Assignment 6 Problem 2

a) Let $d_i(u,v)$ = weight of shortest path from u to v using exactly i edges

Then $d_0(u,v) =$

- 0 if $u = v$
- ∞ if $u \neq v$

And we want $d_h(u,v)$.

Compute d_i from d_{i-1}: $d_i(u,v) = min_x\big(d_{i-1}(u,x) + w(x,v)\big)$ if x exists

Where x are vertices that exists an edge that points from x to v.

Otherwise, $d_i(u,v) = \infty$

To prove the correctness, we consider all possibilities of d_i, then correct by induction on i.

Base case, for d_0, only $u = v$ there exists a path with length 0, otherwise no path exists. Thus, the base case holds.

Induction hypothesis: if d_{i-1} correct.

Induction steps, since d_{i-1} is correct, then we just need to find d_i from d_{i-1}, consider $d_i(u,v)$. To get this, we should consider all adjacent edges that points to v. Say the starting point of one of such edge is x, and the end point is v. Then the lightest path from u which contains x to v should be $d_{i-1}(u,x) + w(x,v)$. By comparing all such edges and find the minimum one, then we get the lightest path from u to v with i edges. Also, if such edges do not exist, then this means that point v is isolated, and no shortest path from u to v exist, then $d_i(u,v)=\infty$.

Therefore, d_i is correct. By induction, the algorithm is correct.

Pseudo code:

Initialize $d_0(u,v)$ for all u,v as above.

for i from 1 to h do

  for $v \in V$ do

    for $u \in V$ do

      $d_i(u,v)=\infty$

      for each edge (x,v) do

        $d_i(u,v)=\min\big(d_i(u,v), d_{i-1}(u,x)+w(x,v)\big)$

      od

    od

  od

od

The runtime is obviously $O(n^3h)$

b) $A_{h_1+h_2}[u,v]=min_x\big(A_{h_1}[u,x]+A_{h_2}[x,v]\big)$ for all $x \in V$

This is true, because consider a h_1+h_2 length path, after first h_1 edges, we must at some vertex. So we just need to minimize the sum of weights of the first h_1 edges and the last h_2 edges. By considering all vertices after h_1 edges, we have considered all possibilities.

The runtime is obviously $O(n^3)$, $O(n^2)$ for all u,v, another $O(n)$ for trying all vertices.