

# Math/CS 714: Assignment 4

## Problem 1

**Beam-Warming method (4 points).** The Beam-Warming method for the linear advection equation  $u_t + au_x = 0$  is given by

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{a^2k^2}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n), \quad (1)$$

where  $U_j^n$  is the approximation of  $u(jh, nk)$ .

- Use Taylor series to show that this method is second-order accurate.
- For a given plane wave solution  $U_j^0 = e^{ijh\xi}$ , compute the amplification factor  $g(\xi)$ , and hence determine the stability restriction for this method.

a) For brevity, define  $U_j^n := U$ . Then we can Taylor expand  $U$  about  $(j, n)$  and we have

$$\begin{aligned} U_{j-1}^n &= U - hU_x + \frac{h^2}{2}U_{xx} \\ U_{j-2}^n &= U - 2hU_x + 2h^2U_{xx} \\ U_j^{n+1} &= U + kU_t + \frac{k^2}{2}U_{tt} + O(k^3) \end{aligned}$$

Substituting these into the Beam-Warming method gives

$$\begin{aligned} U + kU_t + \frac{k^2}{2}U_{tt} + O(k^3) &= U - \frac{ak}{2h} \left( 3U - 4 \left( U - hU_x + \frac{h^2}{2}U_{xx} \right) + (U - 2hU_x + 2h^2U_{xx}) \right) \\ &\quad + \frac{a^2k^2}{2h^2} \left( U - 2 \left( U - hU_x + \frac{h^2}{2}U_{xx} \right) + (U - 2hU_x + 2h^2U_{xx}) \right) + O(h^3) \\ &= U - \frac{ak}{2h} (3U - 4U + 4hU_x - 2h^2U_{xx} + U - 2hU_x + 2h^2U_{xx}) \\ &\quad + \frac{a^2k^2}{2h^2} (U - 2U + 2hU_x - h^2U_{xx} + U - 2hU_x + 2h^2U_{xx}) + O(h^3) \\ &= U - \frac{ak}{2h} (2hU_x) + \frac{a^2k^2}{2h^2} (h^2U_{xx}) + O(h^3) \\ &= U - akU_x + \frac{a^2k^2}{2}U_{xx} + O(h^3) \end{aligned}$$

To reiterate, we have

$$kU_t + \frac{k^2}{2}U_{tt} + O(k^3) = -akU_x + \frac{a^2k^2}{2}U_{xx} + O(h^3)$$

Dividing by  $k$  gives

$$U_t + \frac{k}{2}U_{tt} + O(k^2) = -aU_x + \frac{a^2k}{2}U_{xx} + O(h^2)$$

Then we can use the fact that  $u_t = -au_x$  and  $u_{tt} = a^2u_{xx}$ , and substitute for  $U_{tt} = a^2U_{xx}$  which gives

$$U_t + aU_x = O(k^2) + O(h^2)$$

Thus the Beam-Warming method is second-order accurate in both space and time.

- Given that  $U_j^0 = e^{ijh\xi}$ , we can calculate the amplification factor with  $g(\xi)e^{ijh\xi} = U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{a^2k^2}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$ . Plugging in the plane wave solution and letting  $\nu = \frac{ak}{h}$  we have:

$$\begin{aligned} g(\xi)e^{ijh\xi} &= e^{ijh\xi} - \frac{\nu}{2} \left( 3e^{ijh\xi} - 4e^{i(j-1)h\xi} + e^{i(j-2)h\xi} \right) + \frac{\nu^2}{2} \left( e^{ijh\xi} - 2e^{i(j-1)h\xi} + e^{i(j-1)h\xi} \right) \\ \implies g(\xi) &= \frac{e^{ijh\xi} - \frac{\nu}{2} (3e^{ijh\xi} - 4e^{i(j-1)h\xi} + e^{i(j-2)h\xi}) + \frac{\nu^2}{2} (e^{ijh\xi} - 2e^{i(j-1)h\xi} + e^{i(j-1)h\xi})}{e^{ijh\xi}} \\ &= 1 - \frac{\nu}{2} (3 - 4e^{-ih\xi} + e^{-2ih\xi}) + \frac{\nu^2}{2} (1 - 2e^{-ih\xi} + e^{-2ih\xi}) \end{aligned}$$

Then, noticing that  $e^x = 1 + x + \frac{x^2}{2} + \dots$  we have:

$$\begin{aligned}
g(\xi) &= 1 - \frac{\nu}{2} \left( 3 - 4 \left( 1 - ih\xi + \frac{(ih\xi)^2}{2} \right) + (1 - 2ih\xi + 2(ih\xi)^2) \right) + \frac{\nu^2}{2} \left( 1 - 2 \left( 1 - ih\xi + \frac{(ih\xi)^2}{2} \right) + (1 - 2ih\xi + 2(ih\xi)^2) \right) \\
&= 1 - \frac{\nu}{2} (0 + 2ih\xi - (ih\xi)^2) + \frac{\nu^2}{2} (0 + 0 + (ih\xi)^2) + O((h\xi)^3) \\
&= 1 - i\nu h\xi + \frac{\nu}{2} (h\xi)^2 + \frac{\nu^2}{2} (h\xi)^2 + O((h\xi)^3) \\
&= 1 - i\nu h\xi + \frac{\nu(1+\nu)}{2} (h\xi)^2 + O((h\xi)^3)
\end{aligned}$$

Then we can compute  $|g(\xi)|^2$ :

$$\begin{aligned}
|g(\xi)|^2 &= \left( 1 + \frac{\nu(1+\nu)}{2} (h\xi)^2 + O((h\xi)^3) \right)^2 + (-\nu h\xi + O((h\xi)^3))^2 \\
&= 1 + \nu(1+\nu)(h\xi)^2 + \nu^2 (h\xi)^2 + O((h\xi)^3) \\
&= 1 + \nu(1+2\nu)(h\xi)^2 + O((h\xi)^3)
\end{aligned}$$

For stability, we require that  $|g(\xi)|^2 \leq 1$  for all  $\xi$ . Therefore we need  $\nu(1+2\nu) \leq 0$ . This gives the stability restriction of  $-\frac{1}{2} \leq \nu \leq 0$ .

## Problem 2

**(9 points).** Dropping the last term in the Beam-Warming method from Eq. (1) gives

$$U_j^{n+1} = U_j^n - \frac{ak}{2h} (3U_j^n - 4U_{j-1}^n + U_{j-2}^n), \quad (2)$$

which corresponds to forward Euler method in time, and a second-order one-sided derivative in space. Define  $\nu = ak/h$ .

- Calculate the amplification factor  $g(\xi)$  for a plane wave solution  $U_j^0 = e^{ijh\xi}$ .
- Define  $A(\xi) = |g(\xi)|^2$  and calculate a Taylor series for  $A$  at  $\xi = 0$  up to second order. Using the Taylor series, explain why we consider the numerical scheme of Eq. (2) to be unstable regardless of the choice of timestep.
- Make two plots of  $A(\xi)$  for  $\nu = 1/100$  using two different axis ranges:
  - $0 \leq h\xi \leq 2\pi$  and  $0.91 \leq A \leq 1.01$ ,
  - $0 \leq h\xi \leq 0.17$  and  $1 - 10^{-6} \leq A \leq 1 + 10^{-6}$ .
- Write a program to simulate Eq. (2) on a periodic interval  $[0, 2\pi)$  using  $N = 40$  grid points and a grid spacing of  $h = 2\pi/N$ . Use the initial condition  $u = \exp(2 \sin x)$  and  $\nu = 1/100$ . Plot the solution for  $n = 0, 1000, 2000, 4000$ . Define the root mean squared value of the solution,

$$R(n) = \sqrt{\frac{1}{N} \sum_{j=0}^{N-1} (U_j^n)^2}. \quad (3)$$

Make a plot of  $R$  over the range from  $n = 0$  to  $n = 10000$ . You should find that  $R$  does not grow over time, indicating that the method is stable.

- Using the discrete Fourier transform, it can be shown that an arbitrary initial condition on the periodic interval can be written as

$$U_j^0 = \sum_{l=0}^{N-1} \alpha_l e^{ijlh} \quad (4)$$

for some constants  $\alpha_l$ . Write down an expression for the general solution  $U_j^n$ . Using your answer, explain why your result in part (d) does not contradict the result in part (b).

- We can calculate the amplification factor with  $g(\xi)e^{ijh\xi} = U_j^{n+1} = U_j^n - \frac{ak}{2h} (3U_j^n - 4U_{j-1}^n + U_{j-2}^n)$ . Plugging in the plane wave solution and simplifying as in the previous problem gives  $g(\xi) = 1 - \frac{\nu}{2} (3 - 4e^{-ih\xi} + e^{-2ih\xi})$ .

- b) Taking  $A(\xi) = |g(\xi)|^2$  and expanding in a Taylor series about  $\xi = 0$ , we can use the fact that  $e^x = 1 + x + \frac{x^2}{2} + \dots$  to find that

$$\begin{aligned} A(\xi) &= \left| 1 - \frac{\nu}{2}(3 - 4(1 - ih\xi + \frac{(ih\xi)^2}{2}) + (1 - 2ih\xi + 2(ih\xi)^2)) \right|^2 \\ &= \left| 1 - \frac{\nu}{2}(0 + 2ih\xi - (ih\xi)^2) \right|^2 \end{aligned}$$

Thus we have

$$\begin{aligned} A(\xi) &= \left( 1 + \frac{\nu}{2}(h\xi)^2 \right)^2 + (-\nu h\xi)^2 \\ &= 1 + \nu(h\xi)^2 + \frac{\nu^2}{4}(h\xi)^4 + \nu^2(h\xi)^2 \\ &= 1 + \nu(1 + \nu)(h\xi)^2 + O((h\xi)^4) \end{aligned}$$

Since  $\nu(1 + \nu) > 0$  for all  $\nu > 0$ , we have that  $A(\xi) > 1$  for sufficiently small but nonzero  $\xi$ . This indicates that the method is unstable regardless of the choice of timestep.

- c) In Figure 1 and Figure 2 we have the two plots of  $A(\xi)$  for  $\nu = \frac{1}{100}$ .

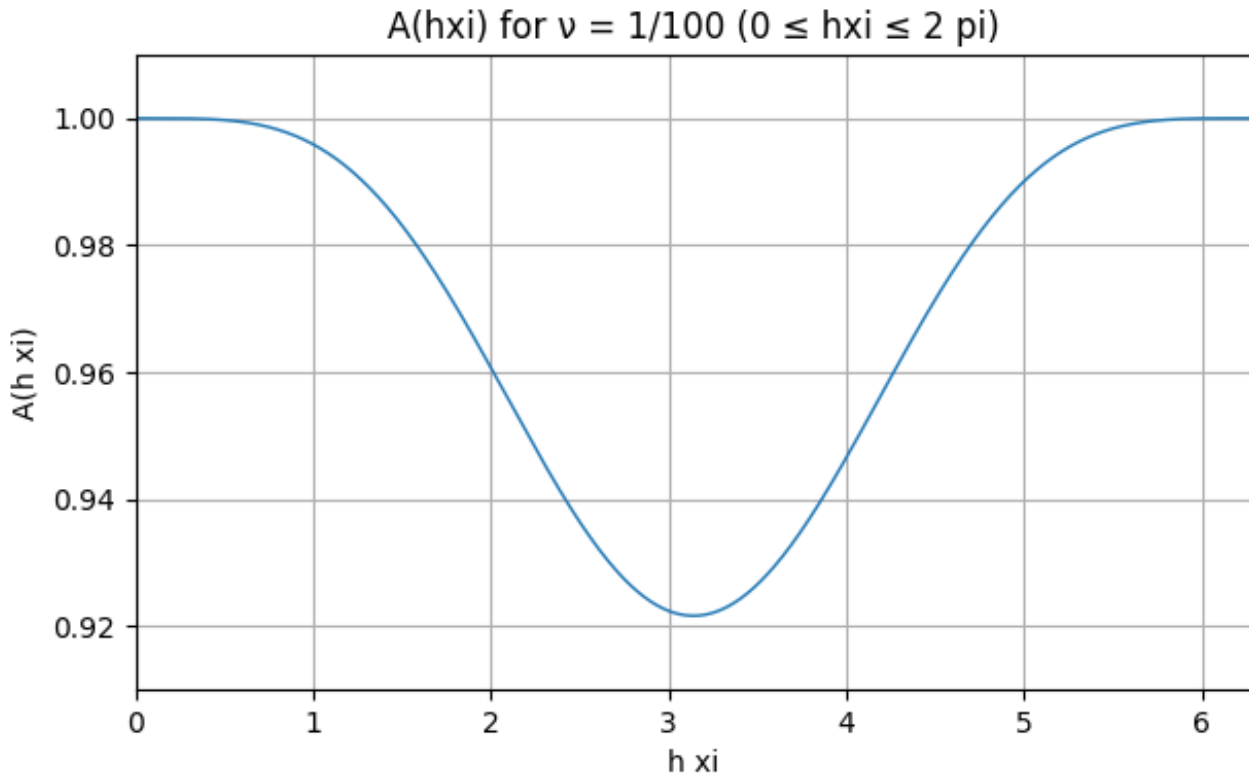


Figure 1:  $A(\xi)$  for  $0 \leq h\xi \leq 2\pi$  and  $0.91 \leq A \leq 1.01$ .

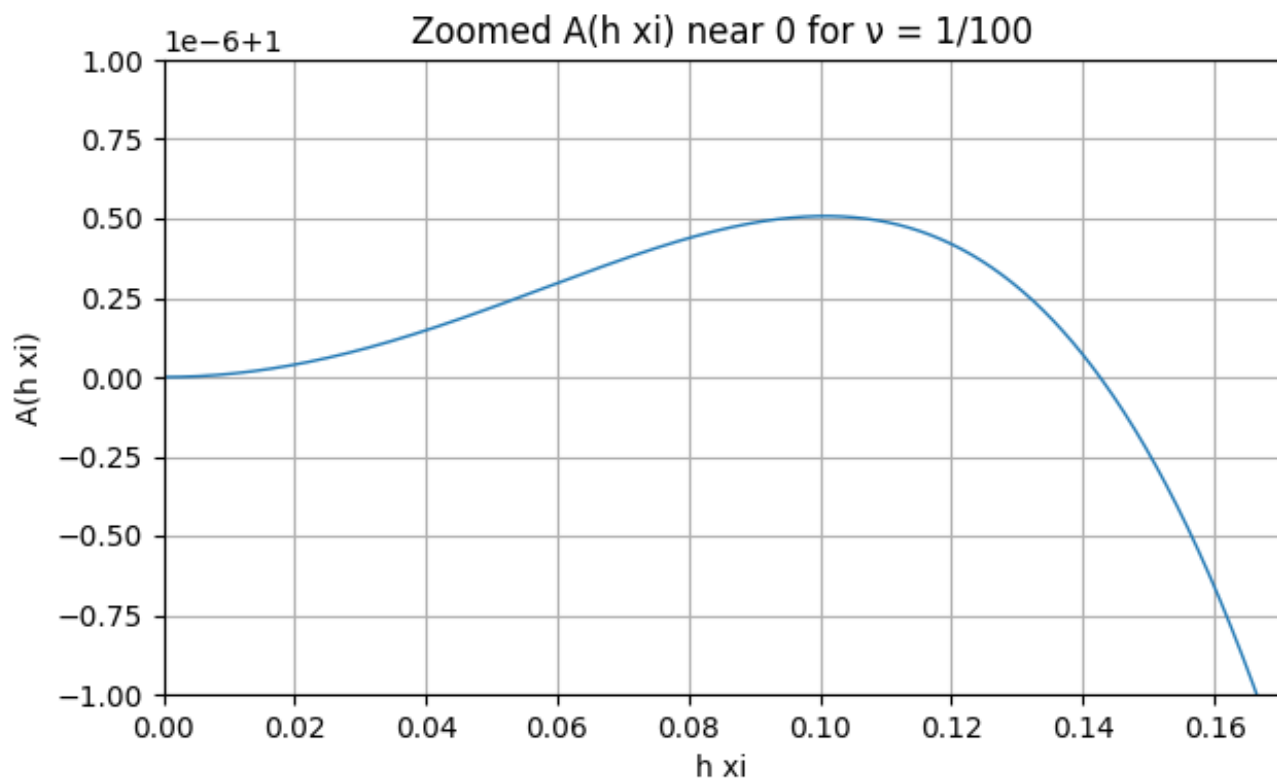


Figure 2:  $A(\xi)$  for  $0 \leq h\xi \leq 0.17$  and  $1 - 10^{-6} \leq A \leq 1 + 10^{-6}$ .

- d) See Figure 3 for the plots of the solution at  $n = 0, 1000, 2000, 4000$  and Figure 4 for the plot of  $R$  over the range from  $n = 0$  to  $n = 10000$ .

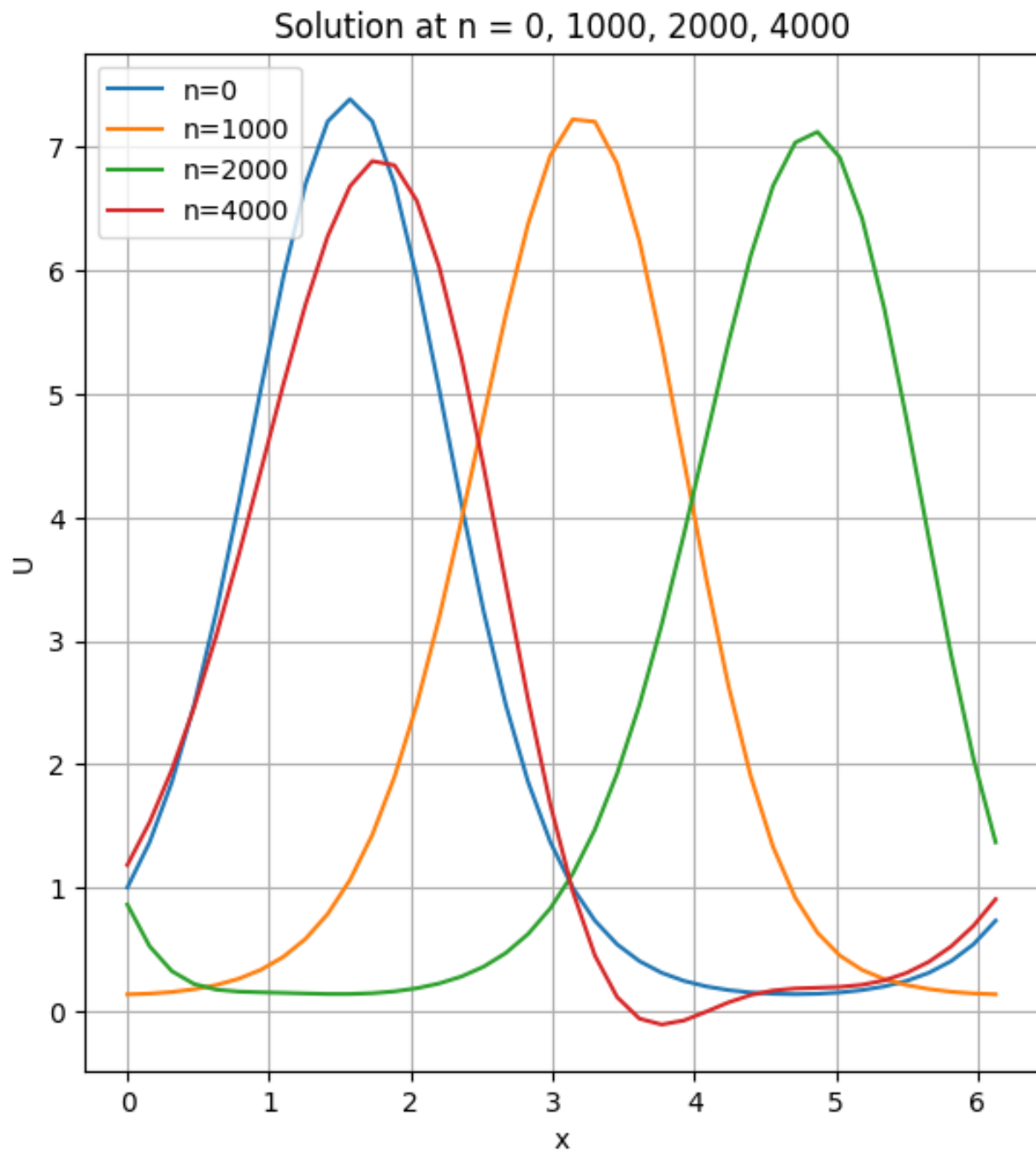


Figure 3: Plots of the solution at  $n = 0, 1000, 2000, 4000$ .

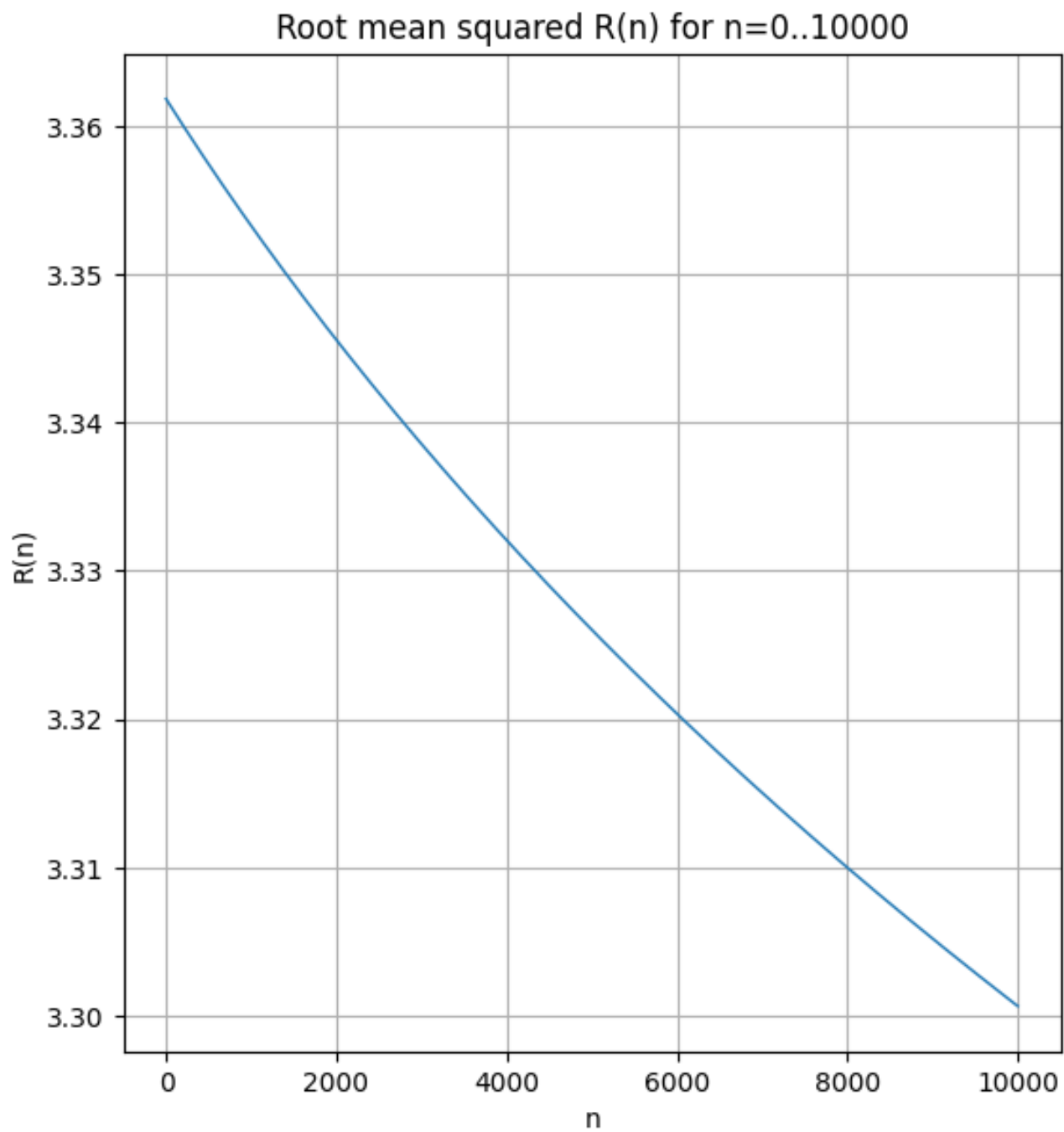


Figure 4: Plot of  $R$  over the range from  $n = 0$  to  $n = 10000$ .

### Problem 3

**Lax–Wendroff method (7 points).** Consider the hyperbolic conservation equation

$$q_t + [A(x)q]_x = 0 \quad (5)$$

for a function on  $q(x, t)$  on the periodic interval  $[0, 2\pi)$ . Let  $A(x) = 2 + \frac{4}{3} \sin x$ . Following the finite volume approach, divide the intervals into  $m$  domains  $\mathcal{C}_i$  of length  $h = \frac{2\pi}{m}$ , for  $i = \{0, 1, \dots, m-1\}$ . Let  $Q_i^n \approx q((i + 1/2)h, n\Delta t)$  be the discretized solution at the center of each  $\mathcal{C}_i$ . The generalized Lax–Wendroff scheme for this equation is given by

$$Q_i^{n+1} = Q_i - \frac{\Delta t}{h} \left[ \mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right] \quad (6)$$

where the fluxes are

$$\mathcal{F}_{i-1/2}^n = \frac{A_{i-1}Q_{i-1}^n + A_iQ_i^n}{2} - \frac{A_{i-1/2}\Delta t}{2h} [A_iQ_i^n - A_{i-1}Q_{i-1}^n]. \quad (7)$$

Here,  $A_i = A((i + 1/2)h)$  and  $A_{i-1/2} = A(ih)$ . It can be shown that the solution to Eq. (5) is time-periodic so that  $q(x, t + T) = q(x, t)$  where  $T = 3\pi/\sqrt{5}$ .

- The CFL condition requires that  $\Delta t \leq \frac{h}{c}$  for stability. What is  $c$  in this case?
- Implement Eq. (7) and set  $\Delta t = \frac{h}{3c}$ . Use the initial condition

$$q(x, 0) = \exp(\sin x + \frac{1}{2} \sin 4x). \quad (8)$$

For  $m = 512$ , plot snapshots of the solution for  $t = 0, \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}, T$ .<sup>a</sup>

- By considering a range of  $m$  (e.g. 256 and upward) with the initial condition in Eq. (8), calculate the  $L_2$  norm between the numerical solution at  $t = T$  and the exact answer. Determine the order of convergence.<sup>b</sup>
- Repeat parts (b) and (c) for the initial condition

$$q(x, 0) = \max\{\frac{\pi}{2} - |x - \pi|, 0\}. \quad (9)$$

- Optional.** By the considering the characteristics, or otherwise, derive the result that  $q$  is time-periodic with period  $T$ .

<sup>a</sup>Since multiples of  $\Delta t$  do not exactly match these snapshot times, you may need to make a small adjustment to the timestep.

<sup>b</sup>When determining the order of convergence, you are interested in the asymptotic properties of error as  $m$  gets large. You can ignore initial transients in error.

- The wave (characteristic) speed is  $A(x)$ , so

$$c = \max_{x \in [0, 2\pi)} |A(x)| = \max_x \left( 2 + \frac{4}{3} \sin x \right) = 2 + \frac{4}{3} = \frac{10}{3}.$$

(Here  $A(x) \geq 2 - \frac{4}{3} = \frac{2}{3} > 0$ , hence the maximum absolute value is  $10/3$ .) Thus the CFL condition is  $\Delta t \leq h/c = \frac{3h}{10}$ .