7 Theta

In this section, we focus on perhaps the most standard application of semidefinite program to graph theory.

Let $G = (V, E)$ be a graph on n vertices. Recall: a *coclique* (or stable set, or independent set) S is a subset of V that induces no edge from $E(G)$. A *clique* is the complement of a clique. A *colouring* of G is a partition of V into cocliques (having a colour given to each one of them). A *clique partition* is a partition of G into cliques. With this:

(i) $\alpha(G)$ is the size of the largest coclique in G, called the *independence number* or *coclique number*.

(ii) $\omega(G)$ is the size of the largest clique, called the *clique number*.

Note that $\alpha(G) = \omega(\overline{G})$.

(iii) $\chi(G)$ is the minimum number of colours needed to colour $V(G)$ with cocliques.

(iv) $\theta(G)$ is the minimum number of cliques needed to partition $V(G)$.

Note that $\theta(G) = \chi(\overline{G})$.

7.1 Theta and its dual

From the end of last section, we have defined the parameter

$$\begin{aligned} \vartheta(G) = \qquad\qquad \max \quad & \langle \mathbf{J}, \mathbf{X} \rangle \\ \text{subject to} \quad & \mathrm{X}_{ij} = 0 \;\forall\, ij \in E(G), \\ & \operatorname{tr} \mathbf{X} = 1, \\ & \mathbf{X} \succcurlyeq \mathbf{0}. \end{aligned}$$

It was easy to see that $\alpha(G) \leq \vartheta(G)$, because if $\mathbf{z}$ denote the characteristic vector of a coclique S, then $\mathbf{Z} = (1/|S|)\mathbf{z}\mathbf{z}^{\mathsf{T}}$ is feasible for the formulation above, with objective value equal to $|S|$.

From SDP duality, we also obtained that

$$\begin{aligned} \vartheta(G) = \qquad\qquad \min \quad & \lambda \\ \text{subject to} \quad & \mathrm{Y}_{ij} = 0 \;\forall\, ij \notin E(G) \\ & \lambda \mathbf{I} + \mathbf{Y} - \mathbf{J} \succcurlyeq \mathbf{0}. \end{aligned}$$

Our goal now is to introduce other alternative ways to look at this parameter $\vartheta(G)$. We would like to understand more precisely why $\vartheta(G) \leq \chi(\overline{G})$, for example, or better yet, what is the connection between ϑ and Hoffman lower bound for the chromatic number.

7.2 Largest eigenvalue

Theorem 7.1. *For any graph G, it follows that*

$$\vartheta(G) = \max\{\lambda_{\max}(\mathbf{B}) : \operatorname{diag}(\mathbf{B}) = \mathbf{1},\ \mathbf{B} \succcurlyeq \mathbf{0},\ \mathbf{B}_{ij} = 0\ \forall\, ij \in E(G)\}.$$

Proof. We will show that each feasible solution to the formulation above gives one for the max formulation of the ϑ parameter with the same objective value, and vice-versa.

Let $\mathbf{B}$ be feasible to the program above, and let $\mathbf{Y}$ be the projection onto an eigenline corresponding to $\lambda_{\max}(\mathbf{B})$. Then $\langle \mathbf{B}, \mathbf{Y}\rangle = \lambda_{\max}(\mathbf{B})$. On the other hand,

$$\langle \mathbf{B}, \mathbf{Y}\rangle = \langle \mathbf{B} \circ \mathbf{Y}, \mathbf{J}\rangle.$$

Now observe that $\mathbf{B} \circ \mathbf{Y}$ has trace equal to the trace of $\mathbf{Y}$, which is 1. It is also positive semidefinite (one of Schur's theorems), and from the definition of $\mathbf{B}$, we also have $(\mathbf{B}\circ\mathbf{Y})_{ij} = 0$ for all $ij \in E(G)$. Thus $\mathbf{B} \circ \mathbf{Y}$ os feasible for the max formulation of ϑ, with objective value $\lambda_{\max}(\mathbf{B})$.

Now consider an optimum solution in the max formulation of ϑ. If any of its diagonal entries is 0, make them equal to 1. Call this new matrix $\mathbf{X}$. Then, let $\mathbf{D}$ be the diagonal matrix that contains its diagonal. Look at

$$\mathbf{B} = \mathbf{D}^{-1/2}\mathbf{X}\mathbf{D}^{-1/2}.$$

It has constant diagonal equal to 1, it is positive-semidefinite, and it is so that $\mathbf{B}_{ij} = 0$ for all $ij \in E(G)$. If $\mathbf{d} = \operatorname{diag}(\mathbf{X}')$, then

$$\frac{\sqrt{\mathbf{d}}^{\mathsf{T}}\mathbf{B}\sqrt{\mathbf{d}}}{\sqrt{\mathbf{d}}^{\mathsf{T}}\sqrt{\mathbf{d}}} = \frac{\vartheta(G)}{1} = \vartheta(G).$$

□

With this result, we can show that the best possible Hoffman ratio is actually equal to $\vartheta(\overline{G})$ (and therefore that $\vartheta(\overline{G}) \leq \chi(G)$).

Corollary 7.2. *Given a matrix $\mathbf{A}$ with 0 diagonal, define*

$$\tilde{\mathbf{A}} = \begin{cases} [-\lambda_{\min}(\mathbf{A})]^{-1}\,\mathbf{A} & \text{if } \mathbf{A} \neq 0 \\ \mathbf{A} & \text{if } \mathbf{A} = 0 \end{cases}.$$

Given G, we have

$$\vartheta(\overline{G}) = \max\{1 + \lambda_{\max}(\tilde{\mathbf{A}}) : \mathrm{A}_{ij} = 0\ \forall\ ij \notin E(G).\}.$$

Proof. From the Theorem, we have

$$\vartheta(\overline{G}) = \max\{\lambda_{\max}(\mathbf{I} + \mathbf{A}) : \mathbf{A} \succcurlyeq -\mathbf{I}, \mathrm{A}_{ij} = 0\ \forall\ ij \notin E(G)\}.$$

For any $\mathbf{A}$ feasible, just note that the best scaling so that $\succcurlyeq -\mathbf{I}$ is maintained is always obtained from dividing $\mathbf{A}$ by $-\lambda_{\min}$. Hence the result follows.

□

7.3 Conic optimization for cocliques and colourings

In this section, we will see how to formulate the problems of finding α and χ as conic programs. Second, we will introduce more parameters between α and χ. They will be motivated by the well known vector colourings.

A matrix $\mathbf{M}$ is called *completely positive* if, for some k, there are nonnegative vectors $\mathbf{x}_1, ..., \mathbf{x}_k \in \mathbb{R}^n_+$ so that

$$\mathbf{M} = \mathbf{x}_1\mathbf{x}_1^{\mathsf{T}} + \ldots + \mathbf{x}_k\mathbf{x}_k^{\mathsf{T}}$$

Clearly, completely positive matrices are positive semidefinite, but the converse does not hold.

Exercise 7.3. Why not?

We will denote

$$\mathbb{S}^n_{\text{clyp}} = \{\mathbf{M} \in \mathbb{S}^n : \mathbf{M} \text{ is completely positive.}\}$$

The completely positive rank of a completely positive matrix $\mathbf{M}$ is the minimum k so that $\mathbf{M}$ writes as a sum of k nonnegative rank-1 projectors, just as above.

Exercise 7.4. If $\mathbf{M}$ is a $n \times n$ completely positive, show that its completely positive rank is upper bounded by $\binom{n+1}{2}$. (Hint: this is actually the dimension of the space of symmetric $n \times n$ matrices.)

A symmetric matrix $\mathbf{M}$ is called *copositive* if, for all $\mathbf{x} \in \mathbb{R}^n_+$, we have

$$\mathbf{x}^{\mathsf{T}}\mathbf{M}\mathbf{x} \geq 0.$$

Clearly, all positive semidefinite matrices are copositive, but the converse does not hold.

Exercise 7.5. Why not?

We will denote

$$\mathbb{S}^n_{\text{copo}} = \{\mathbf{M} \in \mathbb{S}^n : \mathbf{M} \text{ is copositive.}\}$$

Theorem 7.6. *The sets $\mathbb{S}^n_{copo}$ and $\mathbb{S}^n_{clyp}$ are closed, convex conves, and duals of each other.*

Proof. I leave it as an exercise to show that both sets are convex cones. If $\mathbf{M} \notin \mathbb{S}^n_{\text{copo}}$, then there is $\mathbf{x} \geq \mathbf{0}$ with $\mathbf{x}^{\mathsf{T}}\mathbf{M}\mathbf{x} < 0$. Any small variation around $\mathbf{M}$ will not change this fact, thus $\mathbb{S}^n_{\text{copo}}$ is closed. To see that $\mathbb{S}^n_{\text{clyp}}$ is closed, we will show that the limit of any convergent sequence in it belongs to it. Let $(\mathbf{M}^{(k)})_{k \geq 0} \in \mathbb{S}^n_{\text{clyp}}$ be a convergent sequence, converging to $\mathbf{M}$. Let $\mathbf{A}^{(k)} \geq \mathbf{0}$ be so that

$$\mathbf{M}^{(k)} = \mathbf{A}^{(k)}(\mathbf{A}^{(k)})^{\mathsf{T}}.$$

Thus the ith row of $\mathbf{A}^{(k)}$, for $k \to \infty$, form a bounded sequence of vectors. Thus this sequence has a convergent subsequence, say to $\mathbf{a}_i$, with $\mathbf{a}_i \geq \mathbf{0}$. If $\mathbf{A}$ has these as its columns, it follows that

$$\mathbf{M} = \mathbf{A}\mathbf{A}^{\mathsf{T}}.$$

Finally, if $\mathbf{M} \in \mathbb{S}^n_{\text{copo}}$, and $\mathbf{N} = \sum_{i=1}^k \mathbf{x}_k \mathbf{x}_k^\mathsf{T}$ with $\mathbf{x}_i \geq \mathbf{0}$, then

$$\langle \mathbf{M}, \mathbf{N} \rangle = \sum_{i=1}^{k} \langle \mathbf{M}, \mathbf{x}_i \mathbf{x}_i^\mathsf{T} \rangle \geq 0,$$

hence $\mathbf{M} \in (\mathbb{S}^n_{\text{clyp}})^*$. For the converse, if $\mathbf{M} \notin \mathbb{S}^n_{\text{copo}}$, there is $\mathbf{x} \geq \mathbf{0}$ with $\mathbf{x}^\mathsf{T}\mathbf{M}\mathbf{x} < 0$. Consequently, $\mathbf{M} \notin (\mathbb{S}^n_{\text{clyp}})^*$. Because they are closed convex cones, we also get

$$(\mathbb{S}^n_{\text{copo}})^* = \mathbb{S}^n_{\text{clyp}}.$$

□

Exercise 7.7. Describe as best as you can the cones $\mathbb{S}^2_{\text{copo}}$ and $\mathbb{S}^2_{\text{clyp}}$.

7.4 Conic program for the independence number

We now consider the following primal-dual pair of conic programs.

$$\text{(P)} \quad \begin{array}{rl} \max & \langle \mathbf{J}, \mathbf{X} \rangle \\ \text{s.t.} & \mathrm{X}_{ij} = 0 \ \forall \ ij \in E(G) \\ & \operatorname{tr} \mathbf{X} = 1 \\ & \mathbf{X} \in \mathbb{S}^n_{\text{clyp}} \end{array} \qquad \Bigg| \qquad \text{(D)} \quad \begin{array}{rl} \min & \lambda \\ \text{s.t.} & \lambda \mathbf{I} - \mathbf{J} - \mathbf{Y} \in \mathbb{S}^n_{\text{copo}} \\ & \mathrm{Y}_{ij} = 0 \ \forall \ ij \notin E(G). \end{array}$$

We will now show that the optimum of both programs above is $\alpha(G)$.

Lemma 7.8. *The optimum of (P) is $\geq \alpha(G)$.*

Proof. Let S be a maximum sized coclique, with characteristic vector $\mathbf{x}$. Simply note now that

$$\mathbf{X} = \frac{1}{\alpha(G)} \mathbf{x}\mathbf{x}^\mathsf{T}$$

is feasible for (P), with objective value $\alpha(G)$. □

It remains to show that the optimum of (D) is $\leq \alpha(G)$. To achieve this, we will use an influential result due to Motzkin and Straus.

Theorem 7.9. *For any graph G,*

$$\frac{1}{\alpha(G)} = \min\{\mathbf{x}^\mathsf{T}(\mathbf{A} + \mathbf{I})\mathbf{x} : \mathbf{1}^\mathsf{T}\mathbf{x} = 1, \ \mathbf{x} \geq \mathbf{0}\}.$$

Proof. Let $f(G)$ be the optimum of this program. If S is a coclique, with characteristic vector $\mathbf{x}$, then $(1/|S|)\mathbf{x}$ is feasible, and

$$\frac{1}{|S|^2} \mathbf{x}^\mathsf{T}(\mathbf{A} + \mathbf{I})\mathbf{x} = \frac{1}{|S|},$$

thus $f(G) \leq \alpha^{-1}$.

The other direction is more involved, and goes by induction on $n = |V(G)|$. If $n = 1$, the result is trivial. Assume now $f(H) \geq \alpha(G)^{-1}$ for all proper induced subgraphs H of G. Let $\mathbf{y}$ be an optimum giving $f(G)$. If one entry $\mathrm{y}_i = 0$, then we are lucky, and setting $H = G - i$, and $\overline{\mathbf{y}}$ the restriction of $\mathbf{y}$ to H, we have

$$\alpha(G)^{-1} = \alpha(H)^{-1} \leq f(H) \leq \overline{\mathbf{y}}^{\mathsf{T}}(\mathbf{A}(H) + \mathbf{I})\overline{\mathbf{y}} = \mathbf{y}^{\mathsf{T}}(\mathbf{A}(G) + \mathbf{I})\mathbf{y} = f(G).$$

Let $\mathbf{y}$ be a minimizer for $f(G)$, and $\mathbf{y} > \mathbf{0}$. Pick $ij \in E(G)$ and define $\mathbf{z}$ with $\mathrm{z}_i = \mathrm{y}_i + \varepsilon$, $\mathrm{z}_j = \mathrm{y}_j - \varepsilon$, and otherwise $\mathbf{z}_\ell = \mathbf{y}_\ell$. Clearly

$$\mathbf{z}^{\mathsf{T}}(\mathbf{A}+\mathbf{I})\mathbf{z} = f(G) + \varepsilon\big[\mathbf{y}^{\mathsf{T}}(\mathbf{A}+\mathbf{I})(\mathbf{e}_i - \mathbf{e}_j) + (\mathbf{e}_i - \mathbf{e}_j)^{\mathsf{T}}(\mathbf{A}+\mathbf{I})\mathbf{y}\big] + \varepsilon^2(\mathbf{e}_i - \mathbf{e}_j)^{\mathsf{T}}(\mathbf{A}+\mathbf{I})(\mathbf{e}_i - \mathbf{e}_j)$$

The term on ε^2 vanishes, as $ij \in E(G)$. The term on ε vanishes because $\mathbf{y}$ is a minimizer. Thus, we can choose ε so that $\mathbf{z}$ becomes 0 at a coordinate, and apply induction on the graph with the vertex corresponding to this coordinate removed. □

Exercise 7.10. Show that

$$\max\{\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} : \mathbf{1}^{\mathsf{T}}\mathbf{x} = 1,\ \mathbf{x} \geq \mathbf{0}\} = \frac{1}{2}\left(1 - \frac{1}{\omega(G)}\right).$$

Exercise 7.11. In the program

$$\begin{array}{rl} \min & \lambda \\ \text{subject to} & \lambda\mathbf{I} - \mathbf{J} - \mathbf{Y} \in \mathbb{S}^n_{\text{copo}} \\ & \mathrm{Y}_{ij} = 0 \ \forall\ ij \notin E(G), \end{array}$$

argue why there is an optimum solution with $\mathbf{Y}$ having all entries corresponding to edges equal to a constant.

Theorem 7.12. *The optimum of (D) is at most $\alpha(G)$.*

Proof. From the exercise above, we need to show that

$$\alpha(G) \geq \min\{\lambda : \lambda\mathbf{I} - \mathbf{J} - z\mathbf{A} \in \mathbb{S}^n_{\text{copo}},\ \lambda, z \in \mathbb{R}\}.$$

From Motzkin Straus, it follows that, for all $\mathbf{x}$ with $\mathbf{1}^{\mathsf{T}}\mathbf{x} = 1$ and $\mathbf{x} \geq \mathbf{0}$, we have

$$\alpha(G)\mathbf{x}^{\mathsf{T}}(\mathbf{A} + \mathbf{I})\mathbf{x} \geq 1 = \mathbf{x}^{\mathsf{T}}\mathbf{J}\mathbf{x}.$$

Therefore, for all $\mathbf{x} \geq \mathbf{0}$, we have

$$\mathbf{x}^{\mathsf{T}}(\alpha\mathbf{A} + \alpha\mathbf{I} - \mathbf{J})\mathbf{x} \geq 0.$$

Thus the matrix $(\alpha\mathbf{A} + \alpha\mathbf{I} - \mathbf{J})$ is copositive, and thus a feasible solution to the program above, with value $\alpha(G)$. □

Exercise 7.13. Show that

$$\alpha(G)^{-1} = \min\{\langle \mathbf{A} + \mathbf{I}, \mathbf{X}\rangle : \langle \mathbf{J}, \mathbf{X}\rangle = 1,\ \mathbf{X} \in \mathbb{S}^n_{\text{clyp}}\}.$$

7.5 Conic programming for the fractional chromatic number

We now consider the following primal-dual pair of conic programs.

$$
(\mathrm{P}) \quad \begin{array}{rl} \max & \langle \mathbf{J}, \mathbf{X} \rangle \\ \text{s.t.} & \mathrm{X}_{ij} + \mathrm{Z}_{ij} = 0 \ \forall \ ij \in E(G) \\ & \operatorname{tr}(\mathbf{X} + \mathbf{Z}) = 1 \\ & \mathbf{X} \in \mathbb{S}_+^n, \ \mathbf{Z} \in \mathbb{S}_{\text{copo}}^n \end{array}
\qquad\Bigg|\qquad
(\mathrm{D}) \quad \begin{array}{rl} \min & \lambda \\ \text{s.t.} & \lambda \mathbf{I} - \mathbf{J} - \mathbf{Y} \in \mathbb{S}_+ \\ & \lambda \mathbf{I} - \mathbf{Y} \in \mathbb{S}_{\text{clyp}} \\ & \mathrm{Y}_{ij} = 0 \ \forall \ ij \notin E(G). \end{array}
$$

We will now show that the optimum of both programs above is $\chi_f(\overline{G})$.

Lemma 7.14. *The optimum of (D) is* $\leq \chi_f(\overline{G})$.

Proof. Assume $\mathbf{y}$ is an optimum for

$$\min\{\mathbf{1}^T \mathbf{y} : \mathbf{M}\mathbf{y} = \mathbf{1}, \ \mathbf{y} \geq \mathbf{0}\}.$$

Recalling that $\overline{\chi}_f = \mathbf{1}^\mathsf{T}\mathbf{y}$, consider the matrix

$$\mathbf{W} = \overline{\chi}_f \sum_i \mathrm{y}_i \mathbf{x}_{K_i} \mathbf{x}_{K_i}^\mathsf{T} = \sum_{i,j} \mathrm{y}_i \mathrm{y}_j \mathbf{x}_{K_i} \mathbf{x}_{K_i}^\mathsf{T},$$

where the sums run over all cliques of the graph. This matrix is clearly completely positive, and, moreover, its diagonal is constant equal to $\overline{\chi}_f$. We also have $\mathrm{W}_{ij} = 0$ for all $ij \in E(\overline{G})$. So it remains to show that $\mathbf{W} - \mathbf{J} \succcurlyeq \mathbf{0}$. Recalling that $\mathbf{M}\mathbf{y} = \mathbf{1}$, it follows that

$$\mathbf{W} - \mathbf{J} = \sum_{i,j} \mathrm{y}_i \mathrm{y}_j \mathbf{x}_{K_i} \mathbf{x}_{K_i}^\mathsf{T} - \sum_{i,j} \mathrm{y}_i \mathrm{y}_j \mathbf{x}_{K_i} \mathbf{x}_{K_j}^\mathsf{T}.$$

Thus, given any $\mathbf{x} \in \mathbb{R}^n$, and having $a_i = \mathbf{x}_{K_i}^\mathsf{T}\mathbf{x}$, it follows that

$$\mathbf{x}^\mathsf{T}(\mathbf{W} - \mathbf{J})\mathbf{x} = \sum_{i,j} \mathrm{y}_i \mathrm{y}_j (a_i^2 - a_i a_j) = \sum_{i<j} \mathrm{y}_i \mathrm{y}_j (a_i^2 - 2a_i a_j + a_j^2) \geq 0.$$

□

Lemma 7.15. *The optimum of (D) is* $\geq \chi_f(\overline{G})$.

Proof. Consider the dual formulation for $\overline{\chi}_f$, and let $\mathbf{x}$ be an optimum solution, that is, $\mathbf{x} \geq \mathbf{0}$, $\mathbf{M}^\mathsf{T}\mathbf{x} \leq \mathbf{1}$, and

$$\overline{\chi}_f = \mathbf{1}^\mathsf{T}\mathbf{x}.$$

Let $\lambda, \mathbf{Y}$ be optimum solution for (D). Note that

$$\lambda \mathbf{I} - \mathbf{Y} = \sum_{i=1}^{k} \mathbf{z}_i \mathbf{z}_i^\mathsf{T},$$

for some $\mathbf{z}_i \geq \mathbf{0}$. Thus, each $\mathbf{z}_i$ is supported in a clique of the graph (because $\mathrm{Y}_{ij} = 0$ for all non edges). As $\lambda \mathbf{I} - \mathbf{Y} \succcurlyeq \mathbf{J}$, by applying $\mathbf{x}$ on both sides, we obtain

$$\chi_f^2 \leq \sum_{i=1}^{k} (\mathbf{x}^\mathsf{T}\mathbf{z}_i)^2.$$

Assume $\mathbf{z}_i$ is supported in the clique C. Let $\mathbf{x}_C$ be the vector obtained form $\mathbf{x}$ upon making all entries outside of C to be zero. Note that $\mathbf{1}^\mathsf{T}\mathbf{x}_C \leq 1$. We now write $\mathbf{x}^\mathsf{T}\mathbf{z}_i = \sqrt{\mathbf{x}_C}^\mathsf{T}(\sqrt{\mathbf{x}} \circ \mathbf{z}_i)$, and apply Cauchy-Schwarz, to obtain

$$(\mathbf{x}^\mathsf{T}\mathbf{z}_i)^2 \leq (\mathbf{1}^\mathsf{T}\mathbf{x}_C) \cdot [\mathbf{x}^\mathsf{T}(\mathbf{z}_i \circ \mathbf{z}_i)] \leq \mathbf{x}^\mathsf{T}(\mathbf{z}_i \circ \mathbf{z}_i).$$

Thus

$$\chi_f^2 \leq \sum_{i=1}^k (\mathbf{x}^\mathsf{T}\mathbf{z}_i)^2 \leq \mathbf{x}^\mathsf{T}\left(\sum_{i=1}^k (\mathbf{z}_i \circ \mathbf{z}_i)\right) = \mathbf{x}^\mathsf{T}(\lambda\mathbf{1}) = \lambda \cdot \chi_f.$$

□

Exercise 7.16. Find a feasible solution for (P) with objective value $\chi_f(\overline{G})$.

7.6 Vector colourings and theta variants

A graph G has an γ-vector colouring if there is an assignment of unit vectors $\{\mathbf{v}_a : a \in V(G)\} \subseteq \mathbb{R}^d$ so that, if $a \sim b$, then

$$\langle \mathbf{a}, \mathbf{b} \rangle \leq \frac{1}{1-\gamma}.$$

The smallest $\gamma > 1$ so that G has an γ-vector colouring is called the *vector chromatic number* of G, to be denoted by $\chi_{\text{vec}}(G)$ (if G has no edge, this is defined to be equal to 1). The existence of an γ-vector colouring is equivalent to the existence of a matrix $\mathbf{W}$ so that

(i) $\mathbf{W} \succcurlyeq \mathbf{0}$.

(ii) $\operatorname{diag} \mathbf{W} = \mathbf{1}$.

(iii) $W_{ij} \leq 1/(1-\gamma)$ for all $ij \in E(G)$.

The existence of such $\mathbf{W}$ is equivalent to the existence of $\mathbf{Y}$ so that

$$(\gamma - 1)\mathbf{W} = \gamma\mathbf{I} - \mathbf{Y} - \mathbf{J},$$

where $Y_{ij} \geq 0$ if $ij \in E(G)$, and $Y_{ii} = 0$. Given that $\gamma > 1$, it follows that the vector chromatic number of G is the optimum of the SDP

$$\begin{aligned} \min \quad & \gamma \\ \text{subject to} \quad & \gamma\mathbf{I} - \mathbf{Y} - \mathbf{J} \succcurlyeq \mathbf{0} \\ & Y_{ij} \geq 0 \quad \text{for all } ij \notin E(\overline{G}). \end{aligned}$$

This formulation is dual to

$$\begin{aligned} \max \quad & \langle \mathbf{J}, \mathbf{X} \rangle \\ \text{subject to} \quad & \operatorname{tr} \mathbf{X} = 1 \\ & X_{ij} = 0 \quad \text{for all } ij \in E(\overline{G}) \\ & \mathbf{X} \geq \mathbf{0},\ \mathbf{X} \succcurlyeq \mathbf{0}. \end{aligned}$$

Given both formulations above, and recalling the formulation for $\gamma(G)$ as a conic program over $\mathbb{S}^n_{\text{clyp}}$, which in particular belongs to the set of matrices which are non-negative and positive semidefinite, the corollary below follows immediately:

Corollary 7.17. *For any graph G,*

$$\omega(G) \leq \chi_{vec}(G) \leq \vartheta(\overline{G}).$$

Note that $\chi_{\text{vec}}(G)$ can be computed efficiently.

Exercise 7.18. Prove in an elementary way that $\omega(G) \leq \chi_{\text{vec}}(G)$. Hint: consider an optimal γ-vector colouring, and sum the vectors corresponding to a maximum clique. Then consider the norm of this resulting vector.

If we had enforced that the vector colouring satisfies inner product inequality with an equality, we would have arrived at what is known as the *strict vector chromatic number*. This is denoted by $\chi_{\text{svec}}(G)$, and formally defined as the lest $\gamma > 1$ so that there is an assignment of unit vectors $\{\mathbf{v}_a : a \in V(G)\} \subseteq \mathbb{R}^d$ so that, if $a \sim b$, then

$$\langle \mathbf{a}, \mathbf{b} \rangle = \frac{1}{1-\gamma}.$$

Exercise 7.19. Verify that $\chi_{\text{svec}}(G) = \vartheta(\overline{G})$. Hint: look at the SDP formulations above.

An γ-strict vector colouring which further satisfies

$$\langle \mathbf{a}, \mathbf{b} \rangle \geq \frac{1}{1-\gamma}.$$

for all $a, b \in V(G)$ is called a *rigid vector colouring*. The least γ so that G has a rigid vector colouring is denoted by $\chi_{\text{rvec}}(G)$. Note that

$$\chi_{\text{vec}}(G) \leq \chi_{\text{svec}}(G) \leq \chi_{\text{rvec}}(G),$$

as these are all minimization problems with increasing degrees of restrictions. Moreover, it is immediate to verify that $\chi_{\text{rvec}}(G)$ is defined by the following pair of primal-dual SDPs:

$$\text{(P)} \quad \begin{array}{rl} \min & \gamma \\ \text{s.t.} & \gamma \mathbf{I} - \mathbf{Y} - \mathbf{J} \succcurlyeq \mathbf{0} \\ & Y_{ij} = 0 \quad \text{for all } ij \notin E(\overline{G}) \\ & \mathbf{Y} \leq \mathbf{0}. \end{array} \qquad \Bigg| \qquad \text{(D)} \quad \begin{array}{rl} \max & \langle \mathbf{J}, \mathbf{X} \rangle \\ \text{s.t.} & \operatorname{tr} \mathbf{X} = 1 \\ & X_{ij} \leq 0 \quad \text{for all } ij \in E(\overline{G}) \\ & \mathbf{X} \succcurlyeq \mathbf{0}. \end{array}$$

Exercise 7.20. Check the definition of $\chi_f(\overline{G})$ as a conic program, and prove that $\chi_f(G) \geq \chi_{\text{rvec}}(G)$.

Historically, the vector chromatic number was introduced as a variant of ϑ by Schrijver, and it was originally denoted by ϑ' (and called, later, by Schrijver's Theta). The parameter χ_{rvec} was introduced by Szegedy also as a ϑ variant. If one decides to denote $\chi_{\text{vec}}(G) = \vartheta^-(\overline{G})$ and $\chi_{\text{rvec}}(G) = \vartheta^+(\overline{G})$, the inequalities seen above take the pleasant form

$$\gamma(G) \leq \vartheta^-(G) \leq \vartheta(G) \leq \vartheta^+(G) \leq \chi_f(\overline{G}).$$