Proof. A linear operator is defined and determined by its action on a basis. The first equality follows from the fact that both sides act equally on the canonical basis of $\mathbb{R}^n$. The second follows from

$$\mathbf{M} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T,$$

and, by definition of matrix product, $\mathbf{M} = \mathbf{v}_1(\lambda_1\mathbf{v}_1^T) + ... + \mathbf{v}_n(\lambda_n\mathbf{v}_n^T)$. $\square$

You should recall right now that, because $\mathbf{v}_i$ is normalized, then $\mathbf{P}_i = \mathbf{v}_i\mathbf{v}_i^T$ is the matrix that represents the orthogonal projection onto the line spanned by $\mathbf{v}_i$, that is, $\mathbf{P}_i$ is a projection as $\mathbf{P}_i^2 = \mathbf{P}_i$, and it is an orthogonal projection as $\mathbf{P}_i$ is symmetric. Note that $\mathbf{P}_i\mathbf{P}_j = \mathbf{0}$ whenever $i \neq j$, and so any sum of the $\mathbf{P}_i$s for distinct indices will correspond to the orthogonal projection onto the space spanned by the $\mathbf{v}_i$s of the same indices. In particular $\sum_{i=1}^n \mathbf{P}_i = \mathbf{I}$.

Exercise 1.8. Assume $\mathbf{P}_i$s are orthogonal projections. Show that $\mathbf{P}_1 + \mathbf{P}_2$ is an orthogonal projection if and only if $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$.

Show now that $\mathbf{P}_1 + ... + \mathbf{P}_k$ is an orthogonal projection if and only if $\mathbf{P}_i\mathbf{P}_j = \mathbf{0}$ for $i \neq j$.

Say $\mathbf{M}$ is an $n \times n$ symmetric matrix with distinct eigenvalues $\theta_0, ..., \theta_d$. When we write the second equation from the statement of Corollary 1.7, we can collect the terms corresponding to equal eigenvalues, and have

$$\mathbf{M} = \sum_{r=0}^{d} \theta_r \mathbf{E}_r, \tag{1}$$

where, according to the discussion above, each $\mathbf{E}_r$ corresponds to the orthogonal projection onto the θ_r eigenspace. Equation (1) is usually referred to the as the *spectral decomposition* of the matrix $\mathbf{M}$.

Exercise 1.9. Find the spectral decomposition of

$$\mathbf{M} = \begin{pmatrix} 1+\sqrt{2} & 0 & 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} & 0 & 1-\sqrt{2} \\ 1-\sqrt{2} & 0 & 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} & 0 & 1+\sqrt{2} \end{pmatrix}$$

Hint: do not try to compute the characteristic polynomial. It is easier to simply try to look and guess which are the eigenvectors and eigenvalues.

Note that the $\mathbf{E}_r$ are symmetric matrices satisfying $\mathbf{E}_r\mathbf{E}_s = \delta_{rs}\mathbf{E}_r$, and $\sum_{r=0}^d \mathbf{E}_r = \mathbf{I}$.

Exercise 1.10. Prove (or at least convince yourself) that for any polynomial $p(x)$, it follows that

$$p(\mathbf{M}) = \sum_{r=0}^{d} p(\theta_r)\mathbf{E}_r.$$

Exercise 1.11. Let $\mathbf{M}$ be a symmetric matrix, with spectral decomposition as in (1).
(A) What is the minimal polynomial of $\mathbf{M}$? (B) Prove that for each $\mathbf{E}_r$, there is a polynomial p_r of degree d so that $p_r(\mathbf{M}) = \mathbf{E}_r$. Describe this polynomial as explicitly as you can.