Theorem 1.37. *Let* $\mathbf{A}$ *be the adjacency matrix of a connected graph* G*, and* $\lambda_1 \geq ... \geq \lambda_n$ *its spectrum.*

(a) G *is* k*-regular if and only if* $(1/n)(\lambda_1^2 + ... + \lambda_n^2) = \lambda_1$*, and, in this case,* $k = \lambda_1$*.*

(b) G *is bipartite if and only if* $\lambda_1 = -\lambda_n$*. If this is the case, then for all* λ_i*,* $-\lambda_i$ *is also an eigenvalue.*

Proof.

(a) Let $\mathbf{1}$ be the all 1s vector. The equality is equivalent to

$$R_{\mathbf{A}}(\mathbf{1}) = \lambda_1,$$

which, as we saw, is equivalent to $\mathbf{1}$ being an eigenvector of λ_1. This vector is eigenvector if and only if all row sums of $\mathbf{A}$ are equal, or, equivalently, all vertices have the same degree, which is going to be precisely equal to the eigenvalue λ_1.

(b) If G is bipartite, its adjacency matrix can always be written as

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{pmatrix}.$$

If $\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$ is eigenvector for λ_i, then it is easy to see that $\begin{pmatrix} \mathbf{v}_1 \\ -\mathbf{v}_2 \end{pmatrix}$ is eigenvector for $-\lambda_i$. On the other hand, if $-\lambda_1$ is eigenvalue, then, from Lemma 1.36, it follows that $\mathbf{A}^2$ is not irreducible. Thus there are at least two vertices you can never walk from one to another with an even number of steps. Therefore there can be no odd cycles in this graph.

□

Corollary 1.38. *Let* λ *be the largest eigenvalue of* $\mathbf{A}(G)$*. Let* Δ *be the largest degree of* G*, and let* ∂ *be its average degree. Then*

$$\partial \leq \lambda \leq \Delta.$$

Proof. The first inequality follows from the fact that

$$\partial = R_{\mathbf{A}}(\mathbf{1}) \leq \lambda.$$

(Note in particular that this implies $\lambda \geq \delta$, where δ is the smallest degree of G). For the second, we have $\mathbf{A1} \leq \Delta\mathbf{1}$, and with $\mathbf{v}$ eigenvector for λ, we can multiply by $\mathbf{v}^T$ on the left. As $\mathbf{v} > \mathbf{0}$, the sign is preserved, and

$$\lambda\mathbf{v}^T\mathbf{1} = \mathbf{v}^T\mathbf{A1} \leq \Delta\mathbf{v}^T\mathbf{1},$$

so $\theta \leq \Delta$. □

Exercise 1.39. Prove that $\lambda \geq \sqrt{\Delta}$. (Hint: look at $\mathbf{A}^2$ and the proof above).