Applied Stochastic Processes (FIN 514)
Midterm Exam

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BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. You can use the following functions in your answers without further evaluation,

$$\text{Standard normal PDF: } n(x) = e^{-x^2/2}/\sqrt{2\pi}$$
$$\text{Standard normal CDF: } N(x) = \textstyle\int_{-\infty}^{x} n(s)ds.$$

1. (4 points) **(Spread option)** Compute the price of the call option on the spread between two stocks. The payout at maturity T is given as

$$\text{Payout} = \max(S_1(T) - S_2(T),\, 0).$$

Assume that $S_1(0) = S_2(0) = 100$, $r = q = 0$, $\sigma_1 = 20\%$, $\sigma_2 = 10\%$, and $T = 1$ year. Also assume that the BMs driving the two stocks are correlated by 89%. You may use the following values for $N(z)$.

z	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$N(z)$	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564

Solution: We use Margrabe's formula:

$$C = S_1(0)N(d_+) - S_2(0)N(d_-),$$
$$\text{where} \quad d_\pm = \frac{\log(S_1(0)/S_2(0))}{\sigma_R\sqrt{T}} \pm \frac{1}{2}\sigma_R\sqrt{T} \quad \text{and} \quad \sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2},$$

we get

$$\sigma_R = \frac{1}{100}\sqrt{400 + 100 - 2\times 0.89 \times 200} = 12\%,$$
$$d_1 = \frac{\sigma_\mathrm{R}}{2} = 0.06, \quad d_2 = -0.06,$$
$$C = S_0 N(d_1) - KN(d_2) = 100N(0.06) + 100(1 - N(0.06)) = 4.8$$

2. (4 points) **(Option vega under the BSM model)** Derive that the vega of a call option (i.e., sensitivity with respect to the volatility σ) is

$$V = \frac{\partial C}{\partial \sigma} = S_0\, n(d_1)\sqrt{T} = Ke^{-rT} n(d_2)\sqrt{T}.$$

Remind that the call option price under the BSM model is

$$C = S_0 N(d_1) - e^{-rT} K N(d_2) \quad \text{where} \quad d_{1,2} = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}$$

Since the terms d_1 and d_2 are implicit functions of σ, you should also differentiate d_1 and d_2.

Solution: Using the properties

$$\frac{\partial d_{1,2}}{\partial \sigma} = -\frac{\log(S_0\, e^{rT}/K)}{\sigma^2\sqrt{T}} \pm \frac{1}{2}\sqrt{T} = -\frac{d_{2,1}}{\sigma}$$

and

$$d_1^2 - d_2^2 = (A+B)^2 - (A-B)^2 = 4AB = 2\log(S_0\, e^{rT}/K) \quad \Rightarrow \quad \frac{n(d_2)}{n(d_1)} = \frac{S_0\, e^{rT}}{K}$$

we compute the vega as

$$\begin{aligned} V &= \frac{\partial}{\partial \sigma}\left(S_0 N(d_1) - e^{-rT} K N(d_2)\right) = S_0 n(d_1)\frac{-d_2}{\sigma} - e^{-rT} K n(d_2)\frac{-d_1}{\sigma} \\ &= S_0 n(d_1)\left(-\frac{d_2}{\sigma} + \frac{K n(d_2)}{S_0 e^{rT}\, n(d_1)}\frac{d_1}{\sigma}\right) = S_0 n(d_1)\left(-\frac{d_2}{\sigma} + \frac{d_1}{\sigma}\right) \\ &= S_0 n(d_1)\sqrt{T} = K e^{-rT} n(d_2)\sqrt{T}. \end{aligned}$$

3. (6 points) **(Simulation of BM path)** Exotic derivatives often depend on the 'path' of the underlying stock price. Assume that we need to generate the Monte-Carlo paths of standard BM W_t at $t = 1, 3, 5$, and 9. We are going to generate the paths using two approaches, which are eventually same. Assume z_k, for $k = 1, \cdots, 4$ are independent standard normal RV.

   (a) Using the incremental property of BM, i.e., $W_t - W_s \sim N(0, t-s)$, generate RNs for W_1, $W_3 - W_1$, $W_5 - W_3$, and $W_9 - W_5$, using z_k's. Finally, how can you generate RNs for W_1, W_3, W_5, and W_9?

   (b) Now we use covariance matrix approach: Let $\boldsymbol{\Sigma}$ be the covariance matrix of correlated multivariate normal variables and $\boldsymbol{L}$ (lower-triangular matrix) be its Cholesky decomposition, which satisfy $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$. Then, the simulation of the normal variables can obtained as $\boldsymbol{L}\boldsymbol{z}$, where $\boldsymbol{z}$ is the vector of independent standard normal RVs. What is the covariance matrix $\boldsymbol{\Sigma}$ for our case? (Hint: you may use $\mathrm{Cov}(W_s, W_t) = \min(t, s)$ without proof.)

   (c) From (a) and (b), what is the Cholesky decomposition $\boldsymbol{L}$? Verify that $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$ by direct computation.

Solution:

(a)

$$\begin{aligned} W_1 &= z_1, \\ W_3 - W_1 &= \sqrt{2} z_2 \\ W_5 - W_3 &= \sqrt{2} z_3 \\ W_9 - W_5 &= 2 z_4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} W_1 &= z_1, \\ W_3 &= z_1 + \sqrt{2} z_2 \\ W_5 &= z_1 + \sqrt{2} z_2 + \sqrt{2} z_3 \\ W_9 &= z_1 + \sqrt{2} z_2 + \sqrt{2} z_3 + 2 z_4 \end{aligned}$$

(b)

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix}$$

(c)

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}.$$

$$\boldsymbol{L}\boldsymbol{L}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 9 \end{pmatrix} = \boldsymbol{\Sigma}$$

4. (6 points) **(Simulation of CIR process)** In the Heston stochastic volatility model, the stochastic variance $V(t)$ $(= \sigma(t)^2)$ follows the SDE:

$$dV(t) = \kappa(V_\infty - V(t))dt + \alpha\sqrt{V(t)}dZ_t.$$

We want to Monte-Carlo simulate $V(T)$ for some T by discretizing time as $t_k = (k/N)T$ for $k = 1, \cdots, N$ and $\Delta t = T/N$.

(a) Write the formula to compute $V(t_{k+1})$ from $V(t_k)$. Assume z is a standard normal RV.

(b) Instead of simulating V_t, we may consider simulating $\sigma(t) = \sqrt{V(t)}$. Using Itô's lemma, drive the SDE for σ_t.

(c) From the result of (b), write the formula to update $\sigma(t_{k+1})$ from $\sigma(t_k)$. After replacing $\sigma(t)^2$ with $V(t)$, compare the answer to the result from (a). Are they same?

Solution:

(a)

$$V(t_{k+1}) = V(t_k) + \kappa(V_\infty - V(t_k))\Delta t + \alpha\sqrt{V(t_k)\Delta t}\, z$$

(b) Applying Itô's lemma, we get

$$\begin{aligned} d\sigma(t) = d\sqrt{V(t)} &= \frac{dV(t)}{2\sigma(t)} - \frac{(dV(t))^2}{8\sigma(t)^3} \\ &= \frac{\kappa(V_\infty - \sigma(t)^2)dt}{2\sigma(t)} + \frac{\alpha}{2}dZ_t - \frac{\alpha^2 dt}{8\sigma(t)} \\ &= \frac{4\kappa(V_\infty - \sigma(t)^2) - \alpha^2}{8\sigma(t)}dt + \frac{\alpha}{2}dZ_t. \end{aligned}$$

(c) The discretization rule for $\sigma(t)$ is given as

$$\sigma(t_{k+1}) = \sigma(t_k) + \frac{4\kappa(V_\infty - \sigma(t_k)^2) - \alpha^2}{8\sigma(t_t)}\Delta t + \frac{\alpha}{2}\sqrt{\Delta t}\, z.$$

By taking the square of both sides,

$$
\begin{aligned}
V(t_{k+1}) = \sigma(t_{k+1})^2 &= \left(\sigma(t_k) + \frac{4\kappa(V_\infty - \sigma(t_k)^2) - \alpha^2}{8\sigma(t_k)}\Delta t + \frac{\alpha}{2}\sqrt{\Delta t}\, z\right)^2 \\
&= V(t_k) + \frac{4\kappa(V_\infty - V(t_k)) - \alpha^2}{4}\Delta t + \frac{\alpha^2}{4}\Delta t\, z^2 + \alpha\sqrt{V(t_k)\Delta t}\, z + O(\Delta t^2) \\
&= V(t_k) + \kappa(V_\infty - V(t_k))\Delta t + \alpha\sqrt{V(t_k)\Delta t}\, z + \frac{\alpha^2}{4}\Delta t\,(z^2 - 1) + o(\Delta t),
\end{aligned}
$$

where $o(\Delta t)$ is the terms smaller than Δt in order.

This result is differ from (a) by the two terms in red above. Even after ignoring $o(\Delta t)$, the term $\alpha^2 \Delta t\,(z^2-1)/4$ remains. So the two discretization methods are different. The discretization method we applied to $V(t)$ and $\sigma(t)$ (that we learned from class) is called Euler-Maruyama method (WIKIPEDIA). The discretization for $V(t)$ derived via $\sigma(t)$ is called Milstein method (WIKIPEDIA). If we apply Milstein method to $V(t)$, we directly get the same result. Milstein method is known to be more accurate than Euler-Maruyama method.