IMPLEMENTATION DETAILS

0.1. **Notation.**

- For the tuple $\boldsymbol{a} = (a_1, \ldots, a_m)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$, the concatenation $(a_1, \ldots, a_m, b_1, \ldots, b_n)$ is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$.
- Fix an positive integer N.

0.2. **The set $X^{\otimes d}$.** Let X_d be the set of the formal sum of tuples $(u_1, \ldots, u_d, l_1, \ldots, l_d) \in \{0, 1, \ldots, N-1\}^d \times \mathbb{Z}_{\geq 0}^d$. We identify $X^{\otimes d}$ and X_d by

$$\left[\!\!\left[\begin{matrix} \zeta_N^{u_1}, \ldots, \zeta_N^{u_d} \\ l_1, \ldots, l_d \end{matrix} \right]\!\!\right] \mapsto (u_1, \ldots, u_d, l_1, \ldots, l_d).$$

0.3. **The map D_d^{iter}.**

Definition 1. The map

$$\mathsf{push} : \mathbb{Z}_{\geq 0}^d \times \{0, \ldots, d-2\} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}^d$$

is defined by

$$\mathsf{push}((l_1, \ldots, l_d), i, r) = (l_1, \ldots, l_i, l_{i+1} + l_{i+2} - r, l_{i+3}, \ldots, l_d)$$

for $i \geq 0$.

Definition 2. The map

$$\mathsf{Diterpre} : (\mathbb{Z}/N\mathbb{Z})^e \times \mathbb{Z}_{\geq 0}^e \times (\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}_{\geq 0}^d \to \mathsf{X}_d$$

is defined by

$$\begin{aligned}
&\mathsf{Diterpre}(\boldsymbol{v} = (v_1, \ldots, v_e), \boldsymbol{m} = (m_1, \ldots, m_e), \boldsymbol{u} = (u_1, \ldots, u_d), \boldsymbol{l} = (l_1, \ldots, l_d)) \\
&= \mathsf{Diterpre}(\boldsymbol{v} \cdot (u_1 - u_2), \boldsymbol{m} \cdot (l_1), (u_2, \ldots, u_d), (l_2, \ldots, l_d)) \\
&\quad + \sum_{i=2}^{d-1} \sum_{r=l_i}^{l_{i-1}+l_i} (-1)^{r-l_i} \binom{r}{l_i} \mathsf{Diterpre}(\boldsymbol{v} \cdot (u_i - u_{i+1}), \boldsymbol{m} \cdot (r), (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_d), \mathsf{push}(\boldsymbol{l}, i-2, r)) \\
&\quad - \sum_{i=1}^{d-1} \sum_{r=l_i}^{l_i+l_{i+1}} (-1)^{l_i} \binom{r}{l_i} \mathsf{Diterpre}(\boldsymbol{v} \cdot (u_{i+1} - u_i), \boldsymbol{m} \cdot (r), (u_1, \ldots, u_i, u_{i+2}, \ldots, u_d), \mathsf{push}(\boldsymbol{l}, i-1, r)) \\
&\quad + \sum_{r=l_d}^{l_{d-1}+l_d} (-1)^{r-l_d} \binom{r}{l_d} \mathsf{Diterpre}(\boldsymbol{v} \cdot (u_d), \boldsymbol{m} \cdot (r), (u_1, \ldots, u_{d-1}), \mathsf{push}(\boldsymbol{l}, d-2, r)).
\end{aligned}$$

for $d \geq 2$ and

$$\mathsf{Diterpre}((v_1, \ldots, v_e), (m_1, \ldots, m_e), (u), (l)) = (v_1, \ldots, v_e, u, m_1, \ldots, m_e, l).$$

Definition 3. The map

$$\mathsf{Diter} : (\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}_{\geq 0}^d \to \mathsf{X}_d$$

is defined by

$$\mathsf{Diter}(\boldsymbol{u}, \boldsymbol{l}) = \mathsf{Diterpre}((), (), \boldsymbol{u}, \boldsymbol{l}).$$

Proposition 4. *Under the identification $X^{\otimes d} \simeq \mathsf{X}_d$, $D_d^{\mathrm{iter}}((\boldsymbol{u}, \boldsymbol{l})) \in Y^{\otimes d}$ coincides with the image of $\mathsf{Diter}(\boldsymbol{u}, \boldsymbol{l}) \in X^{\otimes d}$ in $Y^{\otimes d}$.*

0.4. The kernel $X^{\otimes d} \to Y^{\otimes d}$.

Proposition 5. *Under the identification $X^{\otimes d} \simeq \mathsf{X}_d$, the kernel of $X^{\otimes d} \to Y^{\otimes d}$ is spanned by the following elements:*

- $(u_1, \dots, u_d, l_1, \dots, l_d)$ *for* $u_1, \dots, u_d \in \{0, \dots, N-1\}^d$ *and* $l_1, \dots, l_d \in \mathbb{Z}_{\geq 0}$ *with* $(u_i, l_i) = (0, 0)$ *for some* i.
- $(u_1, \dots, u_d, l_1, \dots, l_d) - (-1)^{l_i}(u_1, \dots, u_{i-1}, -u_i, u_{i+1}, \dots, u_d, l_1, \dots, l_d)$ *for* $u_1, \dots, u_d \in \{0, \dots, N-1\}^d$, $l_1, \dots, l_d \in \mathbb{Z}_{\geq 0}$ *and* $i \in \{1, \dots, d\}$
- $(u_1, \dots, u_d, l_1, \dots, l_d) - M^{l_i} \sum_{s=0}^{M-1} (u_1, \dots, u_{i-1}, u_i/M + sN/M, u_{i+1}, \dots, u_d, l_1, \dots, l_d)$ *for* $u_1, \dots, u_d \in \{0, \dots, N-1\}^d$, $l_1, \dots, l_d \in \mathbb{Z}_{\geq 0}$, $i \in \{1, \dots, d\}$ *and divisor* M *of* N *with* $u_i \equiv 0 \pmod{M}$ *and* $(u_i, l) \neq (0, 0)$. *(We can omit the caes* $M = 1$*).*

References

[1] M. Hirose, On the motivic fundamental group of the multiplicative group minus N-th roots of unity, in preparation.