Exercise 1.12. Prove that two symmetric matrices $\mathbf{M}$ and $\mathbf{N}$ commute if and only if they can be simultaneously diagonalized by the same set of orthonormal eigenvectors. Is it true that if $\mathbf{M}$ and $\mathbf{N}$ commute, then there is always a polynomial p so that $p(\mathbf{M}) = \mathbf{N}$? Characterize what else you need to observe to guarantee that such polynomial exists.

Exercise 1.13. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices (not necessarily squared shaped), so that both products $\mathbf{AB}$ and $\mathbf{BA}$ are defined. Prove that

$$\operatorname{tr} \mathbf{AB} = \operatorname{tr} \mathbf{BA},$$

and conclude that if $\mathbf{M}$ is a symmetric matrix with eigenvalues $\lambda_1, ..., \lambda_n$, then $\operatorname{tr} \mathbf{M}$ is equal to $\lambda_1 + ... + \lambda_n$. How about $\operatorname{tr} \mathbf{M}^2$?

1.2 The adjacency matrix of a graph

Given a graph G on a vertex set V, one can always define an arbitrary ordering to the vertices, that is, let $V = \{a_1, ..., a_n\}$, and encode the graph as a symmetric 01-matrix as follows. *The adjacency matrix* $\mathbf{A}$ of G is defined as $\mathbf{A}_{ij} = 1$ of $a_i \sim a_j$, and $\mathbf{A}_{ij} = 0$ otherwise (including the diagonal elements).

The field of spectral graph theory concerns itself with the main problem of relating spectral properties of matrices that encode adjacency in a graph (such as $\mathbf{A}$) with the combinatorial properties of the graph. We shall see several examples of such relations.

Exercise 1.14. Let G be a graph, suppose the vertices V are ordered, and let $\mathbf{A}$ be the corresponding adjacency matrix of G. Suppose you reorder the vertices by means of a permutation. Let $\mathbf{P}$ be the 01 matrix representing this permutation. Show that the new adjacency matrix obtained from this re-ordering is $\mathbf{PAP}^T$. Conclude that the eigenvalues are the same, and the only change in the eigenvectors is a permutation of its entries.

Because of this exercise, we shall simply ignore the underlying ordering, and speak of "the" adjacency matrix of G.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ on the same number of vertices, a very natural question is whether or not they encode the same combinatorial structure, which can be translated as: is there a function $f : V_1 \to V_2$ that maps edges to edges and non-edges to non-edges? Such a function, if it exists, is called a *graph isomorphism*. You can think of an isomorphism like this: draw both graphs in the plane, and try to move the vertices of one of them (without creating or destroying edges) so that the two drawings look exactly the same.

Example 1.15. Graphs G_1, G_2 and G_3 are all isomorphic, but G_4 is "different".