Problem 1

Diagonalizability - geometric definition.
Let V be an n-dimensional vector space over a field $\mathbb{F}$ and let $T : V \to V$ a transformation.

(a) Write down the geometric definition (that we gave in class interms of direct sum decomposition of V) for when T is diagonalizable.

(b) Define what does it mean for $\lambda \in \mathbb{F}$ to be an eigenvalue of T. Denote $\mathrm{Spec}(T)$ the set of eigenvalues of T in $\mathbb{F}$. For each $\lambda \in \mathrm{Spec}(T)$, define the eigenspace V_λ. Show that the following are equivalent:

  (i) T is diagonalizable.
  (ii) $V = \bigoplus_{\lambda \in \mathrm{Spec}(T)} V_\lambda$.

(c) A linear transformation $P : V \to V$ is called a projector of $P^2 = P$. Show that any projector is diagonalizable.

1. We say that T is diagonalizable if there exists $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ distinct and subspaces $V_1, \dots, V_k < V$ such that

$$V = \bigoplus_{i=1}^{k} V_i,$$

and T preserves each V_i, and $T|_{V_i} = \lambda_i Id_{V_i}$ for all $i = 1, \dots, k$.

Problem 2

Diagonalizability - computational definition.

(a) Let V be an n-dimensional vector space over a field $\mathbb{F}$ and let $T : V \to V$ a linear transformation. Write down the computational definition (that we gave in class in terms of a basis $\mathscr{B}$ and the corresponding matrix $[T]_{\mathscr{B}}$) for when T is diagonalizable.

(b) For a matrix $A \in M_n(\mathbb{F})$, consider the linear transformation $T_A : \mathbb{F}^n \to \mathbb{F}^n$ given by $v \mapsto Av$. Show that the following are equivalent:

  (i) T_A is diagonalizable (in this case we also say that A is diagonalizable).
  (ii) There exists a diagonal matrix $D \in M_n(\mathbb{F})$ and an invertible matrix $C \in M_n(\mathbb{F})$ such that $C^{-1}AC = D$.

(c) Consider the operator $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

  (a) Show that T_A is diagonalizable
  (b) find its eigenvalues
  (c) find the direct sum decomposition for eigenspaces
  (d) find a basis of eigenvectors
  (e) find $D, C \in M_2(\mathbb{R})$ with D diagonal and C invertible such that $D = C^{-1}AC$.