Problem 1

Cayley-Hamilton Theorem: Let $T : V \to V$ be a linear transformation on a finite-dimensional vector space V over a field $\mathbb{F}$, with the characteristic polynomial $p_T(x)$. Then, $p_T(T) = 0$.

(a) Show that the theorem is true for $T_U : \mathbb{F}^n \to \mathbb{F}^n$ where

$$U = \begin{pmatrix} \lambda_1 & * & \dots & * \\ 0 & \lambda_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Hint: Apply $p_{T_U}(U)$ on the standard basis vectors $e_1, \dots, e_n$.

(b) Show that the theorem is true if there exists a "Flag" in V invariant under T, i.e., a sequence of subspaces

$$\{0_V\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V,$$

such that $\dim(V_i) = i$ and $T(V_i) \subset V_i$ for all $1 \le i \le n$. *Hint:* Take a basis $\mathscr{B} = \{v_1, \dots, v_n\}$ of V with $\mathscr{B}_i = \{v_1, \dots, v_i\}$ a basis of V_i for all $1 \le i \le n$, and consider $[T]_{\mathscr{B}} = U$.

(c) Show by induction on $\dim V = n$ that such a flag exists for any linear transformation $T : V \to V$. The case $n = 1$ is trivial. Now assume the theorem holds for any linear transformation on a vector space of dimension less than n. Complete the inductions step in the following way:

1. Show that $p_T(x)$ has a root λ_1 and an associated eigenvector v_1. Construct the subspace $V_1 = \operatorname{span}\{v_1\}$ and note that it is T-invariant.
2. consider the induces linear transformation $\overline{T} : V/V_1 \to V/V_1$ and note that $\dim(V/V_1) = n - 1$. By the induction assumption, there is a flag of $\overline{T}$-invariant subspaces

$$\{0_{V/V_1}\} \subset \overline{V_2} \subset \overline{V_3} \subset \cdots \subset \overline{V_n} = V/V_1.$$

Recall, there is a canonical projection $\pi : V \to V/V_1$ and for any subset $S \subset V/V_1$, we have $\pi^{-1}(S) = \{v \in V : \pi(v) \in S\}$. Show that the sequence

$$\{0_V\} \subset V_1 \subset \pi^{-1}(\overline{V_2}) \subset \pi^{-1}(\overline{V_3}) \subset \cdots \subset \pi^{-1}(\overline{V_n}) = V$$

is a T-invariant flag in V.

(a) *Proof.* Let $p_{T_U}(x) = \prod_{i=1}^n (x - \lambda_i)$. Let $e_1, e_2, \dots, e_n$ be the standard basis vectors of $\mathbb{F}^n$. We will show that $p_{T_U}(U)(e_i) = 0$ for all $1 \le i \le n$. We proceed by induction on i.
Base Case: For $i = 1$, we have

$$p_{T_U}(U)(e_1) = \left(\prod_{j=1}^n (U - \lambda_j I) \right)(e_1).$$

Note that $(U - \lambda_1 I)(e_1) = 0$ since $U(e_1) = \lambda_1 e_1$. Therefore, $p_{T_U}(U)(e_1) = 0$.
Inductive Step: Assume that $p_{T_U}(U)(e_k) = 0$ for all $1 \le k < i$. We need to show that $p_{T_U}(U)(e_i) = 0$. We have

$$p_{T_U}(U)(e_i) = \left(\prod_{j=1}^n (U - \lambda_j I) \right)(e_i).$$

Note that $(U - \lambda_i I)(e_i) = 0$ since $U(e_i) = \lambda_i e_i +$ (linear combination of $e_1, \dots, e_{i-1}$). By the inductive hypothesis, applying the remaining factors $(U - \lambda_j I)$ for $j < i$ to this linear combination will yield zero. Thus, $p_{T_U}(U)(e_i) = 0$.
Therefore by induction, $p_{T_U}(U)(e_i) = 0$ for all $1 \le i \le n$. Since the e_i form a basis of $\mathbb{F}^n$, it follows that $p_{T_U}(U) = 0$. □

(b) *Proof.* Let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ be a basis of V such that $\mathscr{B}_i = \{v_1, v_2, \dots, v_i\}$ is a basis of V_i for all $1 \le i \le n$. Since

$T(V_i) \subset V_i$, the matrix representation of T with respect to the basis $\mathscr{B}$ is upper triangular:

$$[T]_{\mathscr{B}} = U = \begin{pmatrix} \lambda_1 & * & \dots & * \\ 0 & \lambda_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

By part (a), we know that $p_T(T) = p_{T_U}(U) = 0$. Therefore, the Cayley-Hamilton theorem holds for T. □

(c) *Proof.* We proceed by induction on $n = \dim V$.
Base Case: For $n = 1$, any linear transformation $T : V \to V$ trivially has a flag:

$$\{0_V\} \subset V.$$

Inductive Step: Assume that for any vector space W with $\dim W < n$, any linear transformation $S : W \to W$ has a S-invariant flag. We need to show that any linear transformation $T : V \to V$ with $\dim V = n$ also has a T-invariant flag.
1. Since $p_T(x)$ is a polynomial of degree n over the field $\mathbb{F}$, it has at least one root λ_1. Let v_1 be an associated eigenvector, i.e., $T(v_1) = \lambda_1 v_1$. Define the subspace $V_1 = \operatorname{span}\{v_1\}$. Note that V_1 is T-invariant since for any $c \in \mathbb{F}$,

$$T(cv_1) = cT(v_1) = c\lambda_1 v_1 \in V_1.$$

2. Consider the induced linear transformation $\overline{T} : V/V_1 \to V/V_1$ defined by

$$\overline{T}(v + V_1) = T(v) + V_1.$$

Note that $\dim(V/V_1) = n - 1$. By the induction hypothesis, there exists a flag of $\overline{T}$-invariant subspaces:

$$\{0_{V/V_1}\} \subset \overline{V_2} \subset \overline{V_3} \subset \cdots \subset \overline{V_n} = V/V_1.$$

Let $\pi : V \to V/V_1$ be the canonical projection. For each $2 \le i \le n$, define

$$V_i = \pi^{-1}(\overline{V_i}) = \{v \in V : \pi(v) \in \overline{V_i}\}.$$

We claim that the sequence

$$\{0_V\} \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n = V$$

is a T-invariant flag in V.
To see that each V_i is T-invariant, let $v \in V_i$. Then $\pi(v) \in \overline{V_i}$. Since $\overline{V_i}$ is $\overline{T}$-invariant, we have

$$\overline{T}(\pi(v)) = T(v) + V_1 \in \overline{V_i}.$$

Thus, $T(v) \in V_i$, showing that V_i is T-invariant.
Therefore, we have constructed a T-invariant flag in V. By induction, the result holds for all finite-dimensional vector spaces V. □