5   Eigenvalues and the structure of graphs

5.1   Rayleigh quotients and Interlacing

Given a symmetric matrix $\mathbf{M}$, we recall the definition of the Rayleigh quotient of $\mathbf{M}$ with respect to a non-zero vector $\mathbf{v}$:

$$R_{\mathbf{M}}(\mathbf{v}) = \frac{\mathbf{v}^T \mathbf{M} \mathbf{v}}{\mathbf{v}^T \mathbf{v}}.$$

We will always assume vectors whose Rayleigh quotient is being taken are non-zero. As we have seen, if $\mathbf{v}$ is an eigenvector with corresponding eigenvalue θ, then

$$R_{\mathbf{M}}(\mathbf{v}) = \theta.$$

We also saw that if $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of $\mathbf{M}$ with corresponding eigenprojectors E_rs, and assuming $\mathbf{v}$ is normalized, then

$$R_{\mathbf{M}}(\mathbf{v}) = \mathbf{v}^T \left(\sum_{r=1}^{n} \lambda_r E_r \right) \mathbf{v} = \sum_{r=1}^{n} \lambda_r (\mathbf{v}^T E_r \mathbf{v}) \leq \lambda_1 \left(\sum_{r=0}^{d} \mathbf{v}^T E_r \mathbf{v} \right) = \lambda_1$$

for all vectors $\mathbf{v}$, and equality holds if and only if $\mathbf{v}$ belongs to the λ_1 eigenspace.

Lemma 5.1. *Let $\mathbf{M}$ be a symmetric matrix, with largest eigenvalue λ_1 and smallest eigenvalue λ_n. Then*

$$\lambda_1 = \max_{\mathbf{v} \in \mathbb{R}^n} R_{\mathbf{M}}(\mathbf{v}) \quad \text{and} \quad \lambda_n = \min_{\mathbf{v} \in \mathbb{R}^n} R_{\mathbf{M}}(\mathbf{v}).$$

$\square$

Examining more carefully how we bounded the Rayleigh quotient, it is not hard to see that all eigenvalues can be defined as a max or min of the Rayleigh quotient over certain subspaces. Let L_r denote the orthogonal complement to the sum of the eigenlines corresponding to the largest eigenvalues all the way to λ_r, that is

$$L_r = \text{null}\,(E_1 + E_1 + \ldots + E_{r-1}).$$

Likewise, define S_r to correspond to the orthogonal complement to the sum of the eigenlines corresponding to the smallest eigenvalues all the way to λ_{r+1}, that is

$$S_r = \text{null}\,(E_{r+1} + E_{r+2} + \ldots + E_n).$$

It follows immediately that

$$\lambda_r = \max_{\mathbf{v} \in L_r} R_{\mathbf{M}}(\mathbf{v}) = \min_{\mathbf{v} \in S_r} R_{\mathbf{M}}(\mathbf{v}).$$

The expression of λ_r can be made with the subspaces L_r and S_r implicitly defined, via a min-max formula.

Lemma 5.2 (Courant–Fischer–Weyl min-max principle)**.** *Let $\mathbf{M}$ be a symmetric matrix, with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Then*

$$\lambda_k = \min_{\substack{subspace\ U \\ \dim U = n-k+1}} \max_{\mathbf{v} \in U} R_{\mathbf{M}}(\mathbf{v}) = \max_{\substack{subspace\ U \\ \dim U = k}} \min_{\mathbf{v} \in U} R_{\mathbf{M}}(\mathbf{v}).$$

Proof. I will show the first equality only, as the second is analogous. Note that we have already seen that there is a subspace U of dimension $n - k + 1$ so that

$$\lambda_k = \max_{\mathbf{v} \in U} R_{\mathbf{M}}(\mathbf{u}),$$

this subspace is simply the orthogonal complement of the sum of the eigenlines corresponding to the largest $k - 1$ eigenvalues. The result will now follow if we verify that, for all subspaces U of dimension $n - k + 1$, we have

$$\lambda_k \leq \max_{\mathbf{v} \in U} R_{\mathbf{M}}(\mathbf{u}).$$

To see this, let U be a subspace of dimension $n - k + 1$, and let V be the sum of the eigenlines corresponding to the largest k eigenvalues. As $\dim U + \dim V$ exceeds n, it follows that $U \cap V \neq \emptyset$. Let $\mathbf{v}$ belong to this intersection. Then

$$R_{\mathbf{M}}(\mathbf{v}) \geq \lambda_k \sum_{r=1}^{k} \mathbf{v}^T E_r \mathbf{v} \geq \lambda_k,$$

as we wanted.   □

Exercise 5.3. We've seen that $\Delta \geq \lambda_1$, the largest eigenvalue of $\mathbf{A}$. If the $(d_1, ..., d_n)$ is the degree sequence in decreasing order, then you can now show that $d_i \geq \lambda_i$.

Such min-max formula provides an alternative and meaningful definition of eigenvalues. For graph theory, it is hard to find interesting applications of this formula by itself. We can use it however to prove a strong result.

Theorem 5.4 (Cauchy's Interlacing). *Let $\mathbf{A}$ be a symmetric $n \times n$ matrix and $\mathbf{S}$ be an $n \times m$ matrix satisfying $\mathbf{S}^T \mathbf{S} = \mathbf{I}$. Let $\mathbf{B} = \mathbf{S}^T \mathbf{A} \mathbf{S}$. Let $\theta_1 \geq ... \geq \theta_n$ be the eigenvalues of $\mathbf{A}$ and $\lambda_1 \geq ... \geq \lambda_m$ be those of $\mathbf{B}$. Then*

(a) For all k with $1 \leq k \leq m$,
$$\theta_{n-(m-k)} \leq \lambda_k \leq \theta_k$$

(b) If equality holds in either of the inequalities above for some λ_k eigenvalue of $\mathbf{B}$, then there is a λ_k-eigenvector $\mathbf{v}$ of $\mathbf{B}$ so that $\mathbf{S}\mathbf{v}$ is an eigenvector for λ_k in $\mathbf{A}$.

(c) Let $\mathbf{v}_1, ..., \mathbf{v}_m$ be an orthogonal basis of eigenvectors of $\mathbf{B}$, with $\mathbf{v}_i$ corresponding to λ_i. If for some $\ell \in \{1, ..., m\}$ we have that $\lambda_k = \theta_k$ for all $k = 1, ..., \ell$ (or $\lambda_k = \theta_{n-(m-k)}$ for all $k = \ell, ..., n$), then $\mathbf{S}\mathbf{v}_k$ is an θ_k eigenvector for $\mathbf{A}$ for $k = 1, ..., \ell$ (respectively for $k = \ell, ..., n$).

(d) If there is an $\ell \in \{1, ..., m\}$ so that $\lambda_k = \theta_k$ for all $k = 1, ..., \ell$, and $\lambda_k = \theta_{n-(m-k)}$ for all $k = \ell + 1, ..., n$, then $\mathbf{S}\mathbf{B} = \mathbf{A}\mathbf{S}$. In this case, interlacing is called tight.

Proof. Let $\mathbf{u}_1, ..., \mathbf{u}_n$ be the eigenvectors of $\mathbf{A}$ corresponding to the θ_ks. The key thing now is to observe that, for all k, the subspace

$$\langle \mathbf{v}_1, ..., \mathbf{v}_k \rangle \cap \langle \mathbf{S}^T \mathbf{u}_1, ..., \mathbf{S}^T \mathbf{u}_{k-1} \rangle^\perp$$

contains at least one vector. Let $\mathbf{w}$ be such vector, which, in particular, implies $\mathbf{Sw} \in \langle \mathbf{u}_1, ..., \mathbf{u}_{k-1} \rangle^\perp$. Then, by Lemma 5.2, we have

$$\theta_k \geq \frac{(\mathbf{Sw})^T \mathbf{A}(\mathbf{Sw})}{(\mathbf{Sw})^T (\mathbf{Sw})} \geq \frac{\mathbf{w}^T \mathbf{Bw}}{\mathbf{w}^T \mathbf{w}} \geq \lambda_k.$$

If $\theta_k = \lambda_k$, then $\mathbf{w}$ and $\mathbf{Sw}$ are eigenvectors for $\mathbf{B}$ and $\mathbf{A}$ respectively. Item (iii) follows easily by induction. Finally, with tight interlacing, we can guarantee that $\mathbf{Sv}_1, ..., \mathbf{Sv}_m$ are all eigenvectors for $\mathbf{A}$ with the same eigenvalues they have in $\mathbf{B}$. Therefore $\mathbf{SBv}_k = \mathbf{ASv}_k$ for all k, and as the set of eigenvectors form a basis, the two matrices are equal. $\square$

The basic principle for applying interlacing is to carefully chose the matrix $\mathbf{S}$.

Exercise 5.5. Let $\mathbf{A}$ be a $n \times n$ symmetric matrix, with eigenvalues $\theta_1 \geq ... \geq \theta_n$. Let $\mathbf{B}$ be a principal submatrix, size $(n-1) \times (n-1)$, with eigenvalues $\lambda_1 \geq ... \geq \lambda_{n-1}$. Show that, for all k,

$$\theta_{k+1} \leq \lambda_k \leq \theta_k.$$

(This is actually the reason why the result above is called interlacing.)

The stability number of a graph is the size of the largest subset of vertices which contains no edge inside of it — known as an independent or stable set.

Exercise 5.6. Let $\alpha(G)$ be the stability number of a graph. Then

$$\alpha(G) \leq |\{k : \theta_k \geq 0\}| \quad \text{and} \quad \alpha(G) \leq |\{k : \theta_k \leq 0\}|.$$

This follows easily if you note that an independent set corresponds to a block of 0s in $\mathbf{A}(G)$. Write the details.

Consider now the Petersen graph. By using interlacing, we will show two interesting facts about it. The Petersen graph has eigenvalues $3, 1^{(5)}, -2^{(4)}$. If its incidence matrix is $\mathbf{N}$, then the adjacency matrix is $\mathbf{NN}^T - 3\mathbf{I}$, and the adjacency matrix of its line graph is $\mathbf{N}^T \mathbf{N} - 2\mathbf{I}$.

Exercise 5.7. Find the spectrum of its line graph.

If the Petersen graph contains a Hamilton cycle, then its line graph contains an induced cycle C_{10}. This means that we can delete 5 vertices of its line graph, and find C_{10}. The eigenvalues of C_{10} are

$$2, \pm \left(\frac{1 \pm \sqrt{5}}{2} \right), -2$$

whereas 2 and -2 are simple, and the others each have multiplicity 2.

Exercise 5.8. Use interlacing now to show that the Petersen graph does not have a Hamilton cycle.

Finally, an application of the method of finding a vector in an intersection of subspace by looking at the dimension. The graph K_{10} has 45 edges, and therefore it would be possible for the edges of K_{10} to be partitioned into copies of the Petersen graph. However, this is not possible. To see that, assume K_{10} contains already two disjoint copies of Pete, say G and H. The eigenspace corresponding to 1 in G has dimension 5, and the same for that of 1 in H. As both eigenspaces are orthogonal to the line spanned by $\mathbf{1}$, then there must be at least one vector, say $\mathbf{w}$, who is simultaneously an eigenvector for $\mathbf{A}(G)$ and $\mathbf{A}(H)$. Thus

$$\mathbf{A}(\mathbf{J} - \mathbf{I} - \mathbf{A}(G) - \mathbf{A}(H))\mathbf{w} = -3\mathbf{w}.$$

This means that the complement of G and H in K_{10} has eigenvalue -3, and therefore cannot be isomorphic to the Petersen graph.

5.2   Partitions - cliques, cocliques, colourings

Consider a partition of the vertex set of a graph G, with characteristic matrix $\mathbf{P}$ (meaning, rows of $\mathbf{P}$ are vertices, and columns are parts of the partition, with 1s and 0s indicating whether a vertex belongs or not to a part).

A partition of the vertex set of a graph with characteristic matrix $\mathbf{P}$ is called equitable with respect to $\mathbf{A}(G)$ if the column space of $\mathbf{P}$ is $\mathbf{A}(G)$-invariant, that is, if there is a matrix $\mathbf{B}$ so that

$$\mathbf{AP} = \mathbf{PB}.$$

Combinatorially, this means that the number of neighbours a vertex a has in a class C of the partition is determined uniquely by the class that contains a. In other words, any two vertices in a given class have the same number of neighbours in any other given class (including their own).

All graphs contain at least one equitable partition of its vertex set: that in which all classes are singletons. If the graph is regular, then the partition that contains only one class is also equitable.

Given any partition with characteristic matrix $\mathbf{P}$, it is always possible to scale each column of $\mathbf{P}$ so that it becomes a normal vector. If $\mathbf{S}$ is the matrix obtained in this manner, it is immediate to verify that

$$\mathbf{S}^T\mathbf{S} = \mathbf{I},$$

and therefore $\mathbf{S}$ is suitable to be used as in Theorem 5.4.

For a first example of this property, we derive another bound to the independence number of a graph.

Corollary 5.9 (Ratio bound for independent sets)**.** *Let G be k-regular on n vertices, with smallest eigenvalue θ_n. Then*

$$\alpha(G) \le \frac{n(-\theta_n)}{k - \theta_n}.$$

If equality holds, then the partition of the vertex set into any maximum independent set and its complement is equitable, and in particular, there is a τ eigenvector which is constant in each class of this partition.

Proof. Let $\mathbf{P}$ be the characteristic matrix of a partition that contains two class: one is a maximum independent set, and the other is its complement. Let $\mathbf{S}$ be the normalized characteristic matrix. Then

$$\mathbf{S}^T\mathbf{A}\mathbf{S} = \begin{pmatrix} 0 & \frac{\alpha k}{\sqrt{\alpha}\sqrt{n-\alpha}} \\ \frac{\alpha k}{\sqrt{\alpha}\sqrt{n-\alpha}} & \frac{(n-\alpha)k-k\alpha}{n-\alpha} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sqrt{\alpha}k}{\sqrt{n-\alpha}} \\ \frac{\sqrt{\alpha}k}{\sqrt{n-\alpha}} & k - \frac{k\alpha}{n-\alpha} \end{pmatrix}.$$

Clearly, the eigenvalues are k and $(-k\alpha)/(n-\alpha)$. Due to interlacing, it follows that

$$(-k\alpha)/(n-\alpha) \geq \theta_n,$$

which rearranges to

$$\alpha \leq \frac{n(-\theta_n)}{k-\theta_n}.$$

If you can't compute the eigenvalues easily, you can simply compare the determinant of $\mathbf{S}^T\mathbf{A}\mathbf{S}$ with the product of the largest and smallest eigenvalues of $\mathbf{A}$.

If equality holds, and because the largest eigenvalue of $\mathbf{A}$ and $\mathbf{S}^T\mathbf{A}\mathbf{S}$ are also equal, we have that (iv) in Theorem 5.4 applies. Moreover, (ii) of said theorem implies the assertion about the τ-eigenvector. $\square$

It is quite surprising that this bound is met in several interesting cases, although it is not a good approximation for α in the general case (no such hope exists).

Exercise 5.10. Let δ be the smallest degree of G. If G is any graph (not necessarily regular), with largest eigenvalue θ_1 and smallest eigenvalue θ_n. Show that

$$\alpha \leq \frac{n(-\theta_1\theta_n)}{\delta^2 - \theta_1\theta_n}.$$

Hint: let k be the average degree in the independent set, and proceed as above.

Exercise 5.11. Let G be k-regular on n vertices, with eigenvalues $\theta_1 \geq ... \geq \theta_n$. Assume G contains an induced subgraph H with n' vertices and m' edges. Show that

$$\theta_2 \geq \frac{2m'n - (n')^2k}{n'(n-n')} \geq \theta_n.$$

Characterize what happens if equality holds in either side.

Exercise 5.12. Let $\omega(G)$ be the size of a maximum clique in G, that is, the size of the largest subgraph of G which is isomorphic to a complete graph. Assume G is k-regular. Find an upper bound to ω using the eigenvalues of G.

We now devote our attention to the chromatic number of G. A colouring of $V(G)$ is an assignment of colours to the vertex set of G so that any two neighbours receive different colours. It is always possible to colour a graph with n colours. A graph is 2-colourable if and only if it is bipartite. The chromatic number of a graph $\chi(G)$ is the minimum number of colours necessary to colour the vertices of G.

Just like α and ω, χ is hard to approximate, so any simple formulas using the spectrum of G can only bound, and even so not that well in the general case. However, this is quite significant to the best one could do.

Exercise 5.13. Explain why

$$\alpha \cdot \chi \geq n \quad \text{and also} \quad \omega \leq \chi.$$

The first inequality in the exercise above immediately implies a spectral lower bound to χ in regular graphs, using the upper bound to α. As it turns out, we can ignore the requirement of the graph to be regular.

Theorem 5.14 (Hoffman). *Let G be a graph with chromatic number χ, largest eigenvalue θ_1 and smallest eigenvalue θ_n. Then*

$$\chi(G) \geq 1 - \frac{\theta_1}{\theta_n}.$$

Proof. Let $\mathbf{P}$ be the characteristic matrix of a colouring. To prove this result, it won't be enough to simply scale the columns of $\mathbf{P}$ and proceed with interlacing (in fact, try to do this). Instead, we shall first scale the rows of $\mathbf{P}$. Let $\mathbf{D}$ be a diagonal matrix whose diagonal entries are taken from the Perron eigenvector $\mathbf{v}$ of G. Let $\mathbf{S}$ be the obtained from $\mathbf{DP}$ upon multiplying from the right by a diagonal matrix $\mathbf{E}$ which effects to normalizing its columns. Thus $\mathbf{S}^T\mathbf{S} = \mathbf{I}$, and we proceed with interlacing now. We have $\mathbf{B} = \mathbf{S}^T\mathbf{A}\mathbf{S}$ with 0s in the diagonal, as the support of $\mathbf{S}$ corresponds to a colouring of G. Note that $\mathbf{B}$ is $m \times m$ with $m = \chi$. We also note that θ is an eigenvalue of $\mathbf{B}$, because $\mathbf{S}^T\mathbf{A}\mathbf{S}(\mathbf{E}^{-1}\mathbf{1}) = \mathbf{S}^T\mathbf{A}\mathbf{v} = \theta_1\mathbf{E}^{-1}\mathbf{1}$. Hence, by interlacing,

$$0 = \operatorname{tr}\mathbf{B} = \lambda_1 + \lambda_2 + ... + \lambda_m \geq \theta_1 + (\chi - 1)\theta_n.$$

$\square$

Exercise 5.15. What can you say if equality holds in this bound?

Note that in the last line of the proof, our bound was quite crude. An immediate improvement is to say

Corollary 5.16. *Let G be a graph with chromatic number χ, and eigenvalues $\theta_1 \geq ... \geq \theta_n$. Then*

$$\theta_1 + \theta_n + \theta_{n-1} + ... + \theta_{n-(\chi-2)} \leq 0.$$

$\square$

Exercise 5.17. In this exercise, you will show that if $\theta_2 > 0$, then

$$\chi(G) \geq 1 - \frac{\theta_{n-\chi+1}}{\theta_2}.$$

I will give you a hint. Let $\mathbf{P}$ be the partition matrix of an optimal colouring. Let $\mathbf{v}_1$ be the Perron eigenvector, and $\mathbf{D}$ the diagonal matrix which contains its entries in the diagonal. Consider

$$\ker(\mathbf{P}^T\mathbf{D}) \cap \langle \mathbf{v}_n, ..., \mathbf{v}_{n-\chi+1} \rangle.$$

Prove that this intersection contains a vector, define a diagonal matrix with this vector, and also define $\mathbf{A}' = \mathbf{A} - (\theta_1 - \theta_2)\mathbf{v}_1\mathbf{v}_1^T$. Now proceed as in the proof of Hoffman's theorem.

Exercise 5.18. If m_n is the multiplicity of θ_n, verify now that

$$\chi \geq \min\{1 + m_m, 1 - \frac{\theta_n}{\theta_2}\}.$$

It is quite remarkable that in the results above, while exploring the connection between the eigenvalues of $\mathbf{A}$ and the independence number of G or its chromatic number, nowhere the fact that the entries of $\mathbf{A}$ are restricted to 1s and 0s was used. The only real constraint is that the non-zero entries are restricted to positions corresponding to edges in the graph. In fact, one can vary the entries of $\mathbf{A}$, and as long as the eigenvalue expressions increase (for chromatic number) or decrease (for independence number), better bounds will be obtained. This leads to the interesting topic of applications of semidefinite programming to algebraic graph theory.

There is a famous upper bound for χ:

Theorem 5.19 (Brooks). *Let G be a graph with maximum degree Δ. Then $\chi(G) \leq \Delta$, unless G is a complete graph or an odd cycle, in which cases $\Delta + 1$ colours suffice.* $\qquad\square$

This is one of the classical theorems in graph theory. Its proof is certainly not trivial (only purely combinatorial proofs are known, so you will have to research that on your own). I am sure you remember that $\theta_1 \leq \Delta$. It turns out, we can somehow strengthen the statement of Brooks theorem for several graphs.

Theorem 5.20 (Wilf). *If G is a graph with chromatic number χ and largest eigenvalue θ_1, then*

$$\chi \leq 1 + \theta_1.$$

Equality holds if and only if G is an odd cycle or the complete graph.

Proof. Let G' be a subgraph of G which is χ-critical, meaning, the subgraph whose removal of any vertex decreases the chromatic number. In this subgraph, the degree of any vertex is at least $\chi - 1$ (why?). Thus its largest eigenvalue is at least $\chi - 1$. By interlacing, the largest eigenvalue of G is at least $\chi - 1$. $\qquad\square$

Exercise 5.21. Finish the proof of the theorem above.

Exercise 5.22. One final exercise here: show that $\theta_n \geq -n/2$, for any connected graph.

Problem 5.1. *Find a spectral proof of Brooks theorem.*

5.3   Other eigenvalues

The second largest eigenvalue of $\mathbf{A}^2$ seems to be related to how "random" the graph looks like. This is very vague, I know, reason why it will be better explained looking at the results. Let us assume G is a regular graph of degree k, with eigenvalues $\theta_1 \geq \theta_2 \geq ... \geq \theta_n$. Let $\lambda > 0$ be such that λ^2 is the second largest eigenvalue of A^2, that is, $\lambda = \max\{|\theta_2|, |\theta_n|\}$.

Theorem 5.23. *Let G be k-regular, and λ as before. Let S and T be two subsets of $V(G)$, of respective sizes s and t. Let $e(S,T)$ be the number of edges from S to T. Then*

$$\left| e(S,T) - \frac{kst}{n} \right| \le \lambda \sqrt{st\left(1 - \frac{s}{n}\right)\left(1 - \frac{t}{n}\right)} \le \lambda\sqrt{st}.$$

Proof. Let

$$\mathbf{A} = \sum_{r=1}^{n} \theta_i E_i$$

be the spectral decomposition of $\mathbf{A}$ into 1-dimensional eigenspaces. Let χ_S and χ_T be the characteristic vectors of sets S and T. It follows that

$$e(S,T) = \chi_S^T \mathbf{A} \chi_T = \sum_{r=1}^{n} \theta_i (\chi_S^T E_i \chi_T).$$

Note that $(\chi_S^T E_1 \chi_T) = st/n$ therefore

$$\left| e(S,T) - \frac{kst}{n} \right| = \left| \sum_{r=2}^{n} \theta_i (\chi_S^T E_i \chi_T) \right| \le \lambda \sum_{r=2}^{n} |\chi_S^T E_i \chi_T|.$$

By Cauchy Schwarz (applied twice),

$$\left| e(S,T) - \frac{kst}{n} \right| \le \lambda \sum_{r=2}^{n} \sqrt{\chi_S^T E_i \chi_S} \sqrt{\chi_T^T E_i \chi_T}$$

$$\le \sqrt{\sum_{r=2}^{n} \chi_S^T E_i \chi_S} \sqrt{\sum_{r=2}^{n} \chi_T^T E_i \chi_T}$$

$$= \sqrt{s - \frac{s^2}{n}} \sqrt{t - \frac{t^2}{n}}.$$

$\square$

That is, if λ is small compared to k, then between any two subsets of vertices of the graph, the number of edges tends to be the "expected" number, had every edge been put randomly and independently in the graph.

Exercise 5.24. Can you use the result above (or its proof method?) to show the ratio bound for cocliques without going through interlacing?

If $u \in V(G)$, let $N(u)$ denote the neighbourhood of u.

Exercise 5.25. Let T be a subset of $V(G)$ of size t. Show that

$$\sum_{u \in V(G)} \left(|N(u) \cap T| - \frac{kt}{n} \right)^2 \le \frac{t(n-t)}{n}\lambda^2.$$

From Theorem 5.23, you would indeed expect that a large ratio k/λ implies that the graph "looks" random. If that is indeed the case, the diameter would also be relatively small. We can turn this intuition into a result.

Theorem 5.26. *Let G be a k-regular connected graph on n vertices, $n \geq 2$. Let d be its diameter, $\theta_1 = k$ its largest eigenvalue, and $\lambda = \max\{\theta_2, |\theta_n|\}$. Assume G is not bipartite, thus $\lambda < k$. Then*

$$d \leq \left\lfloor \frac{\log(n-1)}{\log(k/\lambda)} \right\rfloor + 1.$$

Proof. Let m be an integer, with

$$m > \frac{\log(n-1)}{\log(k/\lambda)}.$$

Thus, $k^m > (n-1)\lambda^m$. We will show that this implies that all entries of A^m are positive, therefore $d \leq m$. To see that, we will again apply Cauchy-Schwarz twice. Note that

$$
\begin{aligned}
(A^m)_{a,b} = \mathbf{e}_a^T A^m \mathbf{e}_b &= \frac{k^m}{n} + \sum_{r=2}^n \theta_r^m (\mathbf{e}_a^T \mathbf{E}_r \mathbf{e}_b) \\
&\geq \frac{k^m}{n} - \lambda^m \sum_{r=2}^n |\mathbf{e}_a^T \mathbf{E}_r \mathbf{e}_b| \\
&\geq \frac{k^m}{n} - \lambda^m \sum_{r=2}^n \sqrt{\mathbf{e}_a^T \mathbf{E}_r \mathbf{e}_a} \sqrt{\mathbf{e}_b^T \mathbf{E}_r \mathbf{e}_b} \\
&\geq \frac{k^m}{n} - \lambda^m \sqrt{\sum_{r=2}^n \mathbf{e}_a^T \mathbf{E}_r \mathbf{e}_a} \sqrt{\sum_{r=2}^n \mathbf{e}_b^T \mathbf{E}_r \mathbf{e}_b} \\
&= \frac{k^m}{n} - \lambda^m \left(1 - \frac{1}{n}\right).
\end{aligned}
$$

This last term is positive if $k^m > (n-1)\lambda^m$. □

We now proceed to show how to associate the third eigenvalue of a graph to matchings. First, a result due to Tutte whose proof is purely combinatorial, and therefore we skip:

Theorem 5.27. *A graph G has no perfect matching if and only if there is a subset $S \subseteq V(G)$ so that the subgraph of G induced by $V\backslash S$ has more than $|S|$ odd components (that is, a connected component with an odd number of vertices).*

(Note however that one direction of the Theorem is very easy to show).
Again, we will be dealing with regular graphs (for the last time).

Theorem 5.28. *A connected k-regular graph G on n vertices, n even, and eigenvalues $\theta_1 \geq ... \geq \theta_n$, has a perfect matching if*

$$
\theta_3 \leq \begin{cases} k - 1 + \frac{3}{k+1} & \text{if } k \text{ is even,} \\ k - 1 + \frac{3}{k+2} & \text{if } k \text{ is odd.} \end{cases}
$$

Proof. Assume there is no perfect matching. By (the difficult direction of) Tutte's theorem, there is a set S of size s so that $V \setminus S$ has at least $s + 2$ odd components (why not $s + 1$ only?). Let $G_1, ..., G_q$ be each of one these, each of size n_i. Then

$$\sum_{i=1}^{q} e(G_i, S) \leq ks.$$

As $s \geq 1$, $e(G_i, S) \geq 1$, this implies $e(G_i, S) < k$ and $n_i > 1$ for at least three values of i. Say $i = 1, 2, 3$, ordered in such way that the largest eigenvalues of $\mathbf{A}(G_i)$, say λ_i, satisfy $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Upon taking the union of these three graphs, we find $\theta_3 \geq \lambda_3$.

We now look at G_3. We have that its average degree is

$$\partial_3 = \frac{2|E(G_3)|}{n_3} = \frac{kn_3 - e(G_3, S)}{n_3} = k - \frac{e(G_3, S)}{n_3}.$$

Note that $e(G_3, S) < k$, and $n_3 > 1$, so $k < n_3$. If k is even, $e(G_3, S)$ is even. If k is odd, then $k \leq n_3 - 2$. Thus

$$\partial_3 \geq \begin{cases} k - \frac{k-2}{k+1} & \text{if } k \text{ is even,} \\ k - \frac{k-1}{k+2} & \text{if } k \text{ is odd.} \end{cases}$$

As G_3 is not regular, because $e(G_3, S)$, we have $\partial_3 < \lambda_3 \leq \theta_3$, as wished. $\square$

As we have seen over several previous results, the hypothesis of a graph being regular comes in very handy when dealing with the eigenvalues of the adjacency matrix. Our goal is to introduce another type of adjacency matrix that shall overcome this necessity.

5.4 References

Here is the set of references used to write the past few pages.

W. Haemers's paper "Interlacing Eigenvalues of Graphs" is a standard reference for applications of interlacing to combinatorics.

More interlacing resources are Brouwer and Haemers's textbook "Spectra of Graphs", and Godsil and Royle's "Algebraic Graph Theory", Chapter 9.

For the theorem associating eigenvalues and matchings, the reference is Brouwer and Haemers's paper "Eigenvalues and Perfect Matchings".

The diameter bound is due to Fan Chung "Diameters and Eigenvalues".