Problem 1: Order notation

a) $f(n) = 3.14159n^3 + 4000n, g(n) = 0.0001n^3 - 2^{2^{10}}n^2$
$f(n) \in \Theta(g(n))$

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{3.14159}{0.0001} = 31415.9$ which between 0 and infinity

b) $f(n) = \log(n!)\,, g(n) = n(\log n)^2$

$$f(n) = \sum_{i=1}^{n} \log n < \sum_{i=1}^{n} (\log n)^2 = n(\log n)^2 = g(n) \quad \forall\, n > 3$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\sum_{i=1}^{n} \log n}{\sum_{i=1}^{n} (\log n)^2} = 0$$

Therefore, $f(n) \in o(g(n))$

c) $f(n) = (\log n)^{\log\log n}, g(n) = 2^{\sqrt{\log n}}$

Since $\log n \ll \sqrt{n}, \quad \log\log n \ll \sqrt{\log n}$

Let $x = \log n$, then $f(n) = x^{\log x}, g(n) = 2^{\sqrt{x}}$

$f(n)^{\log x} = x^{(\log x)^2}, g(n)^{\log x} = 2^{\sqrt{x}\cdot\log x} = x^{\sqrt{x}}$
Since $(\log x)^2 \ll \sqrt{x}$, we can say that $f(n) \in o(g(n))$

d) $f(n) = (n \, mod \, 3) \cdot n \;,\; g(n) = 0.1n$
Since $0 \le (n \, mod \, 3) \le 2$, we have $0 \le f(n) \le 2n$
Since f(n) can be 0 periodically, then the theta-notation does not apply.
Also, little-o does not work since both of f and g is linear, and $f(n) \in \Theta(g)$ if $f(n) \neq 0$ since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ where c is between 0 and infinity when $f(n) \neq 0$.

Therefore, $f(n) \in O(g(n))$