Theorem 1.43 (Delsarte)**.** *Let G be a k-regular graph, eigenvalues $k = \lambda_1 \geq .... \geq \lambda_n$. Say the size of the largest coclique of G is α. Then*

$$\alpha \leq \frac{n(-\lambda_n)}{k - \lambda_n}.$$

Proof. Let $\mathbf{x}$ be the characteristic vector of a largest coclique, meaning, $\mathbf{x}_i = 1$ if and only if i belongs to the coclique, and 0 otherwise. Note that

$$\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = 0, \text{ thus } \mathbf{x}^\mathsf{T}(\mathbf{A} - \lambda_n\mathbf{I})\mathbf{x} = -\lambda_n \cdot \alpha.$$

On the other hand,

$$\mathbf{x}^\mathsf{T}(\mathbf{A} - \lambda_n\mathbf{I})\mathbf{x} = \sum_{i=1}^{n}(\lambda_i - \lambda_n)\mathbf{x}^\mathsf{T}\mathbf{E}_i\mathbf{x}.$$

All terms in the sum are non-negative, as $(\lambda_i - \lambda_n) \geq 0$ and $\mathbf{x}^\mathsf{T}\mathbf{E}_i\mathbf{x} \geq 0$. We can discard all of them but the first, and recalling that $\mathbf{1}$ is the eigenvector for λ_1, we have that $\mathbf{E}_1 = (1/n)\mathbf{J}$. Thus

$$-\lambda_n \cdot \alpha \geq \frac{\lambda_1 - \lambda_n}{n}\alpha^2,$$

and the result follows. □

1.7 Graphs with largest eigenvalue at most 2

It is still a common topic of research in spectral graph theory to classify all graphs whose spectrum lie within a constant given interval. One of the earliest and simplest results classifies all graphs whose largest eigenvalue does not surpass 2.

Lemma 1.44. *Let G be a connected graph, and assume H is a proper subgraph of G. Let $\lambda(G)$ be the largest eigenvalue of G, and $\lambda(H)$ the largest eigenvalue of H. Then*

$$\lambda(H) < \lambda(G).$$

Proof. Exercise! □

Exercise 1.45. Argue why any graph with largest eigenvalue at most 2 is either a tree or a cycle.

Exercise 1.46. Show that the following graphs all have largest eigenvalue equal to 2, by exhibiting a suitable eigenvector.