Midterm Problem 1: Power of a Multi-Precision Integer

We only consider the worst case, that is each recursive call have k odd.

Notice that in each recursion call with X and k,

- If k = 1, i.e. $k \in O(1)$, then $Power(X,k) \in O(1)$ since directly return X.
- Otherwise

$$T(X,k) = T\left(X, \left\lfloor \frac{k}{2} \right\rfloor\right) + O\left(\max(\lg Y, \lg Y)\min(\lg Y, \lg Y)^{\log_2 3-1}\right)$$

$$+ O\left(\max(\lg Y, \lg X)\min(\lg Y, \lg X)^{\log_2 3-1}\right)$$

Where Y = $Power\left(X, \left\lfloor \frac{k}{2} \right\rfloor\right) = X^{\left\lfloor \frac{k}{2} \right\rfloor}$.

Thus,

$$T(X,k) = T\left(X, \left\lfloor \frac{k}{2} \right\rfloor\right) + O\left((\frac{\mathrm{k}}{2}\lg X)^{\log_2 3}\right) + O\left(\frac{\mathrm{k}}{2}\lg X^{\log_2 3}\right)$$

$$T(X,k) = T\left(X, \left\lfloor \frac{k}{2} \right\rfloor\right) + O\left((\frac{\mathrm{k}^{\log_2 3}}{2} + \frac{k}{2})(\lg X)^{\log_2 3}\right)$$

Consider the recursion tree, since only one recursion call occurs, each level of the tree only has one node.

$T(X,k)$	$(\frac{k}{2}\lg x)^{\log_2 3} + \frac{k}{2}(\lg x)^{\log_2 3}$
	$\vert$
$T(X,\frac{k}{2})$	$(\frac{k}{4}\lg x)^{\log_2 3} + \frac{k}{4}(\lg x)^{\log_2 3}$
	$\vert$
$T(x,\frac{k}{4})$	$(\frac{k}{8}\lg x)^{\log_2 3} + \frac{k}{8}(\lg x)^{\log_2 3}$
$\vdots$	$\vdots$
$T(X,2)$	$(2\lg x)^{\log_2 3} + 2\lg x^{\log_2 3}$
	$\vert$
$T(X,1)$	1

$$\sum_{i=1}^{\log_2 k-1} (\frac{k}{i}\lg x)^{\log_2 3} + \frac{k}{i}(\lg x)^{\log_2 3}$$

Also, note that the floor symbol is ignored.

We have the total running time $O\left(\sum_{i=1}^{\log_2 k-1} (\lg X)^{\log_2 3} \left(\frac{k}{i} + \left(\frac{k}{i}\right)^{\log_2 3}\right)\right) =$ $O\left(\log X^{\log_2 3} \sum_{i=1}^{\log_2 k} \left(\frac{k}{i}\right)^{\log_2 3}\right)$, since $\frac{k}{i} + \left(\frac{k}{i}\right)^{\log_2 3} < 2\left(\frac{k}{i}\right)^{\log_2 3} = O\left(\left(\frac{k}{i}\right)^{\log_2 3}\right)$

Thus, $T(X,k) \in O\big((K \log X)^{\log_2 3}\big)$