

**Lemma 1.3.** *Let  $\mathbf{M}$  be a real symmetric matrix. If  $U$  is  $\mathbf{M}$ -invariant, then  $U^\perp$  is also  $\mathbf{M}$ -invariant.*

*Proof.* Note that  $\mathbf{v} \in U^\perp$ , by definition, if  $\mathbf{v}^T \mathbf{u} = 0$  for all  $\mathbf{u} \in U$ . For all  $\mathbf{u} \in U$  and  $\mathbf{v} \in U^\perp$ , note that

$$(\mathbf{M}\mathbf{v})^T \mathbf{u} = \mathbf{v}^T \mathbf{M}\mathbf{u} = \mathbf{v}^T (\mathbf{M}\mathbf{u}) = 0,$$

because  $\mathbf{u} \in U$ ,  $U$  is  $\mathbf{M}$ -invariant, and so  $\mathbf{M}\mathbf{u} \in U$ , and  $\mathbf{v} \in U^\perp$ . Thus  $\mathbf{M}\mathbf{v} \in U^\perp$ , as we wanted.  $\square$

Let  $\lambda$  be such that  $\det(\lambda \mathbf{I} - \mathbf{M}) = 0$ . Then  $\lambda \mathbf{I} - \mathbf{M}$  is singular, and therefore it contains at least one non-zero vector in its kernel. This is saying that all square matrices  $\mathbf{M}$  contain at least one eigenvector for each root of  $\phi_{\mathbf{M}}(x) = \det(x\mathbf{I} - \mathbf{M})$ . As  $\mathbf{M}$  is symmetric, we now know that all possible roots of  $\phi_{\mathbf{M}}$  are real.

**Lemma 1.4.** *Let  $U$  be an  $\mathbf{M}$ -invariant subspace with dimension  $\geq 1$ . Then there is one eigenvector of  $\mathbf{M}$  in  $U$ .*

*Proof.* Let  $\mathbf{P}$  be a matrix whose columns form an orthonormal basis for  $U$ . As  $U$  is  $\mathbf{M}$ -invariant, it follows that there is a matrix  $\mathbf{N}$  so that

$$\mathbf{M}\mathbf{P} = \mathbf{P}\mathbf{N}.$$

(Stop now and think carefully why this equality is true.) In particular,  $\mathbf{N} = \mathbf{P}^T \mathbf{M}\mathbf{P}$ , so  $\mathbf{N}$  is symmetric. Let  $\mathbf{u}$  be one eigenvector of  $\mathbf{N}$  with eigenvalue  $\lambda$ . Then

$$\mathbf{M}\mathbf{P}\mathbf{u} = \mathbf{P}\mathbf{N}\mathbf{u} = \lambda \mathbf{P}\mathbf{u},$$

and, moreover  $\mathbf{P}\mathbf{u} \neq \mathbf{0}$ , as the columns of  $\mathbf{P}$  are linearly independent. Thus  $\mathbf{P}\mathbf{u}$  is an eigenvector for  $\mathbf{M}$  in  $U$ .  $\square$

These four lemmas above are all you need to prove the following result by induction as an exercise.

**Theorem 1.5.** *Let  $\mathbf{M}$  be a real symmetric matrix. Then  $\mathbf{M}$  is diagonalizable by set of orthogonal eigenvectors, all of them corresponding to real eigenvalues.*

**Exercise 1.6.** Write the proof of this theorem as an exercise.

**Corollary 1.7.** *Let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be an orthonormal basis of eigenvectors for  $\mathbf{M}$ , each corresponding to an eigenvalue  $\lambda_1, \dots, \lambda_n$  (these are not necessarily distinct). Let  $\mathbf{P}$  be the matrix whose  $i$ th column is  $\mathbf{v}_i$ , and  $\Lambda$  the diagonal matrix whose  $i$ th diagonal element is  $\lambda_i$ . Then*

$$\mathbf{P}^T \mathbf{M} \mathbf{P} = \Lambda,$$

and

$$\mathbf{M} = \lambda_1(\mathbf{v}_1 \mathbf{v}_1^T) + \dots + \lambda_n(\mathbf{v}_n \mathbf{v}_n^T).$$