Problem 1

The discrete Fourier transform. Let V and W be two n-dimensional vector spaces over a field F, and $T : V \to W$ a linear transformation.

(a) Define when we say that T is invertible.

(b) Suppose V and W are finite dimensional. Show that TFAE:

1. T is invertible.
2. T maps a basis $\mathscr{B}$ of V to a basis $\mathscr{C} = \{T(v); v \in \mathscr{B}\}$ for W.

(c) Let $N > 1$, be an integer, and consider the set $\mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\}$, with the addition and multipication is defined modulo N. Inside the vector space $\mathscr{H} = \mathbb{C}(\mathbb{Z}_N)$ of all functions from $\mathbb{Z}_N$ to $\mathbb{C}$, consider the subset

$$\mathscr{D} = \{\delta_t : t \in \mathbb{Z}_N\},$$

where δ_t is the delta function at t, $\delta_t(s) = 1$ if $s = t$, and 0 otherwise, and consider the subset

$$\mathscr{E} = \{e_w : w \in \mathbb{Z}_N\},$$

where e_w is the function given by

$$e_w(s) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} ws}, \quad s \in \mathbb{Z}_N.$$

1. Show that $\mathscr{E}$ is a basis for $\mathscr{H}$. You can do it using the following facts.
   - The dimenson of $\mathscr{H}$ is N.
   - The elements of $\mathscr{E}$ are linearly independent. To show this you can use the fact that on $\mathscr{H}$ we have the so called "inner product"

     $$\langle , \rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C},$$

     given by

     $$\langle f, g \rangle = \sum_{s \in \mathbb{Z}_N} f(s)\overline{g(s)},$$

     where $\overline{g(s)}$ is the complex conjugate of $g(s)$. Then we have,
     Fact. The collection $\mathscr{E}$ satisfies

     $$\langle e_w, e_{w'} \rangle = \begin{cases} 1, & w = w' \\ 0, & w \neq w'. \end{cases}$$

     In particular, using the fact that $\langle , \rangle$ is linear in the first coordinate, i.e., $\langle f + f', g \rangle = \langle f, g \rangle + \langle f', g \rangle$ and $\langle af, g \rangle = a\langle f, g \rangle$ for every $f, f' \in \mathscr{H}, a \in \mathbb{C}$, it is easy to show the linear independency of $\mathscr{E}$.
2. The operator $F_N : \mathscr{H} \to \mathscr{H}$ that is given by

   $$F_N[\delta_t] = e_{-t}$$

   is called the discrete Fourier transform modulo N. For $f \in \mathscr{H}$, denote $\hat{f} = F_N(f)$. Show that
   1. we have the formula

      $$\hat{f}(w) = \frac{1}{\sqrt{N}} \sum_{t \in \mathbb{Z}_N} f(t) e^{-\frac{2\pi i}{N} wt},$$

      for $w \in \mathbb{Z}_N$
3. The operator F_N is invertible.

(a) We say that T is invertible if there exists a linear transformation $S : W \to V$ such that $S \circ T = I_V$ and $T \circ S = I_W$, where I_V and I_W are the identity transformations on V and W, respectively.

(b) *Proof.* ($1 \implies 2$) Suppose T is invertible. Then there exists a linear transformation $S : W \to V$ such that $S \circ T = I_V$ and $T \circ S = I_W$. Let $\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. We will show that $\mathscr{C} = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W. First, we show that $\mathscr{C}$ spans W. Let $w \in W$. Since S is a linear transformation from W to V, we can write

$S(w)$ as a linear combination of the basis vectors in $\mathscr{B}$:

$$S(w) = a_1v_1 + a_2v_2 + \ldots + a_nv_n,$$

for some scalars $a_1, a_2, \ldots, a_n \in F$. Applying T to both sides, we get:

$$T(S(w)) = T(a_1v_1 + a_2v_2 + \ldots + a_nv_n) = a_1T(v_1) + a_2T(v_2) + \ldots + a_nT(v_n).$$

But since $T \circ S = I_W$, we have $T(S(w)) = w$. Therefore, w can be expressed as a linear combination of the vectors in $\mathscr{C}$, showing that $\mathscr{C}$ spans W.

Next, we show that $\mathscr{C}$ is linearly independent. Suppose there exist scalars $b_1, b_2, \ldots, b_n \in F$ such that:

$$b_1T(v_1) + b_2T(v_2) + \ldots + b_nT(v_n) = 0.$$

Applying S to both sides, we get:

$$S(b_1T(v_1) + b_2T(v_2) + \ldots + b_nT(v_n)) = b_1S(T(v_1)) + b_2S(T(v_2)) + \ldots + b_nS(T(v_n)) = 0.$$

But since $S \circ T = I_V$, we have $S(T(v_i)) = v_i$ for each i. Therefore, we have:

$$b_1v_1 + b_2v_2 + \ldots + b_nv_n = 0.$$

Since $\mathscr{B}$ is a basis for V, the vectors $v_1, v_2, \ldots, v_n$ are linearly independent. Thus, the only solution to the above equation is $b_1 = b_2 = \ldots = b_n = 0$. This shows that $\mathscr{C}$ is linearly independent.

Since $\mathscr{C}$ spans W and is linearly independent, it is a basis for W.

($2 \implies 1$) Suppose T maps a basis $\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$ of V to a basis $\mathscr{C} = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ for W. We will show that T is invertible by constructing an inverse linear transformation $S : W \to V$. Since $\mathscr{C}$ is a basis for W, every vector $w \in W$ can be uniquely expressed as a linear combination of the vectors in $\mathscr{C}$:

$$w = c_1T(v_1) + c_2T(v_2) + \ldots + c_nT(v_n),$$

for some scalars $c_1, c_2, \ldots, c_n \in F$. We define the linear transformation $S : W \to V$ by:

$$S(w) = c_1v_1 + c_2v_2 + \ldots + c_nv_n.$$

We need to show that $S \circ T = I_V$ and $T \circ S = I_W$.

First, we show that $S \circ T = I_V$. Let $v \in V$. Since $\mathscr{B}$ is a basis for V, we can express v as a linear combination of the basis vectors:

$$v = a_1v_1 + a_2v_2 + \ldots + a_nv_n,$$

for some scalars $a_1, a_2, \ldots, a_n \in F$. Applying T to both sides, we get:

$$T(v) = a_1T(v_1) + a_2T(v_2) + \ldots + a_nT(v_n).$$

Now, applying S to both sides, we have:

$$S(T(v)) = S(a_1T(v_1) + a_2T(v_2) + \ldots + a_nT(v_n)) = a_1S(T(v_1)) + a_2S(T(v_2)) + \ldots + a_nS(T(v_n)).$$

By the definition of S, we have $S(T(v_i)) = v_i$ for each i. Therefore, we get:

$$S(T(v)) = a_1v_1 + a_2v_2 + \ldots + a_nv_n = v.$$

This shows that $S \circ T = I_V$.

Since inverses are unique, we conclude that T is invertible. □

(c) 1. We show that $\mathscr{E}$ is a basis for $\mathscr{H}$.

Proof. Since the dimension of $\mathscr{H}$ is N, it suffices to show that the elements of $\mathscr{E}$ are linearly independent. Suppose there exist scalars $a_w \in \mathbb{C}$ for each $w \in \mathbb{Z}_N$ such that:

$$\sum_{w \in \mathbb{Z}_N} a_w e_w = 0.$$

Taking the inner product of both sides with $e_{w'}$ for some fixed $w' \in \mathbb{Z}_N$, we get:

$$\left\langle \sum_{w \in \mathbb{Z}_N} a_w e_w, e_{w'} \right\rangle = 0.$$

By linearity of the inner product, this becomes:

$$\sum_{w \in \mathbb{Z}_N} a_w \langle e_w, e_{w'} \rangle = 0.$$

Using the fact that $\langle e_w, e_{w'} \rangle = 1$ if $w = w'$ and 0 otherwise, we have:

$$a_{w'} = 0.$$

Since this holds for every $w' \in \mathbb{Z}_N$, we conclude that all coefficients a_w must be zero. Therefore, the elements of $\mathscr{E}$ are linearly independent, and hence $\mathscr{E}$ is a basis for $\mathscr{H}$. □

2. 1. To show the formula for $\hat{f}(w)$, we start with the definition of the discrete Fourier transform:

$$\hat{f} = F_N(f) = F_N\left(\sum_{t \in \mathbb{Z}_N} f(t)\delta_t\right) = \sum_{t \in \mathbb{Z}_N} f(t)F_N(\delta_t) = \sum_{t \in \mathbb{Z}_N} f(t)e_{-t}.$$

   Evaluating $\hat{f}$ at $w \in \mathbb{Z}_N$, we have:

$$\hat{f}(w) = \sum_{t \in \mathbb{Z}_N} f(t)e_{-t}(w) = \sum_{t \in \mathbb{Z}_N} f(t)\frac{1}{\sqrt{N}}e^{-\frac{2\pi i}{N}wt} = \frac{1}{\sqrt{N}}\sum_{t \in \mathbb{Z}_N} f(t)e^{-\frac{2\pi i}{N}wt}.$$

   This proves the desired formula.

   2. *Proof.* To show that F_N is invertible, we will construct an inverse operator $F_N^{-1} : \mathscr{H} \to \mathscr{H}$. We define F_N^{-1} on the basis elements e_w as follows:

$$F_N^{-1}(e_w) = \delta_{-w}.$$

   We extend F_N^{-1} linearly to all of $\mathscr{H}$. Now, we need to verify that $F_N^{-1} \circ F_N = I_{\mathscr{H}}$ and $F_N \circ F_N^{-1} = I_{\mathscr{H}}$. First, we show that $F_N^{-1} \circ F_N = I_{\mathscr{H}}$. Let $f \in \mathscr{H}$. Then:

$$F_N^{-1}(F_N(f)) = F_N^{-1}\left(\sum_{t \in \mathbb{Z}_N} f(t)e_{-t}\right) = \sum_{t \in \mathbb{Z}_N} f(t)F_N^{-1}(e_{-t}) = \sum_{t \in \mathbb{Z}_N} f(t)\delta_t = f.$$

   Next, we show that $F_N \circ F_N^{-1} = I_{\mathscr{H}}$. Let $g \in \mathscr{H}$. Then:

$$F_N(F_N^{-1}(g)) = F_N\left(\sum_{w \in \mathbb{Z}_N} g(w)\delta_{-w}\right) = \sum_{w \in \mathbb{Z}_N} g(w)F_N(\delta_{-w}) = \sum_{w \in \mathbb{Z}_N} g(w)e_w = g.$$

   Since both compositions yield the identity operator on $\mathscr{H}$, we conclude that F_N is invertible. □

Problem 2

Diagonalization. Let T be an operator on a vector space V over a field $\mathbb{F}$.

(a) We say that

1. a scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if
2. a vector $v \in V, v \neq 0$ is an eigenvector, with eigenvalue $\lambda \in \mathbb{F}$, if

(b) Show that if $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ is an ordered basis of V, consisting of eigenvectors of T, then there exists scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{F}$ such that

$$[T]_{\mathscr{B}} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Remark. The process (if possible) of finding a basis $\mathscr{B}$ of V consisting of eigenvectors of T, and the corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, is called a diagonalization of T.

(c) Consider the space $\mathscr{H} = \mathbb{C}(\mathbb{Z}_3)$.

1. we have an operator called time shift

$$\begin{cases} L : \mathscr{H} \to \mathscr{H}, \\ L[f](t) = f(t-1), \end{cases}$$

for every $f \in \mathscr{H}, t \in \mathbb{Z}_3$. Find a diagonalization of L, and write the corresponding diagonal matrix

$$D = [L]_{\mathscr{B}}.$$

2. in addition, we have an operator called frequency shift

$$\begin{cases} M : \mathscr{H} \to \mathscr{H}, \\ M[f](t) = e^{\frac{2\pi i}{3} t} f(t), \end{cases}$$

for every $f \in \mathscr{H}, t \in \mathbb{Z}_3$. Find a diagonalization of M, and write the corresponding diagonal matrix

$$D = [M]_{\mathscr{B}}.$$

(a) 1. A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if there exists a non-zero vector $v \in V$ such that $T(v) = \lambda v$.
2. A vector $v \in V, v \neq 0$ is an eigenvector, with eigenvalue $\lambda \in \mathbb{F}$, if $T(v) = \lambda v$.

(b) *Proof.* Let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ be an ordered basis of V, consisting of eigenvectors of T. By definition of eigenvectors, we have:

$$T(v_i) = \lambda_i v_i,$$

for some scalars $\lambda_i \in \mathbb{F}$ and for each $i = 1, 2, \dots, n$. To find the matrix representation of T with respect to the basis $\mathscr{B}$, we compute the action of T on each basis vector and express the result in terms of the basis vectors. The j-th column of the matrix $[T]_{\mathscr{B}}$ corresponds to the coefficients of $T(v_j)$ expressed in terms of the basis $\mathscr{B}$. Since $T(v_j) = \lambda_j v_j$, we have:

$$[T]_{\mathscr{B}} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Thus, the matrix representation of T with respect to the basis $\mathscr{B}$ is a diagonal matrix with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ on the diagonal. $\square$

(c) 1. To find a diagonalization of the time shift operator L, we first identify its eigenvalues and eigenvectors. The operator L shifts the input function by one unit to the left. We can represent the action of L on the standard basis functions $\delta_0, \delta_1, \delta_2$ of $\mathscr{H}$ as follows:

$$L[\delta_0] = \delta_2, \quad L[\delta_1] = \delta_0, \quad L[\delta_2] = \delta_1.$$

To find the eigenvalues and eigenvectors, we look for solutions to the equation $L[f] = \lambda f$. By solving this equation, we find that the eigenvalues of L are the cube roots of unity: $1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$. The corresponding eigenvectors can be constructed as linear combinations of the basis functions. After finding the eigenvectors, we can form a basis $\mathscr{B}$ consisting of these eigenvectors. The diagonal matrix $D = [L]_{\mathscr{B}}$ will then have the eigenvalues on its diagonal:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}.$$

2. Similarly, for the frequency shift operator M, we analyze its action on the standard basis functions:

$$M[\delta_0] = \delta_0, \quad M[\delta_1] = e^{\frac{2\pi i}{3}}\delta_1, \quad M[\delta_2] = e^{\frac{4\pi i}{3}}\delta_2.$$

The eigenvalues of M are also the cube roots of unity: $1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$. The corresponding eigenvectors can be constructed similarly. After finding the eigenvectors, we can form a basis $\mathscr{B}$ consisting of these eigenvectors. The diagonal matrix $D = [M]_{\mathscr{B}}$ will then have the eigenvalues on its diagonal:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}.$$

Problem 3

Heisenberg's commutation relations. consider the vector space $\mathscr{H} = \mathbb{C}(\mathbb{Z}_N)$.

(a) For every $\tau \in \mathbb{Z}_N$, we have an operator $L_\tau : \mathscr{H} \to \mathscr{H}$, called time shift, given by

$$L_\tau[f](t) = \text{-----}$$

and, for every $\omega \in \mathbb{Z}_N$, we have an operator $M_\omega : \mathscr{H} \to \mathscr{H}$, called frequency shift, given by

$$M_\omega[f](t) = \text{-----}$$

(b) Show that $M_\omega \circ L_\tau = e^{-\frac{2\pi i}{N}\omega\tau} L_\tau \circ M_\omega$, for every $\tau, \omega \in \mathbb{Z}_N$.

(c) Show that for every $\tau, \omega \in \mathbb{Z}_N$,

$$L_\tau \circ F_N = F_N \circ M_\tau,$$

where F_N is the discrete Fourier transform modulo N described in Problem 1.

(a) For every $\tau \in \mathbb{Z}_N$, we have an operator $L_\tau : \mathscr{H} \to \mathscr{H}$, called time shift, given by

$$L_\tau[f](t) = f(t - \tau),$$

and, for every $\omega \in \mathbb{Z}_N$, we have an operator $M_\omega : \mathscr{H} \to \mathscr{H}$, called frequency shift, given by

$$M_\omega[f](t) = e^{\frac{2\pi i}{N}\omega t} f(t).$$

(b) *Proof.* To show that $M_\omega \circ L_\tau = e^{-\frac{2\pi i}{N}\omega\tau} L_\tau \circ M_\omega$, we compute both sides. Let $f \in \mathscr{H}$ and $t \in \mathbb{Z}_N$.

$$(M_\omega \circ L_\tau)[f](t) = M_\omega[L_\tau[f]](t) = M_\omega[f(t-\tau)] = e^{\frac{2\pi i}{N}\omega t} f(t-\tau),$$
$$(L_\tau \circ M_\omega)[f](t) = L_\tau[M_\omega[f]](t) = L_\tau[e^{\frac{2\pi i}{N}\omega t} f(t)] = e^{\frac{2\pi i}{N}\omega(t-\tau)} f(t-\tau).$$

Now, we can see that:

$$e^{-\frac{2\pi i}{N}\omega\tau}(L_\tau \circ M_\omega)[f](t) = e^{-\frac{2\pi i}{N}\omega\tau} e^{\frac{2\pi i}{N}\omega(t-\tau)} f(t-\tau) = e^{\frac{2\pi i}{N}\omega t} f(t-\tau).$$

Thus, we have shown that:

$$M_\omega \circ L_\tau = e^{-\frac{2\pi i}{N}\omega\tau} L_\tau \circ M_\omega.$$

□

(c) *Proof.* To show that $L_\tau \circ F_N = F_N \circ M_\tau$, we compute both sides. Let $f \in \mathscr{H}$ and $t \in \mathbb{Z}_N$.

$$(L_\tau \circ F_N)[f](t) = L_\tau[F_N[f]](t) = L_\tau\left[\frac{1}{\sqrt{N}}\sum_{s\in\mathbb{Z}_N} f(s)e^{-\frac{2\pi i}{N}st}\right] = \frac{1}{\sqrt{N}}\sum_{s\in\mathbb{Z}_N} f(s)e^{-\frac{2\pi i}{N}s(t-\tau)},$$
$$(F_N \circ M_\tau)[f](t) = F_N[M_\tau[f]](t) = F_N[e^{\frac{2\pi i}{N}\tau t}f(t)] = \frac{1}{\sqrt{N}}\sum_{s\in\mathbb{Z}_N} e^{\frac{2\pi i}{N}\tau s}f(s)e^{-\frac{2\pi i}{N}st}.$$

Now, we can see that:

$$(F_N \circ M_\tau)[f](t) = e^{\frac{2\pi i}{N}\tau t}(L_\tau)[F_N[f]](t).$$

Thus, we have shown that:

$$L_\tau \circ F_N = F_N \circ M_\tau.$$

$\square$