Axion Electrodynamics

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1 Strong CP Problem and Axion

The strong-CP violating term $\bar{\theta}$ (naively speaking, which explains the different behavior of matter and antimatter) appears as an input of the standard model.

$$\bar{\theta} = \frac{g^2}{32\pi^2} G_{\alpha\mu\nu} \tilde{G}^{\alpha\mu\nu} \tag{1.1}$$

Theoretically, this $\bar{\theta}$ value lies between 0 and 2π. However, the large $\bar{\theta}$ value would induce a large electric dipole moment of a neutron (nEDM), which is NOT observed in precision physics experiments. Thus, experimental constraints on currently observed nEDM value implies that the strong-CP violation must be extremely tiny. But **why** is it so tiny? This is the problem of naturalness, so-called Strong CP problem.

In 1977, Peccei-Quinn theory introduced a way to resolve the Strong CP problem. This theory effectively promote $\bar{\theta}$ to a field, not a steady value, by adding a new symmetry (so-called Peccei Quinn symmetry) that becomes spontaneously broken. It results a new particle naturally relaxing the CP violation parameter to 0. This particle is called **Axion**. Frank Wilczek named this new hypothesized particle the axion after a brand of laundry detergent because it cleaned up a problem.

2 Classical Electrodynamics

In this section, I want to briefly summarize the field representation of the Maxwell's equations.

2.1 Maxwell's equations

The Maxwell's equations are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{2.1}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.2}$$

$$\nabla \times \vec{E} + \partial_t \vec{B} = 0 \tag{2.3}$$

$$\nabla \times \vec{B} - \epsilon_0 \mu_0 \partial_t \vec{E} = \mu_0 \vec{J}_e \ . \tag{2.4}$$

Four vector potentials and sources are

$$A^\mu = (\phi/c, \ \vec{A}) \tag{2.5}$$

$$\vec{J}_e^\mu = (c\rho_e, \ \vec{J}_e) \ . \tag{2.6}$$

The followings are the anti symmetric field tensor and its dual.

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \tag{2.7}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & -E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{bmatrix} \tag{2.8}$$

2.2 Properties of the Field Tensor

Here, I list several useful properties of the field tensor.

- The field tensor, which is anti-symmetric, exhibits only 6 independent components.

$$F^{\mu\nu} = -F^{\nu\mu} \tag{2.9}$$

- The inner product is Lorentz invariant.

$$F_{\mu\nu}F^{\mu\nu} = 2\left(B^2 - \frac{E^2}{c^2}\right) \tag{2.10}$$

- The field components can be extracted.

$$E^i = cF^{i0} = -\frac{d\phi}{dx^i} - \frac{dA^i}{dt} \tag{2.11}$$

$$B^i = \frac{1}{2}\epsilon_{ijk}F_{jk} = (\nabla \times \vec{A})^i \tag{2.12}$$

- Maxwell's equation is simplified. (2.2), (2.3) and (2.1), (2.4) are coupled.

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{2.13}$$

$$\partial_\mu F^{\mu\nu} = \mu_0 J_e^\nu \tag{2.14}$$

- The Lagrangian is given by following.

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - A_\mu J_e^\mu \\ &= -\frac{1}{4\mu_0}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - J_e^\mu A_\mu \\ &- \frac{1}{2\mu_0}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) - J_e^\mu A_\mu\end{aligned} \tag{2.15}$$

- Using the Lagrangian, the Euler-Lagrange equation can be obtained.

$$\begin{aligned}&\partial_\mu\left(\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}_0}{\partial A_\nu} = 0 \\ \Longrightarrow &-\partial_\mu\left(\frac{1}{\mu_0}(\partial^\mu A^\nu - \partial^\nu A^\mu)\right) + J_e^\nu = 0\end{aligned} \tag{2.16}$$

A key mathematical trick for the derivation is following differentiation.

$$\begin{aligned}\frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\kappa A^\lambda \partial_\kappa A_\lambda) &= \partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\kappa A_\lambda) + \partial_\kappa A_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\kappa A^\lambda) \\ &= \partial^\kappa A^\lambda \delta_\kappa^\mu \delta_\lambda^\nu + \eta^{\kappa\alpha}\eta^{\lambda\beta}\partial_\kappa \partial_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\alpha A_\beta) \\ &= \partial^\kappa A^\lambda \delta_\kappa^\mu \delta_\lambda^\nu + \eta^{\kappa\alpha}\eta^{\lambda\beta}\partial_\kappa \partial_\lambda \delta_\alpha^\mu \delta_\beta^\nu \\ &= 2\partial^\mu A^\nu\end{aligned} \tag{2.17}$$

Where η and δ stand for the matrix tensor and Kronecker delta respectively. Similarly,

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\kappa A^\lambda \partial_\lambda A_\kappa) = 2\partial^\nu A^\mu \ . \tag{2.18}$$

2.3 Duality: Magnetic Source

The duality symmetry of the Maxwell's equations is more clearly demonstrated when the existence of magnetic monopole is considered.

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{2.19}$$

$$\nabla \cdot \vec{B} = \mu_0 \rho_m \tag{2.20}$$

$$\nabla \times \vec{E} + \partial_t \vec{B} = -\mu_0 \vec{J_m} \tag{2.21}$$

$$\nabla \times \vec{B} - \epsilon_0 \mu_0 \partial_t \vec{E} = \mu_0 \vec{J_e} \tag{2.22}$$

By introducing the magnetic source four vector $J_m^\mu = (c\rho_m,\ \vec{J_m})$, the field tensor Maxwell's equations are modified:

$$\partial_\mu F^{\mu\nu} = \mu_0 J_e^\nu \tag{2.23}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = \mu_0 \frac{J_m^\nu}{c} \tag{2.24}$$

Note that the conservation of the charges is simply $\partial_\mu J_e^\mu = \partial_\mu J_m^\mu = 0$.

It is well known that the Maxwell's equations are invariant under the duality transformation: mixing of $\vec{E}$ and $\vec{B}$. This field mixing can be easily understood as **SO(2)** rotation [1],

$$U(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{2.25}$$

Field strength and its dual are invariant under the rotation of the sources.

$$\begin{pmatrix} J_e'^{\mu} \\ J_m'^{\mu}/c \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} J_e^\mu \\ J_m^\mu/c \end{pmatrix} \tag{2.26}$$

Since neither magnetic monopole nor magnetic current are observed, we can regard $J_m'^{\mu} = 0$, which leads to the constraint for ϕ.

$$0 = -\sin\phi J_e^\mu + \cos\phi J_m^\mu/c \tag{2.27}$$

3 Maxwell's Equations With Axion

3.1 Axion Lagrangian & Maxwell's Equations

In this section, I show how axion field changes the Maxwell's equations. I introduce axion-like term,

$$\mathcal{L}_\theta = -\frac{\kappa}{\mu_0 c}\theta \vec{E}\cdot\vec{B} = \frac{\kappa}{4\mu_0}\theta F_{\mu\nu}\tilde{F}^{\mu\nu} \ . \tag{3.1}$$

θ is pseudo-scalar field, representing that the intrinsic nature of axion (pseudo-scalar Boson). κ is a coupling constant. Note that the presence of the axion-like term directly breaks the SO(2) symmetry unless we are at $th\bar{e}ta = 0$, the CP-preserving configuration. The Lagrangian for the axion electrodynamics is given as below, combining the lagrangian of the electromagnetic field, axion-like term, and axion lagrangian with potential $U(\theta)$: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\theta + \mathcal{L}_a$.

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - A_\mu J_e^\mu + \frac{\kappa}{4\mu_0}\theta F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}(\partial^\mu\theta)(\partial_\mu\theta) - U(\theta) \tag{3.2}$$

where

$$\mathcal{L}_a = \frac{1}{2}(\partial^\mu\theta)(\partial_\mu\theta) - U(\theta) \ . \tag{3.3}$$

The Euler-Lagrange equation associated with A for the Lagrangian is

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \ , \tag{3.4}$$

where the Lagrangian is simplified as

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2}(\partial^\mu\theta)(\partial_\mu\theta) - U(\theta) + \frac{\kappa}{4\mu_0}\theta F_{\mu\nu}\tilde{F}^{\mu\nu} \ . \tag{3.5}$$

From (2.16), we know how to deal with $\mathcal{L}_0$. This term contributes

$$-\partial_\mu\left(\frac{1}{\mu_0}(\partial^\mu A^\nu - \partial^\nu A^\mu)\right) + J_e^\nu = -\partial_\mu\left(\frac{1}{\mu_0}F^{\mu\nu}\right) + J_e^\nu \ . \tag{3.6}$$

to the first term of the Euler-Lagrange equation. It is obvious that the second and third term of (3.5) contributes nothing. However, the last term is little tricky. Let me expand $F_{\mu\nu}\tilde{F}^{\mu\nu}$ first.

$$\begin{aligned} F_{\mu\nu}\tilde{F}^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu\partial_\alpha A_\beta - \partial_\mu A_\nu\partial_\beta A_\alpha - \partial_\nu A_\mu\partial_\alpha A_\beta - \partial_\nu A_\mu\partial_\beta A_\alpha) \\ &= \epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu\partial_\alpha A_\beta - \partial_\mu A_\nu\partial_\beta A_\alpha) \end{aligned} \tag{3.7}$$

Obeying the property of $\epsilon^{\mu\nu\alpha\beta}$, μ, ν, α, and β cannot be equivalent to others. It leads to the point that only 2 terms are considered with given μ and ν. For instance, if $\mu = 0$ and $\nu = 1$, $(\alpha,\ \beta) = (2,\ 3)$ or $(3,\ 2)$ are possible.

Otherwise, $\epsilon^{\mu\nu\alpha\beta} = 0$. Then,

$$\begin{aligned} \frac{\partial}{\partial_\mu A_\nu}(F_{\mu\nu}\tilde{F}^{\mu\nu}) &= \frac{\partial}{\partial_\mu A_\nu}(\epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu \partial_\alpha A_\beta - \partial_\mu A_\nu \partial_\beta A_\alpha)) \\ &= \epsilon^{\mu\nu\alpha\beta}(2\partial_\alpha A_\beta - 2\partial_\beta A_\alpha) = 4\tilde{F}^{\mu\nu} \end{aligned} \tag{3.8}$$

Therefore, the $\mathcal{L}_\theta$ term contributes

$$\partial_\mu \left(\frac{\kappa\theta}{\mu_0} \tilde{F}^{\mu\nu} \right) \ . \tag{3.9}$$

Combining (3.6) and (3.9), the Euler-Lagrange equation associated with A can be written as

$$\partial_\mu F^{\mu\nu} - \kappa \partial_\mu (\theta \tilde{F}^{\mu\nu}) = \mu_0 J_e^\nu \ . \tag{3.10}$$

Now, based on the Euler-Lagrange equation, modification of the Maxwell's equations can be obtained. First, by substituting $\nu = 0$ to (3.10),

$$\begin{aligned} \partial_\mu F^{\mu 0} - \kappa \partial_\mu (\theta \tilde{F}^{\mu 0}) &= \mu_0 J_e^0 = \mu_0 c \rho_e \\ \Longrightarrow \frac{1}{c}(\nabla \cdot \vec{E}) - \kappa \vec{B} \cdot \nabla\theta &= \mu_0 c \rho_e \\ \Longrightarrow \nabla \cdot \vec{E} &= \frac{\rho_e}{\epsilon_0} + \kappa c \vec{B} \cdot \nabla\theta \ . \end{aligned} \tag{3.11}$$

The Gauss' law for the electric charge (2.1) modified as below.

$$\nabla \cdot (\vec{E} - c\kappa\theta\vec{B}) = \frac{\rho_e}{\epsilon_0} \tag{3.12}$$

by substituting $\nu = 3$ to the same equation,

$$\begin{aligned} &\partial_\mu F^{\mu 3} - \kappa \partial_\mu (\theta \tilde{F}^{\mu 3}) = \mu_0 J_e^3 = \mu_0 J_{e,z} \\ &\Longrightarrow -\frac{1}{c^2}\partial_t E_z + \partial_x B_y - \partial_y B_x - \frac{\kappa}{c}(\partial_t\theta)(-B_z) - \kappa(-\partial_x\theta)\frac{E_y}{c} - \kappa\partial_y\theta\frac{E_x}{c} = \mu_0 J_e, z \ . \end{aligned} \tag{3.13}$$

Since $\nu = 3$ is the z component of the whole equation, the vector from is given by

$$\begin{aligned} \frac{1}{c^2}\partial_t \vec{E} + \nabla \cdot \vec{B} &= \mu_0 J_{e,z} - \frac{\kappa}{c}(\partial_t\theta)\vec{B} - \kappa\nabla\theta \times \frac{\vec{E}}{c} \\ \Longrightarrow \nabla \times \vec{B} - \mu_0\epsilon_0\partial_t\vec{E} &= \mu_0(\vec{J}_e - \frac{\kappa}{\mu_0 c}((\partial_t\theta)\vec{B} + \nabla\theta \times \vec{E})) \ , \end{aligned} \tag{3.14}$$

which leads to the modification of the Ampere's law equation (2.4) as

$$\nabla \times (c\vec{B} + \kappa\theta\vec{E}) = \frac{1}{c}\partial_t(\vec{E} - c\kappa\theta\vec{B}) + c\mu_0\vec{J}_e \ . \tag{3.15}$$

Applying the SO(2) rotation with $\phi = \pi/2$, the dual equation of (3.10) can be obtained.

$$\begin{pmatrix} F'^{\mu\nu} \\ \\ \tilde{F}'^{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F^{\mu\nu} \\ \\ \tilde{F}^{\mu\nu} \end{pmatrix} \tag{3.16}$$

$$\begin{pmatrix} J'^{\mu}_{e} \\ \\ J'^{\mu}_{m}/c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \\ -1 & 0 \end{pmatrix} \begin{pmatrix} J^{\mu}_{e} \\ \\ J^{\mu}_{m}/c \end{pmatrix} \tag{3.17}$$

$$\Longrightarrow \partial_\mu \tilde{F}^{\mu\nu} + \kappa \partial_\mu (\theta F^{\mu\nu}) = \frac{J^{\mu}_{m}}{c} \tag{3.18}$$

Sticking to the symmetry of (3.10) and (3.18), the modified form of (2.2) and (2.3) can be also written.

$$\nabla \cdot (c\vec{B} + \kappa\theta\vec{E}) = c\mu_0 \rho_m \tag{3.19}$$

$$\nabla \times (\vec{E} - c\kappa\theta\vec{B}) = -\frac{1}{c}\partial_t (c\vec{B} + \kappa\theta\vec{E}) - \mu_0 \vec{J}_m \tag{3.20}$$

Note that the electric field and magnetic field are coupled through the axion field. Meanwhile, the Euler-Lagrange equation associated with θ for the Lagrangian (3.2) is

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \theta)} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \ . \tag{3.21}$$

The differentiation of the terms is much simpler than the first equation. The equation yields

$$\partial_\mu (\partial^\mu \theta) - \left(\frac{\kappa}{4\mu_0} F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{\partial U}{\partial \theta} \right) = 0 \ , \tag{3.22}$$

which is simplified using d'Alembert operator $\Box$ as below.

$$\Box \theta = \frac{\kappa}{4\mu_0} F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{\partial U}{\partial \theta} = \frac{\kappa}{4\mu_0} \vec{E} \cdot \vec{B} - \frac{\partial U}{\partial \theta} \tag{3.23}$$

Usually, the form of the axion potential is assumed as quadratic potential:

$$U(\theta) = \frac{1}{2} m_a^2 \theta^2 \ , \tag{3.24}$$

where m_a is the axion mass. Then (3.23) can be represented as below.

$$(\Box + m_a^2)\theta = -\kappa \vec{E} \cdot \vec{B} \tag{3.25}$$

This form of equation is so-called **Klein-Gordon equation**, that is, the relativistic wave equation. Now we can write the set of Maxwell's equation assuming the existence of the magnetic monopole and axion.

$$\begin{cases} \nabla \cdot (\vec{E} - c\kappa\theta\vec{B}) = \dfrac{\rho_e}{\epsilon_0} \\ \nabla \cdot (c\vec{B} + \kappa\theta\vec{E}) = c\mu_0\rho_m \\ \nabla \times (\vec{E} - c\kappa\theta\vec{B}) = -\dfrac{1}{c}\partial_t(c\vec{B} + \kappa\theta\vec{E}) - \mu_0\vec{J_m} \\ \nabla \times (c\vec{B} + \kappa\theta\vec{E}) = \dfrac{1}{c}\partial_t(\vec{E} - c\kappa\theta\vec{B}) + c\mu_0\vec{J_e} \\ (\Box + m_a^2)\theta = -\kappa\vec{E} \cdot \vec{B} \end{cases} \tag{3.26}$$

3.2 Propagation of Waves

In this section, I derive the propagation of the electromagnetic wave with axion field. Assume the free space condition $\vec{J_e} = \vec{J_m} = 0$. Then the Maxwell's equations (3.26) is modified as below. For simplicity, $c = 1$ is applied.

$$\begin{cases} \nabla \cdot (\vec{E} - c\kappa\theta\vec{B}) = 0 \\ \nabla \cdot (c\vec{B} + \kappa\theta\vec{E}) = 0 \\ \nabla \times (\vec{E} - c\kappa\theta\vec{B}) + \partial_t(\vec{B} + \kappa\theta\vec{E}) = 0 \\ \nabla \times (\vec{B} + \kappa\theta\vec{E}) - \partial_t(\vec{E} - \kappa\theta\vec{B}) = 0 \\ (\Box + m_a^2)\theta = -\kappa\vec{E} \cdot \vec{B} \end{cases} \tag{3.27}$$

Let $\hat{\vec{E}} = \vec{E} - c\kappa\theta\vec{B}$ and $\hat{\vec{B}} = \vec{B} + \kappa\theta\vec{E}$. Applying $\nabla\times$ to the third and forth equation and using $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F$,

$$\nabla(\nabla \cdot \hat{\vec{E}}) - \nabla^2\hat{\vec{E}} + \partial_t\vec{\hat{\vec{B}}} = -\nabla^2\hat{\vec{E}} + \partial_t\vec{\hat{\vec{B}}} = 0 \tag{3.28}$$

$$\nabla(\nabla \cdot \hat{\vec{B}}) - \nabla^2\hat{\vec{B}} + \partial_t\vec{\hat{\vec{E}}} = -\nabla^2\hat{\vec{B}} + \partial_t\vec{\hat{\vec{E}}} = 0\ . \tag{3.29}$$

These are the free space wave equations.

$$\begin{cases} \Box\hat{\vec{E}} = 0 \\ \Box\hat{\vec{B}} = 0 \end{cases} \tag{3.30}$$

One solution to this set of equations are

$$\begin{cases} \hat{\vec{E}} = I_+ \cos{(kz - \omega t)}\mathbf{x} - I_- \sin{(kz - \omega t)}\mathbf{y} \\ \hat{\vec{B}} = I_- \sin{(kz - \omega t)}\mathbf{x} + I_+ \cos{(kz - \omega t)}\mathbf{y} \end{cases}, \tag{3.31}$$

the transverse wave propagating along **z** direction. Also, simply by definition,

$$\vec{E} = \frac{\hat{\vec{E}} + \kappa\theta\hat{\vec{B}}}{1+\kappa^2\theta^2} \ , \quad \vec{B} = \frac{\hat{\vec{B}} - \kappa\theta\hat{\vec{E}}}{1+\kappa^2\theta^2} \ . \tag{3.32}$$

Since $\hat{\vec{E}} \cdot \hat{\vec{B}} = \vec{E} \cdot \vec{B} = 0$, the electric and magnetic fields propagate orthogonally in the absence of charge sources, just as in the familiar electrodynamics without magnetic charge and axion.

Let me summarize the discussion with axion. A solution of the last equation of (3.27) is

$$\theta = \theta_0 \cos\left(\vec{p} \cdot \vec{r} - \Omega t\right) \ , \tag{3.33}$$

where Ω satisfies the dispersion relation $\Omega^2 = |\vec{p}|^2 + m_a^2$. This equation describes the propagation of a free scalar particle. Therefore, the electromagnetic field propagates as a transverse wave while the axion wave behaves as a longitudinal wave.

Remark

In some of axion literature, following convention for electromagnetic fields is used.

$$\vec{E}_a = -c\kappa\theta\vec{B}_0$$
$$\vec{B}_a = \kappa\theta\vec{E}_0$$

The subscript a stands for the axion-induced fields, while the subscript 0 means the background fields. Also, normally $\vec{a}$ and $g_{a\gamma\gamma}$ are used for the axion field and axion-photon coupling instead of θ and κ, respectively. Therefore, assuming no free source, in the presence of a coherently oscillating axion field $a(t)$, the Maxwell's equations for the background fields in natural units are modified to [2]

$$\nabla \cdot \vec{E}_0 = g_{a\gamma\gamma} a \vec{B}_0$$
$$\nabla \cdot \vec{B}_0 = 0$$
$$\nabla \times \vec{E}_0 = -\partial_t \vec{B}_0$$
$$\nabla \times (\vec{B}_0 + g_{a\gamma\gamma} a \vec{E}_0) = \partial_t(\vec{E}_0 - g_{a\gamma\gamma} a \vec{B}_0) \ .$$

References

[1] Luca Visinelli. Axion-electromagnetic waves. *Modern Physics Letters A*, 28(35):1350162, 2013.

[2] Dongok Kim, Junu Jeong, SungWoo Youn, Younggeun Kim, and Yannis K Semertzidis. Revisiting the detection rate for axion haloscopes. *Journal of Cosmology and Astroparticle Physics*, 2020(03):066–066, 2020.