(d) We provide one way of finding this now. The other will come later as an exercise. Look at the cycle C_{2n+2}. Let ω be a $(2n+2)$th root of unity. Then

$$\begin{pmatrix} 1 \\ \omega \\ \vdots \\ \omega^{2n+1} \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ \omega^{-1} \\ \vdots \\ \omega^{-(2n+1)} \end{pmatrix}$$

are both eigenvalues of $\mathbf{A}(C_{2n+2})$ for $\omega + \omega^{-1}$, and so is any linear combination of them. In particular

$$\begin{pmatrix} 1 \\ \omega \\ \vdots \\ \omega^{2n+1} \end{pmatrix} - \begin{pmatrix} 1 \\ \omega^{-1} \\ \vdots \\ \omega^{-(2n+1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \omega - \omega^{-1} \\ \vdots \\ \omega^{2n+1} + \omega^{-2n-1} \end{pmatrix}.$$

Note that there will be another 0 at position $n+2$, corresponding to $\omega^{n+1} - \omega^{-n-1} = -1 - (-1) = 0$. The n non-zero entries (only when $\omega \neq 1$) from positions 2 to $n+1$ are part of an eigenvector of C_{2n+2} which do not get interfered by the rest of the graph (those 0s at positions 1 and $n+2$ "disconnect" the eigenvector). Hence this part of the eigenvector is also an eigenvector for P_n (subgraph of C_{2n+2} from positions 2 to $n+1$). Therefore the spectrum of $\mathbf{A}(P_n)$ is

$$\omega + \omega^{-1} = 2\cos\left(\pi \frac{k}{n+1}\right) \quad \text{for } k = 1, ..., n.$$

1.5 Bounds to the largest and smallest eigenvalues

Later in this course we will study a powerful technique called interlacing, which allows for several interesting results relating eigenvalues and combinatorics. For now, however, we will see some examples of results one could obtain by means of ad-hoc ideas, applied to the largest and smallest eigenvalues.

Exercise 1.40. Let G be a graph with m edges, and let $\theta_1, ..., \theta_n$ be the eigenvalues of $\mathbf{A}$. Assume that they are numbered so that $\theta_i^2 \geq \theta_{i+1}^2$. Show that

$$\theta_i \leq \sqrt{\frac{2m}{i}}.$$

Recall now our usual notation for the eigenvalues of $\mathbf{A}$ with $\lambda_1 \geq ... \geq \lambda_n$. A consequence of the exercise above is that $\lambda_1 \leq \sqrt{2m}$. We can do better.

Theorem 1.41 (Hong)**.** *For a graph with n vertices and m edges,*

$$\lambda_1 \leq \sqrt{2m - (n-1)}.$$

Proof. Let $\mathbf{x}$ be an unit length eigenvector corresponding to λ_1, and let $\mathbf{x}(i)$ be the vector obtained from $\mathbf{x}$ by erasing all entries which do not correspond to neighbours of i. In particular, the ith entry of $\mathbf{x}(i)$ is also zeroed.