6 Optimization for cliques and colourings

6.1 Geometry of inequalities

We are in $\mathbb{R}^n$, and given a few vectors, we will concern ourselves with some special linear combinations of these vectors.

If $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ are vectors and $\alpha_1, ..., \alpha_m$ are scalars, then

$$\alpha_1 \mathbf{u}_1 + ... + \alpha_m \mathbf{u}_m$$

is a linear combination of the vectors, defined to be

(i) *affine combination* if $\sum \alpha_i = 1$,

(ii) *conical combination* if $\alpha_i \geq 0$ for all i,

(iii) *convex combination* if $\sum \alpha_i = 1$ and $\alpha_i \geq 0$ for all i.

The *affine hull* of the set $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ is the set of all affine combinations, and, similarly, the *convex hull* is the set of all convex combinations.

A subset of $\mathbb{R}^n$ is called *convex* if the segment between any two points of the set is entirely contained in the set. A *cone* is a set subset C of $\mathbb{R}^n$ so that if $\mathbf{v} \in C$, then $\alpha \mathbf{v} \in C$ for all $\alpha \geq 0$.

Exercise 6.1. Show that the convex hull of a set of vectors is precisely equal to the smallest convex set that contains those points.

A *hyperplane* $\mathcal{H}$ in $\mathbb{R}^n$ is the set of vectors that lie in the kernel of a linear functional, meaning, there is $\mathbf{a}$ so that

$$\mathcal{H} = \{\mathbf{x} : \mathbf{a}^\mathsf{T}\mathbf{x} = 0\}.$$

If instead of 0 we put a non-zero scalar β, then we call it an *affine hyperplane*. If we replace the equality sign by $\leq$, then the region defined is called a *half-space*.

Exercise 6.2. Verify that the intersection of convex sets is convex.

Exercise 6.3. Prove that a half-space is a convex set.

The intersection of a finite number of half-spaces is called a *polyhedron*, that is, any polyhedron $\mathbb{P}$ can be determined by a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ as follows:

$$\mathbb{P} = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}.$$

A *polytope* is a bounded polyhedron. It is not hard to believe that a set is a polytope if and only if it is the convex hull of a finite number of points, though the proof is less straightforward.

Given $\mathbf{A}$ and $\mathbf{b}$, a major and important problem is to decide whether the polyhedron determined by the inequalities $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ is empty or not. To that avail, we introduce the Theorem of Alternatives.

Theorem 6.4. *Given* $\mathbf{A}$ *and* $\mathbf{b}$*, exactly one of the following holds.*

(1) The system $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ *has a solution.*

(2) There is $\mathbf{z}$ *with* $\mathbf{z} \geq \mathbf{0}$*,* $\mathbf{z}^{\mathsf{T}}\mathbf{A} = \mathbf{0}$*, and* $\mathbf{b}^{\mathsf{T}}\mathbf{z} < 0$*.*

Proof. Clearly both cannot be true at the same time. So we must show both cannot be false at the same time. The proof is by induction on the number of columns of $\mathbf{A}$. Within the induction there is an implicit algorithm known as the Fourier-Motzkin elimination procedure.

For the base case, note that if $\mathbf{A}$ has zero columns, then (1) simply states that $\mathbf{b} \geq \mathbf{0}$. If that fails, one coordinate is negative, say ith, so simply choose $\mathbf{z} = \mathbf{e}_i$.

Assume the theorem holds for any matrix with $n-1$ columns. Consider now $\mathbf{A}$ a $m \times n$ matrix. Assume wlog that the inequalities were scaled so that the magnitude of the non-zero coefficients of x_n is 1. Let I_+ be the row indices corresponding to positive coefficients of x_n, I_- to negative, and I_0 to the 0 coefficients. Therefore the system writes as

$$\sum_{k=1}^{n-1} \mathrm{A}_{ik}\mathrm{x}_k + \mathrm{x}_n \leq \mathrm{b}_i, \quad \text{for all } i \in I_+,$$
$$\sum_{k=1}^{n-1} \mathrm{A}_{ik}\mathrm{x}_k - \mathrm{x}_n \leq \mathrm{b}_i, \quad \text{for all } i \in I_-,$$
$$\sum_{k=1}^{n-1} \mathrm{A}_{ik}\mathrm{x}_k \leq \mathrm{b}_i, \quad \text{for } i \in I_0.$$

This implies

$$\max_{i \in I_-}\left(\sum_{k=1}^{n-1} \mathrm{A}_{ik}\mathrm{x}_k - \mathrm{b}_i\right) \leq \min_{i \in I_+}\left(\mathrm{b}_i - \sum_{k=1}^{n-1} \mathrm{A}_{ik}\mathrm{x}_k\right),$$

so the original system has a solution if and only if the following system also does

$$\sum_{k=1}^{n-1}(\mathrm{A}_{ik} + \mathrm{A}_{jk})\mathrm{x}_k \leq \mathrm{b}_i + \mathrm{b}_j, \quad \text{for } i \in I_- \text{ and } j \in I_+,$$
$$\sum_{k=1}^{n-1} \mathrm{A}_{ik}\mathrm{x}_k \leq \mathrm{b}_i, \quad \text{for } i \in I_0.$$

If the original has no solution, the one above also is not, and, by induction, there are w_{ij}, $i \in I_i$ and $j \in I_+$, and , v_i, $i \in I_0$, so that, for all indices, $w_{ij} \geq 0$, $v_i \geq 0$, and

$$\sum_{i \in I_-, j \in I_+} (\mathrm{A}_{ik} + \mathrm{A}_{jk})w_{ij} + \sum_{i \in I_0} \mathrm{A}_{ik}v_i = 0, \quad \text{for all } k,$$

and

$$\sum_{i \in I_-, j \in I_+} (\mathrm{b}_i + \mathrm{b}_j)w_{ij} + \sum_{i \in I_0} \mathrm{b}_i v_i < 0.$$

Now, simply define the vector $\mathbf{z}$ with

$$\begin{aligned} \mathrm{z}_i &= \sum_{j \in I_+} w_{ij}, &\quad \text{for } i \in I_-, \\ \mathrm{z}_j &= \sum_{i \in I_-} w_{ij}, &\quad \text{for } j \in I_+, \\ \mathrm{z}_i &= v_i, &\quad \text{for } i \in I_0. \end{aligned}$$

It satisfies $\mathbf{z} \geq \mathbf{0}$, $\mathbf{z}^\mathsf{T}\mathbf{A} = \mathbf{0}$, and $\mathbf{b}^\mathsf{T}\mathbf{z} < \mathbf{0}$. □

Exercise 6.5. Farka's lemma says that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ has no solution if and only if there is $\mathbf{y}$ so that $\mathbf{y}^\mathsf{T}\mathbf{A} \geq \mathbf{0}$ and $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$. Prove this (using the Theorem of the Alternatives).

6.2 Linear programs

A *linear program* is an optimization problem that attempts to find the maximum or minimum of a linear functional whose viability region is a polyhedron. Upon certain operations, an equivalent formulation to any LP has the form

$$\begin{aligned} \max \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

The vector $\mathbf{x}$ corresponds to variables. The objective function is linear, and so must be the remaining constraints.

Optimization of combinatorial structures can be modelled with LPs provided integrality constraints are added to the variables $\mathbf{x}$. In this case, the optimization problem is called an *integer program*. For example, let $\mathbf{N}$ be the incidence matrix of a graph, with rows corresponding to vertices, and columns to edges. Then

$$\begin{aligned} \max \quad & \mathbf{1}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & \mathbf{Nx} \leq \mathbf{1} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{x} \in \mathbb{Z}^m \end{aligned}$$

yields an optimum solution that corresponds to the edges of a matching of maximum size.

A linear program can satisfy one of three possibilities. Either there is a vector $\mathbf{x}$ that satisfies the constraints and attains a maximum, or there is not, and this happens either because there are vectors satisfying the constraints of arbitrarily large objective value, or because there is no vector at all satisfying the constraints (this is because the objective function is continuous and the region of feasibility is a closed set).

To any linear program, one can define a dual program, as follows

$$\text{(P)} \quad \begin{aligned} \max \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \qquad \Bigg| \qquad \text{(D)} \quad \begin{aligned} \min \quad & \mathbf{b}^\mathsf{T}\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^\mathsf{T}\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

This dual program was written carefully so that one can always guarantee that any feasible solution to the primal has objective value less or equal than any feasible solution to the dual. In fact, if $\mathbf{x}$ and $\mathbf{y}$ are a pair of primal-dual solutions, it follows that

$$\mathbf{c}^\mathsf{T}\mathbf{x} \leq (\mathbf{y}^\mathsf{T}\mathbf{A})\mathbf{x} = \mathbf{y}^\mathsf{T}(\mathbf{A}\mathbf{x}) \leq \mathbf{y}^\mathsf{T}\mathbf{b}.$$

This is known as **weak duality**, and it implies, in particular, that if any of the primal or dual is unbounded, then the other must be infeasible.

Perhaps surprisingly, if both have feasible solutions, then their respective optima have same objective value.

Theorem 6.6 (Strong duality)**.** *Assume (P) and (D) are feasible. Then their optima solutions have the same objective values.*

Proof. Consider the following big system of equations

$$\begin{pmatrix} \mathbf{A} & \\ & -\mathbf{A}^\mathsf{T} \\ -\mathbf{c}^\mathsf{T} & \mathbf{b}^\mathsf{T} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \leq \begin{pmatrix} \mathbf{b} \\ -\mathbf{c} \\ 0 \end{pmatrix}, \quad \text{with } \mathbf{x}, \mathbf{y} \geq \mathbf{0}.$$

Assume the theorem is false. Then, because of weak duality, the system above has no solution. By a variant of the Theorem of the Alternatives, there is a vector $\begin{pmatrix}\mathbf{u} & \mathbf{v} & q\end{pmatrix}^\mathsf{T} \geq \mathbf{0}$ so that

$$\begin{pmatrix}\mathbf{u} & \mathbf{v} & q\end{pmatrix}^\mathsf{T} \begin{pmatrix} \mathbf{A} & \\ & -\mathbf{A}^\mathsf{T} \\ -\mathbf{c}^\mathsf{T} & \mathbf{b}^\mathsf{T} \end{pmatrix} \geq \mathbf{0} \quad \text{and} \quad \begin{pmatrix}\mathbf{u} & \mathbf{v} & q\end{pmatrix}^\mathsf{T} \begin{pmatrix} \mathbf{b} \\ -\mathbf{c} \\ 0 \end{pmatrix} < 0.$$

Thus

$$\mathbf{u}^\mathsf{T}\mathbf{A} \geq q\mathbf{c}^\mathsf{T}, \quad \mathbf{A}\mathbf{v} \leq q\mathbf{b}, \quad \mathbf{u}^\mathsf{T}\mathbf{b} - \mathbf{v}^\mathsf{T}\mathbf{c} < 0.$$

This immediately leads to a contradiction. □

Exercise 6.7. Prove that it is only needed to assume that either the primal or the dual is feasible to get that the other is also feasible.

Exercise 6.8. Prove the complementary slackness conditions, ie, if $\mathbf{x}$ and $\mathbf{y}$ are a pair of respective optima solutions for a primal-dual pair of LPs as above, then it is not possible that a variable is non-zero while the corresponding inequality is not satisfied with equality. (Hint: write extra variables $\mathbf{u}$ so that $\mathbf{A}\mathbf{x} + \mathbf{u} = \mathbf{b}$ and variables $\mathbf{v}$ with $\mathbf{A}^\mathsf{T}\mathbf{y} - \mathbf{v} = \mathbf{c}$.)

There are two main reasons why we are discussing linear programs. The first is that eventually I will say something like "Remember the results we had for variables whose corresponding vector lies in the positive orthant of $\mathbb{R}^n$? They work pretty much the same for this other cone over here".

6.3 Fractional chromatic number

The other reason is that we can already associate this theory to some of the concepts we saw before

Let $\mathbf{M}$ be a matrix whose rows are indexed by the vertices of a graph G, and the columns by all cliques of the graph. The integer program

$$\begin{aligned} \max \quad & \mathbf{1}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{M}^{\mathsf{T}}\mathbf{x} \leq \mathbf{1} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{x} \in \mathbb{Z}^n \end{aligned}$$

is asking for the maximum number of vertices of the graph so that no two of them belong to the same clique, ie, a maximum coclique of the graph. If we drop the integrality constraints, we obtain an LP, to which we can write the dual

$$\text{(P)} \quad \begin{aligned} \max \quad & \mathbf{1}^{\mathsf{T}}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{M}^{\mathsf{T}}\mathbf{x} \leq \mathbf{1} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad \Bigg| \qquad \text{(D)} \quad \begin{aligned} \min \quad & \mathbf{1}^{\mathsf{T}}\mathbf{y} \\ \text{s.t.} \quad & \mathbf{M}\mathbf{y} \geq \mathbf{1} \\ & \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Note that if we add integrality constraints to the dual, we will be finding the smallest number of cliques necessary to cover the vertices of G, meaning, $\chi(\overline{G})$. For this reason, the optimum of these LPs is denoted by $\chi_f(\overline{G})$, with "f" standing for fractional.

It is an immediate consequence that $\alpha(G) \leq \chi_f(\overline{G})$ (or, if you prefer, $\omega(G) \leq \chi_f(G)$.

Exercise 6.9. Show that

$$\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

Show that equality occurs if the graph is vertex-transitive (meaning: there is automorphism mapping any vertex to any vertex).

It seems plausible to believe optimization over certain subsets of vector spaces plays a big role in the theory behind α or ω, and χ or $\overline{\chi}$. We saw another example of this connection before: Hoffman's bound $\chi(G) \geq 1 - \theta_1(\mathbf{A})/\theta_n(\mathbf{A})$ held true even if we varied the non-zero entries of $\mathbf{A}$. Our goal over the next sections is to understand how these seemingly unrelated approaches are in fact very much related.

6.4 Positive semidefinite matrices

A real matrix $\mathbf{M}$ is positive semidefinite if it satisfies the following properties:

- $\mathbf{M}$ is symmetric.
- $\mathbf{v}^T\mathbf{M}\mathbf{v} \geq 0$ for all $\mathbf{v}$.

If the inequality is strict for all non-zero $\mathbf{v}$, then $\mathbf{M}$ is called positive definite. The only thing we want now is a characterization.

This is probably one of the most famous "exercises" in linear algebra.

Theorem 6.10. *Let* $\mathbf{M}$ *be a symmetric matrix. The following are equivalent.*

(a) $\mathbf{M}$ *is positive semidefinite.*

(b) The eigenvalues of $\mathbf{M}$ *are non-negative.*

(c) There exists a matrix $\mathbf{B}$ *so that* $\mathbf{M} = \mathbf{B}^T\mathbf{B}$.

(d) For all positive semidefinite matrices $\mathbf{A}$*, we have* $\langle \mathbf{M}, \mathbf{A} \rangle \geq 0$.

Proof. Assume (a). Let $\mathbf{M}\mathbf{v} = \theta\mathbf{v}$. Then $0 \leq \mathbf{v}^T\mathbf{M}\mathbf{v} = \theta\mathbf{v}^T\mathbf{v}$, thus $\theta \geq 0$. Assume (b). We diagonalize $\mathbf{M}$ as

$$\mathbf{M} = \mathbf{P}^T\mathbf{D}\mathbf{P}.$$

As $\mathbf{D} \geq 0$, we have

$$\mathbf{M} = \mathbf{P}^T\sqrt{\mathbf{D}}\sqrt{\mathbf{D}}\mathbf{P} = (\sqrt{\mathbf{D}}\mathbf{P})^T(\sqrt{\mathbf{D}}\mathbf{P}).$$

Assume (c). Then

$$\langle \mathbf{M}, \mathbf{A} \rangle = \operatorname{tr} \mathbf{M}\mathbf{A} = \operatorname{tr} \mathbf{B}^T\mathbf{B}\mathbf{A} = \operatorname{tr} \mathbf{B}\mathbf{A}\mathbf{B}^T.$$

As $\mathbf{A}$ is psd, we have $\operatorname{tr} \mathbf{B}\mathbf{A}\mathbf{B}^T \geq 0$. Finally, assume (d). Take $\mathbf{A} = \mathbf{v}\mathbf{v}^T$, which is clearly psd for any $\mathbf{v}$. We have $0 \leq \langle \mathbf{M}, \mathbf{v}\mathbf{v}^T \rangle = \mathbf{v}^T\mathbf{M}\mathbf{v}$, as wished. □

Exercise 6.11. Show that $\mathbf{M}$ is positive semidefinite if and only if its principal minors are non-negative (use interlacing?). Recall, a principal minor is a determinant of a square submatrix symmetric about the main diagonal.

Exercise 6.12. Let $\mathbf{M} \succcurlyeq \mathbf{0}$. Given vectors $\mathbf{u}$ and $\mathbf{v}$, and $t \in \mathbb{R}$, write $(\mathbf{u} + t\mathbf{v})^\mathsf{T}\mathbf{M}(\mathbf{u} + t\mathbf{v})$ explicitly. You know that this is ≥ 0 for all t. Use this to conclude that

$$(\mathbf{u}^\mathsf{T}\mathbf{M}\mathbf{v})^2 \leq (\mathbf{u}^\mathsf{T}\mathbf{M}\mathbf{u})(\mathbf{v}^\mathsf{T}\mathbf{M}\mathbf{v}).$$

(This is probably our favourite proof of *Cauchy-Schwarz*'s).

Suppose the symmetric matrix $\mathbf{M}$ is written as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix},$$

with $\mathbf{A}$ invertible. This leads to the factorization

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C}\mathbf{A}^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

The matrix $\mathbf{S} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is called the Schur complement of $\mathbf{A}$ in $\mathbf{M}$.

Theorem 6.13. *Given* $\mathbf{M}$ *in block form as above, with* $\mathbf{A}$ *invertible, then* $\mathbf{M} \succcurlyeq \mathbf{0}$ *if and only if* $\mathbf{A}$ *and* $\mathbf{S}$ *are also.*

Proof. Follows immediately from noting that $\mathbf{M} = \mathbf{P}^\mathsf{T} \begin{pmatrix} \mathbf{A} & \\ & \mathbf{S} \end{pmatrix} \mathbf{P}$, with a matrix $\mathbf{P}$ that is non-singular. □

This result gives the famous *Cholesky decomposition* of a matrix:

Theorem 6.14. *If* $\mathbf{A}$ *is positive semidefinite, there is a lower triangular matrix* $\mathbf{L}$ *so that* $\mathbf{A} = \mathbf{L}\mathbf{L}^\mathsf{T}$.

Proof. By induction on n. If $\mathrm{A}_{1,1} = 0$, then the first row and column of $\mathbf{A}$ are zero (why?), thus we can apply induction to the submatrix of $\mathbf{A}$ obtained upon deleting them. Otherwise, we have

$$\mathbf{A} = \begin{pmatrix} a & \mathbf{b}^\mathsf{T} \\ \mathbf{b} & \mathbf{A}_1 \end{pmatrix},$$

thus, there is a lower triangular $\mathbf{P}$ with

$$\mathbf{P}^{-\mathsf{T}}\mathbf{A}\mathbf{P}^{-1} = \begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_1 - a^{-1}\mathbf{b}\mathbf{b}^\mathsf{T} \end{pmatrix},$$

where both diagonal blocks are positive semidefinite. By induction, both admit a Cholesky decomposition, which regrouped with $\mathbf{P}$ gives the Cholesky decomposition of $\mathbf{A}$. □

Exercise 6.15. How to use this decomposition to decide (efficiently) whether a given matrix is positive semidefinite?

6.5 Kronecker and Schur

If $\mathbf{A}$ and $\mathbf{B}$ are matrices, their *Kronecker product* $\mathbf{A}\otimes\mathbf{B}$ is the matrix obtained from replacing the ij entry of $\mathbf{A}$ by $\mathrm{A}_{i,j}\mathbf{B}$. You are invited to verify that the Kronecker product is bilinear, but, most notably, it satisfies

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = \mathbf{AC}\otimes\mathbf{BD},$$

provided all products are well defined. In particular, if $\mathbf{u}$ is eigenvector of $\mathbf{A}$ and $\mathbf{v}$ is one of $\mathbf{B}$, then $\mathbf{u}\otimes\mathbf{v}$ is eigenvector of $\mathbf{A}\otimes\mathbf{B}$ — its corresponding eigenvalue is the product of the original eigenvalues. This fact alone immediately implies the following:

Lemma 6.16. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive semidefinite, then so it is* $\mathbf{A}\otimes\mathbf{B}$.

The operator $\mathrm{vec}(\mathbf{A})$ takes $\mathbf{A}$, with n columns, to the column vector

$$\begin{pmatrix} \mathbf{A}\mathbf{e}_1 \\ \vdots \\ \mathbf{A}\mathbf{e}_n \end{pmatrix}.$$

Exercise 6.17. Convince yourself that

$$\mathrm{vec}(\mathbf{A}\mathbf{M}\mathbf{B}^\mathsf{T}) = (\mathbf{A}\otimes\mathbf{B})\,\mathrm{vec}(\mathbf{M}).$$

In other words, the linear map $\mathbf{M}\mapsto\mathbf{A}\mathbf{M}\mathbf{B}^\mathsf{T}$ is represented by $\mathbf{A}\otimes\mathbf{B}$.

The *Schur product* $\mathbf{A}\circ\mathbf{B}$ of two matrices with the same shape is defined as the entry-wise product (your high school dream).

Exercise 6.18. Prove that if $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite, then $\mathbf{A} \circ \mathbf{B}$ is also. (Kronecker and Schur are together in this section and it's not a coincidence).

Our goal in the next section is to look at

$$\begin{array}{rl} \max & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

and replace $\mathbf{x}$, vector in $\mathbb{R}^n$ with non-negative entries, by a positive semidefinite matrix. Of course, we need to build some theory to grasp how each component of the program above will find its analagous. Hopefully, we will gain more power in optimizing, while still maintaining the possibility to solve the programs.

6.6 Cones

An Euclidean space is a vector space over the reals of finite dimension and equipped with an inner product. For example, $\mathbb{R}^n$ and $\mathbb{S}^n$ are Euclidean spaces, and the respective inner products are given by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\mathsf{T}\mathbf{y}$, and $\langle \mathbf{A}, \mathbf{B} \rangle = \operatorname{tr} \mathbf{A}\mathbf{B}$. A cone $\mathbb{K}$ is a subset so that for all $\mathbf{x} \in \mathbb{K}$, we have $\alpha\mathbf{x} \in \mathbb{K}$ for all $\alpha > 0$.

We introduce four properties about cones that shall be useful to us. A cone $\mathbb{K}$ is called...

- *...closed* if no point that is the limit of a sequence of points in $\mathbb{K}$ is outside of $\mathbb{K}$ — for example, $\mathbb{R}^n_+$ is closed, but $\mathbb{R}^n_{++}$ is not;
- *...pointed* if $\mathbb{K}$ contains no line about the origin, equivalently, if $\mathbf{x} \in \mathbb{K}$ and $-\mathbf{x} \in \mathbb{K}$, then $\mathbf{x} = \mathbf{0}$;
- *...convex* if $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$, i.e., the line segment between any two points in $\mathbb{K}$ belongs entirely to $\mathbb{K}$ — for example, $\mathbb{S}^n_+$ is convex (a fact that is trivial only if you use the correct condition from Theorem **??**...);
- *...a cone with non-empty interior* if there is at least one point in $\mathbb{K}$ so that a ball of positive radius around that point is entirely contained in $\mathbb{K}$ — for example, the vectors $\mathbf{x}$ in $\mathbb{R}^n$ with $\mathbf{x} \geq \mathbf{0}$ and $\mathrm{x}_1 = 0$ for a cone that contains no interior point.

Exercise 6.19. Consider the *Lorentz cone*, defined as

$$\mathbb{K} = \{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \|\mathbf{x}\| \leq t\}.$$

Verify if it satisfies each of the four properties above. Then, (try to) do the same for the cone of *copositive matrices*, defined as

$$\mathbb{P} = \{\mathbf{M} \in \mathbb{S}^n : \mathbf{x}^\mathsf{T}\mathbf{M}\mathbf{x} \geq 0 \text{ for all } \mathbf{x} \geq 0\}.$$

If $\mathbb{V}_1$ and $\mathbb{V}_2$ are Euclidean spaces, let $\mathbf{c} \in \mathbb{V}_1$, $\mathcal{A} : \mathbb{V}_1 \to \mathbb{V}_2$ be linear, and $\mathbf{b} \in \mathbb{V}_2$. Let $\mathbb{K}$ be a closed, pointed and convex cone, with non-empty interior. A *conic program* is an optimization problem that can be written in the form

$$\begin{array}{rl} \max & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \mathcal{A}(\mathbf{x}) = \mathbf{b} \\ & \mathbf{x} \in \mathbb{K}. \end{array}$$

(In some contexts, a conic program is presented with "sup" instead of "max", to highlight the fact that it can be feasible and bounded while having no optimal solution. I will present all programs with "max" and "min", understanding that those might not exist even if the program is feasible and bounded).

For a linear program, $\mathbb{K} = \mathbb{R}^n_+$. For a semidefinite program, $\mathbb{K} = \mathbb{S}^n_+$. In both cases you can view $\mathbb{V}_1$ or $\mathbb{V}_2$ as isomorphic to $\mathbb{R}^k$ for some k, though in the second case this has to be made explicit via an isomorphism such as $\mathrm{vec} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n^2}$, as we discussed.

In order to better understand conic programs (and their duals), let us introduce some more theory. Given $\mathcal{A} : \mathbb{V}_1 \to \mathbb{V}_2$, linear, its *adjoint* is the (unique) linear transformation $\mathcal{A}^* : \mathbb{V}_2 \to \mathbb{V}_1$ that satisfies

$$\langle \mathcal{A}(\mathbf{x}), \mathbf{y} \rangle_{\mathbb{V}_2} = \langle \mathbf{x}, \mathcal{A}^*(\mathbf{y}) \rangle_{\mathbb{V}_1}.$$

Note that when $\mathbb{V}_1 = \mathbb{R}^n$ and $\mathbb{V}_2 = \mathbb{R}^m$, with the conventional inner product, and $\mathcal{A}$ is represented by a $m \times n$ matrix $\mathbf{A}$, then $\mathcal{A}^*$ is given by $\mathbf{A}^\mathsf{T}$.

Exercise 6.20. Consider the linear operator $\mathrm{diag} : \mathbb{R}^{n \times n} \to \mathbb{R}^n$, that acts as

$$\mathrm{diag}(\mathbf{X}) = \sum_{i=1}^{n} \mathbf{X}_{ii} \mathbf{e}_i.$$

Describe its adjoint (hint: we will denote it by Diag).

Given $\mathbf{x}, \mathbf{y} \in \mathbb{V}$, we say that

$$\mathbf{x} \succcurlyeq_{\mathbb{K}} \mathbf{y} \iff \mathbf{x} - \mathbf{y} \in \mathbb{K}.$$

Exercise 6.21. You are invited to prove (or at least think about and convince yourself of) the following properties:

(1) $\mathbf{x} \succcurlyeq_{\mathbb{K}} \mathbf{y} \iff \mathbf{y} \succcurlyeq_{-\mathbb{K}} \mathbf{x}$

(2) $\succcurlyeq_{\mathbb{K}}$ is a partial order, meaning, for all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathbb{V}$, we have $\mathbf{x} \succcurlyeq_{\mathbb{K}} \mathbf{x}$; $\mathbf{x} \succcurlyeq_{\mathbb{K}} \mathbf{y}$ and $\mathbf{y} \succcurlyeq_{\mathbb{K}} \mathbf{x}$ imply $\mathbf{x} = \mathbf{y}$; and $\mathbf{x} \succcurlyeq_{\mathbb{K}} \mathbf{y}$ and $\mathbf{y} \succcurlyeq_{\mathbb{K}} \mathbf{z}$ imply $\mathbf{x} \succcurlyeq_{\mathbb{K}} \mathbf{z}$.

(3) $\mathbf{x} \succcurlyeq_{\mathbb{K}} \mathbf{y}$ imply $\alpha\mathbf{x} \succcurlyeq_{\mathbb{K}} \alpha\mathbf{y}$ for all $\alpha > 0$.

(4) $\mathbf{x} \succcurlyeq_{\mathbb{K}} \mathbf{y}$ and $\mathbf{u} \succcurlyeq_{\mathbb{K}} \mathbf{v}$ imply $\mathbf{x} + \mathbf{u} \succcurlyeq_{\mathbb{K}} \mathbf{y} + \mathbf{v}$.

Exercise 6.22. If $\mathcal{A}$ is a linear map from $\mathbb{V}_1$ to $\mathbb{V}_2$, and $\mathbb{K}$ is a convex cone in $\mathbb{V}_1$, show that $\mathcal{A}\mathbb{K}$ is a convex cone in $\mathbb{V}_2$.

The *dual of a cone* $\mathbb{K} \subseteq \mathbb{V}$ is defined as

$$\mathbb{K}^* = \{\mathbf{x} \in \mathbb{V} : \langle \mathbf{x}, \mathbf{s} \rangle \geq 0 \text{ for all } \mathbf{s} \in \mathbb{K}\}.$$

Lemma 6.23. *If $\mathbb{K}$ is a cone, then $\mathbb{K}^*$ is a closed convex cone.*

Proof. First, if $\mathbf{x} \in \mathbb{K}^*$, then $\alpha\mathbf{x} \in \mathbb{K}^*$ for all $\alpha > 0$, as $\langle \alpha\mathbf{x}, \mathbf{s} \rangle = \alpha\langle \mathbf{x}, \mathbf{s} \rangle \geq 0$ for all $\mathbf{s} \in \mathbb{K}$.

Second, $\mathbb{K}^*$ is closed because it is the intersection of closed sets (research this fact from analysis if you want).

Third, $\mathbb{K}^*$ is convex because it is the intersection of convex sets (remember that you already showed that half-spaces are convex). □

Exercise 6.24. Which are the cones $(\mathbb{R}^n_+)^*$ and $(\mathbb{S}^n_+)^*$? Could you also describe the duals of the two cones appearing in Exercise 6.19 ?

A cone that is equal to its dual is called a *self-dual cone*.

We end this section stating an important result about cones and their duals. Its proof is beyond the scope of this course.

Theorem 6.25. *If $\mathbb{K}$ is a closed, convex cone, then so it is $\mathbb{K}^*$, and $\mathbb{K}^{**} = \mathbb{K}$. Moreover,*

(i) If K is pointed, then $\mathbb{K}^*$ has non-empty interior.

(ii) If K has non-empty interior, then $\mathbb{K}^*$ is pointed.

6.7 Duality

Say $\mathbb{V}_1$ and $\mathbb{V}_2$ are Euclidean spaces, let $\mathbf{c} \in \mathbb{V}_1$, $\mathcal{A} : \mathbb{V}_1 \to \mathbb{V}_2$ be linear, and $\mathbf{b} \in \mathbb{V}_2$. Let $\mathbb{K} \subseteq \mathbb{V}_1$ and $\mathbb{L} \subseteq \mathbb{V}_2$ be closed, convex cones. Consider the conic program

$$\begin{array}{rl} \max & \langle \mathbf{c}, \mathbf{x} \rangle_{\mathbb{V}_1} \\ \text{subject to} & \mathcal{A}(\mathbf{x}) \preccurlyeq_{\mathbb{L}} \mathbf{b} \\ & \mathbf{x} \in \mathbb{K}. \end{array}$$

Recall linear program duality, where from

$$\begin{array}{crl|crl} & \max & \mathbf{c}^\mathsf{T}\mathbf{x} & & \min & \mathbf{b}^\mathsf{T}\mathbf{y} \\ \text{(P)} & \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} & \text{(D)} & \text{s.t.} & \mathbf{A}^\mathsf{T}\mathbf{y} \geq \mathbf{c} \\ & & \mathbf{x} \geq \mathbf{0} & & & \mathbf{y} \geq \mathbf{0} \end{array}$$

we obtained immediately that

$$\mathbf{c}^\mathsf{T}\mathbf{x} \leq \mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} \leq \mathbf{y}^\mathsf{T}\mathbf{b}.$$

Let us now define a conic program crafted in such way that the objective value of any of its feasible solutions is an upper bound on the objective value of the conic program above.

- We know that $\mathbf{b} - \mathcal{A}(\mathbf{x}) \in \mathbb{L}$, thus $\langle \mathbf{b} - \mathcal{A}(\mathbf{x}), \mathbf{y} \rangle_{\mathbb{V}_2} \geq 0$ for all $\mathbf{y} \in \mathbb{L}^*$.
- Therefore, $\langle \mathbf{b}, \mathbf{y} \rangle_{\mathbb{V}_2} \geq \langle \mathcal{A}(\mathbf{x}), \mathbf{y} \rangle_{V_2} = \langle \mathbf{x}, \mathcal{A}^*(\mathbf{y}) \rangle_{V_1}$.

- Now, if we require $\mathcal{A}^*(\mathbf{y}) - \mathbf{c} \in \mathbb{K}^*$, it follows that, for all $\mathbf{x} \in \mathbb{K}$, we have $\langle \mathcal{A}^*(\mathbf{y}) - \mathbf{c}, \mathbf{x}\rangle_{\mathbb{V}_1} \geq 0$, hence $\langle \mathcal{A}^*(\mathbf{y}), \mathbf{x}\rangle_{\mathbb{V}_1} \geq \langle \mathbf{c}, \mathbf{x}\rangle_{\mathbb{V}_1}$.

We arrive at the following primal-dual formulation for conic programs:

$$\text{(P)} \quad \begin{array}{rl} \max & \langle \mathbf{c}, \mathbf{x}\rangle_{\mathbb{V}_1} \\ \text{s.t.} & \mathcal{A}(\mathbf{x}) \preccurlyeq_{\mathbb{L}} \mathbf{b} \\ & \mathbf{x} \in \mathbb{K}. \end{array} \qquad \Bigg| \qquad \text{(D)} \quad \begin{array}{rl} \min & \langle \mathbf{b}, \mathbf{y}\rangle_{\mathbb{V}_2} \\ \text{s.t.} & \mathcal{A}^*(\mathbf{y}) \succcurlyeq_{\mathbb{K}^*} \mathbf{c} \\ & \mathbf{y} \in \mathbb{L}^*. \end{array}$$

The discussion above has shown the weak duality theorem:

Theorem 6.26. *For a pair of primal-dual conic programs such as above, if $\mathbf{x}$ is feasible for (P) and $\mathbf{y}$ is feasible for (D), then $\langle \mathbf{c}, \mathbf{x}\rangle_{\mathbb{V}_1} \leq \langle \mathbf{b}, \mathbf{y}\rangle_{\mathbb{V}_2}$.* □

Strong duality for linear programs guarantees that optimum solutions for the primal and dual, if they exist, must be equal. A similar theorem holds true also for semidefinite programs: under some natural and mild extra conditions, equality also holds for the optimum of the primal and dual formulations. I will not show this theorem in this course: you are however invited the research it on your own.

All semidefinite programs we will study in this course satisfy these mild conditions to guarantee strong duality, therefore from here on we assume that for a pair of primal-dual SDPs, if they are bounded, they achieve their optimum values, which coincide.

Exercise 6.27. Write the dual of the conic program

$$\begin{array}{rl} \max & \langle \mathbf{c}, \mathbf{x}\rangle_{\mathbb{V}_1} \\ \text{subject to} & \mathcal{A}(\mathbf{x}) = \mathbf{b} \\ & \mathbf{x} \in \mathbb{K}. \end{array}$$

Exercise 6.28. Using Theorem 6.25, verify that the dual of the dual is the primal.

Let us now come back to something more concrete. Assume $\mathbb{V}_1 = \mathbb{S}^n$, and $\mathbb{V}_2 = \mathbb{R}^m$. Have $\mathbb{K} = \mathbb{S}^n_+$. How do linear functions from $\mathbb{S}^n$ to $\mathbb{R}^m$ look like?

Well, if $f : \mathbb{S}^n \to \mathbb{R}$ is a linear functional, then there exists a unique $\mathbf{M}$ so that, for all $\mathbf{X} \in \mathbb{S}^n$, $f(\mathbf{X}) = \langle \mathbf{X}, \mathbf{M}\rangle$. This can be easily obtained — in fact, if $\{\mathbf{M}_1, ..., \mathbf{M}_\ell\}$ form an orthonormal basis, then

$$\mathbf{M} = \sum f(\mathbf{M}_i)\mathbf{M}_i.$$

Therefore, if $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ is linear, there are matrices $\mathbf{A}_1$,..., $\mathbf{A}_m$ so that, for all $\mathbf{X}$,

$$\mathcal{A}(\mathbf{X}) = \begin{pmatrix} \langle \mathbf{A}_1, \mathbf{X}\rangle \\ \vdots \\ \langle \mathbf{A}_m, \mathbf{X}\rangle \end{pmatrix}.$$

Exercise 6.29. Given $\mathcal{A}$ with matrices $\mathbf{A}_i$ as above, describe who is $\mathcal{A}^*(\mathbf{y})$ for $\mathbf{y} \in \mathbb{R}^m$.

Exercise 6.30. Given the positive semidefinite program

$$\begin{aligned} \max \quad & \langle \mathbf{C}, \mathbf{X} \rangle \\ \text{subject to} \quad & \langle \mathbf{A}_i, \mathbf{X} \rangle \leq \mathrm{b}_i \text{ for } i = 1, ..., m \\ & \mathbf{X} \succcurlyeq \mathbf{0}, \end{aligned}$$

write its dual.

Exercise 6.31. Assume a positive semidefinite program (P) and its dual (D) have feasible solution with the same objective value. What would complementary slackness conditions say about them?

6.8 Three SDP formulations

Lovász Theta

We now introduce the Lovász Theta parameter of a graph. Given a graph G, with adjacency matrix $\mathbf{A}$, let $\mathbf{z}$ denote the characteristic vector of an independent set S. If $\mathbf{Z} = (1/|S|)\mathbf{z}\mathbf{z}^\mathsf{T}$, then $\operatorname{tr} \mathbf{Z} = 1$, $\mathbf{Z} \succcurlyeq \mathbf{0}$, and $\mathbf{Z}_{ij} = 0$ for all edges ij of the graph. As such, it is a feasible solution to the semidefinite program

$$\begin{aligned} \max \quad & \langle \mathbf{J}, \mathbf{X} \rangle \\ \text{subject to} \quad & \mathrm{X}_{ij} = 0 \text{ for all edges } ij \text{ of the graph,} \\ & \operatorname{tr} \mathbf{X} = 1, \\ & \mathbf{X} \succcurlyeq \mathbf{0}, \end{aligned}$$

and its objective value is equal to $|S|$.

Exercise 6.32. You are invited to check that all constraints are of the form $\langle *, \mathbf{X} \rangle = *$.

Exercise 6.33. Write the dual of the program above.

The common optimum value of the program above and its dual is defined as the Lovász theta parameter of the graph, denoted by $\vartheta(G)$.

Eigenvalues

Recall that given $\mathbf{M} \in \mathbb{S}^n$ with largest eigenvalue θ and smallest eigenvalue τ, we have

$$\theta = \max_{\mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\|=1} \mathbf{x}^\mathsf{T}\mathbf{M}\mathbf{x},$$

and

$$\tau = \min_{\mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\|=1} \mathbf{x}^\mathsf{T}\mathbf{M}\mathbf{x},$$

Consider now that SDP

$$\begin{aligned} \max \quad & \langle \mathbf{M}, \mathbf{X} \rangle \\ \text{subject to} \quad & \operatorname{tr} \mathbf{X} = 1, \\ & \mathbf{X} \succcurlyeq \mathbf{0}. \end{aligned}$$

Exercise 6.34. Write its dual. Conclude (trivially!) that the dual has an optimal solution with objective value θ. Knowing this, conclude now that the primal also has an optimal solution with objective value θ.

Exercise 6.35. Write an SDP formulation for τ and repeat the steps of the previous exercise.

Sum of eigenvalues

Largest (or smallest) eigenvalues are optima of SDPs. The same can be said about the sum of the k largest eigenvalues. In this section, assume $\lambda_j(\mathbf{X})$ stands for the jth largest eigenvalue of $\mathbf{X}$.

Theorem 6.36. *Let $\mathbf{X} \in \mathbb{S}^n$, $\mu \in \mathbb{R}$, $k \in [n]$. The following are equivalent.*

(1) $\sum_{j=1}^{k} \lambda_j(\mathbf{X}) \leq \mu$.

(2) There are $\mathbf{Y} \in \mathbb{S}^n_+$ and $\eta \in \mathbb{R}$ so that

$$\mu - k\eta \geq \operatorname{tr} \mathbf{Y} \quad \textit{and} \quad \mathbf{Y} - \mathbf{X} + \eta \mathbf{I} \succcurlyeq \mathbf{0}.$$

Proof. From (2) to (1), we have

$$\mathbf{Y} - \mathbf{X} \succcurlyeq -\eta \mathbf{I} \implies \lambda_j(\mathbf{Y}) - \lambda_j(\mathbf{X}) \geq -\eta.$$

(This implication is not trivial — due to Ky Fan.) Thus, immediately, $\sum_{j=1}^{k} \lambda_j(\mathbf{X}) \leq \mu$.

From (1) to (2), have $\eta = \lambda_k(\mathbf{X})$, and

$$\mathbf{Y} = \sum_{j=1}^{k} (\lambda_j(\mathbf{X}) - \eta) \mathbf{v}_j \mathbf{v}_j^{\mathsf{T}},$$

where $\mathbf{v}_j$ are the eigenvectors of $\mathbf{X}$ corresponding to λ_j. Thus $\mu - k\eta - \operatorname{tr}(\mathbf{Y}) \geq 0$, and $\mathbf{Y} - \mathbf{X} + \eta \mathbf{I} \succcurlyeq \mathbf{0}$. □

Exercise 6.37. Given $\mathbf{M}$, show that $\lambda_1(\mathbf{M}) + ... + \lambda_k(\mathbf{M})$ is equal to

$$\begin{array}{rl} \max & \langle \mathbf{M}, \mathbf{X} \rangle \\ \text{subject to} & \langle \mathbf{I}, \mathbf{X} \rangle = k \\ & \mathbf{X} \preccurlyeq \mathbf{I} \\ & \mathbf{X} \succcurlyeq \mathbf{0}. \end{array}$$

(Use weak duality!).