Problem 1

Fix n, and denote X_k the set of all k-element subsets of $\{1,\dots,n\}$ $(k \le n)$. It carries an action of S_n, and we can consider the corresponding representation V_k of S_n, where V_k is the space of $\mathbb{C}$-valued functions on X_k.
Show that $V_k \cong V_{n-k}$.

Proof. We will show that there is an isomorphism of representations between V_k and V_{n-k}. Let $A \in X_k$ be a k-element subset of $\{1,\dots,n\}$. Define a map $\phi : V_k \to V_{n-k}$ by sending a function $f \in V_k$ to a function $\phi(f) \in V_{n-k}$ defined as follows:

$$\phi(f)(B) = f(\{1,\dots,n\} \setminus B)$$

for every $(n-k)$-element subset $B \in X_{n-k}$. Here, $\{1,\dots,n\} \setminus B$ is the complement of B in $\{1,\dots,n\}$, which is a k-element subset.

To show that ϕ is a representation isomorphism, we need to verify two things: 1. ϕ is linear. 2. ϕ commutes with the action of S_n.

1. **Linearity**: For any $f_1, f_2 \in V_k$ and scalars $a, b \in \mathbb{C}$, we have

$$\phi(af_1 + bf_2)(B) = (af_1 + bf_2)(\{1,\dots,n\} \setminus B) = af_1(\{1,\dots,n\} \setminus B) + bf_2(\{1,\dots,n\} \setminus B) = a\phi(f_1)(B) + b\phi(f_2)(B).$$

Thus, ϕ is linear.

2. **Commuting with the action of S_n**: For any $\sigma \in S_n$, we need to show that

$$\phi(\sigma \cdot f) = \sigma \cdot (\phi(f)).$$

By definition of the action on functions,

$$(\sigma \cdot f)(A) = f(\sigma^{-1}(A)).$$

Therefore,

$$\phi(\sigma \cdot f)(B) = (\sigma \cdot f)(\{1,\dots,n\} \setminus B) = f(\sigma^{-1}(\{1,\dots,n\} \setminus B)).$$

On the other hand,

$$(\sigma \cdot (\phi(f)))(B) = \phi(f)(\sigma^{-1}(B)) = f(\{1,\dots,n\} \setminus \sigma^{-1}(B)).$$

Since σ is a bijection, we have

$$\sigma^{-1}(\{1,\dots,n\} \setminus B) = \{1,\dots,n\} \setminus \sigma^{-1}(B).$$

Thus,

$$\phi(\sigma \cdot f)(B) = f(\{1,\dots,n\} \setminus \sigma^{-1}(B)) = (\sigma \cdot (\phi(f)))(B).$$

This shows that ϕ commutes with the action of S_n. Since ϕ is a linear bijection that commutes with the action of S_n, it is an isomorphism of representations. Therefore, we conclude that $V_k \cong V_{n-k}$. ∎