2: Basic Algebra

EN1106 - Introductory Mathematics

Level I - Semester 1

Algebraic Terminology

Algebraic Notations

- Algebra is the study of symbols and the rules that govern the manipulation of the symbols.
  - Constants: Symbols that take fixed or unchanging values
  - Variables: Symbols that represent quantities that can vary
- Symbols are usually English or Greek and are case-sensitive
- Position of a symbol in relation to other symbols is important
  - e.g. xy, x^y, y_x

Algebraic Notations

- Symbols are usually English or Greek and are case-sensitive

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Y	υ	upsilon
E	ε	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Algebraic Notations

- Position of a symbol in relation to other symbols is important
  - e.g. xy, x^y, y_x

- When a symbol is placed to the right and slightly higher than another symbol it is referred to as a superscript. (x^y)
- If a symbol is placed to the right and slightly lower than another symbol it is called a subscript (y_x)

Algebraic Expression

- A quantity made up of symbols together with +, −, ×, or ÷ is called an algebraic expression.
  - Addition $x + y$
  - Subtraction $x - y$
  - Multiplication $x \times y$
  - Division $x \div y$

Powers or Indices

- To abbreviate quantity $a \times a \times a$, the notation a^3, pronounced ‘a cubed’, is used.
  - The superscript “3” is called a power or index and the letter “a” is called the base.
- Similarly $a \times a$ is written a^2, pronounced ‘a squared’.
- When the BODMAS rule is applied, powers should be given higher priority than any other operation

Exercises

5. Write the following expressions compactly using indices:
   (a) $xxxyyx$ (b) $xxyyzzz$
   (c) $xyzxyz$ (d) $abccba$

9. Without using a calculator find
   (a) $(-6)^2$ (b) $(-3)^2$ (c) $(-4)^3$
   (d) $(-2)^3$

6. Using a calculator, evaluate
   (a) 7^4 (b) 7^5 (c) $7^4 \times 7^5$ (d) 7^9
   (e) 8^3 (f) 8^7 (g) $8^3 \times 8^7$ (h) 8^{10}

Substitution and Formulae

- Substitution - replacing letters by actual numerical values.
  - e.g. Find the value of a^3 when $a = 3$.

    $a^3 = a \times a \times a$. Substitute the number 3 in place of the letter a as $3 \times 3 \times 3 = 27$.
- A formula is used to relate two or more quantities
  - e.g. Use the formula $A = pq$ to find A when $p = 5$ and $q = 2$.

    $A = p \times q = 5 \times 2 = 10$.

Exercises

Evaluate $3x^2y$ when $x = 2$ and $y = 5$.

Use the formula $y = \frac{x^3}{2} + 3x^2$ to find y when

(a) $x = 0$ (b) $x = 2$ (c) $x = 3$

(d) $x = -1$

Addition and Subtraction of Like Terms

- Like terms are multiples of the same quantity.
  - E.g. y, $7y$ and $0.5y$ are all multiples of y and so are like terms.

- Like terms can be collected together and added or subtracted in order to simplify them.
  - E.g. $y + 7y + 0.5y = 8.5y$

    $3x + 7x - x^2 = 10x - x^2$

Exercises

1. Simplify, if possible,

   (a) $5p - 10p + 11q + 8q$ (b) $-7r - 13s + 2r + z$ (c) $181z + 13r - 2$

   (d) $x^2 + 3y^2 - 2y + 7x^2$ (e) $4x^2 - 3x + 2x + 9$

2. Simplify

   (a) $5y + 8p - 17y + 9q$ (b) $7x^2 - 11x^3 + 14x^2 + y^3$ (c) $4xy + 3xy + y^2$

   (d) $xy + yx$ (e) $xy - yx$

Expansion and Factorization of Algebraic Expressions

Multiplying Algebraic Expressions

- When multiplying three or more numbers together the order in which we carry out the multiplication is also irrelevant.
- E.g. (1) Simplify $(2x)(5x)$

  $(2x)(5x)$ could be written as $(2 \times x) \times (5 \times x)$, and

  then as $(2 \times 5) \times (x \times x)$, which evaluates to $10x^2$.

- E.g. (2) Simplify $(a)\times(-2b)$.

  $(a)\times(-2b) = (-2ab)$

Exercises

1. Simplify each of the following:
   (a) (4)(3)(7) (b) (7)(4)(3) (c) (3)(4)(7)

2. Simplify
   (a) $5 \times (4 \times 2)$ (b) $(5 \times 4) \times 2$

3. Simplify each of the following:
   (a) $7(2z)$ (b) $15(2y)$ (c) $(2)(3)x$
   (d) $9(3a)$ (e) $(11)(5a)$ (f) $2(3x)$

4. Simplify each of the following:
   (a) $5(4x^2)$ (b) $3(2y^3)$ (c) $11(2u^2)$
   (d) $(2 \times 4) \times u^2$ (e) $(13)(2z^2)$

7. Simplify each of the following:
   (a) $(abc)(a^2bc)$ (b) $x^2y(xy)$
   (c) $(xy^2)(xy^2)$

8. Explain the distinction, if any, between $(xy^2)(xy^2)$ and xy^2xy^2.

9. Explain the distinction, if any, between $(xy^2)(xy^2)$ and $(xy^2) + (xy^2)$.
   In both cases simplify the expressions.

10. Simplify
    (a) $(3z)(-7z)$ (b) $3z - 7z$

Removing Brackets

- Expression (5 - 4) + 7 is different from 5 - (4 + 7) because of the position of the brackets. In order to simplify an expression it is often necessary to remove brackets.
- e.g. Remove the brackets from a(b + c)

  ab + ac

- e.g. Remove the brackets from a(b - c)

  ab - ac

Removing Brackets

- E.g. Remove the brackets from $(a + b)(c + d)$

  $a(c+ d) + b(c+ d)$

  $ac + ad + bc + bd$

- E.g. Remove the brackets from $(a + b)(c - d)$

  $a(c - d) + b(c - d)$

  $ac – ad + bc – bd$

Exercises

1. Remove the brackets from
   (a) $4(x+1)$ (b) $-4(x+1)$
   (c) $4(x-1)$ (d) $-4(x-1)$

2. Remove the brackets from
   (a) $5(x-y)$ (b) $19(x+3y)$
   (c) $8(a+b)$ (d) $(5+x)y$
   (e) $12(x+4)$ (f) $17(x-9)$
   (g) $-(a-2b)$ (h) $\frac{1}{2}(2x+1)$
   (i) $-3m(-2+4m+3n)$

3. Remove the brackets and simplify the expressions:
   (a) $18-13(x+2)$ (b) $x(x+y)$

5. Remove the brackets and simplify the following expressions:
   (a) $(x+3)(x-7)$ (b) $(2x-1)(3x+7)$
   (c) $(4x+1)(4x-1)$
   (d) $(x+3)(x-3)$ (e) $(2-x)(3+2x)$

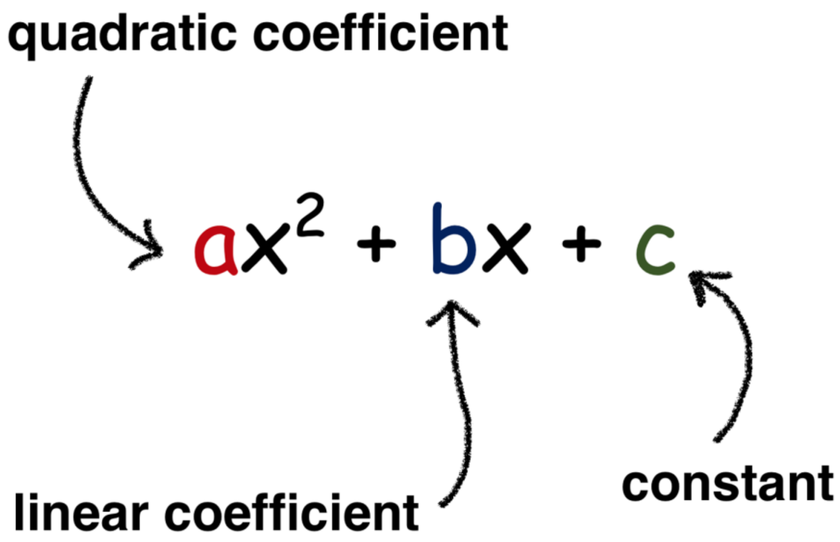
6. Remove the brackets and simplify the following expressions:
   (a) $\dfrac{1}{2}(x+2y)+\dfrac{7}{2}(4x-y)$
   (b) $\dfrac{3}{4}(x-1)+\dfrac{1}{4}(2x+8)$

Quadratic Expressions

- Expressions of the form $ax^2 + bx + c$, where a, b and c are numbers, are called quadratic expressions.
- The coefficient of x^2 is “a”, and “a” cannot be zero
- The coefficient of x is “b”, and it can be zero
- The constant term is “c”, and it can be zero

Examples of Quadratic Expressions

- $x^2 + 6x + 8$
- $x^2 - x - 30$
- $x^2 - 5x + 6$
- $2x^2 + 5x - 3$
- $2y^2 - 5y + 2$



Factorizing Quadratic Expressions

- To factorise a quadratic expression means to express it as a product of two or more terms.

(1) Quadratic expressions where the coefficient of x^2 is 1

$$x^2 + bx + c$$

The factorisation of $x^2 + bx + c$ will be of the form $(x + m)(x + n)$. This means that mn must equal c and $m + n$ must equal b.

$$\begin{aligned}(x+m)(x+n) &= (x+m)x + (x+m)n \\ &= x^2 + mx + nx + mn \\ &= x^2 + (m+n)x + mn\end{aligned}$$

Factorizing Quadratic Expressions(2)

Example for factorising quadratic expressions where the coefficient of x^2 is 1

$x^2 + 9x + 18$ mn = 18 and m + n must equal 9

$x^2 + 6x + 3x + 18$ therefore, m and n must be values 6 and 3

$x(x + 6) + 3(x + 6)$

$(x + 6)(x + 3)$

The factors of $x^2 + 9x + 18$ is $(x + 6)(x + 3)$

Factorizing Quadratic Expressions (3)

(2) Quadratic expressions where the coefficient of x^2 is not 1

$2x^2 + 11x + 12$

All possible factors of the first and last terms must be found.

The factors of the first term, $2x^2$, are 2x and x.

The factors of the last term, 12, are 12,1 -12,-1 6,2 -6,-2 and 4,3 -4,-3.

Try each combination in turn to find which gives a coefficient of x of 11.

The only one producing a middle term of 11x is 3 & 4 where $(2x + 3)(x + 4)$.

$2x^2 + 8x + 3x + 12$

Therefore, the factors of $2x^2 + 11x + 12$ are $(2x + 3)(x + 4)$

Exercises

1. Factorise the following quadratic expressions:
   (a) $x^2 + 3x + 2$ (b) $x^2 + 13x + 42$
   (c) $x^2 + 2x - 15$ (d) $x^2 + 9x - 10$
   (e) $x^2 - 11x + 24$ (f) $x^2 - 100$
   (g) $x^2 + 4x + 4$ (h) $x^2 - 36$
   (i) $x^2 - 25$ (j) $x^2 + 10x + 9$
   (k) $x^2 + 8x - 9$ (l) $x^2 - 8x - 9$
   (m) $x^2 - 10x + 9$ (n) $x^2 - 5x$

2. Factorise the following quadratic expressions:
   (a) $2x^2 - 5x - 3$ (b) $3x^2 - 5x - 2$
   (c) $10x^2 + 11x + 3$ (d) $2x^2 + 12x + 16$
   (e) $2x^2 + 5x + 3$ (f) $3s^2 + 5s + 2$
   (g) $3z^2 + 17z + 10$ (h) $9x^2 - 36$
   (i) $4x^2 - 25$

Evaluation of Algebraic Expressions

Algebraic Fractions

- One algebraic expression divided by another is called an algebraic fraction.
- e.g. (1) $\frac{3x}{5x}$ (2) $\frac{3x^2}{6x}$

- Note: Rules for determining the sign of the answer when dividing positive and negative algebraic expressions are the same as those used for dividing numeric.

Cancelling common factors

- In numerical fractions, any factors which appear in both the numerator and the denominator are called common factors.
- These can be cancelled.
- e.g. (1) (2)

$$\frac{3}{12} = \frac{1 \times 3}{4 \times 3} = \frac{1 \times \cancel{3}}{4 \times \cancel{3}} = \frac{1}{4}$$

$$\frac{\overset{1}{\cancel{3}}}{\underset{3}{\cancel{15}}} \times \frac{\overset{1}{\cancel{5}}}{\underset{2}{\cancel{6}}} = \frac{1}{6}$$

Simplify Algebraic Fractions

- A fraction is expressed in its simplest form by factorising the numerator and denominator and cancelling any common factors.
- e.g.

(1) $$\frac{x+4}{(x-3)(x+4)} = \frac{1\cdot\cancel{(x+4)}}{(x-3)\cancel{(x+4)}} = \frac{1}{x-3}$$

(2) $$\frac{x+2}{x^2+3x+2} = \frac{1(x+2)}{(x+2)(x+1)} = \frac{1}{x+1}$$

Exercises

1. Simplify

(a) $\dfrac{9x}{3y}$ (b) $\dfrac{9x}{x^2}$ (c) $\dfrac{9xy}{3x}$ (d) $\dfrac{9xy}{3y}$

(e) $\dfrac{9xy}{xy}$ (f) $\dfrac{9xy}{3xy}$

2. Simplify

(a) $\dfrac{15x}{3y}$ (b) $\dfrac{15x}{5y}$ (c) $\dfrac{15xy}{x}$ (d) $\dfrac{15xy}{xy}$

(e) $\dfrac{x^5}{-x^3}$ (f) $\dfrac{-y^3}{y^7}$ (g) $\dfrac{-y}{-y^2}$ (h) $\dfrac{-y^{-3}}{-y^4}$

3. Simplify the following algebraic fractions:

(a) $\dfrac{4}{12+8x}$ (b) $\dfrac{5+10x}{5}$ (c) $\dfrac{2}{4+14x}$

(d) $\dfrac{2x}{4+14x}$ (e) $\dfrac{2x}{2+14x}$ (f) $\dfrac{7}{49x+7y}$

(g) $\dfrac{7y}{49x+7y}$ (h) $\dfrac{7x}{49x+7y}$

(n) $\dfrac{x+3}{x^2+7x+12}$

(c) $\dfrac{2x+8}{x^2+2x-8}$ (d) $\dfrac{7ab}{a^2b^2+9ab}$

Multiplication of Algebraic Fractions

- To multiply two algebraic fractions: multiply the numerators together and multiply the denominators together.
- e.g. $\frac{5}{7x} \times \frac{4}{3y}$ $\frac{a}{a+b} \times \frac{b}{5a^2} = \frac{ab}{5a^2(a+b)}$

$\frac{20}{21xy}$

Division of Algebraic Fractions

- Division is performed by inverting the second fraction and multiplying.

- e.g. (1) $\dfrac{10a}{b} \div \dfrac{a^2}{3b} = \dfrac{10a}{b} \times \dfrac{3b}{a^2} = \dfrac{30ab}{a^2b} = \dfrac{30}{a}$

  (2) $\dfrac{x^2y^3}{z} \div \dfrac{y}{x} = \dfrac{x^2y^3}{z} \times \dfrac{x}{y} = \dfrac{x^3y^3}{zy} = \dfrac{x^3y^2}{z}$

Exercises

Simplify

(a) $\frac{5}{4} \times \frac{a}{25}$ (b) $\frac{5}{4} \times \frac{a}{b}$ (c) $\frac{8a}{b^2} \times \frac{b}{16a^2}$

(d) $\frac{9x}{3y} \times \frac{2x}{y^2}$ (e) $\frac{3}{5a} \times \frac{b}{a}$ (f) $\frac{1}{4} \times \frac{x}{y}$

(b) $\frac{x-2}{4} \div \frac{x}{16}$ (c) $\frac{12ab}{5ef} \div \frac{4ab^2}{f}$

(d) $\frac{x+3y}{2x} \div \frac{y}{4x^2}$ (e) $\frac{3}{x} \times \frac{3}{y} \times \frac{1}{z}$

Simplify

$$\frac{x+1}{x+2} \times \frac{x^2+6x+8}{x^2+4x+3}$$

Addition and subtraction of algebraic fractions

- Similar to adding and subtracting numerical fractions.
- e.g.

$$\frac{4}{x+y} - \frac{3}{y}$$

$$\frac{4}{x+y} = \frac{4}{x+y} \times \frac{y}{y} = \frac{4y}{(x+y)y}$$

$$\frac{3}{y} = \frac{3}{y} \times \frac{x+y}{x+y} = \frac{3(x+y)}{(x+y)y}$$

$$\frac{4}{x+y} - \frac{3}{y} = \frac{4y}{(x+y)y} - \frac{3(x+y)}{(x+y)y} = \frac{4y - 3(x+y)}{(x+y)y} = \frac{y - 3x}{(x+y)y}$$

Proper and Improper Fractions

- <u>Proper Fractions</u>
  - Highest power of the numerator < Highest power of the denominator
  - e.g. $\frac{x+5}{x^3}$, $\frac{x^2+3x-9}{2x^3+7}$
- <u>Improper Fractions</u>
  - Highest power of the numerator >= Highest power of the denominator
  - e.g. $\frac{9x^2}{3x}$, $\frac{x^3+x^2+x+100}{100x^3-5x^2}$

Partial fractions

- A single fraction can be expressed as the sum of two or more simpler fractions.
- Each of these simpler fractions is known as a partial fraction.

Single Fraction

$$\frac{5x+2}{(x-2)(x+4)}$$

Partial Fractions

$$\frac{3}{x+4} \qquad \frac{2}{x-2}$$

Formulae

Rearranging a Formula

- Transposing a formula involves rearrange the formula.
- To transpose a formula:
  a. add the same quantity to both sides
  b. subtract the same quantity from both sides
  c. multiply or divide both sides by the same quantity
  d. perform operations on both sides

Rearranging a Formula Example

- Transpose the formula for “F”

$$C = \frac{5\,(F - 32)}{9}$$

Multiply both sides by 9 to get:

$$9C = 5(F - 32)$$

Write backwards to get F on to the left:

$$5(F - 32) = 9C$$

Divide both sides by 5 to get:

$$F - 32 = \frac{9C}{5}$$

Add 32 to both sides to get:

$$F = \frac{9C}{5} + 32$$

Exercises

. Transpose each of the following formulae to make x the subject:

(a) $y = 3x$ (b) $y = \frac{1}{x}$ (c) $y = 7x - 5$

(d) $y = \frac{1}{2}x - 7$ (e) $y = \frac{1}{2x}$

(f) $y = \frac{1}{2x+1}$ (g) $y = \frac{1}{2x} + 1$

(h) $y = 18x - 21$ (i) $y = 19 - 8x$

Make x the subject of the following formulae:

(a) $y = 1 - x^2$ (b) $y = \frac{1}{1 - x^2}$

(c) $y = \frac{1 - x^2}{1 + x^2}$