1 : Theory of Matrices, Vector spaces and Linear Transformations

IT5506 – Mathematics for Computing II

Level III - Semester 5

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1.5 The Determinant of a square matrix

Determinants

Every square matrix A is associated with a scalar called the **determinant of A**, and is denoted by |A|.

1.5.1 Defining the determinant of a square matrix through its basic properties (through elementary operation)

Determinants of matrices of order one

Let A = (a_{ij}) be a square matrix of order one.

Then we define **|A| =** a_{11}

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1.5.2 Calculating the determinant of any square matrix using elementary operations

Determinants of matrices of order two

Let A = (a_{ij}) be a square matrix of order two. i.e. ,

A = $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ Then we define **|A| =** $a_{11}a_{22} - a_{12}a_{21}$

Example:

Let A = $\begin{pmatrix} 5 & 1 \\ 2 & -3 \end{pmatrix}_{2\times 2}$

Then $|A|$ = $5 \times (-3) - 1 \times 2 = -17$

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1.5.2 Calculating the determinant of any square matrix using elementary operations

Determinants of matrices of order three

Let A = (a_{ij}) be a square matrix of order three. i.e. ,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then we define

$$|A| = a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13}$$

Where m_{ij} is the determinant of the matrix of order 2 obtained by deleting the row and column containing a_{ij} .

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1.5.2 Calculating the determinant of any square matrix using elementary operations

Determinants of matrices of order three

Example:

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$

$|A| = 1 \text{ x } \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - 0 \text{ x } \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} + 2 \text{ x } \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}$

$|A| = 1 \text{ x } (-1) – 0 \text{ x } (0) + 2 \text{ x } (1)$

$= 1$

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1.5.3 Properties of determinant

Properties of Determinants

- Let A be a matrix of order n. Then $|A^T| = |A|$.

  Eg. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$ = -2

- Let A be a square matrix of order n. Then $|kA| = k^n|A|$ where k is a scalar.

  Eg. $\left|2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right| = \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} = -8 = 2^2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

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1.5.3 Properties of determinant

Properties of Determinants

- Let A be a square matrix of order n. If any two rows (or columns) of A are identical, then |A|= 0

Eg. $\begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 5 & 3 \end{vmatrix} = 0$

- If A = (a_{ij}) is a diagonal matrix or a triangular matrix of order n, then

$|A| = a_{11}a_{22}\ldots\ldots\ldots a_{nn}$

Eg. $\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 6 \end{vmatrix} = 1.2.6 = 12$

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1.5.3 Properties of determinant

Properties of Determinants

- Let I be the identity matrix of order n. Then $|I| = 1$

Eg. Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad |I| = 1(1-0) - 0(0-0) + 0(0-0) = 1$

- Let A be a square matrix of order n. If B is obtained from A by interchanging any two rows (or columns) of A, then $|B| = -|A|$.

Eg. $\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 3 \\ 0 & 0 & 6 \\ 0 & 2 & 8 \end{vmatrix}$

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1.5.3 Properties of determinant

Properties of Determinants

- Let A be a square matrix of order n. If B is obtained from A by multiplying a row (or column) by a nonzero scalar k,

  then |B| = k|A|

  Eg. $\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$

- Let A be a square matrix of order n. If B is obtained from A by adding a scalar multiple of a row (or column) of A to another row (or column) of A, then |B| = |A|.

  Eg. $\begin{vmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 1+0.k & 1+2k & 4+5k \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix}$

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1.5.3 Properties of determinant

Properties of Determinants

- If any row (or column) of a square matrix A is the sum of two or more elements, then the determinant can be expressed as the sum of two or more determinants.

Eg. $$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1+3 \\ 2 & 3 & 2+3 \\ 3 & 5 & 4+2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 5 & 2 \end{vmatrix}$$

- Let A be a square matrix of order n. If B is also a square matrix of order n, then

$$|AB| = |A||B|.$$

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1.5.4 The big formula for calculating the determinant and inverse of a square matrix (if it exists).

Singular and Non-singular Matrices

If $|A| \neq 0$, then A is said to be a **nonsingular** matrix; otherwise it is said to be **singular**.

Note:

If A^{-1} exists, $|AA^{-1}| = |A||A^{-1}| = 1$

i.e., $|A^{-1}| = 1/|A|$

That is, If A is invertible, it is non-singular.

Example: Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $|A| = 4 – 6 = -2 \neq 0$.

Therefore, A^{-1} exists and $A^{-1} = \dfrac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

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Adjoint of a Square Matrix

Let A = (a_{ij}) be a square matrix of order n. The **cofactor** c_{ij} of a_{ij} is defined as

$$c_{ij} = (-1)^{i+j} \, m_{ij}$$

Where m_{ij} is the determinant of the matrix of order n-1 obtained by deleting the row and column containing a_{ij} .

Example:

Let A = $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$ then then the **cofactor** c_{23} of a_{23} is derived as:

$$c_{23} = (-1)^{2+3} \, m_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = (1)(0) - (0)(-1) = 0$$

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Adjoint of a Square Matrix

Similarly, we can find the cofactors of all the elements a_{ij} of the matrix A and the cofactor matrix C of A can be given as:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

Let A = (a_{ij}) be a square matrix of order n and let C denote its matrix of cofactors. The **adjoint of A** denoted by adj A, is C^T, the transpose of the matrix of cofactors.

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Adjoint of a Square Matrix

Example:

Let A = $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ Then C = $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

Therefore adj A = C^T = $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

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1.5.4 The big formula for calculating the determinant and inverse of a square matrix (if it exists).

Finding the Inverse of a Matrix

If A is a non-singular matrix, then

$A^{-1} = \dfrac{1}{|A|}$ adj A

Example:

Let A = $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

Then |A| = 1(2 – 0) – 0 + (-1)(0 – 1) = 3. Thus A is invertible

$A^{-1} = \dfrac{1}{3}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

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