Problem 1

How you might use a computational solution for diagonalization? Let V be an n-dimensional vector space over a field $\mathbb{F}$, and $\mathscr{B} = \{v_1, \dots, v_n\}$ be an ordered basis of V.

(a) Show that there is a unique matrix $[T]_{\mathscr{B}} \in M_n(\mathbb{F})$ such that

$$[T(v)]_{\mathscr{B}} = [T]_{\mathscr{B}}[v]_{\mathscr{B}}.$$

The matrix $[T]_{\mathscr{B}}$ is called the matrix representation of T with respect to the basis $\mathscr{B}$.

(b) Do the following:

1. Show that if $[v]_{\mathscr{B}}$ is an eigenvector of $[T]_{\mathscr{B}}$ with eigenvalue λ, then v is an eigenvector of T with eigenvalue λ.
2. Explain how from eigenbasis of $[T]_{\mathscr{B}}$ you can get an eigenbasis of T.

(c) Consider the space V, of complex valued functions on $\mathbb{Z}_4 = \{0, 1, 2, 3\}$,

$$V = \mathbb{C}(\mathbb{Z}_4),$$

and subspace

$$U = \mathrm{Span}(\{\delta_1, \delta_2\}),$$

where δ_1, δ_2 are the delta functions on 1, and 2, respectively.
We define the operator $T : V \to V$, given by

$$T[f](x) = \begin{cases} f(0) + f(3), & x = 0; \\ f(0) + f(2) + f(3), & x = 1; \\ f(0) + f(1) + f(3), & x = 2; \\ f(0) - f(3), & x = 3. \end{cases}$$

1. Show that T takes the vector space U to itself, *i.e.*, $T(u) \in U$ for every $u \in U$.
   Consider the operator
   $$\begin{cases} \overline{T} : V/U \to V/U, \\ \overline{T}(f + U) = T(f) + U. \end{cases}$$
2. Compute the matrix $A \in M_2(\mathbb{C})$
   $$A = [\overline{T}]_{\mathscr{B}},$$
   where $\mathscr{B}$ is the basis $\mathscr{B} = \{\delta_0 + U, \delta_3 + U\} \subset V/U$.
3. Find eigenvalues and corresponding eigenvectors of A.
4. Using the spectral (i.e., eigenvalues and eigenvectors) results you obtained in 3 above, compute eigenvalues and eigenvectors of $\overline{T}$.

(a) **Statement:** There exists a unique matrix $[T]_{\mathscr{B}} \in M_n(\mathbb{F})$ such that $[T(v)]_{\mathscr{B}} = [T]_{\mathscr{B}}[v]_{\mathscr{B}}$ for all $v \in V$.

Proof. Let $\mathscr{B} = \{v_1, \dots, v_n\}$ be an ordered basis of V. For each basis vector v_j, we can write

$$T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$$

for some unique scalars $a_{ij} \in \mathbb{F}$. Define the matrix $[T]_{\mathscr{B}} = (a_{ij})_{i,j=1}^{n}$, where a_{ij} is the i-th coordinate of $T(v_j)$ in the basis $\mathscr{B}$.

Now let $v \in V$ be arbitrary. Write $v = \sum_{j=1}^{n} c_j v_j$, so $[v]_{\mathscr{B}} = (c_1, \dots, c_n)^t$. Then

$$T(v) = T\left(\sum_{j=1}^{n} c_j v_j\right) = \sum_{j=1}^{n} c_j T(v_j) = \sum_{j=1}^{n} c_j \sum_{i=1}^{n} a_{ij} v_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} c_j\right) v_i.$$

Thus $[T(v)]_{\mathscr{B}} = \left(\sum_{j=1}^{n} a_{ij} c_j\right)_{i=1}^{n} = [T]_{\mathscr{B}}[v]_{\mathscr{B}}$.

For uniqueness, suppose $A \in M_n(\mathbb{F})$ also satisfies $[T(v)]_{\mathscr{B}} = A[v]_{\mathscr{B}}$ for all $v \in V$. In particular, for each basis vector v_j, we have $[T(v_j)]_{\mathscr{B}} = A[v_j]_{\mathscr{B}} = Ae_j$, where e_j is the j-th standard basis vector. This means the j-th column of A equals the j-th column of $[T]_{\mathscr{B}}$. Thus $A = [T]_{\mathscr{B}}$. $\square$

(b) 1. If $[v]_{\mathscr{B}}$ is an eigenvector of $[T]_{\mathscr{B}}$ with eigenvalue λ, then

$$[T(v)]_{\mathscr{B}} = [T]_{\mathscr{B}}[v]_{\mathscr{B}} = \lambda[v]_{\mathscr{B}} = [\lambda v]_{\mathscr{B}}.$$

Since the coordinate map $v \mapsto [v]_{\mathscr{B}}$ is an isomorphism, it follows that $T(v) = \lambda v$ with $v \neq 0$. Thus v is an eigenvector of T with eigenvalue λ.

2. Let $\{w_1, \ldots, w_n\}$ be an eigenbasis of $[T]_{\mathscr{B}}$ with $[T]_{\mathscr{B}} w_i = \lambda_i w_i$. Define $v_i \in V$ by $[v_i]_{\mathscr{B}} = w_i$ (equivalently $v_i = C_{\mathscr{B}}^{-1}(w_i)$ where $C_{\mathscr{B}}$ is the coordinate isomorphism). By part 1, $T(v_i) = \lambda_i v_i$. Because $C_{\mathscr{B}}$ is an isomorphism, $\{v_1, \ldots, v_n\}$ is a basis of V. Hence $\{v_i\}$ is an eigenbasis of T.

(c) 1. For $u = a\,\delta_1 + b\,\delta_2 \in U$,

$$T[u](0) = u(0)+u(3) = 0, \quad T[u](1) = u(0)+u(2)+u(3) = b, \quad T[u](2) = u(0)+u(1)+u(3) = a, \quad T[u](3) = u(0)-u(3) = 0.$$

Hence $T[u] = b\,\delta_1 + a\,\delta_2 \in U$. Thus U is T-invariant and $\overline{T} : V/U \to V/U$ is well defined by $\overline{T}(f+U) = T(f)+U$.

2. In V/U use the basis $\mathscr{B} = \{\delta_0 + U,\ \delta_3 + U\}$. Compute

$$T[\delta_0] = \delta_0 + \delta_1 + \delta_2 + \delta_3 \implies \overline{T}(\delta_0 + U) = (\delta_0 + \delta_3) + U,$$

$$T[\delta_3] = \delta_0 + \delta_1 + \delta_2 - \delta_3 \implies \overline{T}(\delta_3 + U) = (\delta_0 - \delta_3) + U.$$

Therefore, relative to $\mathscr{B}$,

$$A = [\overline{T}]_{\mathscr{B}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

3. The characteristic polynomial is $\chi_A(\lambda) = \lambda^2 - 2$, so the eigenvalues are $\lambda_\pm = \pm\sqrt{2}$. Corresponding eigenvectors can be taken as

$$\lambda = \sqrt{2}:\ v_+ = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}, \qquad \lambda = -\sqrt{2}:\ v_- = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix}.$$

4. Using that $A = [\overline{T}]_{\mathscr{B}}$, the eigenvalues of $\overline{T}$ are the eigenvalues of A, namely $\lambda_\pm = \pm\sqrt{2}$. To find eigenvectors in V/U, let

$$w_\alpha := (\delta_0 + \alpha\,\delta_3) + U \in V/U.$$

From the computations in (2),

$$\overline{T}(w_\alpha) = \overline{T}(\delta_0 + U) + \alpha\,\overline{T}(\delta_3 + U) = (\delta_0 + \delta_3) + \alpha(\delta_0 - \delta_3) + U = \big((1+\alpha)\delta_0 + (1-\alpha)\delta_3\big) + U.$$

The eigenvector equation $\overline{T}(w_\alpha) = \lambda\, w_\alpha$ is

$$(1+\alpha)\delta_0 + (1-\alpha)\delta_3 = \lambda(\delta_0 + \alpha\,\delta_3),$$

which gives the system

$$1 + \alpha = \lambda, \qquad 1 - \alpha = \lambda\alpha.$$

Eliminating λ using $\lambda = 1 + \alpha$ yields

$$1 - \alpha = \alpha(1+\alpha) \implies \alpha^2 + 2\alpha - 1 = 0 \implies \alpha = \sqrt{2} - 1 \ \text{ or } \ \alpha = -1 - \sqrt{2}.$$

The corresponding eigenvalues are $\lambda = 1+\alpha = \sqrt{2}$ and $\lambda = 1+\alpha = -\sqrt{2}$, respectively. Hence eigenvectors (cosets) of $\overline{T}$ are

$$w_+ = (\delta_0 + (\sqrt{2}-1)\delta_3) + U \quad (\lambda = \sqrt{2}), \qquad w_- = (\delta_0 - (1+\sqrt{2})\delta_3) + U \quad (\lambda = -\sqrt{2}).$$

Any nonzero scalar multiples in V/U of these representatives are also eigenvectors.

Problem 2

Conjugacy relation.
Let V be a finite-dimensional vector space over a field $\mathbb{F}$.

(a) Two operators (i.e., linear transformations) $S, T : V \to V$ are called conjugates if there is an invertible operator $R : V \to V$, such that $RSR^{-1} = T$. In this case we write $S \sim T$. Show that $\sim$ is an equivalence relation on the space $\mathrm{End}(V)$ of all operators from V to itself. For a given operator the collection of all linear transformation which equivalen to him, is called its conjugacy class.

(b) Suppose S, T are operators on V.

1. Show that if S and T are conjugate and T is diagonalizable then also S is diagonalizable.
   Definition. For $\lambda \in \mathrm{Spect}(T)$, eigenvalue, the dimension $m_\lambda = \dim(V_\lambda)$ is called the multiplicity of λ.
2. Suppose S, T, are diagonalizable. show that the following are equivalent:
   i) S and T have the same eigenvalues and multiplicity of each eigenvalue (i.e., the dimensions of the corresponding eigenspaces are the same for both operators).
   ii) S and T are conjugate.

   Remark. The meaning of the result obtained in 2 above is that, the equivalence class of a diagonalizable operator is completely described by its eigenvalues and their multiplicities.
3. Suppose $A \in M_n(\mathbb{F})$ is diagonalizable. Show that A is conjugate to its transpose A^t (this true in fact for every matrix A, but we do not yet know how to show this).

(a) **Statement:** The relation $\sim$ is an equivalence relation on $\mathrm{End}(V)$.

Proof. We verify the three properties of an equivalence relation:

Reflexivity: For any $T \in \mathrm{End}(V)$, take $R = I_V$ (the identity operator). Then R is invertible and $RTR^{-1} = I_V \circ T \circ I_V = T$. Thus $T \sim T$.

Symmetry: Suppose $S \sim T$. Then there exists an invertible operator R such that $RSR^{-1} = T$. Multiplying on the left by R^{-1} and on the right by R, we get $S = R^{-1}TR = R^{-1}T(R^{-1})^{-1}$. Since R^{-1} is invertible, we have $T \sim S$.

Transitivity: Suppose $S \sim T$ and $T \sim U$. Then there exist invertible operators R_1 and R_2 such that $R_1SR_1^{-1} = T$ and $R_2TR_2^{-1} = U$. Then

$$U = R_2TR_2^{-1} = R_2(R_1SR_1^{-1})R_2^{-1} = (R_2R_1)S(R_2R_1)^{-1}.$$

Since R_2R_1 is invertible, we have $S \sim U$.

Therefore, $\sim$ is an equivalence relation. □

(b) 1. **Statement:** If S and T are conjugate and T is diagonalizable, then S is diagonalizable.

Proof. Since $S \sim T$, there exists an invertible operator R such that $T = RSR^{-1}$, or equivalently, $S = R^{-1}TR$.
Since T is diagonalizable, there exists a basis $\mathscr{B} = \{v_1, \ldots, v_n\}$ of V consisting of eigenvectors of T with eigenvalues $\lambda_1, \ldots, \lambda_n$.
For each i, let $w_i = R^{-1}(v_i)$. Since R^{-1} is invertible, $\{w_1, \ldots, w_n\}$ is a basis of V. We claim that each w_i is an eigenvector of S:

$$S(w_i) = S(R^{-1}(v_i)) = R^{-1}TR(R^{-1}(v_i)) = R^{-1}T(v_i) = R^{-1}(\lambda_i v_i) = \lambda_i R^{-1}(v_i) = \lambda_i w_i.$$

Thus $\{w_1, \ldots, w_n\}$ is a basis of eigenvectors of S, so S is diagonalizable. □

2. **Statement:** For diagonalizable operators S, T, the following are equivalent:
   i) S and T have the same eigenvalues with the same multiplicities.
   ii) S and T are conjugate.

Proof. $(ii) \implies (i)$: Suppose S and T are conjugate via $T = RSR^{-1}$ for some invertible R. If v is an eigenvector of S with eigenvalue λ, then

$$T(R(v)) = RSR^{-1}(R(v)) = RS(v) = R(\lambda v) = \lambda R(v).$$

So $R(v)$ is an eigenvector of T with the same eigenvalue λ. Moreover, R restricts to an isomorphism from the eigenspace $V_\lambda(S)$ to $V_\lambda(T)$, so $\dim V_\lambda(S) = \dim V_\lambda(T)$. Thus the eigenvalues and multiplicities agree.

$(i) \implies (ii)$: Suppose S and T have the same eigenvalues $\lambda_1, \ldots, \lambda_k$ with the same multiplicities $m_1, \ldots, m_k$. Since both are diagonalizable, we can choose eigenbases $\{v_1, \ldots, v_n\}$ of S and $\{w_1, \ldots, w_n\}$ of T, where both bases are ordered so that the first m_1 vectors correspond to λ_1, the next m_2 to λ_2, etc.

Define $R : V \to V$ by $R(v_i) = w_i$ for each i, and extend linearly. Since R maps a basis to a basis, R is invertible. For each i, if v_i is an eigenvector of S with eigenvalue λ_j (for some j), then w_i is an eigenvector of T with the same eigenvalue λ_j. Thus

$$RSR^{-1}(w_i) = RS(v_i) = R(\lambda_j v_i) = \lambda_j R(v_i) = \lambda_j w_i = T(w_i).$$

Since RSR^{-1} and T agree on a basis, $RSR^{-1} = T$, so $S \sim T$. □

3. **Statement:** If $A \in M_n(\mathbb{F})$ is diagonalizable, then A is conjugate to A^t.

   Proof. Since A is diagonalizable, there exists an invertible matrix P such that $PAP^{-1} = D$, where D is diagonal. Taking transposes,
   $$(PAP^{-1})^t = D^t \quad \implies \quad (P^{-1})^t A^t P^t = D^t = D.$$
   Thus $A^t = P^t D (P^{-1})^t = P^t D (P^t)^{-1}$.
   Also, $A = P^{-1} D P$. Since both A and A^t are conjugate to the same diagonal matrix D, and conjugacy is an equivalence relation (by part (a)), we have $A \sim A^t$.
   Explicitly, we can write $A^t = (P^t P^{-1}) A (P^t P^{-1})^{-1}$, so $R = P^t P^{-1}$ gives the conjugacy. □

Problem 3

Projectors. Let V be a vector space over a field $\mathbb{F}$, and $P : V \to V$ an operator.

(a) Recall that:

1. We say that P is a <u>projector</u> if $P^2 = P$.
2. We say that P is a <u>projector onto a subspace $W \subset V$</u>, if image$(P) = W$, and for every $w \in W$, $P(w) = w$.

(b) Show that TFAE:

1. P is a projector.
2. there is a subspace $W \subset V$, such that P is a projector onto W.
3. there are subspaces $U, W \subset V$, such that $V = U \oplus W$, and $P = Pr_W$, the standard projection $Pr_W(u+w) = w$, for every $u \in U, w \in W$.
4. $V = \ker(P) \oplus \text{image}(P)$, and on image$(P)$, P acts as the identity operator.

(c) Consider the operator $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, given by the multiplication by matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Show that T_A is a projector, and onto what subspace.

Problem 4

Direct sum and projectors. Let V be a vector spac, and $U, W \subset V$.

(a) Define when V is a direct sum of U and W, denoted $V = U \oplus W$.

(b) Show that TFAE:

1. $V = U \oplus W$.
2. there exists a projector P_U, P_W, onto U and W, respectively, such that
   i.) $P_U \circ P_W = 0 = P_W \circ P_U$,
   ii.) $Id_V = P_U + P_W$.

(c) Suppose $T : V \to V$, linear transformation. Show that TFAE:

1. $V = V_\lambda \oplus V_\mu$, direct-sum of two eigenspaces, $\lambda \neq \mu$.
2. There are projectors $P_\lambda, P_\mu : V \to V$, such that,
   i.) $P_\lambda \circ P_\mu = 0 = P_\mu \circ P_\lambda$,
   ii.) $Id_V = P_\lambda + P_\mu$,
   iii.) $T = \lambda P_\lambda + \mu P_\mu$.

   Moreover, show that in this case

$$P_\lambda = \frac{1}{\lambda - \mu}(T - \mu Id_V), \quad P_\mu = \frac{1}{\mu - \lambda}(T - \lambda Id_V).$$

(a) **Definition:** V is a direct sum of U and W, written $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{0\}$. Equivalently, every $v \in V$ can be written uniquely as $v = u + w$ with $u \in U$, $w \in W$.

(b) (1) $\implies$ (2): If $V = U \oplus W$, define $P_U(v)$ and $P_W(v)$ by the unique decomposition $v = u + w$ with $u \in U$, $w \in W$, setting $P_U(v) = u$, $P_W(v) = w$. Then P_U, P_W are projectors onto U, W, $Id_V = P_U + P_W$, and $P_U \circ P_W = 0 = P_W \circ P_U$.

(2) $\implies$ (1): If $Id_V = P_U + P_W$, then for any v, $v = P_U v + P_W v \in U + W$. If $v \in U \cap W$, then $v = P_U v = P_U(P_W v) = 0$, so $U \cap W = \{0\}$. Hence $V = U \oplus W$.

(c) (1) $\implies$ (2): If $V = V_\lambda \oplus V_\mu$ with $\lambda \neq \mu$, define P_λ, P_μ as the projections along the complementary eigenspace. Then P_λ, P_μ are projectors with $Id_V = P_\lambda + P_\mu$, $P_\lambda P_\mu = 0 = P_\mu P_\lambda$, and since T acts as λ on V_λ and as μ on V_μ, we have

$$T = \lambda P_\lambda + \mu P_\mu.$$

Moreover, the polynomial formulas

$$P_\lambda = \frac{1}{\lambda - \mu}(T - \mu \, Id_V), \qquad P_\mu = \frac{1}{\mu - \lambda}(T - \lambda \, Id_V)$$

satisfy $P_\lambda^2 = P_\lambda$, $P_\mu^2 = P_\mu$, $P_\lambda P_\mu = P_\mu P_\lambda = 0$, $P_\lambda + P_\mu = Id_V$, and agree with the geometric projections because on V_λ they act as $1, 0$ and on V_μ as $0, 1$.

(2) $\implies$ (1): From $Id_V = P_\lambda + P_\mu$ and $P_\lambda P_\mu = 0 = P_\mu P_\lambda$, by part (b) we get $V = \text{image}(P_\lambda) \oplus \text{image}(P_\mu)$. Also

$$TP_\lambda = (\lambda P_\lambda + \mu P_\mu)P_\lambda = \lambda P_\lambda, \qquad TP_\mu = (\lambda P_\lambda + \mu P_\mu)P_\mu = \mu P_\mu,$$

so $\text{image}(P_\lambda) \subseteq V_\lambda$ and $\text{image}(P_\mu) \subseteq V_\mu$. If $v \in V_\lambda$, then

$$\lambda v = Tv = (\lambda P_\lambda + \mu P_\mu)v = \lambda P_\lambda v + \mu P_\mu v$$

and since $v = P_\lambda v + P_\mu v$, we get $(\lambda - \mu)P_\mu v = 0$, hence $P_\mu v = 0$ and $v = P_\lambda v \in \text{image}(P_\lambda)$. Thus $\text{image}(P_\lambda) = V_\lambda$ and similarly $\text{image}(P_\mu) = V_\mu$, proving $V = V_\lambda \oplus V_\mu$.