Two isomorphic graphs can always be seen as graphs on the same vertex set, and the isomorphism is a re-ordering that preserves adjacency and non-adjacency. Thus:

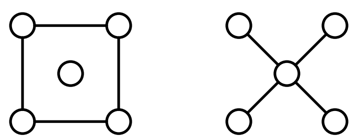
Theorem 1.16. *Let G and H be isomorphic graphs. Order their vertex sets from 1 to n, and let $\mathbf{P}$ be the permutation matrix that corresponds to the isomorphism from G to H. Then*

$$\mathbf{P}\mathbf{A}(G)\mathbf{P}^T = \mathbf{A}(H).$$

As a consequence, $\mathbf{A}(G)$ and $\mathbf{A}(H)$ have the same eigenvalues. □

Exercise 1.17. Order the vertices of G_1 and G_2 equally in terms of their geometric position. Then find the matrix $\mathbf{P}$ so that $\mathbf{P}\mathbf{A}(G_1)\mathbf{P} = \mathbf{A}(G_2)$. Compute the eigenvalues of G_1 and G_4 (using a software?) and conclude that they cannot be isomorphic.

One of the motivations of the development of spectral graph theory was the hope that two graphs would be isomorphic if and only if they had the same eigenvalues. Such a claim would immediately provide an efficient polynomial time algorithm to decide whether two graphs are isomorphic (and yet no such algorithm is known to this day). Two graphs with the same eigenvalues are called *cospectral graphs.* The following pair of graphs are the smallest known cases of cospectral but (clearly) non-isomorphic graphs. They have spectrum $2, 0^{(3)}, -2$.



G_1 G_2

This example also shows that the spectrum of a graph does not determine whether the graph is connected or not. This immediately raises the general question: what graph properties can be determined from the spectrum?

A *walk* of length r in a graph G is a sequence of $r+1$ (possibly repeated) vertices $a_0, ..., a_r$ with the property that $a_i \sim a_j$. A walk is *closed* if $a_0 = a_r$.

Lemma 1.18. *The number of distinct walks of length r from a to b in G is precisely equal to $(\mathbf{A}^r)_{ab}$.*

Exercise 1.19. Verify this result on at least 3 different graphs checking powers $r = 1, 2, 3$ for each. Then, sketch a proof by induction of this result.

Corollary 1.20. *If G has diameter D, then it must have at least $D+1$ distinct eigenvalues.*

Proof. Let

$$\mathbf{A}(G) = \sum_{r=0}^{d} \theta_r \mathbf{E}_r$$

be the spectral decomposition of $\mathbf{A}(G)$. Let W be the subspace of $\mathrm{Sym}_n(\mathbb{R})$ generated by $\{\mathbf{A}^0, \mathbf{A}, \mathbf{A}^2, ...\}$. As we saw in the past section, all powers of $\mathbf{A}$ are a linear combinations of the $\mathbf{E}_r$s, and each $\mathbf{E}_r$ is a polynomial in $\mathbf{A}$. Moreover, the matrices $\mathbf{E}_r$ are pairwise