Consider n number of vectors $v_1, v_2, \cdots, v_n$ in W.

a. If $v_1, v_2, \cdots, v_n$ are linearly independent, then they form a basis for W.

b. If $v_1, v_2, \cdots, v_n$ span W, then they form a basis for W.

Proof: Since $\dim(W) = n$, it has some basis consisting of n vectors $v_1, v_2, \cdots, v_n$.

a. Suppose $u_1, u_2, \cdots, u_m$ are linearly independent vectors in W. Since $u_1, u_2, \cdots, u_m$ are linearly independent in W and $v_1, v_2, \cdots, v_n$ are spanning W, the Exchange Lemma 1.6.4.3 implies that $m \leq n$.

b. Suppose the vectors $u_1, u_2, \cdots, u_s$ span W. Since $v_1, v_2, \cdots, v_n$ are linearly independent and $u_1, u_2, \cdots, u_s$ are spanning W, the Exchange Lemma 1.6.4.3 implies that $n \leq s$.

Existence of basis for subspaces of $\mathbb{R}^{\boldsymbol{n}}$

Theorem 1.6.6.3: Let V be a subspace of $\mathbb{R}^{\boldsymbol{n}}$. Then there exist linearly independent vectors $S = \{u_1, u_2, \cdots, u_k\}$ in V such that $V = Span(S)$.

Proof:

1. If $V = \{0\}$, then V is the empty span, and we are done.
2. Otherwise, there exists a non-zero vector u_1 in V. If $V = Span(\{u_1\})$, we are done and $\{u_1\}$ is a base for V.
3. Otherwise, there exists $u_2 \in V$ such that $u_2 \notin Span(\{u_1\})$.
   If $V = Span(\{u_1, u_2\})$, we are done and $\{u_1, u_2\}$ is a base for V.

Continue in this way. Note that after the j^{th} step of this process, the vectors $u_1, u_2, \cdots, u_j$ are linearly independent. This is because, by construction, no vector is in the span of the previous vectors, and therefore no vector is redundant.

Since there can be at most n linearly independent vectors in $\mathbb{R}^{\boldsymbol{n}}$, the process must stop after k steps for some $k \leq n$.

But then $V = Span(\{u_1, u_2, \cdots, u_k\})$, as desired and $\{u_1, u_2, \cdots, u_k\}$ is a base for V.

Example

Example 1.6.6.1: Let W be sub space of $\mathbb{R}^3$ given by $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x - y + 2z = 0 \right\}$.

Find the $\dim(W)$ and a basis for W.

Solution: W is a subspace (a plane passing through the origin) given by the equation $x - y + 2z = 0$. This can be considered as a system of linear homogenous equations (one equation with three unknowns).

By taking $y = t$ and $z = s$ as the free variables and solve for $x = y - 2z = t - 2s$, an arbitrary element of W is in the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t - 2s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \text{for } t, s \in \mathbb{R}.$$

Hence, $W = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Since the two spanning vectors are linearly independent, they form a basis of W, and thus $\dim(W) = 2$.

Extension of a Linearly independent set to a basis

Theorem 1.6.6.4: Let V be a finite dimensional vector space V, and let $u_1, u_2, \cdots, u_r$ be linearly independent vectors in V. Then it is possible to extend $\{u_1, u_2, \cdots, u_r\}$ to a basis of W. In other words, there exist zero or more vectors $\{w_1, w_2, \cdots, w_s\}$ such that $\{u_1, u_2, \cdots, u_r, w_1, w_2, \cdots, w_s\}$ is a basis of W.

Proof: Let $\mathrm{B} = \{v_1, v_2, \cdots, v_k\}$ be a basis of V. Consider the sequence of $r + k$ vectors $u_1, u_2, \cdots, u_r, w_1, v_1, v_2, \cdots, v_k$.

Since W is spanned by the vectors $v_1, v_2, \cdots, v_k$, it is certainly also spanned by the larger set of vectors $u_1, u_2, \cdots, u_r, w_1, v_1, v_2, \cdots, v_k$.

We know that we can obtain a basis of W by removing the redundant vectors from $u_1, u_2, \cdots, u_r, w_1, v_1, v_2, \cdots, v_k$.

On the other hand, $u_1, u_2, \cdots, u_r$ are linearly independent, so none of them can be redundant. It follows that the resulting basis of V contains all of the vectors $u_1, u_2, \cdots, u_r$.

In other words, we have found a basis of W that is an extension of $u_1, u_2, \cdots, u_r$, which is what had to be shown.

Example

Example 1.6.6.2: Extend $\left\{u_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}\right\}$ to a basis of $\mathbb{R}^{\mathbf{4}}$.

Solution: Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_{\mathbf{3}}, \boldsymbol{e}_{\mathbf{4}}\}$ be e the standard basis of $\mathbb{R}^{\mathbf{4}}$. We obtain the desired basis by applying the casting out algorithm to $u_1, u_2, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_{\mathbf{3}}, \boldsymbol{e}_{\mathbf{4}}$:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} \xRightarrow[\text{opearations}]{\text{elementary row}} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

Therefore, we cast out the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_{\mathbf{4}}$ and keep the rest. The resulting basis is

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{e}_{\mathbf{2}}, \boldsymbol{e}_{\mathbf{3}}\} = \left\{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Spanning set can be shrunk to a basis

Theorem 1.6.6.5: Let V be a finite dimensional vector space V, and let $S = \{u_1, u_2, \cdots, u_r\}$ be a spanning set of V. Then S can be shrunk to a basis. That is; there exists a basis B of V such that $B \subseteq S$.

Proof: This is merely a restatement of Theorem 1.6.5.3. We obtain the linearly independent subset of S by removing the redundant vectors from S, which can be achieved by the casting-out algorithm.

Example

Example 1.6.6.3: Find a subset of $\{u_1, u_2, u_3, u_4\}$ of $\mathbb{R}^4$ that is linearly independent and has the same span as $\{u_1, u_2, u_3, u_4\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -6 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, u_4 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 5 \end{bmatrix}.$$

Solution: Reduce the matrix A whose columns are $u_1, u_2, u_3,$ and u_4 to reduced echelon form using elementary row operations (casting-out algorithm):

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ -2 & 4 & 2 & 2 \\ 3 & -6 & 1 & 5 \end{bmatrix} \xRightarrow[\text{opearations}]{\text{elementary row}} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the redundant vectors are u_2 and u_4. We remove them and are left with u_1 and u_3. Therefore, by Theorem 1.6.5.3, $\{u_1, u_3\}$ is linearly independent and $\langle\{u_1, u_3\}\rangle = \langle\{u_1, u_2, u_3, u_4\}\rangle$.

Hence, $\dim(\langle\{u_1, u_2, u_3, u_4\}\rangle) = 2$.

Activity

- Use one of the basis tests of Theorem 1.6.6.2 to determine whether the vectors

$$u_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \textit{and} \;\; u_4 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

forms a basis for $\mathbb{R}^3$.

- Find a basis and determine the dimension for the following subspace of $\mathbb{R}^4$:

$$W = \left\{ \begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} \in \mathbb{R}^4 : u + v = w + x \text{ and } u + w = v + x \right\}.$$

- Shrink $\{u_1, u_2, u_3, u_4\}$ to a basis of $\mathbb{R}^3$ by removing redundant vectors, where

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \textit{and} \;\; u_4 = \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}.$$