8 The Laplacian matrix

8.1 Basics

Let G be a graph, and define $\mathbf{D}(G)$ to be the diagonal matrix whose entries correspond to the degrees of the vertices of G. Define the Laplacian matrix of G by

$$\mathbf{L} = \mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G).$$

Theorem 8.1. *The Laplacian matrix is positive semidefinite. Moreover, the multiplicity of* 0 *as an eigenvalue of* $\mathbf{L}$ *is equal to the number of connected components of* G.

Proof. To see this, assume G has been oriented, meaning, each edge has been assigned a direction, thus becoming an arc. Let $\mathbf{N}$ be the corresponding vertex by arc incidence matrix, so that an entry is 0 if the arc does not touch the vertex, $+1$ if the vertex is the head of the arc, and -1 if it is the tail. It is immediate to see that

$$\mathbf{L} = \mathbf{N}\mathbf{N}^T.$$

(Note that this does not depend on the choice for the orientation.)

Following, $\mathbf{N}^T\mathbf{v} = 0$ if and only if $\mathbf{L}\mathbf{v} = 0$. It is immediate to see that $\mathbf{N}^T\mathbf{v} = 0$ if and only if $\mathbf{v}$ is constant on each connected component of G, whence the result follows (and describes essentially the unique eigenvector for 0 in a connected graph — the constant vector). $\square$

Exercise 8.2. Assume G is regular, and let $\theta_1 \geq ... \geq \theta_n$ be the eigenvalues of $\mathbf{A}(G)$, with corresponding eigenbasis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. Find an expression of the eigenvalues of $\mathbf{L}(G)$, and find a corresponding eigenbasis.

Exercise 8.3. Let $0 = \lambda_1 \leq ... \leq \lambda_n$ be the eigenvalues of $\mathbf{L}(G)$. Find the eigenvalues of $\mathbf{L}(\overline{G})$. Use this exercises to find the eigenvalues of $\mathbf{L}(K_{n,m})$ (this is the complete bipartite graph with n vertices on one side and m on the other).

As we have seen, $\mathbf{L}(G)$ is positive semidefinite. It follows that $\mathbf{x}^T\mathbf{L}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. We can moreover find a useful and meaningful expression for this. As $\mathbf{L} = \mathbf{N}\mathbf{N}^T$ where $\mathbf{N}$ is the incidence matrix of an orientation of the graph, it follows that

$$\mathbf{x}^T\mathbf{L}\mathbf{x} = (\mathbf{N}^T\mathbf{x})^T(\mathbf{N}^T\mathbf{x}) = \sum_{uv \in E(G)} (\mathbf{x}_u - \mathbf{x}_v)^2.$$

Exercise 8.4. Assume G is connected, on n vertices, and let λ_2 be its second smallest Laplacian eigenvalue. We certainly know (from the minimax principle for eigenvalues) that

$$\lambda_2 = \min_{\mathbf{v} \perp \mathbf{1}} \frac{\sum_{ab \in E(G)} (\mathbf{v}_a - \mathbf{v}_b)^2}{\sum_{a \in V(G)} \mathbf{v}_a^2}.$$

What I want you to show is that

$$\lambda_2 = \min_{\mathbf{v} \neq \gamma \mathbf{1}} \frac{n \sum_{ab \in E(G)} (\mathbf{v}_a - \mathbf{v}_b)^2}{\sum_{a<b} (\mathbf{v}_a - \mathbf{v}_b)^2}.$$

(The minimum is simply being taken over all vectors which are just not constant.)

Also, and without much difficulty now, prove that

$$\lambda_n = \max_{\mathbf{v} \neq \alpha \mathbf{1}} \frac{n \sum_{ab \in E(G)} (\mathbf{v}_a - \mathbf{v}_b)^2}{\sum_{a<b} (\mathbf{v}_a - \mathbf{v}_b)^2}.$$

Exercise 8.5. Can you prove lower and upper bounds to the eigenvalues os $\mathbf{L}$ with regards to the number of vertices, edges, and minimum, average and maximum degrees of the graph?

8.2 Trees

A spanning tree of a connected graph G on n vertices is a subset of its edges that connects all vertices without forming any cycle. Necessarily, any spanning tree will contain $n-1$ edges.

A first result we shall see about the Laplacian matrix is actually a quite surprising one: we can efficiently count how many spanning trees any graph has.

Let $\tau(G)$ denote the number of spanning trees of the graph G. Recall the notation for edge deletion and contraction: $G \backslash e$ is the graph G with e removed, and G/e is the graph G with e removed and its incident vertices identified.

Lemma 8.6. *For any graph G and edge e, we have*

$$\tau(G) = \tau(G \backslash e) + \tau(G/e).$$

Exercise 8.7. Why?

We can now state the Matrix-Tree Theorem (due to Kirchkoff).

Theorem 8.8. *Let G be a graph, Laplacian $\mathbf{L}$. Let $a \in V(G)$, and $\mathbf{L}[a]$ denote the submatrix of $\mathbf{L}$ obtained upon deleting row and column corresponding to a. Then*

$$\tau(G) = \det \mathbf{L}[a].$$

Proof. This will be a proof by induction on the number of edges. You should check a few base cases on your own. Let us now assume G has m edges, and the result holds for any graph on fewer edges. Let $e \in E(G)$, with $e = \{a, b\}$. In G/e, vertices a and b are identified — let c be the name they receive in this case. If we show that

$$\det \mathbf{L}(G)[a] = \det \mathbf{L}(G \backslash e)[a] + \det \mathbf{L}(G/e)[c],$$

then, by induction and the lemma above, we will be done. So this equality above is now our task. In computing $\det \mathbf{L}(G)[a]$, we will perform row expansion in the row corresponding to b. Note that all terms of this expansion coming from an off-diagonal position will appear exactly the same in $\det \mathbf{L}(G \backslash e)[a]$. The only problem is the diagonal position — it is one unit larger in $\mathbf{L}(G)[a]$ then in $\mathbf{L}(G \backslash e)[a]$. Now the submatrix corresponding to excluding row and column b from $\mathbf{L}(G \backslash e)[a]$ is precisely $\mathbf{L}(G/e)[c]$, that is

$$\det \mathbf{L}(G)[a] = \det \mathbf{L}(G \backslash e)[a] + \det \mathbf{L}(G \backslash e)[a, b] = \det \mathbf{L}(G \backslash e)[a] + \det \mathbf{L}(G/e)[c],$$

as wished. □

Exercise 8.9. Very easily now you can verify that the number of spanning trees of K_n vertices is n^{n-2}.

Exercise 8.10. Prove that the number of spanning trees of G that contain a given edge $e = ab$ is equal to $\det \mathbf{L}[a, b]$.

As we have all learned before, for any square matrix $\mathbf{M}$,

$$\mathbf{M}\,\mathrm{adj}(\mathbf{M}) = \det(\mathbf{M})\mathbf{I},$$

where $\mathrm{adj}(\mathbf{M})_{ij} = (-1)^{i+j} \det \mathbf{M}(j, i)$. As we have just seen from above, all diagonal entries of $\mathrm{adj}\, \mathbf{L}(G)$ are equal to $\tau(G)$.

However for any G, $\det \mathbf{L}(G) = 0$. If we now assume G is connected, we know that there is essentially only one eigenvector to the eigenvalue 0, thus the equality

$$\mathbf{L}(G) \cdot \mathrm{adj}\, \mathbf{L}(G) = \mathbf{0}$$

implies that all columns of $\mathrm{adj}\, \mathbf{L}(G)$ are constant, and therefore all entries of $\mathrm{adj}\, \mathbf{L}(G)$ are equal to $\tau(G)$. It is immediate to verify all comments above hold if G is disconnected, in which case $\tau(G) = 0$.

Corollary 8.11. *For any graph G, we have*

$$\mathrm{adj}\, \mathbf{L}(G) = \tau(G)\mathbf{J}.$$

□

Exercise 8.12. Prove that for any graph G with Laplacian eigenvalues $\lambda_1 \leq ... \leq \lambda_n$, it holds that

$$\tau(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i.$$

Hint: let $\psi(x)$ be the characteristic polynomial of $\mathbf{L}$. Arrive at the result realizing that

$$\prod_{i=1}^{n} (x - \lambda_i) = \psi(x) = \det(x\mathbf{I} - \mathbf{L}).$$

8.3 Representation, springs and energy

A *representation of a graph* is a map $\rho : V(G) \to \mathbb{R}^m$ (you can think of it as an m-dimensional drawing of G). You can associate ρ to an $n \times m$ matrix $\mathbf{R}$ — each row is the image of the corresponding vertex. A representation ρ is called balanced if $\sum_{a \in V} \rho(a) = \mathbf{0}$, that is, if $\mathbf{1}^T \mathbf{R} = \mathbf{0}$. Upon assuming we can freely translate a representation, we can always assume it is balanced. Moreover, we shall also assume the columns of $\mathbf{R}$ are linearly independent (otherwise we simply look at the representation to the subspace of $\mathbb{R}^m$ and rewrite ρ upon a change of basis so that $\mathbf{R}$ has fewer columns).

Now imagine a physical model, in which the vertices have been placed in $\mathbb{R}^m$. Some of them, $U \subseteq V(G)$, are "nailed", some of them, $V(G) - U$, are free. The edges are *springs*. For now, identical springs, with spring constant 1. By Hooke's law, the force the spring between a and b exerts in a is equal to $\rho(b) - \rho(a)$. Note that a configuration is in equilibrium if and only if the net force at each vertex in $V - U$ is 0. This is equivalent to requiring, for all $a \in V - U$, that

$$\sum_{b \sim a} \rho(b) - \rho(a) = 0 \iff \deg(a)\rho(a) - \sum_{b \sim a} \rho(b) = \mathbf{0}.$$

In other words, $\mathbf{LR}$ must have a rectangle of 0s in the rows corresponding to the vertices in $V - U$. Once the entries of $\mathbf{R}$ corresponding to vertices in U have been determined, finding the remaining entries of $\mathbf{R}$ so that this holds is equivalent to solving m systems of equation whose coefficient matrix is $\mathbf{L}[U]$. All these systems have unique solutions if the graph is connected and $U \neq$, because $\mathbf{L}[U]$ is positive definite.

Exercise 8.13. Let $\mathbf{L}[U]$ denote the submatrix of $\mathbf{L}$ obtained upon removing rows and columns corresponding to the vertices in subset U. Assume the graph is connected, and U non-empty. Prove that all eigenvalues of $\mathbf{L}[U]$ are positive.

Exercise 8.14. Convince yourself that nothing really changes if we assume the spring between a and b to have spring constant ω_{ab}.

Physics also teaches us that vertices will settle in the position the minimizes the potential energy. The potential energy of a spring with constant ω and stretched to a length ℓ is $(1/2)\omega\ell^2$ (we will ignore the fraction). Thus, the potential energy of a configuration is

$$\mathcal{E}(\rho) = \sum_{ab \in E(G)} \omega_{ab} ||\rho(a) - \rho(b)||^2.$$

Let W be a diagonal matrix, indexed by $E(G)$, whose diagonal entry is equal to ω_{ab}. As before, let $\mathbf{N}$ be the incidence matrix of an orientation of G, and $\mathbf{R}$ the matrix of the representation. It is immediate to verify that

$$\mathcal{E}(\rho) = \operatorname{tr} \mathbf{R}^T \mathbf{N} \mathbf{W} \mathbf{N}^T \mathbf{R}.$$

Note that $\mathbf{NWN}^T$ is simply a weighted Laplacian (and if $\mathbf{W}$ has positive diagonal and the graph is connected, then $\mathbf{NWN}^T$ is positive semidefinite, and 0 is a simple eigenvalue with eigenvector $\mathbf{1}$).

A representation is called *orthogonal* if the columns of $\mathbf{R}$ are orthonormal. In this case, $\mathbf{R}^T\mathbf{R} = \mathbf{I}$. Requiring a representation to be orthogonal is also a way of imposing a shape. We do not need to "nail" vertices in this case, as the following theorem shows.

Theorem 8.15. *Let G be a graph, with a weighted Laplacian matrix $\mathcal{L}$, with eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_n$. The minimum energy of a balanced orthogonal representation into $\mathbb{R}^m$ is equal to*

$$\sum_{r=2}^{m+1} \lambda_r.$$

Proof. To any orthogonal representation into $\mathbf{R}^k$ whose first column is $\mathbf{1}$ corresponds a balanced orthogonal representation in $\mathbb{R}^{k-1}$ with the same energy, obtained upon ignoring this first column. Thus the minimum energy of a balanced orthogonal representation into $\mathbb{R}^m$ is equal to the minimum energy of an orthogonal representation into $\mathbb{R}^{m+1}$ whose first column is $\mathbf{1}$. Let $\mathbf{R}$ be the matrix of one such representation. Its energy is

$$\operatorname{tr} \mathbf{R}^T \mathbf{L} \mathbf{R},$$

which, by interlacing, is at least $\sum_{r=1}^{m+1} \lambda_r$. Recall that $\lambda_1 = 1$. Moreover, one representation meeting this energy exists: simply write the eigenvectors corresponding $\lambda_1, ..., \lambda_{m+1}$ as the columns of $\mathbf{R}$. □

An immediate consequence is a method to draw graphs in $\mathbb{R}^m$ that is balanced and will somehow look "rigid" and "having a volume" — the so called spring embedding. Just pick the eigenvectors of $\mathbf{L}$ corresponding to $\lambda_2, ..., \lambda_{m+1}$, line them up as columns of a matrix, and map each vertex to the corresponding row. You should try this method to draw graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ using your favourite software.

8.4 Electrical currents

We define a physical model of a graph in which each edge corresponds to a wire. Let us say each edge has weight ω_{ab}, and this will mean to us that its resistance is $1/\omega_{ab}$ (a small weight corresponds to a big resistance). Ohm's law says that the potential drop across a resistor is equal to the current flowing times the resistance. If the current from a to b is $i(a, b)$ and the potentials in a and b are $v(a)$ and $v(b)$, then

$$v(a) - v(b) = \frac{i(a, b)}{\omega_{ab}}.$$

If $\mathbf{N}$ is the incidence matrix of an orientation of G, $\mathbf{W}$ the diagonal matrix with edge weights, $\mathbf{v}$ the vector with vertex potentials, and $\mathbf{i}$ the vector of edge currents, we now have

$$\mathbf{i} = \mathbf{W}\mathbf{N}^T\mathbf{v}.$$

Let $\mathbf{j}$ be a vector indexed by vertices whose ath entry denotes the net current entering or leaving the network at a. Recall that by Kirchkoff's law, the current entering a node is equal to the current exiting. Thus

$$\mathbf{j} = \mathbf{N}\mathbf{i}.$$

All together, and again making $\mathbf{L} = \mathbf{N}\mathbf{W}\mathbf{N}^T$ the weighted Laplacian matrix, we have

$$\mathbf{j} = \mathbf{L}\mathbf{v}.$$

As a consequence of this fact, it must be that $\mathbf{1}^T\mathbf{j} = 0$.

On the other hand, assume now that we are given a vector indicating currents entering and leaving the network satisfying $\mathbf{1}^T\mathbf{j} = 0$. Is this enough to find the voltages that correspond to the system?

A solution for $\mathbf{v}$ can be found computing the pseudo-inverse $\mathbf{L}^+$ of $\mathbf{L}$. That is, if $\mathbf{L}$ has spectral decomposition

$$\mathbf{L} = \sum_{i=1}^{n} \lambda_i \mathbf{E}_i,$$

with $0 = \lambda_1 < \lambda_2 \leq ... \leq \lambda_n$, then, given $\mathbf{j}$, with $\mathbf{1}^T\mathbf{j} = 0$, a solution for $\mathbf{v}$ can be found as

$$\mathbf{v} = \left(\sum_{i=2}^{n} \frac{1}{\lambda_i} \mathbf{E}_i \right) \mathbf{j}.$$

(Note that if $\mathbf{v}$ is as above, then $\mathbf{v} + \alpha\mathbf{1}$ also satisfies $\mathbf{L}(\mathbf{v} + \alpha\mathbf{1}) = \mathbf{j}$ for any α).

Now assume a and b are neighbours, and imagine one unit of current is pushed into a, and one unit extracted from b (meaning: $\mathbf{j}_a = -\mathbf{j}_b = 1$, 0 elsewhere, or simply $\mathbf{j} = \mathbf{e}_a - \mathbf{e}_b$). We can solve which potential arrangement at all vertices allows for this, and the difference of potential between a and b is defined as their *effective resistance*. In other words

$$R_{\text{eff}}(a, b) = (\mathbf{e}_a - \mathbf{e}_b)^T \mathbf{L}^+ (\mathbf{e}_a - \mathbf{e}_b).$$

Exercise 8.16. Suppose you have two edges, ab and cd. Prove that the difference of potential between c and d when you push one unit of current at a a and remove it at b is the same as the difference of potential between a and b when you push one unit of current at a c and remove it at d.

8.5 Connectivity and interlacing

Again, $\mathbf{L}$ is the Laplacian matrix, and $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ its eigenvalues.

Over the next few sections, we will learn that λ_2 carries powerful information about a graph. We start with a bound associating λ_2 to the connectivity. Given a graph G, it is k-vertex-connected if it has more than k vertices, and remains connected whenever fewer than k vertices are removed. The *vertex connectivity* of a graph, denoted $\kappa_0(G)$, is the largest k to so that G is k-vertex connected. For all graphs which are not complete, this definition is equivalent to saying that $\kappa_0(G)$ is the smallest size of a subset of vertices whose removal disconnects G.

Computing the vertex connectivity of a graph is not difficult — Menger's theorem says that the size of a minimum cut in a graph is equal to the maximum number of disjoint paths that can be found between any pair of vertices. I invite you to prove this result using linear programming duality.

Nevertheless, an eigenvalue bound can always be useful.

Theorem 8.17. *Suppose $U \subseteq V(G)$. Then*

$$\lambda_2(G) \leq \lambda_2(G \backslash U) + |U|.$$

Proof. Let $\mathbf{v}'$ be a normalized $\lambda_2(G\backslash U)$ eigenvector of $\mathbf{L}(G\backslash U)$. Let $\mathbf{v}$ be the extension of

$\mathbf{v}'$ to $\mathbf{R}^{V(G)}$, adding 0s in the remaining entries. By Courant-Fisher-Weyl, we have

$$\begin{aligned}\lambda_2(G) &\leq \sum_{ab \in E(G)} (\mathbf{v}_a - \mathbf{v}_b)^2 \\ &\leq \sum_{a \in U} \sum_{b \sim a} \mathbf{v}_b^2 + \sum_{ab \in E(G \backslash U)} (\mathbf{v}_a - \mathbf{v}_b)^2 \\ &\leq |U| + \lambda_2(G \backslash U).\end{aligned}$$

□

If U is a cut-set, then $G \backslash U$ is disconnected, thus 0 has multiplicity bigger than 1, and therefore

$$\lambda_2(G) \leq \kappa_0(G).$$

This immediately implies that $\lambda_2(G) \leq \delta(G)$.

Exercise 8.18. Prove that for all trees on more than 2 vertices, $\lambda_2 \leq 1$. Prove that equality holds if and only if the tree is a star.

Interlacing "works" for $\mathbf{L}$, but the problem is that the submatrices of $\mathbf{L}$ are not Laplacian matrices of subgraphs. If we would like to related the eigenvalues of $\mathbf{L}(G)$ with those of the Laplacians of subgraphs, we must use different methods. The following exercise can be proved elementarily, just as we did above.

Exercise 8.19. Let G be a graph, and $e \in E(G)$. Prove that

$$\lambda_2(G \backslash e) \leq \lambda_2(G) \leq \lambda_2(G \backslash e) + 2.$$

Show that equality holds in the second bound if and only if G is complete.

8.6 Partitioning and cuts

Even though finding the minimum cut in a graph amount to an easy task, other problems involving cuts or partitions into clusters are significantly harder. We introduce three problems related to edge cuts:

(a) bipartition width: finding the minimum over all $e(U, \overline{U})$ where $U \subseteq V(G)$, and $|U| = \lfloor n/2 \rfloor$.

(b) maxcut: finding the maximum cut, meaning a non-empty propoer subset U of $V(G)$ so that $e(U, \overline{U})$ is maximized.

(c) finding the conductance, meaning, the minimum over all $e(U, \overline{U})/|U|$, with $U \subseteq V(G)$, $0 < |U| \leq n/2$.

These parameters are all NP-hard to compute, but we can find some interesting bounds or approximations using the eigenvalues λ_2 or λ_n. We start with an easy observation.

Lemma 8.20. *For all $U \subseteq V(G)$, we have*

$$\lambda_2 \frac{|U|(n-|U|)}{n} \leq e(U,\overline{U}) \leq \lambda_n \frac{|U|(n-|U|)}{n}.$$

Proof. Both bounds follow immediately from Exercise 8.4. □

This immediately leads to a lower bound to the bipartition width of the graph, caled $\mathrm{bw}(G)$. We have

$$\mathrm{bw}(G) \geq \frac{1}{4} n \lambda_2(G).$$

It also implies an immediate upper bound to the maxcut, labelled $\mathrm{mc}(G)$. We have

$$\mathrm{mc}(G) \leq \frac{1}{4} n \lambda_n(G).$$

Both these bounds can be made stronger by solving semidefinite programs. I won't get into details, but I will hint where in the expression we are allowed to put "new variables".

Theorem 8.21. *Let G be a graph, of even order n. Then*

$$bw(G) \geq \frac{1}{4} n \max_{\mathbf{v} \perp \mathbf{1}} \min_{\mathbf{u} \perp \mathbf{1}} \frac{\langle (\mathbf{L} + \boldsymbol{diag}(\mathbf{c}))\mathbf{u}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle},$$

Proof. Let S be a set of cardinality $n/2$ with $e(S,\overline{S}) = \mathrm{bw}(G)$, and define $\mathbf{w} \in \mathbb{R}^V$ to be $+1$ in S and -1 in $\overline{S}$. Note that $\mathbf{w} \perp \mathbf{1}$. Also,

$$\langle \mathbf{diag}(\mathbf{v})\mathbf{w}, \mathbf{w} \rangle = 0.$$

Therefore

$$\begin{aligned} \frac{\langle (\mathbf{L} + \mathbf{diag}(\mathbf{v}))\mathbf{w}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle} &= \frac{\langle \mathbf{L}\mathbf{w}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle} = \frac{\sum_{ab \in E} (\mathbf{w}_a - \mathbf{w}_b)^2}{\sum_{a \in V} \mathbf{w}_a^2} \\ &= \frac{4e(S,\overline{S})}{n} = \frac{4}{n} \mathrm{bw}(G). \end{aligned}$$

□

Exercise 8.22. Let $\mathbf{Q}$ be a $n \times (n-1)$ matrix with orthonormal columns and $\mathbf{1}$ in its left kernel. Argue why we also have

$$\mathrm{bw}(G) \geq \frac{1}{4} n \max_{\mathbf{v} \perp \mathbf{1}} \lambda_1(\mathbf{Q}^T(\mathbf{L} + \mathbf{diag}(\mathbf{v})\mathbf{Q}).$$

Exercise 8.23. Prove that

$$\mathrm{mc}(G) \leq \frac{1}{4} n \min_{\mathbf{v} \perp \mathbf{1}} \lambda_n((\mathbf{L} + \mathbf{diag}(\mathbf{v})).$$

(Hint: it is similar to the Theorem above).

For the third parameter we defined, the conductance, denoted by $\Phi(G)$ and also called the isoperimetric number, Lemma 8.20 implies that

$$\Phi(G) \geq \lambda_2/2.$$

For this parameter, we can bound it from the other side as well.

Theorem 8.24. *Given a graph G, we have*

$$\Phi(G) < \sqrt{2\Delta(G)\lambda_2(G)}.$$

Proof. We consider a normalized eigenvector $\mathbf{v}$ for λ_2, and we assume without loss of generality that the vertices are ordered, meaning $V(G) = \{1, 2, ..., n\}$, in such a way that $\mathbf{v}_i \geq \mathbf{v}_{i+1}$ for all i. Let V_+ be the vertices with $\mathbf{v}_i > 0$, and assume $\mathbf{v}$ is signed so that $|V_+| \leq n/2$. Also, define $\mathbf{u}$ vector with $\mathbf{u}_i = \mathbf{v}_i$ if $\mathbf{v}_i > 0$, and $\mathbf{u}_j = 0$ otherwise. We finally define E_+ the set of edges incident to one vertex in V_+.

To each i, $1 \leq i \leq |V(G)|$, we consider the cut

$$C_i = \{\{j, k\} \in E(G) : 1 \leq j \leq i < k \leq n\}.$$

Let

$$\alpha = \min_{1 \leq i \leq n} \frac{|C_i|}{\min\{i, n-i\}},$$

whence $\alpha \geq \Phi(G)$. Let $\mathbf{P}$ be the projection onto the subspace spanned by the characteristic vectors of the vertices in V_+, that is, $\mathbf{Pv} = \mathbf{u}$. We now have

$$\begin{aligned}
\lambda_2 = \frac{\mathbf{v}^T\mathbf{PLv}}{\mathbf{v}^T\mathbf{Pv}} &= \frac{(\mathbf{N}^T\mathbf{Pv})^T(\mathbf{N}^T\mathbf{v})}{\sum_{i\in V_+}\mathbf{v}_i^2} \\
&= \frac{\sum_{ij\in E_+}(\mathbf{u}_i-\mathbf{u}_j)(\mathbf{v}_i-\mathbf{v}_j)}{\sum_{i\in V_+}\mathbf{v}_i^2} \\
&> \frac{\sum_{ij\in E_+}(\mathbf{u}_i-\mathbf{u}_j)^2}{\sum_{i\in V_+}\mathbf{u}_i^2} \\
&= \frac{\sum_{ij\in E_+}(\mathbf{u}_i-\mathbf{u}_j)^2\sum_{ij\in E_+}(\mathbf{u}_i+\mathbf{u}_j)^2}{\sum_{i\in V_+}\mathbf{u}_i^2\sum_{ij\in E_+}(\mathbf{u}_i+\mathbf{u}_j)^2} \\
&\geq \frac{\left(\sum_{i\sim j}|\mathbf{u}_i^2-\mathbf{u}_j^2|\right)^2}{2\Delta\left(\sum_{i\in V_+}\mathbf{u}_i^2\right)^2} \\
&\geq \frac{\left(\sum_i|\mathbf{u}_i^2-\mathbf{u}_{i+1}^2||C_i|\right)^2}{2\Delta\left(\sum_{i\in V_+}\mathbf{u}_i^2\right)^2} \\
&\geq \frac{\left(\sum_i|\mathbf{u}_i^2-\mathbf{u}_{i+1}^2|\alpha i\right)^2}{2\Delta\left(\sum_{i\in V_+}\mathbf{u}_i^2\right)^2} \\
&\geq \frac{\alpha^2}{2\Delta} \\
&\geq \frac{\Phi^2}{2\Delta}
\end{aligned}$$

□

Exercise 8.25. Justify in details each of the steps of the inequality chain above.

Note that not only the result above gives a bound, but it also has an implicit algorithm in its proof. In fact, we were able to efficiently find a set of vertices U so that

$$\Phi \leq \frac{e(U,\overline{U})}{|U|} \leq \sqrt{2\Delta\lambda_2} \leq 2\sqrt{\Delta\Phi}.$$

8.7 Normalized Laplacian

As we now know,

$$\mathbf{L} = \mathbf{D} - \mathbf{A}.$$

We assume G has no isolated vertices. We define $\mathbf{Q}$ as

$$\mathbf{Q} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}.$$

Note that it is positive semidefinite.

Exercise 8.26. Prove that
$$R_{\mathbf{Q}}(\mathbf{u}) = \frac{\sum_{ab \in E} (\mathbf{v}_a - \mathbf{v}_b)^2}{\sum_{a \in V} \mathbf{v}_a^2 d(a)},$$
where $\mathbf{v} = \mathbf{D}^{-1/2}\mathbf{u}$.

We also prove some basic properties.

Theorem 8.27. *Let G be a graph with no isolated vertices, and denote the eigenvalues of $\mathbf{Q}$ by $\mu_1 \leq ... \leq \mu_n$. Then*

(i) $\sum_j \mu_j = n$.

(ii) For $n \geq 2$, $\mu_2 \leq n/(n-1)$, and equality holds if and only if G is the complete graph. Also, $\mu_n \geq n/(n-1)$.

(iii) For a graph not complete, we have $\mu_2 \leq 1$.

(iv) The multiplicity of 0 as an eigenvalue is the number of connected components of G.

(v) We have $\mu_n \leq 2$, and equality holds if and only if G is bipartite. In this case, for all μ eigenvalue of $\mathbf{Q}$, $2 - \mu$ is also eigenvalue.

Exercise 8.28. Prove the properties above.

Exercise 8.29. Let G be a graph, connected, diameter d. Prove that
$$\mu_2 \geq \frac{1}{d \sum_{a \in V} d(a)}.$$

8.8 Random Walks

Let G be a weighted graph (edge weights are given by a function ω). We assume a walker is sitting at a vertex, and then decides to move with certain probability. At each step, this walker hops from one vertex to another with probability proportional to the edge weight of the corresponding edge. This model is equivalent to a Markov chain defined on a finite measurable state space.

We will be specially interested in the expected behaviour of a random walker, rather than on some fixed particular instance of this experiment.

Let $\mathbf{p}_t \in \mathbb{R}^V$ denote the probability distribution of the walker at time t. As such, $\mathbf{p}_t(a) \geq 0$ for all $a \in V$, and
$$\sum_{a \in V} \mathbf{p}_t(a) = 1.$$
Because the probability of arriving at a at $t+1$ steps is only determined by the probability distribution at time t and the edge weights, we have
$$\mathbf{p}_{t+1}(a) = \sum_{b \sim a} \frac{\omega_{ab}}{d_b} \mathbf{p}_t(b).$$

with d_b meaning the sum of the weights of edges incident to b. Equivalently,

$$\mathbf{p}_{t+1} = \mathbf{A}\mathbf{D}^{-1}\mathbf{p}_t,$$

where $\mathbf{A}$ is the weighted adjacency matrix and $\mathbf{D}$ the diagonal matrix of (weighted) degrees. Let $\mathbf{W} = \mathbf{A}\mathbf{D}^{-1}$. It is immediate to verify that $\mathbf{p}_{t+k} = \mathbf{W}^k\mathbf{p}_t$, where $\mathbf{p}_0$ typically stands for the starting distribution. We also see that

$$\mathbf{D}^{-1/2}\mathbf{W}\mathbf{D}^{1/2} = \mathbf{I} - \mathbf{Q},$$

where $\mathbf{Q}$ is the normalized Laplacian. If there is a probability that the walker does not move, say $1/2$, we now have

$$\mathbf{p}_{t+1} = (1/2)\mathbf{I}\mathbf{p}_t + (1/2)\mathbf{A}\mathbf{D}^{-1}\mathbf{p}_t.$$

Let $\mathbf{Z} = (1/2)(\mathbf{I} + \mathbf{A}\mathbf{D}^{-1})$. You can now see that

$$\mathbf{D}^{-1/2}\mathbf{Z}\mathbf{D}^{1/2} = \mathbf{I} - (1/2)\mathbf{Q},$$

Matrices $\mathbf{Z}$ or $\mathbf{W}$ are not symmetric, but they are both similar to a symmetric matrix. This gives that they are diagonalizable with real eigenvectors.

Exercise 8.30. If $\mathbf{v}$ is λ-eigenvector of $\mathbf{Q}$, to which eigenpair of $\mathbf{W}$ or $\mathbf{Z}$ they relate? Later, prove that all eigenvalues of $\mathbf{W}$ are between -1 and 1, and those of $\mathbf{Z}$ lie between 0 and 1.

Exercise 8.31. What do you obtain if you replace $1/2$ by another probability?

A random walk $\mathbf{W}$ converges to a distribution $\mathbf{p}$ if for any given ε and any distribution $\mathbf{q}$, there is an n so that

$$||\mathbf{W}^n\mathbf{q} - \mathbf{p}|| < \varepsilon.$$

Exercise 8.32. Can you show that if $\mathbf{W}$ converges to $\mathbf{p}$, then $\mathbf{p}$ is "stable", meaning, $\mathbf{W}\mathbf{p} = \mathbf{p}$? Later, prove that every graph contains a stable distribution, and if the graph is connected, this is unique (it also does not depend on the probability of staying put).

Exercise 8.33. Show that if there is any probability that a walker stays put (we call these random walks "lazy"), then the random walk will converge to the stable distribution. Describe the graphs for which a non-lazy random walk does not converge.

Example 8.34. Imagine now the following experiment. A deck of n cards c_1,...,c_n is lying on a table. We will shuffle these cards in a very stupid way: at each time step, we select i, j from 1 to n uniformly at random and exchange the positions of cards i and j (that includes choosing $i = j$ and doing nothing). How fast does this procedure produces a good shuffling?

The graph here is the one whose vertex set corresponds to the permutations on n elements. Vertices adjacent if one can be obtained from the other by applying a transposition (this is called the Cayley Graph $\mathrm{Cay}(S_n, T)$).

The weights here are simply determined:

$$\mathbf{p}_{t+1} = (1/n)\mathbf{I}\mathbf{p}_t + [(n-1)/n]\mathbf{A}\mathbf{D}^{-1}\mathbf{p}_t,$$

with $\mathbf{A}_{\sigma\tau} = 1$ if $\sigma\tau^{-1}$ is a transposition, and $= 0$ otherwise.

We thus want to know how fast $\mathbf{p}_t$ becomes the stable distribution.

Exercise 8.35. What is the stable distribution of the example above?

We can now provide a reasonably good estimate.

Theorem 8.36. *Let $\mathbf{W}$ be the transition matrix of a random walk, having laziness probability $1-\rho$, and with eigenvalues $1 = \omega_1 \geq \omega_2 \geq ... \geq \omega_n$. Assume $\omega = \max\{|\omega_2|, |\omega_n|\} < 1$, meaning, the random walk converges, say, to $\mathbf{p}$. Let $\mathbf{p}_0 = \mathbf{e}_a$, meaning, the walk starts at a. Then*

$$|\mathbf{p}_t(b) - \mathbf{p}(b)| < \sqrt{\frac{d(b)}{d(a)}}\omega^t.$$

Proof. Let

$$\mathbf{Q} = \sum_{i=1}^{n} \lambda_i \mathbf{F}_i$$

be the spectral decomposition of the normalized Laplacian. Then

$$\mathbf{W} = \mathbf{I} - \rho \mathbf{D}^{1/2}\mathbf{Q}\mathbf{D}^{-1/2} = \sum_{i=1}^{n}(1-\rho\lambda_i)\mathbf{D}^{1/2}\mathbf{F}_i\mathbf{D}^{-1/2} = \sum_{i=1}^{n} \omega_i \mathbf{E}_i.$$

Note that

$$\mathbf{E}_1 = \frac{1}{\sum_{a\in V} d(a)}\mathbf{D}^{1/2}\mathbf{D}^{1/2}\mathbf{1}\mathbf{1}^T\mathbf{D}^{1/2}\mathbf{D}^{-1/2} = \frac{1}{\sum_{a\in V} d(a)}\mathbf{D}\mathbf{1}\mathbf{1}^T.$$

Then

$$\begin{aligned}
|\mathbf{p}_t(b) - \mathbf{p}(b)| &= |\mathbf{e}_b^T\mathbf{W}^t\mathbf{e}_a - \mathbf{e}_b^T\mathbf{E}_1\mathbf{e}_b| \\
&= \left|\sum_{i=2}^{n} \omega_i^t(\mathbf{e}_b^T\mathbf{E}_i\mathbf{e}_a)\right| \\
&\leq \omega^t \sum_{i=2}^{n} \left|\mathbf{e}_b^T\mathbf{E}_i\mathbf{e}_a\right| \\
&\leq \omega^t \sqrt{\frac{d(b)}{d(a)}} \sum_{i=2}^{n} \left|\mathbf{e}_b^T\mathbf{F}_i\mathbf{e}_a\right| \\
&\leq \omega^t \sqrt{\frac{d(b)}{d(a)}} \sum_{i=2}^{n} \sqrt{\mathbf{e}_b^T\mathbf{F}_i\mathbf{e}_b}\sqrt{\mathbf{e}_a^T\,\mathbf{F}_i\mathbf{e}_a} \\
&\leq \omega^t \sqrt{\frac{d(b)}{d(a)}} \sqrt{\sum_{i=2}^{n} \mathbf{e}_b^T\mathbf{F}_i\mathbf{e}_b}\sqrt{\sum_{i=2}^{n} \mathbf{e}_a^T\,\mathbf{F}_i\mathbf{e}_a} \\
&\leq \omega^t \sqrt{\frac{d(b)}{d(a)}} \sqrt{1-\mathbf{e}_b^T\mathbf{F}_1\mathbf{e}_b}\sqrt{1-\mathbf{e}_a^T\,\mathbf{F}_1\mathbf{e}_a} \\
&< \omega^t \sqrt{\frac{d(b)}{d(a)}}.
\end{aligned}$$

□

Exercise 8.37. Assume G is connected and non-bipartite, with an initial probability distribution $\mathbf{q}$. Let $\mathbf{W}$ be the transition matrix of a non-lazy random walk, and, as before, let $\omega = \max\{|\omega_2|, |\omega_n|\}$. Let $\mathbf{p}$ be the stable distribution. Prove that

$$||\mathbf{p}_t - \mathbf{p}|| < \omega^t.$$

We end this section with a nice exercise.

Exercise 8.38. Let $\mathbf{L}$ be the combinatorial Laplacian, with eigenvalues $\lambda_1 \leq ... \leq \lambda_n$, and $\mathbf{Q}$ the normalized version, with eigenvalues $\mu_1 \leq ... \leq \mu_n$. Let Δ and δ be the largest and smallest degrees of the graph. Verify that

$$\frac{\lambda_i}{\Delta} \leq \mu_i \leq \frac{\lambda_i}{\delta}.$$

(Hint: Use the Courant-Fisher-Weyl theorem — and apply the transformation $\mathbf{D}^{1/2}$).

8.9 References

Here is the set of references used to write the past few pages.

The initial material on Laplacian matrix was mostly based on Godsil and Royle's, Chapter 13.

Fan Chung's book "Spectral Graph Theory" is the standard reference on the Normalized Laplacian.

Bojan Mohar has several articles about Laplacian matrices: "Some Applications of Laplace Eigenvalues of Graphs", "The Laplacian Spectrum of Graphs", "Eigenvalues in combinatorial optimization" (with S. Poljak), and others.

Finally, I also acknowledge D. Spielman's course notes (2018), specially for the last section on random walks.