Problem 1

a) Define when we say that a vector space W is a quotient of V modulo U.

b) Recall the construction (given in class) of the vector space V/U, and the onto map $q : V \to V/U$, with kernel $\ker(q) = U$.

c) Suppose T is a linear transformation between vector spaces V and W,

$$T : V \to W,$$

write down the natural induced map

$$\tilde{T} : V/\ker(T) \to \operatorname{im}(T),$$

and show that it is an isomorphism.

a) A space W is called a quotient space of V modulo U, denoted $W = V/U$, if there exists a surjective linear transformation $e : V \to W$ such that $\ker(e) = U$.

b) For any $v \in V$, consider the left coset $v + U = \{v + u : u \in U\}$. Define $q : V \to V/U$ by $v \mapsto v + U$. Clearly q is surjective. Recall that $\tilde{v} \mapsto \tilde{v} + U = U$ if and only if $\tilde{v} \in U$. Then we have that $\ker(q)$ is precisely the set of all $\tilde{v} \in V$ such that $\tilde{v} + U = U$, which is exactly U. Thus, $q : V \to V/U$ is a surjective linear transformation with $\ker(q) = U$, so V/U is a quotient space of V modulo U.

c) First we write down the natural induced map $\tilde{T} : V/\ker(T) \to \operatorname{im}(T)$ by $\tilde{v} + \ker(T) \mapsto T(\tilde{v})$. To show that this is well-defined, suppose $\tilde{v} + \ker(T) = \tilde{w} + \ker(T)$ for some $\tilde{v}, \tilde{w} \in V$. Then $\tilde{v} - \tilde{w} \in \ker(T)$, so $T(\tilde{v} - \tilde{w}) = 0$, and thus $T(\tilde{v}) = T(\tilde{w})$. Hence, $\tilde{T}$ is well-defined.
Next we show that $\tilde{T}$ is an isomorphism. First, we show that $\tilde{T}$ is linear. For any $\tilde{v}, \tilde{w} \in V$ and $a, b \in \mathbb{F}$, we show that $\tilde{T}(a(\tilde{v} + \ker(T)) + b(\tilde{w} + \ker(T))) = a\tilde{T}(\tilde{v} + \ker(T)) + b\tilde{T}(\tilde{w} + \ker(T))$. Consider

$$\tilde{T}(a(\tilde{v}+\ker(T))+b(\tilde{w}+\ker(T))) = \tilde{T}((a\tilde{v}+b\tilde{w})+\ker(T)) = T(a\tilde{v}+b\tilde{w}) = aT(\tilde{v})+bT(\tilde{w}) = a\tilde{T}(\tilde{v}+\ker(T))+b\tilde{T}(\tilde{w}+\ker(T)).$$

Thus, $\tilde{T}$ is linear.
Next, we show that $\tilde{T}$ is surjective. For any $w \in \operatorname{im}(T)$, there exists some $v \in V$ such that $T(v) = w$. Then $\tilde{T}(v + \ker(T)) = T(v) = w$, so $\tilde{T}$ is surjective.
Finally, we show that $\tilde{T}$ is injective. Suppose $\tilde{T}(\tilde{v} + \ker(T)) = 0$ for some $\tilde{v} \in V$. Then $T(\tilde{v}) = 0$, so $\tilde{v} \in \ker(T)$, and thus $\tilde{v} + \ker(T) = \ker(T)$, the zero element of $V/\ker(T)$. Hence, $\tilde{T}$ is injective.
Thus we have that $\tilde{T}$ is a bijective linear transformation, and hence an isomorphism.

Note: In part (c), I believe that we could also use the First Isomorphism Theorem to show that $\tilde{T}$ is an isomorphism, since we have that $\tilde{T}$ is a linear transformation from $V/\ker(T)$ to $\operatorname{im}(T)$ with kernel $\{0\}$, so by the First Isomorphism Theorem, $V/\ker(T) \cong \operatorname{im}(T)$.

Problem 2

Basis. Let V be a vector space over $\mathbb{F}$, and $\mathscr{B} \subset V$.

a) Complete the following definition:
Definition. We say that $\mathscr{B}$ is a basis of V if ...

b) Suppose V is finite dimensional. Write down the general facts we know about existence and cardinality of bases for V.

c) Suppose V is finite-dimensional and $U < V$, is a subspace. Suppose $\mathscr{B}_U$ is a basis for U. Complete it to a basis $\mathscr{B}$ for V, and consider the set $\mathscr{C} = \mathscr{B} \setminus \mathscr{B}_U$. Show that the set

$$\mathscr{B}_{V/U} = \{q(v); v \in \mathscr{C}\},$$

(where $q : V \to V/U$ is the quotient map) is a basis for the quotient space V/U constructed above.

a) We say that $\mathscr{B}$ is a basis of V if $\mathscr{B}$ is linearly independent and spans V.

b) From Linear Algebra 1, if V is finite dimensional, then V has a basis, and any two bases of V have the same cardinality. We call this cardinality the dimension of V, denoted $\dim(V)$.

c) Suppose V is finite-dimensional and $U < V$, is a subspace. Suppose $\mathscr{B}_U$ is a basis for U. Complete it to a basis $\mathscr{B}$ for V, and consider the set $\mathscr{C} = \mathscr{B} \setminus \mathscr{B}_U$. Show that the set

$$\mathscr{B}_{V/U} = \{q(v); v \in \mathscr{C}\},$$

(where $q : V \to V/U$ is the quotient map) is a basis for the quotient space V/U constructed above.

Proof. First we show that $\mathscr{B}_{V/U}$ spans V/U. For any $\tilde{v} + U \in V/U$, since $\mathscr{B}$ spans V, we can write $\tilde{v} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$ for some $b_i \in \mathscr{B}$ and $a_i \in \mathbb{F}$. We can separate the b_i into those that are in $\mathscr{B}_U$ and those that are in $\mathscr{C}$. Thus, we can write

$$\tilde{v} = c_1 u_1 + c_2 u_2 + \cdots + c_m u_m + d_1 c_1' + d_2 c_2' + \cdots + d_k c_k',$$

where $u_i \in \mathscr{B}_U$, $c_j' \in \mathscr{C}$, and $c_i, d_j \in \mathbb{F}$. Then

$$\tilde{v} + U = (d_1 c_1' + d_2 c_2' + \cdots + d_k c_k') + U = d_1(c_1' + U) + d_2(c_2' + U) + \cdots + d_k(c_k' + U),$$

so $\tilde{v} + U$ is in the span of $\mathscr{B}_{V/U}$. Thus, $\mathscr{B}_{V/U}$ spans V/U.
Next we show that $\mathscr{B}_{V/U}$ is linearly independent. Suppose

$$a_1(q(c_1')) + a_2(q(c_2')) + \cdots + a_k(q(c_k')) = 0,$$

for some $c_i' \in \mathscr{C}$ and $a_i \in \mathbb{F}$. Then

$$q(a_1 c_1' + a_2 c_2' + \cdots + a_k c_k') = 0,$$

so $a_1 c_1' + a_2 c_2' + \cdots + a_k c_k' \in U$. Since $c_i' \in \mathscr{C}$ and $\mathscr{C}$ is linearly independent, we must have $a_i = 0$ for all i. Thus, $\mathscr{B}_{V/U}$ is linearly independent.
We have shown that $\mathscr{B}_{V/U}$ spans V/U and is linearly independent, so it is a basis for V/U. □

Problem 3

Dual Space. Let V be a vector space over $\mathbb{F}$.

a) Define the dual space of V.
We denote the dual space by V^*, and call its elements functionals.

b) Suppose V is finite dimensional and $\mathscr{B}$ is a basis for V. For $v \in \mathscr{B}$ define a functional $\varphi_v \in V^*$ by the following values on every $u \in \mathscr{B}$,

$$\varphi_v(u) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

Show that $\mathscr{B}^* = \{\varphi_v : v \in \mathscr{B}\}$ is a basis for V^* (it is called the dual basis to $\mathscr{B}$).
In particular, $\dim(V^*) = \dim(V)$.

c) Write down a natural isomorphism

$$\mathbb{F}^n_{\text{row}} \to (\mathbb{F}^n_{\text{col}})^*.$$

a) According to the definition given in class, the dual space of V, denoted V^*, is $\operatorname{Hom}(V, \mathbb{F})$, the set of all linear transformations from V to $\mathbb{F}$.

b) *Proof.* We show that $\mathscr{B}^* = \{\varphi_v : v \in \mathscr{B}\}$ is a basis for V^*.
First we show that $\mathscr{B}^*$ spans V^*. For any $\psi \in V^*$, since $\mathscr{B}$ is a basis for V, we can write any $x \in V$ as $x = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$ for some $b_i \in \mathscr{B}$ and $a_i \in \mathbb{F}$. Then we have that

$$\psi(x) = \psi(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) = a_1 \psi(b_1) + a_2 \psi(b_2) + \cdots + a_n \psi(b_n).$$

Since $\psi \in V^*$, we can express $\psi(b_i)$ in terms of the dual basis elements:

$$\psi(b_i) = \varphi_{b_i}(b_i) = 1 \quad \text{and} \quad \psi(b_j) = 0 \text{ for } j \neq i.$$

Thus,

$$\psi(x) = a_i \varphi_{b_i}(b_i) = a_i.$$

This shows that $\mathscr{B}^*$ spans V^*.
Next we show that $\dim(V^*) = \dim(V)$. Since $\mathscr{B}$ is a basis for V, we have that $\dim(V) = |\mathscr{B}|$. Since $\mathscr{B}^*$ is constructed by taking one functional φ_v for each $v \in \mathscr{B}$, we have that $|\mathscr{B}^*| = |\mathscr{B}|$. Thus, $\dim(V^*) = |\mathscr{B}^*| = |\mathscr{B}| = \dim(V)$.

□

c) The natural isomorphism $\Phi : \mathbb{F}^n_{\text{row}} \to (\mathbb{F}^n_{\text{col}})^*$ is given by

$$\Phi((a_1, a_2, \ldots, a_n))((x_1, x_2, \ldots, x_n)^T) = a_1x_1 + a_2x_2 + \cdots + a_nx_n,$$

where $(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n_{\text{row}}$ and $(x_1, x_2, \ldots, x_n)^T \in \mathbb{F}^n_{\text{col}}$. This map is linear, bijective, and thus an isomorphism.

Problem 4

Determinant. Denote $\mathbb{F}^2_{\text{col}}$ the vector space of column vectors of length two over a field $\mathbb{F}$. We assume that $-1 \neq 1$ in $\mathbb{F}$.

a) Complete the definition: The vector space $\Lambda(\mathbb{F}^2_{\text{col}})$, called the <u>determinant</u> of $\mathbb{F}^2_{\text{col}}$, is the collection of functions

$$\mathscr{A} : (\mathbb{F}^2_{\text{col}}) \times (\mathbb{F}^2_{\text{col}}) \to \mathbb{F},$$

that satisfies:

1. *Multilinearity:* Namely,
2. *Skew-symmetry:* Namely,

b) Show that

1. An element, $\mathscr{A} \in \Lambda(\mathbb{F}^2_{\text{col}})$, is completely determined by the value

$$\mathscr{A}((1, 0), (0, 1)).$$

2. Show that $\Lambda(\mathbb{F}^2_{\text{col}})$ is 1-dimensional.
3. Verify that the element $\mathscr{A}_1 \in \Lambda(\mathbb{F}^2_{\text{col}})$ that satisfies

$$\mathscr{A}_1((1, 0), (0, 1)) = 1,$$

has the formula

$$\mathscr{A}_1((x, y), (x', y')) = xy' - x'y.$$

c) Consider the natural action $M[\mathscr{A}]$ of a matrix $M \in M_2(\mathbb{F})$ on an element $\mathscr{A} \in \Lambda(\mathbb{F}^2_{\text{col}})$, where $M(\mathscr{A})$ is given by

$$M[\mathscr{A}]((x, y), (x', y')) = \mathscr{A}((x, y)M, (x', y')M).$$

Compute a formula for the scalar $d(M) \in \mathbb{F}$, such that

$$M[\mathscr{A}] = d(M) \cdot \mathscr{A}.$$

Hint: Since $\dim(\Lambda(\mathbb{F}^2_{\text{col}})) = 1$, the linear transformation on $\Lambda(\mathbb{F}^2_{\text{col}})$ is given by $\mathscr{A} \mapsto M[\mathscr{A}]$ is just multiplication by a scalar $d(M)$. For

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

this scalar can be computed by computing both sides of (1) in the following case

$$M[\mathscr{A}_1]((1, 0), (0, 1)) = d(M) \cdot \mathscr{A}_1((1, 0), (0, 1)),$$

where $\mathscr{A}_1$ is the function defined in the previous section.

a) Complete the definition: The vector space $\Lambda(\mathbb{F}^2_{\text{col}})$, called the <u>determinant</u> of $\mathbb{F}^2_{\text{col}}$, is the collection of functions

$$\mathscr{A} : (\mathbb{F}^2_{\text{col}}) \times (\mathbb{F}^2_{\text{col}}) \to \mathbb{F},$$

that satisfies:

1. *Multilinearity:* Namely, for all $u, v, w \in \mathbb{F}^2_{\text{col}}$ and all $c \in \mathbb{F}$,

$$\mathscr{A}((u + v), w) = \mathscr{A}(u, w) + \mathscr{A}(v, w),$$
$$\mathscr{A}(u, (v + w)) = \mathscr{A}(u, v) + \mathscr{A}(u, w),$$
$$\mathscr{A}((cu), v) = c \cdot \mathscr{A}(u, v),$$
$$\mathscr{A}(u, (cv)) = c \cdot \mathscr{A}(u, v).$$

2. *Skew-symmetry:* Namely, for all $u, v \in \mathbb{F}^2_{\text{col}}$,

$$\mathscr{A}(u, v) = -\mathscr{A}(v, u).$$

b) 1. *Proof.* Let $\mathscr{A} \in \Lambda(\mathbb{F}^2_{\text{col}})$. For any $(x, y), (x', y') \in \mathbb{F}^2_{\text{col}}$, we can write

$$(x, y) = x(1, 0) + y(0, 1),$$

$$(x', y') = x'(1, 0) + y'(0, 1).$$

Then by multilinearity, we have

$$\mathscr{A}((x, y), (x', y')) = \mathscr{A}(x(1, 0) + y(0, 1), x'(1, 0) + y'(0, 1)).$$

Expanding this using multilinearity, we get

$$= xx'\mathscr{A}((1, 0), (1, 0)) + xy'\mathscr{A}((1, 0), (0, 1)) + yx'\mathscr{A}((0, 1), (1, 0)) + yy'\mathscr{A}((0, 1), (0, 1)).$$

By skew-symmetry, we have $\mathscr{A}((1, 0), (1, 0)) = 0$ and $\mathscr{A}((0, 1), (0, 1)) = 0$. Also by skew-symmetry, we have $\mathscr{A}((0, 1), (1, 0)) = -\mathscr{A}((1, 0), (0, 1))$. Thus,

$$\mathscr{A}((x, y), (x', y')) = xy'\mathscr{A}((1, 0), (0, 1)) - yx'\mathscr{A}((1, 0), (0, 1)) = (xy' - yx')\mathscr{A}((1, 0), (0, 1)).$$

This shows that $\mathscr{A}$ is completely determined by the value $\mathscr{A}((1, 0), (0, 1))$. □

2. *Proof.* From part (1), we have that any $\mathscr{A} \in \Lambda(\mathbb{F}^2_{\text{col}})$ is completely determined by the value $\mathscr{A}((1, 0), (0, 1))$. Thus, we can define a linear transformation $\Phi : \Lambda(\mathbb{F}^2_{\text{col}}) \to \mathbb{F}$ by $\Phi(\mathscr{A}) = \mathscr{A}((1, 0), (0, 1))$. This map is linear and surjective. The kernel of this map is the set of all $\mathscr{A}$ such that $\mathscr{A}((1, 0), (0, 1)) = 0$. But from part (1), this means that $\mathscr{A}$ is the zero map. Thus, the kernel is trivial, so Φ is injective. Hence, Φ is an isomorphism. Since $\mathbb{F}$ is 1-dimensional, we have that $\Lambda(\mathbb{F}^2_{\text{col}})$ is also 1-dimensional. □

3. *Proof.* Let $\mathscr{A}_1 \in \Lambda(\mathbb{F}^2_{\text{col}})$ be such that $\mathscr{A}_1((1, 0), (0, 1)) = 1$. For any $(x, y), (x', y') \in \mathbb{F}^2_{\text{col}}$, we have

$$\begin{aligned}
\mathscr{A}_1((x, y), (x', y')) &= \mathscr{A}_1(x(1, 0) + y(0, 1), x'(1, 0) + y'(0, 1)) \\
&= xx'\mathscr{A}_1((1, 0), (1, 0)) + xy'\mathscr{A}_1((1, 0), (0, 1)) + yx'\mathscr{A}_1((0, 1), (1, 0)) + yy'\mathscr{A}_1((0, 1), (0, 1)) \\
&= xy' \cdot 1 + yx' \cdot (-1) \\
&= xy' - yx'.
\end{aligned}$$

Thus, $\mathscr{A}_1((x, y), (x', y')) = xy' - yx'$. □

c) We compute the formula for the scalar $d(M) \in \mathbb{F}$ such that

$$M[\mathscr{A}] = d(M) \cdot \mathscr{A}.$$

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We compute both sides of the equation

$$M[\mathscr{A}_1]((1, 0), (0, 1)) = d(M) \cdot \mathscr{A}_1((1, 0), (0, 1)).$$

First, we compute the left side:

$$\begin{aligned}
M[\mathscr{A}_1]((1, 0), (0, 1)) &= \mathscr{A}_1((1, 0)M, (0, 1)M) \\
&= \mathscr{A}_1((a, c), (b, d)) \\
&= ad - bc.
\end{aligned}$$

Next, we compute the right side:

$$d(M) \cdot \mathscr{A}_1((1, 0), (0, 1)) = d(M) \cdot 1 = d(M).$$

Equating both sides, we have

$$ad - bc = d(M).$$

Thus, the formula for the scalar $d(M)$ is

$$d(M) = ad - bc,$$

which is the determinant of the matrix M.

How to Approach Each Question (Summary)

Problem 1: Quotient Spaces and Induced Maps

(a) Recall that a quotient space V/U is formed by partitioning V into cosets of U, and W is a quotient if it is isomorphic to V/U for some U.

(b) The construction of V/U uses the surjective map $q : V \to V/U$ sending v to $v + U$, with kernel U.

(c) To show the induced map $\tilde{T} : V/\ker(T) \to \operatorname{im}(T)$ is an isomorphism, check that it is well-defined, linear, injective, and surjective.

Problem 2: Bases and Quotients

(a) A basis is a set that is linearly independent and spans the vector space.

(b) In finite dimensions, every vector space has a basis, and all bases have the same number of elements (the dimension).

(c) To show $\mathscr{B}_{V/U}$ is a basis for V/U, show it spans V/U and is linearly independent, using the properties of the quotient map and the way the basis is constructed.

Problem 3: Dual Spaces and Dual Bases

(a) The dual space V^* consists of all linear maps from V to $\mathbb{F}$.

(b) The dual basis is constructed by defining functionals that pick out coordinates with respect to the original basis; show these functionals form a basis for V^*.

(c) The natural isomorphism between $\mathbb{F}^n_{\text{row}}$ and $(\mathbb{F}^n_{\text{col}})^*$ is given by matrix multiplication (dot product).

Problem 4: Determinant as an Alternating Multilinear Map

(a) Define $\Lambda(\mathbb{F}^2_{\text{col}})$ as the space of bilinear, skew-symmetric maps from pairs of vectors to $\mathbb{F}$.

(b) Show that such a map is determined by its value on $((1, 0), (0, 1))$, so the space is 1-dimensional, and compute the explicit formula for the standard determinant.

(c) To find $d(M)$, compute how the determinant map transforms under a linear change of basis (matrix action), and relate this to the usual determinant formula.