1 Symmetric matrices and adjacency of a graph

In this section, we shall introduce the basic theory of symmetric matrices, including a result generally overlooked in a first or second linear algebra course. We shall define the adjacency matrix of a graph, and then make connections between the algebraic properties of this matrix and the combinatorial properties of the graph.

1.1 Symmetric matrices

We shall work over the vector space $\mathbb{R}^n$. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $\langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{v}^T\mathbf{u}$ is an inner product (meaning, it is a positive-definite commutative bilinear form). A linear operator $\mathbf{M} : \mathbb{R}^n \to \mathbb{R}^n$ is self-adjoint if $\langle \mathbf{Mv}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{Mu} \rangle$ for all $\mathbf{u}$ and $\mathbf{v}$, and, because $\mathbf{M}$ can (and will) be seen as a square matrix, it follows that $\mathbf{M}$ is a self-adjoint operator if and only if $\mathbf{M} = \mathbf{M}^T$, that is, $\mathbf{M}$ is a symmetric matrix. Symmetric matrices enjoy two key important properties: they are diagonalizable by orthogonal eigenvectors, and all of their eigenvalues are real. We start proving both properties.

Lemma 1.1. *The eigenvalues of a real symmetric matrix are real numbers.*

Proof. Let $\mathbf{Mu} = \lambda\mathbf{u}$, with $\mathbf{u} \neq \mathbf{0}$. Some of these things could be complex numbers, so we can take the conjugate on both sides, recovering

$$\mathbf{M}\overline{\mathbf{u}} = \overline{\lambda}\overline{\mathbf{u}}.$$

Thus $\overline{\mathbf{u}}$ is an eigenvector with eigenvalue $\overline{\lambda}$. Thus

$$\lambda\mathbf{u}^T\overline{\mathbf{u}} = (\mathbf{Mu})^T\overline{\mathbf{u}} = \mathbf{u}^T(\mathbf{M}\overline{\mathbf{u}}) = \overline{\lambda}\mathbf{u}^T\overline{\mathbf{u}}.$$

Because $\mathbf{u}^T\overline{\mathbf{u}} \neq 0$ if $\mathbf{u} \neq 0$, then $\lambda = \overline{\lambda}$. □

Now simply assume whenever we are dealing with a symmetric matrix, its eigenvalues are real, and any eigenvector can be assumed to be real.

Lemma 1.2. *Let* $\mathbf{M}$ *be a real symmetric matrix, and assume* $\mathbf{u}$ *and* $\mathbf{v}$ *are eigenvectors associated to different eigenvalues. Then* $\mathbf{v}^T\mathbf{u} = 0$*, that is, they are orthogonal.*

Proof. Say $\mathbf{Mu} = \lambda\mathbf{u}$ and $\mathbf{Mv} = \mu\mathbf{v}$, with $\lambda \neq \mu$. It follows that

$$\lambda(\mathbf{v}^T\mathbf{u}) = \mathbf{v}^T\mathbf{Mu} = (\mathbf{v}^T\mathbf{Mu})^T = \mathbf{u}^T\mathbf{M}^T\mathbf{v} = \mathbf{u}^T\mathbf{Mv} = \mu(\mathbf{u}^T\mathbf{v}) = \mu(\mathbf{v}^T\mathbf{u}).$$

As $\lambda \neq \mu$, it must be that $\mathbf{v}^T\mathbf{u} = 0$. □

The lemma above already implies that if $\mathbf{M}$ is diagonalizable, then it is diagonalizable with orthogonal eigenvectors — as, in fact, we eigenvectors corresponding to distinct eigenvalues are orthogonal, and inside each eigenspace we can always find an orthogonal basis. We move forward.

A subspace U of $\mathbb{R}^n$ is said to be $\mathbf{M}$-invariant if, for all $\mathbf{u} \in U$, $\mathbf{Mu} \in U$. This is a key fundamental concept in linear algebra, and several results are proven by noting that certain subspaces are invariant for certain operator.