CS281 Section #5 Notes, v 1.1

Due:

Boring Title: Undirected Graphical Models, Exciting Title: Markov Random Fields

1 Independence properties are simpler than in DAGS

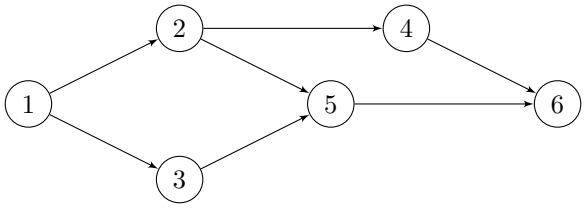
- **global Markov property**: for node sets A, B, C in graph G, $\mathbf{x}_A \perp \mathbf{x}_B \,|\, \mathbf{x}_C$ iff removing all nodes in C leaves no path connecting any node in A to any node in C
- **undirected local Markov property**: given its set of immediate neighbors (i.e. its **Markov blanket**), a node t is conditionally independent of all other nodes
- **pairwise Markov property**: two nodes conditionally independent given the rest if no direct edge between them
- global implies pairwise and vice versa

Example 1. Markov blanket for UGMs is defined above. What does the Markov blanket in DGMs include?
Solution: Parents, children, and co-parents

2 DGM to UGM Conversion

For each pair of parent nodes that share a child, marry them by adding an edge. Then drop the direction of all edges. Finally, use the CI properties of UGMs to answer a CI query more easily than in a DGM. At worst, we lose a few CI assumptions. However, we should not introduce new ones.

Example 2. Convert the following DGM to UGM:



Solution: add edges between 2-3 and 4-5, then drop directions

3 Distributions and Graphs

- graph G is an **I-map** of distribution p if $I(G) \subseteq I(p)$ where $I(.)$ is "the set of conditional independencies encoded by...". **Question:** is a naive bayes DGM with extra connections among the x's an I-map for $p(x, y)$ with the typical naive Bayes assumptions?
- G is a **perfect map** of p if $I(G) = I(p)$
- DGMs, UGMs perfect maps for different (but overlapping) sets of distributions
- Graphs only specify the conditional independence structure, nothing about the parameters, distributions.
- Shape of graph determines difficulty of inference

4 Parameterization of UGMs

Each edge in a DGM represents a normalized conditional probability distribution. What do undirected edges represent?

- A **clique** is a set of nodes such that each node has an edge with all others in the set. A **maximal clique** is the click of the largest size that a node belongs to.
- Each clique has a **potential function** that assigns each possible configuration a score. Considering just 8 nodes with binary values, this is 2^8 states to score for just the one clique.
- We use **log-potentials** $\theta_c(x_c)$. The book uses potentials $\psi_c(x_c) = \exp[\theta_c(x_c)]$
- joint distribution over $\mathbf{x}$'s proportional to product over all potentials

$$\begin{aligned} p(x_1, ..., x_T) &= \exp\big[\sum_c \theta_c(x_c) - A(\theta)\big] \\ &\propto \prod_c \exp\big[\theta_c(x_c)\big] = \prod_c \psi_c(x_c) \end{aligned}$$

Very important: UGMs are **globally normalized** ($A(\theta)$ term) instead of **locally normalized** like each edge of a DGM. More about this below in CRFs.

Example 3. Draw the UGM corresponding to the Naive Bayes DGM and describe how you would form the potential functions
Solution: Take away the edge directions, nothing to moralize, let $\theta_c(x_c, y) = \log p(x_c|y)$. $\exp[\sum_c \theta_c(x_c, y)] = \prod_c \psi_c(x_c, y)$ won't be normalized but $A(\theta)$ will take care of it. Give y its own potential $\theta(y) = \log p(y)$

5 Ising model

Example 4. Let X_s be the Bernoulli random variable associated with node s on a graph $G = (V, E)$ with $|V| = m$. X_s takes on the **spin** value $\{-1, +1\}$. This could correspond to magnet orientation, pixel value, and many other things. X_s and X_t interact if they have a direct edge in the graph (we expect a certain relationship between neighboring pixels). Let $\theta_{st} \in \mathbb{R}$ be the strength of the edge (s, t) and θ_s be the marginal potential for node s (determined by an external field). Write down the joint probability of the set of $\mathbf{x}$ in terms of potential functions that correspond to the model above. It should look like an exponential family. How do you calculate $A(\theta)$? What is the distribution's dimension?
Solution:

$$p_\theta(x) = \exp\big[\sum_{s\in V} \theta_s(x_s) + \sum_{(s,t)\in E} \theta_{st}(x_s, x_t) - A(\theta)\big]$$

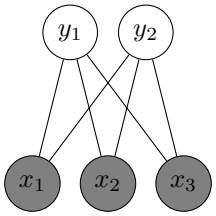
$$A(\theta) = \log \sum_{x\in\{0,1\}^m} \exp\big[\sum_{s\in V} \theta_s(x_s) + \sum_{(s,t)\in E} \theta_{st}(x_s, x_t)\big]$$

the dimension of the family $d = m + |E|, \theta \in \mathbb{R}^d$

6 Conditional Random Fields

- Clique potentials functions over hidden configurations y conditioned on observed features x
- Can be thought of as a **structured output** form of logistic regression
- CRF advantage over MRF like logistic regression vs. naive Bayes, i.e. don't need to model things that we always observe
- **x are not necessarily independent**

An example with pairwise cliques:



See a time series CRF at end of notes.

7 Stereo Depth Reconstruction CRF

Example 5. Given two images $\mathbf{x}_L$ and $\mathbf{x}_R$ taken at a small known angle difference, recover the depth measurement of each pixel by estimating the position of pixel $\mathbf{x}_L(i,j)$ in $\mathbf{x}_R$, which will have a different horizontal value and the same vertical value. This difference, call it y_s, is the **disparity** at i,j. By computing each such disparity, a disparity map shows an estimate of depth. Write down a suitable potential function $\psi_s(y_s, \mathbf{x})$. How could you define a $\psi_{st}(y_s, y_t)$ to enforce similar disparity in neighboring pixels?

Solution:

$$\psi_s(y_s, \mathbf{x}) \propto \exp\big[-\frac{1}{2\sigma^2}(\mathbf{x}_L(i_s, j_s) - \mathbf{x}_R(i_s + y_s, j_s))^2\big]$$
$$\psi_{st}(y_s, y_t) \propto \exp\big(-\frac{1}{2\gamma^2}(y_s - y_t)^2\big)$$

σ and γ are parameters. Demo: http://nghiaho.com/?page_id=1366. We will cover soon, but the general idea will make sense. Datasets available here: http://vision.middlebury.edu/stereo/ and a paper here: http://www.cs.middlebury.edu/~schar/papers/LearnCRFstereo_cvpr07.pdf

8 Time series models

Time-dependent patterns. Be it a generative or discriminative model, we may be interested in questions like: how likely is this (underlying) sequence given another (observed) one. Some examples discussed in class: assigning discrete phoneme labels to continuous speech signals, assigning a continuous label (inferred trajectory of airplane) to noisy continuous signals obtained from radar.

9 Example in detail: Hidden Markov Model

Assume that the underlying process is a discrete Markov process. Each state is connected to the data via an **emission model**. There are K discrete states and the transition probability between them is given by the matrix $\mathbf{A} \in \mathbb{R}^{k \times k}$, which fully parametrize the categorical dist. for y_{t+1}.

Example 6. Memorylessness An important hallmark of a Markov process is that the next state only depends on the current state, and not the previous states. This property, called "memorylessness", is formalized by the Chapman-Kolmogorov equation. If we denote the state of the system at time t y_t, then

$$p(y_{t+1}|y_t, y_{t-1}, ..., y_1) = p(y_{t+1}|y_t).$$

Is this property illustrated in the UGM representation?

9.1 Parameters of the HMM

Given data $\mathbf{X}$, the likelihood of a state sequence $\mathbf{y}$ is given by:

$$P(\mathbf{X}, \mathbf{y}) = p(y_1) \prod_{t=2}^{T} p(y_t|y_{t-1}) \prod_{t=1}^{T} p(\mathbf{x}_t|y_t).$$

Note that the three components are:

- $p(y_1)$ the initial probabilities. Usually it is parameterized by a K-sized vector π.
- $p(y_t|y_{t-1})$ the transition probabilities. It is parameterized by the transition matrix $\mathbf{A}$.
- $p(x_t|y_t)$ are the emission probabilities. In a typical setting, for each y_t there is a multivariate normal (MVN) distribution parameterized by $\{\mathbf{\Sigma}_k, \mu_k\}$.
- Stay within exponential family with categorical and Gaussian distributions

Maximum likelihood estimate (MLE) for HMMs is difficult because the joint distribution does not factorize over T (why?). In fact, if there are K states and T data points, we will have to consider all K^T paths to do maximum likelihood exactly. More on this in a few weeks.

10 HMMs with continuous state space: Kalman filtering

In an HMM, we considered the scenario where continuous observations are generated by a process transitioning in a discrete state space. Expanding on HMMs, **Kalman filtering** considers continuous observations that are generated by continuously changing processes. Specifically, we assume to be getting noise sensor data $\mathbf{X} \in \mathbb{R}^D$ that comes from a source $\mathbf{Y} \in \mathbb{R}^D$. We can use similar notations from our discussion of HMMs. The components are:

- $p(\mathbf{y}_t|\mathbf{y}_{t-1}) = \mathcal{N}(\mathbf{y}_t|\mathbf{A}\mathbf{y}_{t-1}, \mathbf{\Gamma})$ are analogous to transition probabilities in HMMs. Alternatively, we can write this as a linear equation (this model is sometimes called a **linear dynamical system**).

  $$\mathbf{y}_t = \mathbf{A}\mathbf{y}_{t-1} + \mathbf{w}_t,$$

  where $\mathbf{w}_t$ is a zero-mean Gaussian noise term.
- $p(\mathbf{x}_t|\mathbf{y}_t) = \mathcal{N}(\mathbf{x}_t|\mathbf{C}\mathbf{y}_{t-1}, \mathbf{\Sigma})$ are the emission probabilities. It can be similarly written in linear form:

  $$\mathbf{x}_t = \mathbf{C}\mathbf{y}_t + \mathbf{v}_t.$$

Since all the conditional probabilities are Gaussian, and linear transformations of Gaussian distributions are still Gaussians, Kalman filters are relatively easy to analyze.

11 Maximum-entropy Markov Models (MEMMs) and CRFs

Consider a directed MEMM where $p(y|x,w) = \exp\left[\log p(y_1) + \log p(y_1|x_1) + \sum_{t=2}^{T} \log p(y_t|y_{t-1}, x_t, w)\right]$ is the probability of a sequence of labels given some data. MEMMs and the other time series models above are so-called **locally normalized** models. Each conditional probability distribution, such as $p(y_t|y_{t-1})$, sums to one. In a **globally normalized** models, each undirected edge can represent an un-normalized potential function. We can compute a product over clique potentials using a variety of functions, and then use a normalizing term.

$$p(y|x,w) = \exp\left[\sum_t \theta^{obs}(y_t, x_t) + \sum_{t=2}^{T} \theta^{trans}(y_t, y_{t-1}) - A(\theta)\right]$$

Example 7. How do you calculate $A(\theta)$?
Solution Take the expression above without the $-A(\theta)$, and sum over all combinations of $\mathbf{y}$

Example 8. In practical applications, we sometimes need signals that come later to disambiguate previous signals. I.e., we need an information flow from $\mathbf{x}_t$ to $\mathbf{y}_{t-1}$. Analyze the time series models that we have discussed. Is this possible in all the models?