orthogonal, thus they are all linearly independent. As a consequence, $\dim W = d+1$, and $\{\mathbf{E}_0, ..., \mathbf{E}_d\}$ form a basis for W. Now observe that if $r \leq D$, then at least one entry of $\mathbf{A}^r$ is non-zero "for the first time", meaning that it was equal to 0 for all smaller powers of $\mathbf{A}$. Thus $\{\mathbf{A}^0, \mathbf{A}, \mathbf{A}^2, ..., \mathbf{A}^D\}$ form a linearly independent set in W, and $D \leq d$. □

Let us now return to the problem of deciding what can be determined by the spectrum of a graph alone. Clearly the number of vertices in a graph is determined by the spectrum. An immediate consequence of the Lemma 1.18 is that the number of edges is also determined by the spectrum.

Corollary 1.21. *Let G be a graph on n vertices, with m edges, and let $\lambda_1, ..., \lambda_n$ the eigenvalues of $\mathbf{A}(G)$. Then*

$$\lambda_1^2 + ... + \lambda_n^2 = 2m.$$

Proof. Both sides are equal to $\operatorname{tr} \mathbf{A}^2$. □

Exercise 1.22. Find a formula for the number of triangles (cycles of length 3) found as subgraphs of G that depends only on the eigenvalues of G. Explain why the number of cycles of length 4 is not determined by the spectrum alone (as you witnessed in the example above).

Exercise 1.23. Does the spectrum alone determines the length of the shortest odd cycle of a graph? Explain.

Exercise 1.24. If G has n vertices, prove that all eigenvalues lie in the interval $(-n, n)$.

Exercise 1.25. Let G be a k-regular graph (that is, all vertices have k neighbours). Prove that k is an eigenvalue for G by describing a corresponding eigenvector.

Let $\mathbf{J}$ stand for the matrix whose all entries are equal to 1. If G is a graph, let $\overline{G}$ stand for the complement graph of G, that is, the graph whose edges are precisely the non-edges of G. Then, clearly,

$$\mathbf{A}(\overline{G}) = \mathbf{J} - \mathbf{A}(G) - \mathbf{I}.$$

As immediate consequence of the past exercise, we have:

Lemma 1.26. *Let G be a k-regular graph, with eigenvalues $k = \lambda_1, ..., \lambda_n$. Then the eigenvalues of $\overline{G}$ are*

$$n - k - 1,\ -\lambda_2 - 1,\ ...,\ -\lambda_n - 1.$$

Proof. The all 1s vector $\mathbf{1}$ is an eigenvector of G. Let $\mathbf{v}_2, ..., \mathbf{v}_n$ complete a basis of orthogonal eigenvectors. Then

$$(\mathbf{J} - \mathbf{A}(G) - \mathbf{I})\mathbf{1} = (n-k-1)\mathbf{1} \quad \text{and} \quad (\mathbf{J} - \mathbf{A}(G) - \mathbf{I})\mathbf{v}_i = -\lambda_i - 1,$$

as $\mathbf{J}\mathbf{v}_i = \mathbf{0}$ because $\mathbf{1}$ and $\mathbf{v}_i$ are orthogonal. □

Exercise 1.27. Assume G contains a pair of vertices a and b so that the neighbourhood of a is equal to neighbourhood of b (the rest of the graph can be anything). For example: