

ASSIGNMENT 3

DUE: Wednesday, February 2, 11:59 PM. DO NOT COPY. ACKNOWLEDGE YOUR SOURCES.

Please read <http://www.student.cs.uwaterloo.ca/~cs341> for general instructions and policies.

Exercises. The following exercises are for practice only. You may ask about them in office hours. Do not hand them in.

1. You want to create a given number n by starting with the number $t = 1$ and then using steps where you either double t or add 1 to t . For example, you can get 7 by starting with 1 and applying the sequence of steps $+1, \times 2, +1, +1, +1$ or the sequence $\times 2, +1, \times 2, +1$. Give a greedy algorithm to find the shortest possible sequence to produce the number n .
2. **Does greed bring happiness?** The *happiest interval scheduling* problem is defined as follows: Given a set of n intervals defined by start and finish times $(s_1, f_1), \dots, (s_n, f_n)$ and n positive integers h_1, \dots, h_n denoting the *happiness* value of each interval, find a subset $I \subseteq \{1, 2, \dots, n\}$ of disjoint intervals that maximizes $\sum_{i \in I} h_i$.

Determine which of the following greedy algorithms correctly solves the happiest interval scheduling problem. Justify each answer with a proof of correctness or a counter-example.

- (a) **Earliest finish time.** Choose the interval i with the earliest finish time, discard all intervals that overlap with i , and recurse.
- (b) **Highest happiness per minute.** Choose the interval i with the maximum *happiness per minute* ratio $\frac{h_i}{f_i - s_i}$, discard all intervals that overlap with i , and recurse.
- (c) **Kondo's algorithm.** If no intervals overlap, select them all. Otherwise, discard the interval i with the smallest happiness value h_i . Repeat.

Problems. To be handed in.

1. [10 marks] **Greedy Algorithms: Skyline Antenna Optimization.**

Consider a city made up of rectangular buildings. An example of a city with four buildings A, B, C and D is given in the picture below. The rooftops of the buildings are labelled with a, b, c and d. Note that some building will be in front of others, but this does not matter for our purposes.

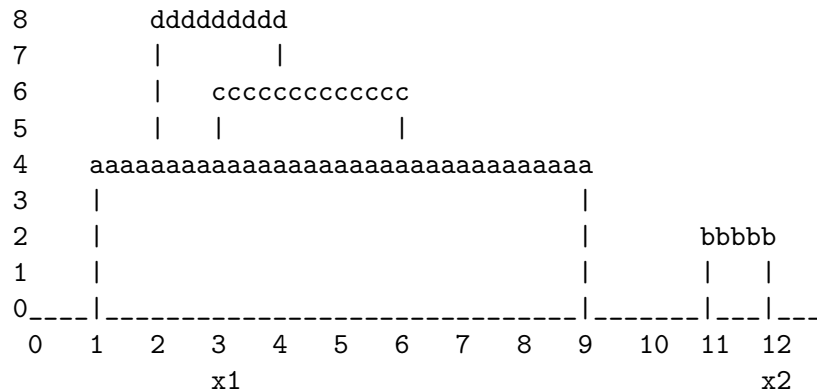
A very clever engineer has designed a super energy efficient antenna that transmits along a plane, instead of in three dimensions. Antennas are positioned along the x -axis. An antenna with x -coordinate x can transmit to a given building only if the horizontal line segment comprising the buildings roof contains x (the height of the building does not matter.) Your goal is minimize the number of antennae that need to be deployed so that every building is covered.

Formally, you are given as input n and two length n arrays:

- $left[i]$ is the x -coordinate of the left edge of the i^{th} building.
- $right[i]$ is the x -coordinate of the right edge of the i^{th} building.

The algorithm should then output a minimal list of (integral) coordinates $[x_1, x_2, \dots, x_k]$ on the x -axis such that for every building, at least one x_i overlaps with the horizontal line segment defining the rooftop of the building.

For example, consider a scenario with four rooftops having $[left, right]$ coordinates $[1, 9]$, $[11, 12]$, $[3, 6]$ and $[2, 4]$.



For this example, one optimal solution is $\{x_1, x_2\} = \{3, 12\}$, of size $k = 2$. The antenna at location 3 covers buildings A, D and C. The antenna at location 12 covers building B.

For full marks your algorithm should have run-time $O(n \log n)$. An $O(n^2)$ algorithm will earn substantial part marks.

2. There are n trails around the campus that you'd like to explore, each will bring you a happiness value of $h[i]$ (but only once, otherwise it gets repetitive).

The melting snow brings one major concern. Data compiled from <https://goose-watch.uwaterloo.ca/> suggest that trail i will become inaccessible $d[i]$ days from today.

Given that you want to explore at most one trail per day (without repetitions), compute the maximum happiness that can be achieved while avoiding geese encounters.

All values in $h[1 \dots n]$ and $d[1 \dots n]$ are positive integers between 1 and n .

- (a) [2 marks] Provide a counter example with n at most 3, and $1 \leq h[i], d[i] \leq n$, to the following greedy algorithm: always explore an accessible trail with maximum happiness.
- (b) [2 marks] Provide a counter example with n at most 3, and $1 \leq h[i], d[i] \leq n$, to the following greedy algorithm: among all trails with the least accessible time remaining, explore the one with maximum happiness.
- (c) [1 mark] Show that the $d[i] \leq n$ restriction is in fact not necessary. That is, if there is some trail with $d[i] > n$, there is some optimum solution where it's explored on day n or earlier.
- (d) [5 marks] Give an $O(n^2)$ time or faster algorithm for finding the maximum total happiness that can be achieved while avoiding geese. Justify the correctness of your algorithm.

Challenge Questions. This is for fun and enrichment only. Do not hand it in.

1. Let (c_1, c_2, \dots, c_n) be a coin system, where c_i denotes the i -th smallest denomination, and it forms a strictly increasing sequence with $c_1 = 1$. Does the greedy algorithm for coin changes in Lecture 6 return an optimal solution if the coin system satisfies one of the following properties?
 - (a) (c_1, c_2, \dots, c_n) satisfies that c_i divides c_{i+1} for all $i = 1, \dots, n - 1$. Notice that, this case includes geometric sequences.
 - (b) (c_1, c_2, \dots) is an infinite arithmetic sequence, i.e. $c_i = 1 + (i - 1) \times d$ for some integer $d \geq 1$, for all $i \in \mathbb{Z}_+$. (How about a finite arithmetic sequence?)