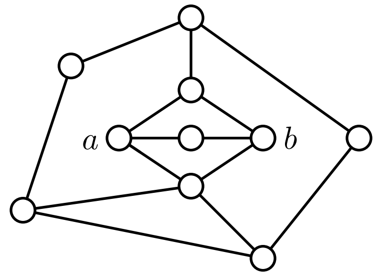


(a) Prove that 0 is an eigenvalue of this graph (Hint: look at a and b and try to produce one eigenvector for 0). If the example looks too complicate, forget about the 5-cycle and focus only on a, b and their neighbours.

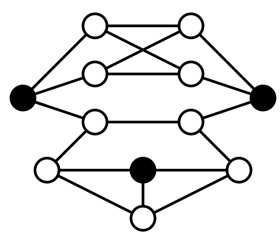
(b) What could you say if a and b shared the same neighbourhood, but were also neighbours themselves?

Exercise 1.28. Assume $G = (V, E)$ is a k-regular graph which contains a subset of vertices $U \subseteq V$ satisfying the following properties:

(a) No two vertices in U are neighbours.

(b) Any vertex in $V \backslash U$ contains exactly one neighbour in U.

Prove that if such U exists, then -1 is an eigenvalue of the graph. (Hint: recall G is assumed to be k-regular, and, again, try to produce one eigenvector. Try first in the example below, where the dark vertices are the vertices in U.)



In this next section, we shall see that two important properties about a graph can be determined from its spectrum alone: whether the graph is regular, and whether the graph is bipartite.

1.3 Perron-Frobenius (a special case)

Let $\mathbf{M}$ be a real $n \times n$ matrix with nonnegative entries. For example, the adjacency matrix of a graph. This matrix is called primitive if, for some integer k, $\mathbf{M}^k > 0$, and it is called irreducible if for all indices i and j, there is an integer k so that $(\mathbf{M}^k)_{ij} > 0$. All primitive matrices are irreducible, but the converse is not necessarily true.