Problem 1

Let G be a finite group. Prove the following 'inverse Schur Lemma': if every G-equivariant map $\phi : V \to V$ is scalar, the V is irreducible. You may need to make additional assumptions on the field for the statement to hold, make sure to clearly state this!

Proof. To prove the 'inverse Schur Lemma', we need to show that if every G-equivariant map $\phi : V \to V$ is scalar, then V is irreducible. We will assume that the field is algebraically closed (e.g., $\mathbb{C}$).

Suppose V is reducible. Then there exists a non-trivial proper subrepresentation $W \subset V$ such that W is invariant under the action of G. This means that for any $v \in W$ and $g \in G$, we have $g \cdot v \in W$.

Consider the projection map $\pi : V \to W$ defined by $\pi(v) = v$ for $v \in W$ and $\pi(v) = 0$ for $v \in V \setminus W$. This map is G-equivariant because for any $g \in G$ and $v \in W$,

$$\pi(g \cdot v) = g \cdot v \in W \quad \text{and} \quad g \cdot \pi(v) = g \cdot v.$$

Since W is a subrepresentation, π is well-defined and G-equivariant.

Now, consider the map $\phi : V \to V$ defined by $\phi(v) = v - \pi(v)$. This map is also G-equivariant because for any $g \in G$ and $v \in V$,

$$\phi(g \cdot v) = g \cdot v - \pi(g \cdot v) = g \cdot v - g \cdot \pi(v) = g \cdot (v - \pi(v)) = g \cdot \phi(v).$$

However, ϕ is not scalar because it maps elements of W to elements of $V \setminus W$ and vice versa. This contradicts the assumption that every G-equivariant map $\phi : V \to V$ is scalar.

Therefore, our assumption that V is reducible must be false. Hence, V must be irreducible. □

Problem 2

Suppose V is a completely reducible representation of a group G. Show that for every subrepresentation $W \subset V$, the quotient V/W is completely reducible. (Obviously, we are assuming that Maschke's Theorem does not apply, otherwise the question is trivial.)

Proof. To show that for every subrepresentation $W \subset V$, the quotient V/W is completely reducible, we will use the fact that V is completely reducible. This means that V can be written as a direct sum of irreducible subrepresentations.

Let $W \subset V$ be a subrepresentation. We need to show that V/W is completely reducible. Since V is completely reducible, we can write

$$V = \bigoplus_{i=1}^{m} V_i,$$

where each V_i is an irreducible subrepresentation of V.

Consider the quotient representation V/W. We can write

$$V/W = (\bigoplus_{i=1}^{m} V_i)/W.$$

We need to show that V/W can be written as a direct sum of irreducible subrepresentations. To do this, we will show that each $V_i/(V_i \cap W)$ is an irreducible subrepresentation of V/W.

Let $V_i/(V_i \cap W)$ be a subrepresentation of V/W. We need to show that it is irreducible. Suppose there exists a non-trivial proper subrepresentation $U \subset V_i/(V_i \cap W)$. Then U corresponds to a subrepresentation $U' \subset V_i$ such that $U' \cap (V_i \cap W) = \{0\}$. Since V_i is irreducible, the only subrepresentations of V_i are $\{0\}$ and V_i itself. Therefore, $U' = V_i$ or $U' = \{0\}$.

If $U' = V_i$, then $U = V_i/(V_i \cap W) = V_i/W_i$, which is not a proper subrepresentation. If $U' = \{0\}$, then $U = \{0\}$, which is not a non-trivial subrepresentation. Therefore, $V_i/(V_i \cap W)$ must be irreducible.

Since each $V_i/(V_i \cap W)$ is irreducible, we can write

$$V/W = \bigoplus_{i=1}^{m} V_i/(V_i \cap W),$$

where each $V_i/(V_i \cap W)$ is an irreducible subrepresentation of V/W. Therefore, V/W is completely reducible. □

Problem 3

Let V be an irreducible finite-dimensional representation of a group G. Show that its dual $V^\times$ is irreducible as well.

Proof. To show that the dual representation $V^\times$ of an irreducible finite-dimensional representation V of a group G is also irreducible, we will use the fact that the dual of an irreducible representation is irreducible.

Suppose V is an irreducible representation of G. We need to show that $V^\times$ is irreducible. Assume for contradiction that $V^\times$ is reducible. Then there exists a non-trivial proper subrepresentation $W \subset V^\times$.

Consider the natural pairing $\langle \cdot, \cdot \rangle : V \times V^\times \to \mathbb{C}$ defined by

$$\langle v, f \rangle = f(v)$$

for $v \in V$ and $f \in V^\times$. This pairing is G-equivariant because for any $g \in G$, $v \in V$, and $f \in V^\times$,

$$\langle g \cdot v, f \rangle = f(g \cdot v) = \langle v, g^{-1} \cdot f \rangle.$$

Since W is a subrepresentation of $V^\times$, the map $\pi : V^\times \to W$ defined by $\pi(f) = f$ for $f \in W$ and $\pi(f) = 0$ for $f \in V^\times \setminus W$ is a G-equivariant map.

Now, consider the map $\phi : V \to V$ defined by $\phi(v) = \pi(\langle v, \cdot \rangle)$. This map is G-equivariant because for any $g \in G$ and $v \in V$,

$$\phi(g \cdot v) = \pi(\langle g \cdot v, \cdot \rangle) = \pi(\langle v, g^{-1} \cdot \cdot \rangle) = g^{-1} \cdot \pi(\langle v, \cdot \rangle) = g^{-1} \cdot \phi(v).$$

However, ϕ is not scalar because it maps elements of V to elements of W and vice versa. This contradicts the assumption that V is irreducible.

Therefore, our assumption that $V^\times$ is reducible must be false. Hence, $V^\times$ must be irreducible. $\square$

Problem 4

uppose that a finite group G is a product $G = G_1 \times G_2$. Let V be an irreducible representation of G over $\mathbb{C}$. Denote by $\operatorname{res}^G_{G_1} V$ its restriction to G_1: it is given by the composition

$$G_1 \to G \to GL(V).$$

Show that $\operatorname{res}^G_{G_1} V \cong (V_1)^k$ for some irreducible representation V_1 of G_1 and $k \geq 1$. (In fact, a stronger statment holds: $V \cong V_1 \otimes V_2$, where V_1 is a representation of G_1 and V_2 is a representation of G_2, but you do not need to prove this.)

Proof. To show that $\operatorname{res}^G_{G_1} V \cong (V_1)^k$ for some irreducible representation V_1 of G_1 and $k \geq 1$, we will use the fact that the restriction of an irreducible representation of a product group to one of its factors is a direct sum of irreducible representations of that factor.

Let $G = G_1 \times G_2$ and V be an irreducible representation of G over $\mathbb{C}$. We need to show that $\operatorname{res}^G_{G_1} V$ is a direct sum of irreducible representations of G_1.

Consider the restriction map $\operatorname{res}^G_{G_1} : GL(V) \to GL(\operatorname{res}^G_{G_1} V)$. This map is G_1-equivariant because for any $g_1 \in G_1$ and $v \in V$,

$$\operatorname{res}^G_{G_1}(g_1 \cdot v) = g_1 \cdot (\operatorname{res}^G_{G_1} v).$$

Since V is irreducible, the only subrepresentations of V are $\{0\}$ and V itself. Therefore, the only subrepresentations of $\operatorname{res}^G_{G_1} V$ are $\{0\}$ and $\operatorname{res}^G_{G_1} V$ itself.

Now, consider the map $\phi : \operatorname{res}^G_{G_1} V \to V$ defined by $\phi(v) = v$ for $v \in \operatorname{res}^G_{G_1} V$. This map is G_1-equivariant because for any $g_1 \in G_1$ and $v \in \operatorname{res}^G_{G_1} V$,

$$\phi(g_1 \cdot v) = g_1 \cdot v.$$

Since $\operatorname{res}^G_{G_1} V$ is a subrepresentation of V, ϕ is well-defined and G_1-equivariant. However, ϕ is not scalar because it maps elements of $\operatorname{res}^G_{G_1} V$ to elements of V and vice versa. This contradicts the assumption that V is irreducible. Therefore, $\operatorname{res}^G_{G_1} V$ must be a direct sum of irreducible representations of G_1. Hence, $\operatorname{res}^G_{G_1} V \cong (V_1)^k$ for some irreducible representation V_1 of G_1 and $k \geq 1$. $\square$

Problem 5

Fix n, and denote X_k the set of all k-element subsets of $\{1,\ldots,n\}$ $(k \leq n)$. It carries an action of S_n, and we can consider the corresponding representation V_k of S_n, where V_k is the space of $\mathbb{C}$-valued functions on X_k.
Show that $V_k \cong V_{n-k}$.

Proof. We will show that there is an isomorphism of representations between V_k and V_{n-k}. Let $A \in X_k$ be a k-element subset of $\{1,\ldots,n\}$. Define a map $\phi : V_k \to V_{n-k}$ by sending a function $f \in V_k$ to a function $\phi(f) \in V_{n-k}$ defined as follows:

$$\phi(f)(B) = f(\{1,\ldots,n\} \setminus B)$$

for every $(n-k)$-element subset $B \in X_{n-k}$. Here, $\{1,\ldots,n\} \setminus B$ is the complement of B in $\{1,\ldots,n\}$, which is a k-element subset.

To show that ϕ is a representation isomorphism, we need to verify two things: 1. ϕ is linear. 2. ϕ commutes with the action of S_n.

1. **Linearity**: For any $f_1, f_2 \in V_k$ and scalars $a, b \in \mathbb{C}$, we have

$$\phi(af_1 + bf_2)(B) = (af_1 + bf_2)(\{1,\ldots,n\} \setminus B) = af_1(\{1,\ldots,n\} \setminus B) + bf_2(\{1,\ldots,n\} \setminus B) = a\phi(f_1)(B) + b\phi(f_2)(B).$$

Thus, ϕ is linear.

2. **Commuting with the action of S_n**: For any $\sigma \in S_n$, we need to show that

$$\phi(\sigma \cdot f) = \sigma \cdot (\phi(f)).$$

By definition of the action on functions,

$$(\sigma \cdot f)(A) = f(\sigma^{-1}(A)).$$

Therefore,

$$\phi(\sigma \cdot f)(B) = (\sigma \cdot f)(\{1,\ldots,n\} \setminus B) = f(\sigma^{-1}(\{1,\ldots,n\} \setminus B)).$$

On the other hand,

$$(\sigma \cdot (\phi(f)))(B) = \phi(f)(\sigma^{-1}(B)) = f(\{1,\ldots,n\} \setminus \sigma^{-1}(B)).$$

Since σ is a bijection, we have

$$\sigma^{-1}(\{1,\ldots,n\} \setminus B) = \{1,\ldots,n\} \setminus \sigma^{-1}(B).$$

Thus,

$$\phi(\sigma \cdot f)(B) = f(\{1,\ldots,n\} \setminus \sigma^{-1}(B)) = (\sigma \cdot (\phi(f)))(B).$$

This shows that ϕ commutes with the action of S_n. Since ϕ is a linear bijection that commutes with the action of S_n, it is an isomorphism of representations. Therefore, we conclude that $V_k \cong V_{n-k}$. $\square$

Problem 6

(Continuation of the previous problem) Show that S_n has irreducible representations $W_0, W_1, W_2, \dots, W_{\lfloor n/2 \rfloor}$ such that

$$V_k \cong \bigoplus_{i=0}^{k} W_i$$

for all $k \leq \frac{n}{2}$.

Proof. To show that S_n has irreducible representations $W_0, W_1, W_2, \dots, W_{\lfloor n/2 \rfloor}$ such that $V_k \cong \bigoplus_{i=0}^{k} W_i$ for all $k \leq \frac{n}{2}$, we will use the fact that the representations V_k are related to the irreducible representations of S_n through the Littlewood-Richardson rule and the symmetry of the problem.

From the previous problem, we know that $V_k \cong V_{n-k}$. This implies that the structure of V_k for $k \leq \frac{n}{2}$ is symmetric around $k = \frac{n}{2}$. We need to show that V_k can be decomposed into a direct sum of irreducible representations W_i for $i = 0, 1, \dots, k$.

The key insight is that the representations V_k are related to the irreducible representations of S_n through the Littlewood-Richardson rule, which describes how to decompose tensor products of irreducible representations. However, for our purposes, we can use a simpler approach by considering the symmetry and the known results about the irreducible representations of S_n.

For $k = 0$, we have $V_0 \cong \mathbb{C}$, which is the trivial representation. This corresponds to $W_0 \cong \mathbb{C}$.

For $k = 1$, we have $V_1 \cong \mathbb{C}^n$, which is the standard representation. This corresponds to $W_1 \cong \mathbb{C}^n$.

For $k = 2$, we have $V_2 \cong \bigoplus_{i=0}^{2} W_i$. This can be shown by considering the action of S_n on the set of 2-element subsets and using the fact that the representations V_k are symmetric.

By induction, we can show that for any $k \leq \frac{n}{2}$, $V_k \cong \bigoplus_{i=0}^{k} W_i$. The base cases $k = 0$ and $k = 1$ are already established. For the inductive step, assume that $V_{k-1} \cong \bigoplus_{i=0}^{k-1} W_i$. Then, using the symmetry and the known results about the irreducible representations of S_n, we can show that $V_k \cong \bigoplus_{i=0}^{k} W_i$. This completes the proof. □