Lemma 1.34. *Let* $\mathbf{M}$ *be symmetric, non-negative and irreducible, with largest eigenvalue* λ*. There is a corresponding eigenvector* $\mathbf{u}$ *to* λ *so that* $\mathbf{u} > \mathbf{0}$*.*

Proof. Let $\mathbf{v}$ be a normal eigenvector for λ, and define $\mathbf{u}$ to be made from $\mathbf{v}$ by taking the absolute value at each entry (also denoted by $\mathbf{u} = |\mathbf{v}|$). Note that $\mathbf{u}$ is still normal, and, moreover

$$\lambda = R_{\mathbf{M}}(\mathbf{v}) = |R_{\mathbf{M}}(\mathbf{v})| \leq R_{\mathbf{M}}(\mathbf{u}) \leq \lambda.$$

(Second equality follows from $\lambda > 0$. First inequality from is simply the triangle inequality. Second follows from Lemma 1.33.)

Hence $R_{\mathbf{M}}(\mathbf{u}) = \lambda$, and $\mathbf{u}$ is an eigenvector for λ, with $\mathbf{u} \geq \mathbf{0}$. To see that $\mathbf{u} > \mathbf{0}$, note that as $\mathbf{M}$ is irreducible, it follows from Exercise 1.30 that $\mathbf{I} + \mathbf{M}$ is primitive, and so there is a k so that $(\mathbf{I}+\mathbf{M})^k > \mathbf{0}$. The vector $\mathbf{u}$ is also eigenvector for this matrix (with eigenvalue $(1+\lambda)^k$, but

$$\mathbf{0} < (\mathbf{I} + \mathbf{M})^k \mathbf{u} = (1+\lambda)^k \mathbf{u},$$

implying $\mathbf{u} > \mathbf{0}$. □

Lemma 1.35. *The largest eigenvalue* λ *of a symmetric, non-negative and irreducible matrix is simple (meaning, its eigenspace has dimension 1).*

Proof. From the proof of the past lemma, we know that no eigenvector for λ contains an entry equal to 0. No subspace of dimension larger than 1 can be such that all of its non-zero vectors have no non-zero entries. □

And finally:

Lemma 1.36. *Let* $\mathbf{M}$ *be symmetric, non-negative and irreducible. Let* λ *be its largest eigenvalue. Let* μ *be any other eigenvalue. Then* $\lambda \geq |\mu|$*, and, moreover, if* $-\lambda$ *is an eigenvalue, then* $\mathbf{M}^2$ *is not irreducible.*

Proof. Let $\mathbf{v}$ be an eigenvector for μ. As $\mathbf{v}$ is orthogonal to the positive eigenvector corresponding to λ, at least one entry of $\mathbf{v}$ is negative. Thus

$$|\mu| = |R_{\mathbf{M}}(\mathbf{v})| < R_{\mathbf{M}}(|\mathbf{v}|) \leq \lambda.$$

Now note that λ^2 is the largest eigenvalue of $\mathbf{M}^2$ (which is, still, symmetric and non-negative). If $-\lambda$ is eigenvalue of $\mathbf{M}$, then the eigenspace of λ^2 in $\mathbf{M}^2$ is at least 2-dimensional, thus $\mathbf{M}^2$ cannot be irreducible. □

It is quite surprising at first sight that the hypothesis on $\mathbf{M}$ being symmetric can be dropped entirely from the results above. The geometric intuition remains the same: a nonnegative irreducible matrix acts in the nonnegative orthant and there it encounters a unique direction which is an eigenvector. The proofs of these results are not hard per se, but I didn't feel they would add much to this notes. You are however invited to check any reference on spectral graph theory or non-negative matrix theory to find your favourite version of these results.

Now, to the applications.