

Lemma 1.3. *Let \mathbf{M} be a real symmetric matrix. If U is \mathbf{M} -invariant, then U^\perp is also \mathbf{M} -invariant.*

Proof. Note that $\mathbf{v} \in U^\perp$, by definition, if $\mathbf{v}^T \mathbf{u} = 0$ for all $\mathbf{u} \in U$. For all $\mathbf{u} \in U$ and $\mathbf{v} \in U^\perp$, note that

$$(\mathbf{M}\mathbf{v})^T \mathbf{u} = \mathbf{v}^T \mathbf{M}\mathbf{u} = \mathbf{v}^T (\mathbf{M}\mathbf{u}) = 0,$$

because $\mathbf{u} \in U$, U is \mathbf{M} -invariant, and so $\mathbf{M}\mathbf{u} \in U$, and $\mathbf{v} \in U^\perp$. Thus $\mathbf{M}\mathbf{v} \in U^\perp$, as we wanted. \square

Let λ be such that $\det(\lambda\mathbf{I} - \mathbf{M}) = 0$. Then $\lambda\mathbf{I} - \mathbf{M}$ is singular, and therefore it contains at least one non-zero vector in its kernel. This is saying that all square matrices \mathbf{M} contain at least one eigenvector for each root of $\phi_{\mathbf{M}}(x) = \det(x\mathbf{I} - \mathbf{M})$. As \mathbf{M} is symmetric, we now know that all possible roots of $\phi_{\mathbf{M}}$ are real.

Lemma 1.4. *Let U be an \mathbf{M} -invariant subspace with dimension ≥ 1 . Then there is one eigenvector of \mathbf{M} in U .*

Proof. Let \mathbf{P} be a matrix whose columns form an orthonormal basis for U . As U is \mathbf{M} -invariant, it follows that there is a matrix \mathbf{N} so that

$$\mathbf{M}\mathbf{P} = \mathbf{P}\mathbf{N}.$$

(Stop now and think carefully why this equality is true.) In particular, $\mathbf{N} = \mathbf{P}^T \mathbf{M} \mathbf{P}$, so \mathbf{N} is symmetric. Let \mathbf{u} be one eigenvector of \mathbf{N} with eigenvalue λ . Then

$$\mathbf{M}\mathbf{P}\mathbf{u} = \mathbf{P}\mathbf{N}\mathbf{u} = \lambda\mathbf{P}\mathbf{u},$$

and, moreover $\mathbf{P}\mathbf{u} \neq \mathbf{0}$, as the columns of \mathbf{P} are linearly independent. Thus $\mathbf{P}\mathbf{u}$ is an eigenvector for \mathbf{M} in U . \square

These four lemmas above are all you need to prove the following result by induction as an exercise.

Theorem 1.5. *Let \mathbf{M} be a real symmetric matrix. Then \mathbf{M} is diagonalizable by set of orthogonal eigenvectors, all of them corresponding to real eigenvalues.*

Exercise 1.6. Write the proof of this theorem as an exercise.

Corollary 1.7. *Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be an orthonormal basis of eigenvectors for \mathbf{M} , each corresponding to an eigenvalue $\lambda_1, \dots, \lambda_n$ (these are not necessarily distinct). Let \mathbf{P} be the matrix whose i th column is \mathbf{v}_i , and Λ the diagonal matrix whose i th diagonal element is λ_i . Then*

$$\mathbf{P}^T \mathbf{M} \mathbf{P} = \Lambda,$$

and

$$\mathbf{M} = \lambda_1(\mathbf{v}_1 \mathbf{v}_1^T) + \dots + \lambda_n(\mathbf{v}_n \mathbf{v}_n^T).$$