Problem 1

Let V be an n-dimensional vector space, and let $\phi : V \to V$ be a linear map. Show that, for any n-linear antisymmetric form $\beta(v_1, \ldots, v_n)$ on n, we have

$$\beta(\phi(v_1), \ldots, \phi(v_n)) = \det(\phi)\beta(v_1, \ldots, v_n).$$

(This formalizes the following idea: any "unit of volume" on V, given by β, gets scaled by $\det(\phi)$ when we apply ϕ. In fact, this can be used as a definition of $\det(\phi)$: this way, some of its properties, such as independence of basis and multiplicativity become clear.)

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Then we can write

$$v_i = \sum_{j=1}^{n} a_{ij} e_j, \quad i = 1, \ldots, n,$$

for some scalars a_{ij}. Then we have

$$\begin{aligned}
\beta(\phi(v_1), \ldots, \phi(v_n)) &= \beta\left(\phi\left(\sum_{j=1}^{n} a_{1j} e_j\right), \ldots, \phi\left(\sum_{j=1}^{n} a_{nj} e_j\right)\right) \\
&= \beta\left(\sum_{j=1}^{n} a_{1j}\phi(e_j), \ldots, \sum_{j=1}^{n} a_{nj}\phi(e_j)\right) \\
&= \sum_{j_1, j_2, \ldots, j_n = 1}^{n} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \beta(\phi(e_{j_1}), \phi(e_{j_2}), \ldots, \phi(e_{j_n})) \\
&= \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \beta(\phi(e_{\sigma(1)}), \phi(e_{\sigma(2)}), \ldots, \phi(e_{\sigma(n)})) \\
&= \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \operatorname{sgn}(\sigma) \beta(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)) \\
&= \det(A)\beta(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)),
\end{aligned}$$

where $A = (a_{ij})$ is the matrix whose columns are the coordinates of v_i in the basis $\{e_1, \ldots, e_n\}$. Now, note that $\beta(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)$ is an n-linear antisymmetric form evaluated at the basis vectors $\{e_1, \ldots, e_n\}$ after applying ϕ. By the definition of determinant, we have

$$\beta(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)) = \det(\phi)\beta(e_1, e_2, \ldots, e_n).$$

Therefore, we conclude that

$$\beta(\phi(v_1), \ldots, \phi(v_n)) = \det(A)\det(\phi)\beta(e_1, e_2, \ldots, e_n).$$

Finally, since β is multilinear, we have

$$\beta(v_1, \ldots, v_n) = \det(A)\beta(e_1, e_2, \ldots, e_n).$$

Combining these results, we obtain

$$\beta(\phi(v_1), \ldots, \phi(v_n)) = \det(\phi)\beta(v_1, \ldots, v_n).$$

$\square$

Problem 2

Let V be the space of polynomials of degree at most n (over some field K; if you want, you can assume $K = \mathbb{R}$). Fix $a, b \in \mathbb{R}$ and consider the linear map

$$\phi : V \to V, \quad p(t) \mapsto p(at + b).$$

Compute $\det(\phi)$.

Proof. Let $\{1, t, t^2, \ldots, t^n\}$ be the standard basis of V. Then we have

$$\phi(t^k) = (at + b)^k = \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} t^j.$$

Thus, the matrix representation of ϕ with respect to this basis is given by

$$[\phi] = \begin{pmatrix} 1 & b & b^2 & \cdots & b^n \\ 0 & a & 2ab & \cdots & nab^{n-1} \\ 0 & 0 & a^2 & \cdots & \binom{n}{2} a^2 b^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a^n \end{pmatrix}.$$

This is an upper triangular matrix, and the determinant of an upper triangular matrix is the product of its diagonal entries. Therefore, we have

$$\det(\phi) = 1 \cdot a \cdot a^2 \cdot a^3 \cdots a^n = a^{\frac{n(n+1)}{2}}.$$

Hence, the determinant of the linear map ϕ is

$$\det(\phi) = a^{\frac{n(n+1)}{2}}.$$

$\square$

Problem 3

Let M be an $n \times n$ matrix (over some field). Let V be the space of $n \times n$ matrices. Consider the linear map $m_M : V \to V$ given by left multiplication by M:

$$A \mapsto MA.$$

Find $\det(m_M)$. (Of course, the answer depends on M.)

Proof. Let $\{E_{ij} \mid 1 \leq i, j \leq n\}$ be the standard basis of V, where E_{ij} is the matrix with a 1 in the (i, j)-th position and 0 elsewhere. Then we have

$$m_M(E_{ij}) = ME_{ij} = \text{the } j\text{-th column of } M \text{ placed in the } i\text{-th column of a zero matrix.}$$

Thus, the action of m_M on the basis elements can be described as follows:

$$m_M(E_{ij}) = \sum_{k=1}^{n} M_{ki} E_{kj}.$$

Therefore, the matrix representation of m_M with respect to this basis is given by a block matrix where each block corresponds to the multiplication by the columns of M. Specifically, the matrix representation can be viewed as an $n^2 \times n^2$ matrix where each block is an $n \times n$ matrix formed by the columns of M.

The determinant of this block matrix can be computed as follows:

$$\det(m_M) = (\det(M))^n,$$

since each of the n blocks contributes a factor of $\det(M)$ to the overall determinant.

Hence, we conclude that

$$\det(m_M) = (\det(M))^n.$$

$\square$

Problem 4

Let K be a field and V be a finite-dimensional vector space. Let

$$\gamma : V \times V \to K$$

be an antisymmetric bilinear form. Show that there exists $k \leq \frac{n}{2}$ and a basis

$$\{a1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_{n-2k}\}$$

of V such that

$$\gamma(a_i, b_i) = 1, \quad \gamma(b_i, a_i) = -1, \quad (i = 1, \ldots, k),$$

and γ vanishes on all other pairs of basis vectors. (When $n = 2k$, this is called a symplectic basis.)

Proof. We proceed by induction on the dimension n of the vector space V.
Base Case: If $n = 0$, the statement is trivially true with $k = 0$ and an empty basis. If $n = 1$, since γ is antisymmetric, we have $\gamma(v, v) = 0$ for any $v \in V$. Thus, we can take $k = 0$ and any basis $\{c_1\}$ of V.
Inductive Step: Assume the statement holds for all vector spaces of dimension less than n. We consider two cases:

i.) If γ is the zero form, then we can take $k = 0$ and any basis $\{c_1, \ldots, c_n\}$ of V.

ii.) If γ is not the zero form, there exist vectors $u, v \in V$ such that $\gamma(u, v) \neq 0$. We can scale u and v such that $\gamma(u, v) = 1$. Now, we can extend $\{u, v\}$ to a basis of V, say $\{u, v, w_1, \ldots, w_{n-2}\}$. Consider the subspace $W = \operatorname{span}\{w_1, \ldots, w_{n-2}\}$. The restriction of γ to W is still an antisymmetric bilinear form. By the inductive hypothesis, there exists a basis of W of the desired form with some $k' \leq \frac{n-2}{2}$. Adding u and v to this basis, we obtain a basis of V of the desired form with $k = k' + 1 \leq \frac{n}{2}$.

Thus, by induction, the statement holds for all finite-dimensional vector spaces V over the field K. □

Problem 5

Consider $\det(A)$ as a multivariable function of the entries of a real matrix A. Compute the directional derivative of this function at the point $A = I$ in the direction of some matrix B. (Equivalently, find the linear approximation for $f(t) = \det(I + tB)$ at $t = 0$.)

Proof. □

Problem 6

Let A be a square $n \times n$ matrix whose characteristic polynomial has n roots in K, counting with multiplicity. Consider the Jordan form of A: suppose that it consists of blocks J_{λ_i, n_i}, where λ_i is the eigenvalue and n_i is the size of the block.
Express the following invariants of A in terms of n_i and λ_i: its characteristic polynomial, its minimal polynomial, the dimension of eigenspace for each λ (this is called "the geometric multiplicity of an eigenvalue") and rank. (No explanation is required.)

Problem 7

Following up on the previous problem, let us go in the opposite direction: explain how to find λ_i and n_i from the data of $\operatorname{rk}(A - \lambda I)^k$ for all $\lambda \in K$ and $k > 0$. In particular, this implies that the Jordan form is unique.