

Note though that we still have

$$\mathbf{e}_i^\top \mathbf{A} \mathbf{x}(i) = \lambda_1 \mathbf{x}_i.$$

By Cauchy-Schwarz,

$$\lambda_1^2 \mathbf{x}_i^2 = |\mathbf{e}_i^\top \mathbf{A} \mathbf{x}(i)|^2 \leq |\mathbf{e}_i^\top \mathbf{A}|^2 |\mathbf{x}(i)|^2 = d_i \left(1 - \sum_{j \neq i} \mathbf{x}_j^2 \right)$$

and summing over all i , we get

$$\lambda_1^2 \leq 2m - \sum_{i=1}^n d_i \left(\sum_{j \neq i} \mathbf{x}_j^2 \right).$$

It is not difficult to show that the subtracted term upper bounds $n - 1$, with equality if and only if $d_i = 1$ or $d_i = n - 1$ for all i . It follows that

$$\lambda_1^2 \leq \sqrt{2m - (n - 1)}.$$

Equality holds if and only if $X = K_n$ or $X = K_{1,n-1}$. \square

We can also provide an interesting lower bound to λ_n .

Theorem 1.42 (Constantine, Hong, Powers). *For a given graph with smallest eigenvalue λ_n ,*

$$\lambda_n \geq -n/2.$$

Proof. Let \mathbf{x} be a unit length eigenvector for λ_n , and suppose there are precisely a vertices whose coordinate is positive and b with negative. Then

$$\begin{aligned} \lambda_n &= \mathbf{x}^\top \mathbf{A} \mathbf{x} \geq \sum_{\mathbf{x}_i \mathbf{x}_j < 0} \mathbf{A}_{ij} \mathbf{x}_i \mathbf{x}_j \\ &\geq \sum_{\mathbf{x}_i \mathbf{x}_j < 0} \mathbf{x}_i \mathbf{x}_j \\ &\geq \lambda_n(K_{a,b}) \\ &= -\sqrt{ab} \\ &\geq -n/2. \end{aligned}$$

\square

1.6 A result for regular graphs

The theorem in this subsection will reappear when we talk about interlacing. But it contains the nice, elementary proof below.

If the graph is regular, the size of the largest coclique can be associated with the smallest eigenvalue.