

Remarks:

- A) Definition is just a definition, there is no need to justify or explain it.
- B) Answers to questions with proofs should be written, as much as you can, in the following format:

- i) Statement
- ii) Main points that will appear in your proof
- iii) The actual proof

Answers to questions with computations should be written, as much as possible, in the following format:

- i) Statement and Result
- ii) Main points that will appear in your computation.
- iii) The actual computation

Problem 1

Vector Spaces. Suppose \mathbb{F} is a field.

- a) Define when we say that a vector space V over a field \mathbb{F} is *finite dimensional*.
- b) Consider the vector space

$$V = \mathbb{F}[x]$$

of all polynomials with coefficients in \mathbb{F} . Show that V is not finite dimensional.

- c) Suppose X is a finite set. Consider the vector space V , of all functions from X to \mathbb{F} ,

$$V = \mathbb{F}(X) := \{\text{all } f : X \rightarrow \mathbb{F}; \text{ s.t. } f \text{ is a function}\},$$

with the standard addition and multiplication by scalars from \mathbb{F} . Show that V is finite dimensional.

- a) We say that a vector space V over a field \mathbb{F} is finite dimensional if there exists $S \subseteq V$ such that $\#(S) < \infty$ and $\text{span}(S) = V$.
- b)
 - i) Statement: Show that $V = \mathbb{F}[x]$ is not finite dimensional.
 - ii) Main Points:
 - Suppose V is finite dimensional with a finite spanning set S .
 - Let m be the maximum degree of the polynomials in S and find a polynomial $q(x)$ with degree greater than m .
 - Show that $q(x)$ cannot be written as a linear combination of the polynomials in S , leading to a contradiction.
 - iii) Proof:

Proof. Suppose V is finite dimensional. Then there exists $S \subseteq V$ such that $\#(S) < \infty$ and $\text{span}(S) = V$. Let $S = \{p_1(x), p_2(x), \dots, p_n(x)\}$. Let $m = \max(\deg(p_i(x)))$ for $1 \leq i \leq n$. Then consider the polynomial $q(x) = x^{m+1}$. Since $\deg(q(x)) > m$, $q(x)$ cannot be written as a linear combination of the polynomials in S . This contradicts the fact that $\text{span}(S) = V$. Thus, V is not finite dimensional. \square

- c)
 - i) Statement: Show that $V = \mathbb{F}(X)$ is finite dimensional.
 - ii) Main Points:
 - Since X is finite, say $X = \{x_1, x_2, \dots, x_n\}$, consider the set of functions $\{\delta_{x_i}\}_{i=1}^n$, where

$$\delta_{x_i}(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- Show that $\{\delta_{x_i}\}_{i=1}^n$ forms a basis for V .

iii) Proof:

Proof. Since X is finite, say $X = \{x_1, x_2, \dots, x_n\}$, consider the set of functions $\{\delta_{x_i}\}_{i=1}^n$, where

$$\delta_{x_i}(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We will show that $\{\delta_{x_i}\}_{i=1}^n$ forms a basis for V . First, we show that they span V . Let $f \in V$. Then we can write

$$f(x) = \sum_{i=1}^n f(x_i) \delta_{x_i}(x)$$

for all $x \in X$. Thus, $\{\delta_{x_i}\}_{i=1}^n$ spans V . Next, we show that they are linearly independent. Suppose

$$\sum_{i=1}^n a_i \delta_{x_i}(x) = 0$$

for some scalars $a_i \in \mathbb{F}$. Evaluating at $x = x_j$, we get

$$a_j = 0$$

for all $j = 1, 2, \dots, n$. Thus, all $a_i = 0$, and $\{\delta_{x_i}\}_{i=1}^n$ are linearly independent. Therefore, $\{\delta_{x_i}\}_{i=1}^n$ forms a basis for V , and hence V is finite dimensional with $\dim(V) = n$. \square

Problem 2

Short exact sequences. Suppose U, V, W are three vector spaces over \mathbb{F} . Consider the following sequence of spaces and linear transformations between them:

$$0 \rightarrow U \xrightarrow{\iota} V \xrightarrow{\epsilon} W \rightarrow 0, \quad (1)$$

where $0 \rightarrow U$, are the obvious maps from the zero space into U , and from the space W onto the zero space, respectively.

- a) Define when we say that the sequence (1) is short exact sequence (s.e.s.).
- b) Given two subspaces $U, V < V$, such that $V = U \oplus W$, Show that there is a natural s.e.s. associated with the spaces of functions $U = \mathbb{F}(U)$, $V = \mathbb{F}(V)$ and $W = \mathbb{F}(Y \setminus X)$, where $Y \setminus X$ denotes set-minus, i.e., the set of elements which are in Y and are not in X .

Problem 3

Dimension. Denote by $\text{Vect } \mathbb{F}^{fd}$ the collection of finite-dimensional vector spaces over \mathbb{F} , with linear transformations between them.

- a) State the fact about uniqueness and existence of unique dimension function

$$\dim : \text{Vect}_{\mathbb{F}}^{fd} \rightarrow \mathbb{N},$$

that satisfies certain desired properties.

Def. For V finite dimensional, the integer $\dim(V)$ is called the dimension of V .

- b) Show that $\dim(M_n(\mathbb{F})) = n^2$.
- c) Suppose $1 + 1 \neq 0$ in \mathbb{F} . Consider the spaces $U = A_n(\mathbb{F})$, $V = M_n(\mathbb{F})$, $W = S_n(\mathbb{F})$, of anti-symmetric matrices ($A^T = -A$), all matrices, and symmetric matrices (satisfy $A^T = A$), respectively.
 - i) Show that, they form in a natural way a s.e.s.
 - ii) Deduce that $\dim(A_n(\mathbb{F})) = \frac{n(n-1)}{2}$ and $\dim(S_n(\mathbb{F})) = \frac{n(n+1)}{2}$.