Example 1.29. Consider

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Verify that the first is primitive, the second and third are both irreducible, but not primitive, and the fourth is neither.

Exercise 1.30. Prove that if $\mathbf{M}$ is irreducible, then $\mathbf{I} + \mathbf{M}$ is primitive.

Exercise 1.31. Let G be a graph. Show that

(a) $\mathbf{A}(G)$ is irreducible if and only if G is connected.

(b) $\mathbf{A}(G)$ is not primitive if G is bipartite.

Over the next few results, we shall actually see, amongst other things, that $\mathbf{A}(G)$ is irreducible but not primitive if and only if G is connected and bipartite. Results below are known as the Perron-Frobenius theory. This theory applies generally to matrices which are assumed to be irreducible and nothing else. We shall however add the hypothesis that the matrices are also symmetric, for the proofs become simpler and more meaningful, and our matrices will almost always be symmetric anyway.

Our first observation.

Lemma 1.32. *Let $\mathbf{M}$ be a nonnegative symmetric matrix, $\mathbf{M} \neq \mathbf{0}$. If λ is the largest eigenvalue of $\mathbf{M}$, then $\lambda > 0$.*

Proof. Follows immediately from $\operatorname{tr} \mathbf{M} \geq 0$. □

For any nonzero vector $\mathbf{u} \in \mathbb{R}^n$, and symmetric matrix $\mathbf{M}$, define

$$R_{\mathbf{M}}(\mathbf{u}) = \frac{\mathbf{u}^T \mathbf{M} \mathbf{u}}{\mathbf{u}^T \mathbf{u}}.$$

This is known as the Rayleigh quotient of $\mathbf{u}$ with respect to $\mathbf{M}$. Note that $R_{\mathbf{M}}(\alpha \mathbf{u}) = R_{\mathbf{M}}(\mathbf{u})$ for all $\alpha \neq 0$, so we shall typically assume $\mathbf{u}$ has been normalized. In a sense, this is a measurement of how much $\mathbf{M}$ displaces $\mathbf{u}$, also proportional to how much $\mathbf{M}$ stretches or shrinks $\mathbf{u}$. Therefore one should expect that this is maximum when $\mathbf{u}$ is an eigenvector of $\mathbf{M}$, corresponding to a large eigenvalue.

Lemma 1.33. *If $\mathbf{u}$ is eigenvector of $\mathbf{M}$ with eigenvalue θ, then $R_{\mathbf{M}}(\mathbf{u}) = \theta$. If λ is the largest eigenvalue of $\mathbf{M}$, then, for all $\mathbf{v} \in \mathbb{R}^n$, $R_{\mathbf{M}}(\mathbf{v}) \leq \lambda$. Equality holds for some $\mathbf{v}$ only if $\mathbf{v}$ is eigenvector for λ.*

Proof. Only the second and third assertions deserve a proof. Let $\mathbf{M} = \sum_{r=0}^{d} \theta_r \mathbf{E}_r$ be the spectral decomposition of $\mathbf{M}$. Assume θ_0 is the largest eigenvalue, and that $\mathbf{v}$ is a normalized vector. Then

$$\begin{aligned} R_{\mathbf{M}}(\mathbf{v}) = \mathbf{v}^T \mathbf{M} \mathbf{v} &= \theta_0(\mathbf{v}^T E_0 \mathbf{v}) + \theta_1(\mathbf{v}^T E_1 \mathbf{v}) + ... + \theta_d(\mathbf{v}^T E_d \mathbf{v}) \\ &\leq \theta_0((\mathbf{v}^T E_0 \mathbf{v}) + (\mathbf{v}^T E_1 \mathbf{v}) + ... + (\mathbf{v}^T E_d \mathbf{v}) = \theta_0. \end{aligned}$$

Equality holds if and only if $(\mathbf{v}^T E_r \mathbf{v}) = 0$ for all $r > 0$, which is the same as saying that $\mathbf{v}$ belongs to the θ_0 eigenspace. □