CS281 Section #4 Notes, v 1.0

1 Neural networks

1.1 Exercises

- See the PyTorch demo here: [walkthrough]

2 Directed Graphical Models

2.1 Review

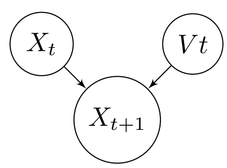
A directed graphical model G over V random variables is a way to factorize a probability distribution $p(x_{1:V})$. They are also called *bayes nets*. They also encode the *conditional independence* structure of the variables, which we'll explain below.

$$p(x_{1:V}|G) = \prod_{i=1}^{V} p(x_i|x_{\text{parents}(i)})$$

Gaussian Bayes Net

Suppose we have a node x_i and its parent nodes $x_{\text{parents(i)}}$ representing continuous variables. Let the parent variables be distributed as Gaussians. Here, we choose to model x_i as a linear function of its parents with Gaussian noise: $p(x_i \,|\, x_{\text{parents(i)}}) = \mathcal{N}(x_i; \text{linear}(x_{\text{parents(i)}}), \sigma_i^2)$

Example 1. X, V are position, velocity. $X(t+1) \sim X(t) + V(t)dt + \epsilon$



Formally,

$$p(x_i \,|\, x_{pa(i)}) = \mathcal{N}(x_i; \mu_i + w_i^T x_{pa(i)}, \sigma_i^2)$$

is a linear Gaussian conditional probability distribution (CPD).

In lecture, we show that multiplying all these CPDs results in a jointly Gaussian distribution,

$$\prod_{i=1}^{V} p(x_i \,|\, x_{pa(i)}) = p(X) = \mathcal{N}(X; \mu, \Sigma)$$

It's straightforward that $X = [\mu_1, ..., \mu_V]$. What remains is to find the covariance matrix. For review from lecture, this is how we do it: for simplicity, rewrite the conditional probability distribution in the following way:

$$x_i = \mu_i + \sum_{j \in x_{pa(i)}} w_{ij}(x_j - \mu_j) + \sigma_i z_i$$

$$z_i \sim \mathcal{N}(0,1) \quad \text{(Gaussian random noise)}$$

Note that w_{ij} can be 0 if x_j is not a parent of x_i. For the "root" nodes, with no parents, all the coefficients are 0. Let $S = \text{diag}(\sigma)$, rewriting this again in matrix-vector form:

$$\begin{aligned}
\mathbf{x} - \mu &= W(\mathbf{x} - \mu) + S\mathbf{z} \\
\mathbf{x} - \mu - W(\mathbf{x} - \mu) &= S\mathbf{z} \\
\mathbf{x} - \mu(I - W) &= S\mathbf{z} \\
\mathbf{x} - \mu &= (I - W)^{-1} S\mathbf{z}
\end{aligned}$$

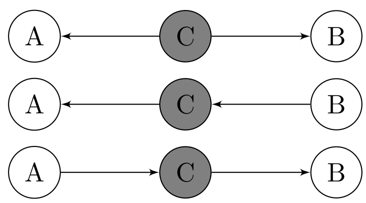
$$\mathbf{\Sigma} = \text{Cov}[\mathbf{x} - \mu] = \text{Cov}[(I - W)^{-1} S\mathbf{z}] = (I - W)^{-1} S \,\text{Cov}[\mathbf{z}] S (I - W)^{-1T}$$

which implies that the variance is $(I - W)^{-1} S^2 (I - W)^{-1T}$

Conditional Independence Statements

Different graph structures can encode the same conditional independence statements.

Example 2. These DAGs for the statement $A \perp B \,|\, C$ are equivalent:



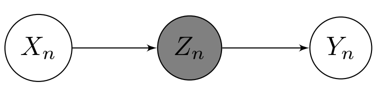
D-Separation

Conditional independencies can disappear or appear with new knowledge (when a random variable is observed). Z is said to *d-separate* X and Y if information about Z renders X and Y conditionally independent: $X \perp Y \,|\, Z$. In general, we can also talk about the *d-separatedness* of two random variables, given a collection.

Formally: Let X, Y be disjoint subsets of nodes in a DAG G. A path between X and Y is given by a sequence of edges that connects a node in X to a node in Y (directionality doesn't matter). Let Z be another subset of nodes disjoint from both X and Y.

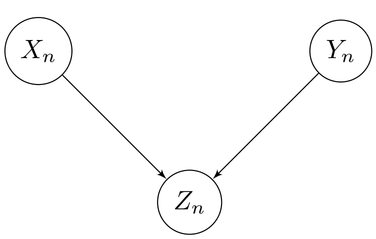
Then $X \perp Y \,|\, Z$ if, for every path between X and Y, one of the following is true:

- $\exists\{\rightarrow C \rightarrow\}$ or $\{\leftarrow C \leftarrow\}$ or $\{\leftarrow C \rightarrow\}, C \in Z$



  Note that in this scenario, when Z_n is unobserved, X_n and Y_n are no longer necessarily conditionally independent. Example: X_n, Y_n, Z_n represents mitochondrial DNA (passed down only from the mother) from grandmother, mother, and an individual. Observing mother's mitochondrial DNA renders X_n and Y_n independent.

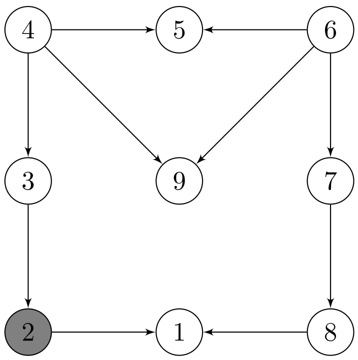
- $\exists\{\rightarrow C \leftarrow\}, C \notin Z$

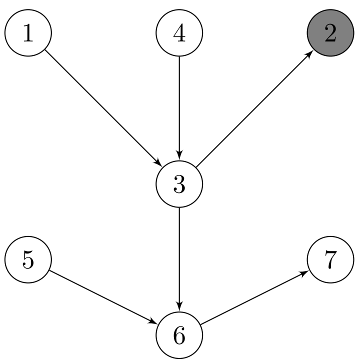


Note that in this scenario, when Z_n becomes observed, X_n and Y_n are no longer conditionally independent. Example: Let X_n, Y_n be the outcomes from 2 fair dice rolls, and let Z_n be the sum. A priori, without observing the sum, the dice rolls are independent. However, if the sum is known, X_n, Y_n are no longer independent.

2.2 Exercises

1. For the following graphs, determine the largest set $X_{i:j}$ such that $X_1 \perp X_{i:j} \mid X_2$

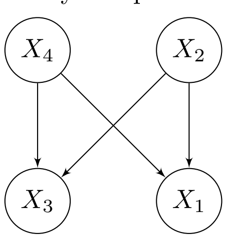


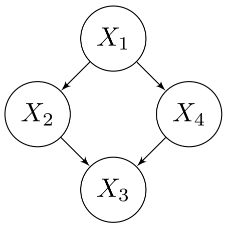


2. Suppose there is a distribution across $X_{1:4}$ such that the only independencies are: $X_1 \perp X_3 \mid \{X_2, X_4\}$ and $X_2 \perp X_4 \mid \{X_1, X_3\}$.

   Example for context: It was just NYFW (New York Fashion Week). Let X_i be the color of model i's shirt. Model 1 and 2 are friends. Model 2 and 3 are friends. Model 3 and 4 are friends. Model 1 and 4 are friends. For sartorial reasons, no model friends can have the same color shirt, lest they be photographed together in the street.

   Can you represent this distribution as a DAG?

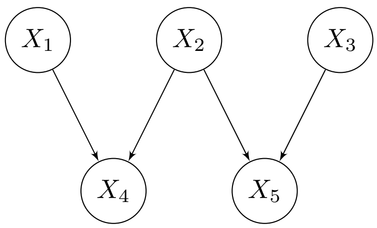




Both of these DAGS $X_1 \perp X_3 \,|\, \{X_2, X_4\}$ but $X_2 \not\perp X_4 \,|\, \{X_1, X_3\}$.

3. Let X_1, X_2, X_3 represent the outcomes of 3 independent binary RV's. Let $X_4 = 1\{X_1 = X_2\}, X_5 = 1\{X_2 = X_3\}$.

   (a) Draw DAG

   (b) Under what circumstance is $X_4 \perp X_5$, if any? (Note $X_4 \perp X_5$ is not implied by this GM)