1.4 Eigenvalues of some classes of graphs

Consider the following classes of graphs:

(a) K_n - complete graphs on n vertices.

(b) $K_{a,b}$ - complete bipartite graphs with a vertices on one side, and b vertices on the other (in particular if $a = 1$, these are the stars).

(c) C_n - cycle graphs on n vertices.

(d) P_n - path graphs on n vertices.

Our goal here is to determine the eigenvalues (and eigenvectors) of these classes.

(a) This is easy. $\mathbf{A}(K_n) = \mathbf{J} - \mathbf{I}$. The eigenvalues of $\mathbf{J}$ are n (simple, with eigenvector $\mathbf{1}$) and 0 (all others). Thus the spectrum of K_n is $n - 1$ and -1.

(b) Write
$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{J}_{a,b} \\ \mathbf{J}_{b,a} & \mathbf{0} \end{pmatrix}.$$
There are $b - 1$ vectors in the kernel of $\mathbf{J}_{a,b}$ and $a - 1$ vectors in the kernel of $\mathbf{J}_{b,a}$. Each corresponding to an eigenvector for the eigenvalue 0 of $\mathbf{A}$. The two eigenvectors remaining are
$$\begin{pmatrix} \sqrt{b}\mathbf{1} \\ \sqrt{a}\mathbf{1} \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt{b}\mathbf{1} \\ -\sqrt{a}\mathbf{1} \end{pmatrix},$$
corresponding to the eigenvalues $\sqrt{ab}$ and $-\sqrt{ab}$ respectively.

(c) This one is trickier. $\mathbf{A}(C_n)$ is the sum of two permutation matrices corresponding to the cycle $(123...n)$ and its inverse, say $\mathbf{P}$ and $\mathbf{P}^{-1}$. An eigenvector for a cyclic matrix can be easily built from an n-root of unity ω:
$$\mathbf{P} \begin{pmatrix} 1 \\ \omega \\ \vdots \\ \omega^{n-1} \end{pmatrix} = \begin{pmatrix} \omega^{n-1} \\ 1 \\ \vdots \\ \omega^{n-2} \end{pmatrix} = \omega^{n-1} \begin{pmatrix} 1 \\ \omega \\ \vdots \\ \omega^{n-1} \end{pmatrix} \text{ and}$$
$$\mathbf{P}^{-1} \begin{pmatrix} 1 \\ \omega \\ \vdots \\ \omega^{n-1} \end{pmatrix} = \begin{pmatrix} \omega \\ \omega^2 \\ \vdots \\ 1 \end{pmatrix} = \omega \begin{pmatrix} 1 \\ \omega \\ \vdots \\ \omega^{n-1} \end{pmatrix},$$
thus the eigenvalues are $\omega^{n-1} = \omega^{-1}$ and ω, hence the eigenvalues of $\mathbf{A}(C_n) = \mathbf{P} + \mathbf{P}^{-1}$ are $\omega^{-1} + \omega$ for all nth roots of unity, that is, $\omega = \mathrm{e}^{2\pi \mathrm{i}(k/n)}$, $k = 0, ..., n - 1$. Thus the eigenvalues of C_n are
$$2 \cos \left(2\pi \frac{k}{n} \right) \quad \text{for } k = 0, ..., n - 1.$$
Note that 2 is always the largest (and simple) eigenvalue, and that -2 is an eigenvalue if and only if n is even. All other eigenvalues have multiplicity 2.