Assignment 3 Problem 3

Which of the following binary trees must have height $O(\log n)$? Justify your answer.

a) There is a constant $c > 0$ such that for all nodes z in T, $z.left.height \leq z.right.height + c$.

This tree does not have height $O(\log n)$. Since there is no restriction on the height of the left subtree of a node, therefore, we can have a tree like all nodes are added on the right for each node. And for such tree, the height is $\Theta(n)$

b) Every node z that is not a leaf in T has exactly two children.

This tree does not have height $O(\log n)$. We can also provide an example that the tree has the height $\Theta(n)$. Considering a tree that except the first level(which only contains a root), all levels has two nodes, one is a leaf, and another one is not. Therefore, let h be the height of the tree, and n be the number of nodes, we have the relation: $2*(h-1)+1 = n$ rearrange and take logarithm on both sides, we have $h-1 = \log(n-1)$ which implies that $h \in \Theta(\log n)$

c) Given a BST T, let $N(z)$ be the number of nodes in the subtree rooted at z. If z is an empty subtree, then $N(z) = 0$.
There is a constant $0 < c \leq 1$ such that for every node z in T, $N(z.left) \geq c \times N(z.right) - 1$ and $N(z.right) \geq c \times N(z.left) - 1$.

This tree has height $O(\log n)$.
We have the relation $N(left) \geq c \times N(right) - 1$ and $N(right) \geq c \times N(left) - 1$.
If we know the N(left), then we can get $c \times N(left) - 1 \leq N(right) \leq \frac{N(left)+1}{c}$
Fix n, let h(n) be the max hight for n-node tree. We have the relation $h(n) = h(left)+1$ if we assume that $N(left) \geq N(right)$
Also, we have that $N(left) + N(right) + 1 = n$ and $N(right) = c \times N(left) - 1$
Simplifying, then we get $N(left) = \frac{n}{c+1}$ and $h(n) = h(\frac{n}{c+1}) + 1$
Note that $1 < c + 1 \leq 2$
when n = 0, h(0) = -1 $\leq \log(n+1) = 0$
when n = 1, h(1) = 0 $\leq \log(n+1) = \log 2 = 1$
assume for $i < n$, we have $h(i) \leq \log(i+1)$
$h(n) = 1 + h(\frac{n}{c+1}) \leq \log(\frac{n}{c+1}) + 1 = \log(\frac{n}{c+1}) + \log 2 = \log(\frac{2n}{c+1}) \in O(\log n)$
By induction, we prove that for such BST with n nodes, the height is $O(\log n)$