Assignment 8 Problem 1 : Hardness of Maximum 2-SAT

a) MAX 2-SAT-DEC should have input with n Boolean variables namely $x_1, x_2, \dots, x_n$ and a list of m clauses which is the same as the 2-SAT instance stated in the question, and a number k.

   The output should be whether there exists an assignment to the n variables x_* that satisfies at least k of these clauses.

b) To convert the input of CLIQUE to MAX 2-SAT-DEC. Consider the graph G(V,E), for each vertex v_i, construct the variable $x_{i1}, x_{i2}, \dots, x_{ik}$ where k is another part of input of CLIQUE. For each variable x_{ij}, it means that whether v_i is the jth vertex in the clique. Then since a single vertex cannot repetitively appear in the clique, we construct the clauses $\neg x_{ia} \vee \neg x_{ib}\ \forall\ i = 1,2, \dots, |V|\ and\ 1 \le a < b \le k$. Also note that any pairs of vertices in the clique should be adjacent, then construct the clauses $\neg x_{ia} \vee \neq x_{jb}\ \forall\ 1 \le a < b \le k\ and\ (v_i, v_j) \notin E(G)$. Then, we complete our convert.

   The input of MAX 2-SAT-DEC should be firstly, $k|V|$ Boolean variables namely $x_{11}, x_{12}, \dots, x_{1k}, x_{21}, x_{22}, \dots, x_{2k}, \dots, x_{|v|1}, x_{|v|2}, \dots, x_{|v|k}$, and all clauses we generated according to the rule stated previously, and lastly, the number k that should be input is the total number of the generated clauses. (This means that we want all clauses to be true. i.e. have to obey the structure property of the clique).

   To prove the correctness, if CLIQUE returns true, then by number the vertices in the k-size clique, MAX 2-SAT-DEC is true by assign variables at the corresponding location to be true according to the way we build the clauses. If MAX 2-SAT-DEC returns true, find the variables that are true, then put corresponding vertex in the corresponding location of the clique we can also get a clique with size k.

Since we are only calling MAX 2-SAT-DEC once, the only thing we need to consider is the time spent to convert the input. Since we are building $k|V|$ variables and the total number of $O(k^2|V|)$ (this is for $\neg x_{ia} \vee \neg x_{ib} \ \forall\ i = 1,2,\dots,|V|\ and\ 1 \leq a < b \leq k$) $+O(k^2|V|^2)$ (this is for $\neg x_{ia} \vee \neq x_{jb} \ \forall\ 1 \leq a < b \leq k\ and\ (v_i, v_j) \notin E(G)$). Therefore, the converting time using should also be polynomial.

Therefore, $CLIQUE \leq_p MAX\ 2 - SAT - DEC$.

c) MAX 2-SAT-DEC is in NP since the certificate is the assignment of Boolean variables, the verification is to check whether there are at least k clauses satisfied. Therefore, $O(m)$ time needed.

Also, since part b) shows that $CLIQUE \leq_p MAX\ 2 - SAT - DEC$. Therefore, MAX 2-SAT-DEC is NP-complete.