Spread and Basket Option Pricing
Applied Stochastic Processes (FIN 514)

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Overview

- Options written on multiple underlying assets
- Extension of Black-Scholes model from uni-variate to multi-variate
- Write a MC pricing routine with control variate
- Analytic approximations (to be used as control variates)
  - Normal model price
  - Kirk's approximation (Margrabe's formula)
  - Geometric basket option

Background: popular derivatives in non-vanilla class

Spread options: $(S_1 - S_2 - K)^+$

- Crack spread option: (P of oil products − P of oil − $K)^+$
- Spark spread option: (P of electricity − P of gas − $K)^+$
- Non-inversion note: digital call on rates term-structure, $(30y - 2y)^+$

Basket options: $(\sum w_k S_k - K)^+$ with $w_k > 0$

- Popular as OTC derivatives in FX and commodities market
- Index options

Asian options (path-dependent): $(\sum_1^N S(t_k)/N - K)^+$

- Efficient hedge over average cost, safe from market manipulation.
- Fed fund swaps: daily compounded trade-averaged FF rate
- China interest swaps: 3-month average of 7-day repo rate as a standard floating rate (cf. 3-month LIBOR)

Problem setup

- N asset prices, $S_k(t)$, following the correlated geometric Brownian motions (GBM)

$$\frac{dS_k(t)}{S_k(t)} = (r - q_k)\,dt + \sigma_k\,dW_k(t) \quad \text{for} \quad 1 \le k \le N$$

  for volatilities σ_k, dividend rates q_k, risk-free rate r and BMs $W_k(t)$, the correlation $\mathbb{E}\{dW_k(t)dW_j(t)\} = \rho_{kj}dt$ ($\rho_{kk} = 1$).
- N observation times: t_k ($1 \le k \le N$), with expiry at T
- The payout of the option at the maturity T

$$C(T) = \Big(\sum_{k=1}^{N} w_k\,S_k(t_k) - K\Big)^+$$

Option types

$$C(T) = \left(\sum_{k=1}^{N} w_k\, S_k(t_k) - K\right)^+$$

- **European basket option**:
  $w_k > 0$ and $t_k = T$ for all k; $N < 10$
- **European spread option**:
  $w_k < 0$ for some k, $t_k = T$ for all k; $N = 2$
- **Discretely monitored Asian option** (covered later):
  $w_k = 1/N$, $0 \le t_1 < \cdots < t_N = T$ and $S_k(t)$'s are identical; $N \gg 10$

Challenges

Mathematical problem

Multi-dimensional integration over the domain of positive payoff

Difficulties

- The lognormal RV sum is neither lognormal nor has analytic distribution.
- Numerical valuation is cursed by dimensionality: $O(M^N)$
  e.g. $h = 0.1$ grid between ± 7 std. dev. for 4 assets: $140^4 \approx 400e6$
- Monte-Carlo simulation is used for pricing in industry and academics.

Quote from Broadie and Detemple (2004, MS Survey)

"Many problems are effectively exponential $\cdots$. Efficient and convergent methods for pricing high-dimensional and path-dependent American securities depend on the development of new algorithms, not faster computers."

Normal model approximation

- Spread Option: $\sigma_{n1} = \sigma_1 S_1(0),\ \sigma_{n2} = \sigma_2 S_2(0)$

$$\text{Var}(S_1(T) - S_2(T)) = (\sigma_{n1}^2 + \sigma_{n1}^2 - 2\rho\sigma_{n1}\sigma_{n2}) \cdot T$$

$$\sigma_n = \sqrt{\sigma_{n1}^2 + \sigma_{n1}^2 - 2\rho\sigma_{n1}\sigma_{n2}}$$

  and use the normal model formula.

- Basket Option: $\sigma_{nj} = \sigma_j S_j(0)$ or $= \sigma_j F_j(0)$

$$\text{Var}(\sum w_k S_k(T)) = \Big(\sum_j w_j^2 \sigma_{nj}^2 + 2\sum_{jk} \rho_{jk} w_j w_k \sigma_{nj}\sigma_{nk}\Big) \cdot T$$

$$\Sigma_{jk} = \rho_{jk}\sigma_{nj}\sigma_{nk}, \quad \sigma_n = \sqrt{\mathbf{w}^T\,\mathbf{\Sigma}\,\mathbf{w}}$$

  and use the normal model formula.

Normal model approximation for Control Variate

- Use the result as a control variate of Spread and Asiain option:

$$C^{CV}(T,K) = C^{MC}(T,K) + \left(C_N^{Exact}(T,K) - C_N^{MC}(T,K)\right)$$

  Use the same sequence of RNs for C^{MC} and C_N^{MC}. [R Demo]

- **Homework Set 3**:
  Implement Monte-Carlo pricer for Spread and Basket options with control variate with normal model price.
- Final project: implement other analytic approximation methods or CV methods (e.g., Kirk's approximation for spread options)

Exchange option: Margrabe's formula

- Option to exchange one asset S_1 for another S_2: $(S_1(T) - S_2(T))^+$
- Spread option with zero strike $K = 0$: $(S_1(T) - S_2(T) - 0)^+$
- Max (best-of) option: $\max(S_1, S_2) = S_2 + \max(S_1 - S_2, 0)$.

Margrabe's formula

$$C_{sw} = S_1(0)N(d_+) - S_2(0)N(d_-),$$

where $d_\pm = \frac{\log(S_1(0)/S_2(0))}{\sigma_R\sqrt{T}} \pm \frac{1}{2}\sigma_R\sqrt{T}$ and $\sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

SDE on S_1/S_2

$$dS_k(t)/S_k(t) = r\,dt + \sigma_k dW_k(t)\ (k=1,2), \quad dW_1 dW_2 = \rho dt$$

Applying Itô's lemma to S_1/S_2,

$$d\left(\frac{S_1}{S_2}\right) = \frac{dS_1}{S_2} - \frac{S_1}{S_2^2}dS_2 + \frac{S_1}{S_2^3}(dS_2)^2 - \boxed{\frac{dS_1 dS_2}{S_2^2}}$$

For $R = S_1/S_2$,

$$\frac{dR}{R} = (\sigma_2^2 - \rho\sigma_1\sigma_2)\,dt + (\sigma_1 dW_1 - \sigma_2 dW_2)$$

Alternatively,

$$d\log R = d\log S_2 - d\log S_1 = -\frac{1}{2}(\sigma_1^2 - \sigma_2^2)\,dt + \sigma_1 dW_1 - \sigma_2 dW_2$$

$$\frac{dR}{R} = (\quad \cdots \quad) + (\sigma_1 dW_1 - \sigma_2 dW_2)^2 = \text{same result}$$

Equivalent Martingale Measure with Numeraire S_2

Decorrelating the SDE on R, for $dW_1' dW_2 = 0$,

$$\frac{dR}{R} = (\sigma_2^2 - \rho\sigma_1\sigma_2)\,dt + \sigma_1(\rho dW_2 + \sqrt{1-\rho^2}\sigma_1 dW_1') - \sigma_2 dW_2.$$

Now we change the measure from P (risk-less saving numeraire) to Q (S_2);

$$\begin{aligned} C_{sw} &= 1 \cdot E^P\left(\frac{(S_1(T)-S_2(T))^+}{e^{rT}}\right) \\ &= S_2(0)E^Q\left(\frac{(S_1(T)-S_2(T))^+}{S_2(T)}\right) = S_2(0)E^Q\left((R(T)-1)^+\right) \end{aligned}$$

Under the new measure, the standard BM is defined as

$$dW_2^P = dW_2^Q + \sigma_2 dt \quad \text{and} \quad dW_1'^P = dW_1'^Q.$$

Then the SDE on R becomes drift-less and can be written with a single BM

$$\begin{aligned} \frac{dR}{R} &= (\sigma_2^2 - \rho\sigma_1\sigma_2)\,dt + \sqrt{1-\rho^2}\sigma_1 dW_1'^P - (\sigma_2 - \rho\sigma_1)(dW_2^Q + \sigma_2\,dt) \\ &= \sqrt{1-\rho^2}\sigma_1 dW_1'^Q + (\sigma_2 - \rho\sigma_1)dW_2^Q = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,dZ^Q \end{aligned}$$

The switch option price is obtained from just another Black-Scholes formula on $R(T)$ with

- $K = 1$
- $\sigma_R = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$
- prefactor (unit of options) $S_2(0)$.

Finally we obtain

$$C_{sw} = S_2(0)\, E^Q\Big((R(T) - 1)^+\Big) = S_2(0)\, \left(\frac{S_1(0)}{S_2(0)} N(d_+) - 1 \cdot N(d_-)\right)$$

where $d_\pm = \frac{\log(S_1(0)/S_2(0))}{\sigma_R\sqrt{T}} \pm \frac{1}{2}\sigma_R\sqrt{T}$ and $\sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.

Kirk's approximation

If we assume S_2 follows displace GBM with $L = K$, $S_2^D = S_2 + K$, then we can apply Margrabe's formula!

$$(S_1 - S_2 - K)^+ = (S_1 - \boxed{(S_2 + K)})^+ = (S_1 - S_2^D)^+$$

The volatility of S^D should be 'calibrated'. We match the local vol at ATM

$$\sigma_2^D (S_2(0) + K) = \sigma_2 S_2(0)$$

Plug in $S_2(0) + K \to S_2(0)$ and $\sigma_2 S_2(0)/(S_2(0) + K) \to \sigma_2$ to Margrabe:

Kirk's approximation

$$C_{Kirk} = S_1(0)N(d_+) - (S_2(0) + K)N(d_-),$$

$$d_\pm = \log\left(\frac{S_1(0)}{S_2(0) + K}\right)/\sigma_R\sqrt{T} \pm \frac{1}{2}\sigma_R\sqrt{T}$$

$$\sigma_R = \sqrt{\sigma_1^2 + \sigma_2'^2 - 2\rho\sigma_1\sigma_2'}, \quad \sigma_2' = \sigma_2 S_2(0)/(S_2(0) + K)$$

Some comments on Kirk's approximation

- The method is fairly accurate for reasonable parameter range: error of 10^{-3}.
- Kirk's formula can be a better control variate than the normal model price.
- What if $K < 0$? Displaced GBM on S_1 instead of S_2?

$$(S_1 - S_2 - K)^+ = (\boxed{S_1 - K} - S_2)^+ = (S_1^D - S_2)^+$$