2 : Linear Programming and Integer Programming

IT5506-Mathematics for Computing II

Level III - Semester 5

Intended Learning Outcomes

At the end of this lesson, you will be able to;

- Formulate a decision problem either as a linear program, or as a integer program, or a mixed- integer program.
- Solve linear and integer programming 2 variable problems graphically and interpret the solutions.
- Solve integer programming problems using the Cutting plane algorithm, Branch and bound algorithm.

List of sub topics

2.2 Introduction to Integer Programming

When there are additional constraints given in a linear programming problem, variables must take integer values. We should assume that all numbers appearing in the statement of problem are integers.

“Integer programming expresses the optimization of a linear function subject to a set of linear constraints over integer variables.”

2.2.2 Graphical Method

Example

Find x_1 and $x_2 \in Z$ to maximise $z = 2x_1 + x_2$

subject to,

$$x_1 + x_2 \leq 7, \quad x_1 - 2x_2 \leq 2, \quad x_1 \geq 0, \quad x_2 \geq 0$$

Solution

Feasible region

$$2x_1 + x_2 = k$$

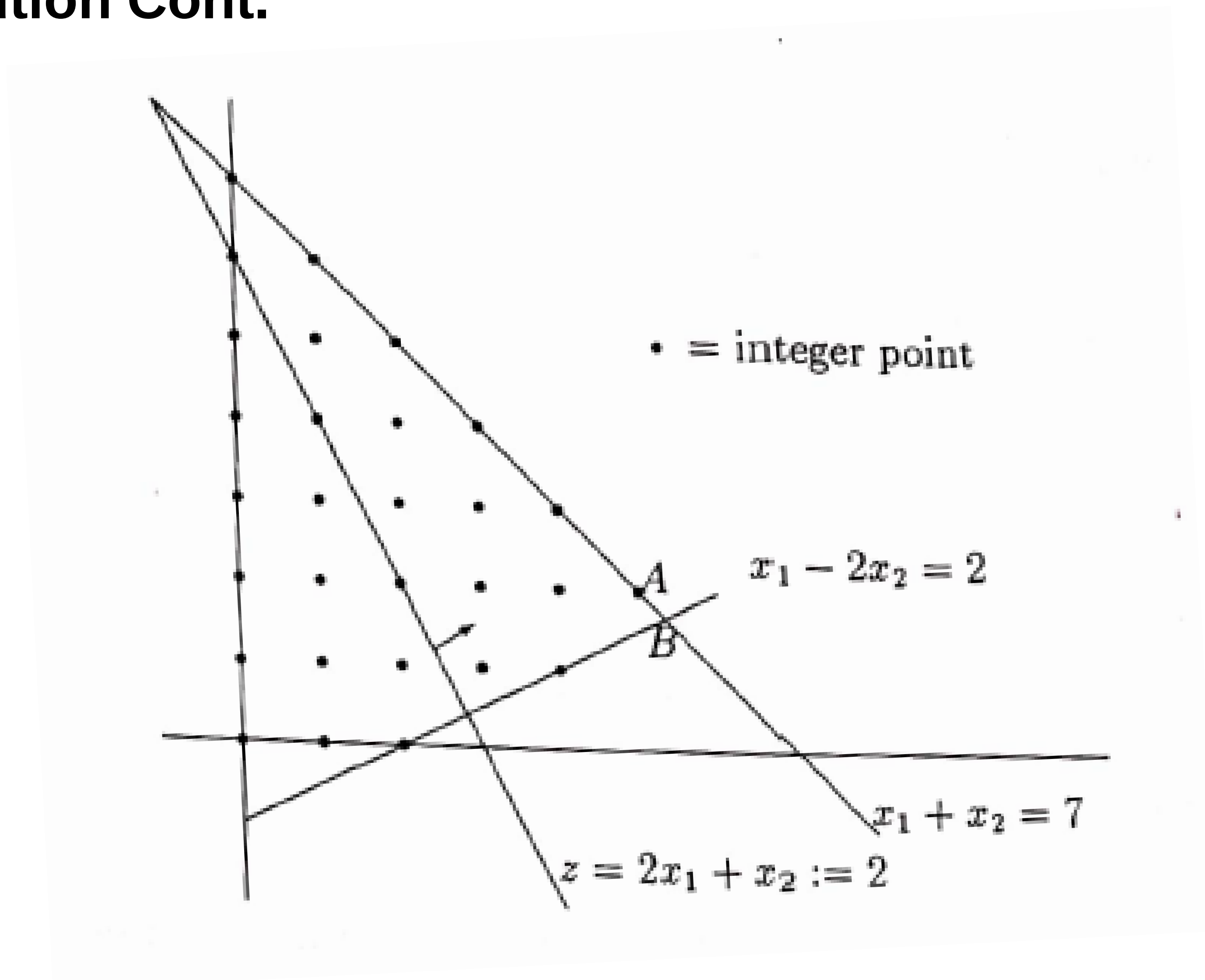
B = (16/3,5/3)

The optimal solution to the integer programming problem is given by the last point with integer coordinates in the feasible region which lies on one of line.

Therefore;

the optimal integer solution is at the point A = (5, 2) and the maximum value of $z = 2x_1 + x_2$ is 12

Solution Cont.



2.2.3 Cutting Plane Algorithm

Consider the following example

Find x_1 and $x_2 \in Z$ to maximise $z = 2x_1 + x_2$

subject to,

$$x_1 + x_2 \le 7, \quad x_1 - 2x_2 \le 2, \quad x_1 \ge 0, \quad x_2 \ge 0$$

Solution

Adding slack variables and obtain feasible dictionary

$x_3 = 7 - x_1 - x_2$

$x_4 = 2 - x_1 \ 2x_2$

$z = -2x_1 - x_2$

Solution

$x_1 = 16/3 - \frac{2}{3} x_3 - \frac{1}{3} x_4$

$x_2 = 5/3 - \frac{1}{3} x_3 + \frac{1}{3} x_4$

$z = 37/3 - 5/3 x_3 - \frac{1}{3} x_4$

Rewriting x_2

$x_2 = 5/3 - \frac{1}{3} x_3 - (-\frac{1}{3})x_4$

$= (1+\frac{2}{3}) - (0+\frac{1}{3})x_3 - (-1+\frac{2}{3})x_4$

$-\frac{2}{3} + 1/3x_3 + 2/3x_4 = 1 - x_2 + x_4$

$-\frac{2}{3} + \frac{1}{3} x_3 + \frac{2}{3} x_4 \geq 0$

$x_5 = -\frac{2}{3} + \frac{1}{3} x_3 + \frac{2}{3} x_4$

Solution Cont.

$x_4 = 1 - ½\, x_3 + 3/2\, x_5$

$x_1 = 5 - ½\, x_3 – ½\, x_5$

$x_2 = 2 - ½\, x_3 + ½\, x_5$

$z = 12 \quad 3/2\, x_3 - ½\, x_5$

This has as integer optimal solution

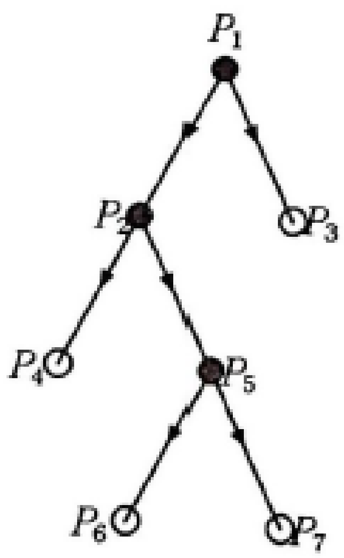
$x_1 = 5$

$x_2 = 2$

$z = 12$

2.2.4 Branch and Bound Algorithm

In branch and bound algorithm searched optimal solution to the integer programming problem P by proceeding along a binary tree of linear programming problems with property that each integer feasible solution to P is a feasible solution to exactly one of linear programming problems which are currently end vertices of the tree. The algorithm terminates when all the tree branches have been terminated as the shown figure.



Consider the following example

Find x_1 and $x_2 \in Z$ to maximise $z = 2x_1 + x_2$

subject to,

$x_1 + x_2 \leq 7$, $x_1 - 2x_2 \leq 2$, $x_1 \geq 0$, $x_2 \geq 0$

Solution

$x_1 \leq 16/3 = 5$

$x_1 \geq 16/3 + 1 = 6$

$x_5 = 5 - x_1 = 5 - (16/3 - \frac{2}{3} x_3 - \frac{1}{3} x_4) = - \frac{1}{3} + \frac{2}{3} x_3 + \frac{1}{3} x_4$

Solution Cont.

$x_4 = 1- 2x_3 + 3x_5$

$x_1 = 5 \qquad + \ x_5$

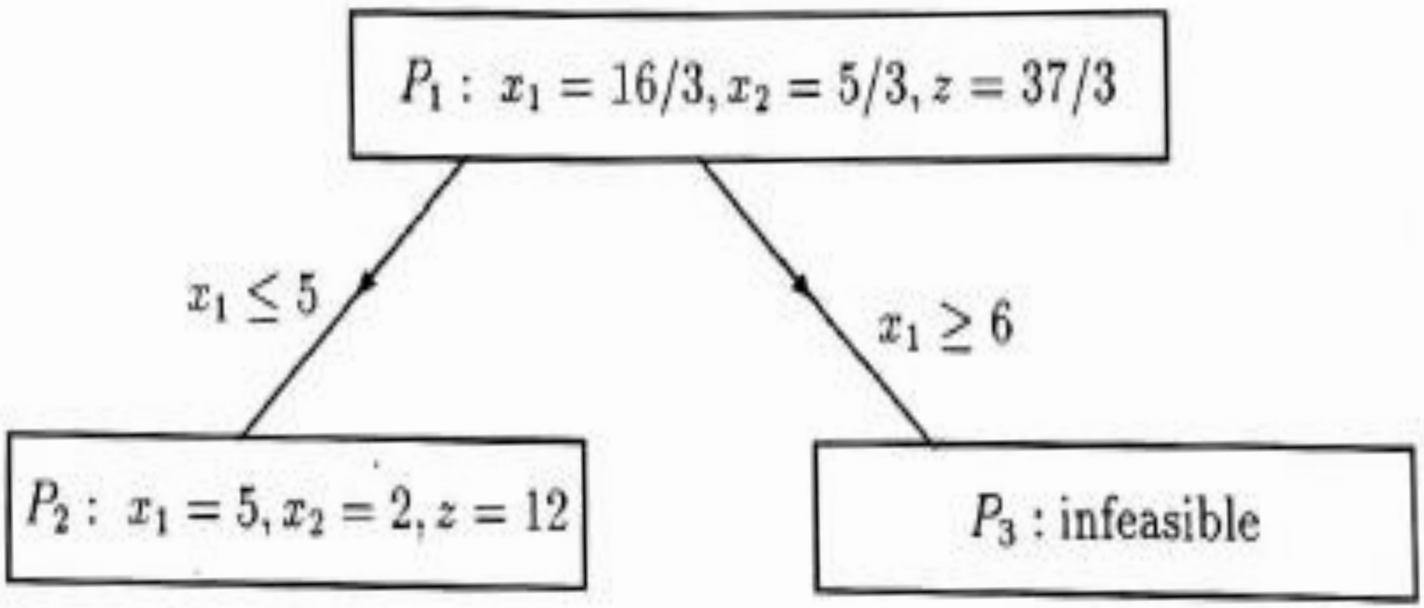
$x_2 = 2 - \ x_3 + \ x_5$

$z = 12 - \ x_3 - \ x_5$

$x_1 = 5$, $x_2 = 2$, $z = 12$

therefore; the integer valued so we terminate the P branch and set to 12.

$x_5 = -6 + x_1 = -6 + (16/3 - 2/3x_3 -1/3x_4) = -⅔ - ⅔ x_3 - ⅓ x_4$


$P_1: \ x_1 = 16/3, x_2 = 5/3, z = 37/3$
$x_1 \leq 5$
$x_1 \geq 6$
$P_2: \ x_1 = 5, x_2 = 2, z = 12$
P_3 : infeasible

2.2.5 Knapsack Problem

The knapsack problem is a particularly simple integer program. It has only one constraint. Furthermore, the coefficients of this constraint and the objective are all non negative.

Example

The following is a knapsack problem.

$$
\begin{array}{ll}
\text{Maximize} & 8x_1 + 11x_2 + 6x_3 + 4x_4 \\
\text{subject to} & 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\
& x_j \in \{0, 1\}.
\end{array}
$$

Knapsack Problem Cont.

To solve the associates linear program, it is simply a matter of determining which variable gives the most “bang for buck”.

If you take c_j / a_j (objective coefficient/constraint coefficient) for each variable, the one with the highest ratio is the best item to place in the knapsack.

Then the item with the second highest ratio is put in and so on until we reach an item that cannot fit.

At this point, a fractional amount of that item is placed in the knapsack to completely fill it.

Activity 06

Select the correct answer.

Consider the example of a manufacturer of animal feed who is producing feed mix for dairy cattle. In our simple example the feed mix contains two active ingredients and a filler to provide bulk. One kg of feed mix must contain a minimum quantity of each of four nutrients as below:

Nutrient	A	B	C	D
gram	90	50	20	2

The ingredients have the following nutrient values and cost

	A	B	C	D	Cost/kg
Ingredient 1 (gram/kg)	100	80	40	10	40
Ingredient 2 (gram/kg)	200	150	20	-	60

Activity 06 Cont.

x_1 = amount (kg) of ingredient 1 in one kg of feed mix

x_2 = amount (kg) of ingredient 2 in one kg of feed mix

x_3 = amount (kg) of filler in one kg of feed mix

(a) $x_1 + x_2 + x_3 > 1$

$100x_1 + 200x_2 >= 90$ (nutrient A)

$80x_1 + 150x_2 >= 50$ (nutrient B)

$40x_1 + 20x_2 >= 20$ (nutrient C)

$10x_1 >= 2$ (nutrient D)

(b) $x_1 + x_2 + x_3 = 1$

$100x_1 + 200x_2 >= 90$ (nutrient A)

$80x_1 + 150x_2 >= 50$ (nutrient B)

$40x_1 + 20x_2 >= 20$ (nutrient C)

$10x_1 >= 2$ (nutrient D)

Activity 06 Cont.

(c) $x_1 + x_2 + x_3 = 1$

$100x_1 + 200x_2 >= 90$ (nutrient A)

$80x_1 - 150x_2 >= 50$ (nutrient B)

$40x_1 + 20x_2 = 20$ (nutrient C)

$10x_1 >= 2$ (nutrient D)

(d) $x_1 + x_2 + x_3 = 1$

$100x_1 + 200x_2 = 90$ (nutrient A)

$80x_1 + 150x_2 >= 50$ (nutrient B)

$40x_1 + 20x_2 >= 20$ (nutrient C)

$10x_1 >= 2$ (nutrient D)

Activity 07

Select the objective function of the given scenario

Consider the example of a manufacturer of animal feed who is producing feed mix for dairy cattle. In our simple example the feed mix contains two active ingredients and a filler to provide bulk. One kg of feed mix must contain a minimum quantity of each of four nutrients as below:

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The ingredients have the following nutrient values and cost

	A	B	C	D	Cost/kg
Ingredient 1 (gram/kg)	100	80	40	10	40
Ingredient 2 (gram/kg)	200	150	20	-	60

Activity 07 Cont.

x_1 = amount (kg) of ingredient 1 in one kg of feed mix

x_2 = amount (kg) of ingredient 2 in one kg of feed mix

x_3 = amount (kg) of filler in one kg of feed mix

(a) maximise $40x_1 + 60x_2$
(b) minimise $40x_1 + 60x_2$
(c) minimise $40x_1 - 60x_2$
(d) minimise $40x_2 + 60x_1$

Activity 08

A toy manufacturer is planning to produce new toys. The setup cost of the production facilities and the unit profit for each toy are given below. :

Toy	Setup cost (£)	Profit (£)
1	45000	12
2	76000	16

The company has two factories that are capable of producing these toys. In order to avoid doubling the setup cost only *one* factory could be used.

The production rates of each toy are given below (in units/hour):

	Toy 1	Toy 2
Factory 1	52	38
Factory 2	42	23

Factories 1 and 2, respectively, have 480 and 720 hours of production time available for the production of these toys. The manufacturer wants to know *which* of the new toys to produce, *where* and *how many* of each (if any) should be produced so as to maximise the total profit.

Activity 08 Cont.

x_{ij} be the number of toys of type j (j=1,2) produced in factory i (i=1,2)

where x_{ij} >=0 and integer

Select the correct constraints

I. $x_{11}/52 + x_{12}/38 <= 480$
   $x_{21}/42 + x_{22}/23 <= 720$

II. $x_{11}/52 + x_{12}/38 = 480$
   $x_{21}/42 + x_{22}/23 <= 720$

Activity 08 Cont.

III. $x_{11}/52 - x_{12}/38 <= 480$

$x_{21}/42 + x_{22}/23 <= 720$

IV. $x_{11}/52 + x_{12}/38 <= 480$

$x_{21}/42 + x_{22}/23 >= 720$

Activity 09

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1	45000	12
2	76000	16

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Activity 09 Cont.

x_{ij} be the number of toys of type j (j=1,2) produced in factory i (i=1,2)

where x_{ij} >=0 and integer

Select the correct objective function.

I. minimise $12(x_{11}+ x_{21}) + 16(x_{12}+ x_{22}) - 45000(f_{11} + f_{21}) - 76000(f_{12} + f_{22})$
II. maximise$12(x_{11}+ x_{21}) + 16(x_{12}+ x_{22}) - 45000(f_{11} + f_{21}) - 76000(f_{12} + f_{22})$
III. maximise$12(x_{11}+ x_{21}) - 16(x_{12}+ x_{22}) - 45000(f_{11} + f_{21}) - 76000(f_{12} + f_{22})$
IV. minimise $12(x_{11}+ x_{21}) + 16(x_{12}+ x_{22}) - 45000(f_{11} + f_{21}) - 76000(f_{12} + f_{22})$

References

- Introduction to Operations Research (7th edition), F S Hillier and G L Liebermann, 2001, McGraw-Hill.