Math/CS 714 **Homework 5** **Stephen Cornelius**

Solution to Problem 1

a) The characteristic velocity is given by the derivative of the flux with respect to the density:

$$c(q) = \frac{df}{dq} = \frac{d}{dq}(q(1-q)) = 1 - 2q.$$

The characteristic velocity is negative when:

$$1 - 2q < 0 \implies q > \frac{1}{2}.$$

b) Since $u(q) = 1 - q$, we have:

$$F^n_{j+\frac{1}{2}} = U^n_{j+\frac{1}{2}} Q^n_{j+\frac{1}{2}} = (1 - Q^n_{j+\frac{1}{2}}) Q^n_{j+\frac{1}{2}} = Q^n_{j+\frac{1}{2}} - (Q^n_{j+\frac{1}{2}})^2,$$

and similarly,

$$F^n_{j-\frac{1}{2}} = Q^n_{j-\frac{1}{2}} - (Q^n_{j-\frac{1}{2}})^2.$$

Substituting these into the following equation,

$$\frac{Q^{n+1}_j - Q^n_j}{\Delta t} + \frac{F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}}{\Delta x} = 0$$

we get:

$$\frac{Q^{n+1}_j - Q^n_j}{\Delta t} + \frac{(Q^n_{j+\frac{1}{2}} - (Q^n_{j+\frac{1}{2}})^2) - (Q^n_{j-\frac{1}{2}} - (Q^n_{j-\frac{1}{2}})^2)}{\Delta x} = 0.$$

Simplifying the numerator:

$$= \frac{Q^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}} - ((Q^n_{j+\frac{1}{2}})^2 - (Q^n_{j-\frac{1}{2}})^2)}{\Delta x} = \frac{Q^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}} - (Q^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}})(Q^n_{j+\frac{1}{2}} + Q^n_{j-\frac{1}{2}})}{\Delta x}.$$

Factoring out $(Q^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}})$, we have:

$$= \frac{(Q^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}})(1 - (Q^n_{j+\frac{1}{2}} + Q^n_{j-\frac{1}{2}}))}{\Delta x}.$$

Thus,

$$\frac{Q^{n+1}_j - Q^n_j}{\Delta t} + \frac{F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}}{\Delta x} = 0,$$

can be rewritten as:

$$\frac{Q^{n+1}_j - Q^n_j}{\Delta t} + \left(1 - Q^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}}\right) \frac{Q^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}}}{\Delta x} = 0,$$

which matches

$$\frac{Q^{n+1}_j - Q^n_j}{\Delta t} + C^n_j \frac{Q^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}}}{\Delta x} = 0,$$

with the discrete characteristic velocity given by

$$C^n_j = 1 - Q^n_{j+\frac{1}{2}} - Q^n_{j-\frac{1}{2}}.$$

c) Implementation can be found in `Homework5.py` in the function `Problem1()`. As mentioned in the problem statement, initial code was implem=ented from this Jupyter notebook. The results for both the red light and green light scenarios are shown in the plots below.

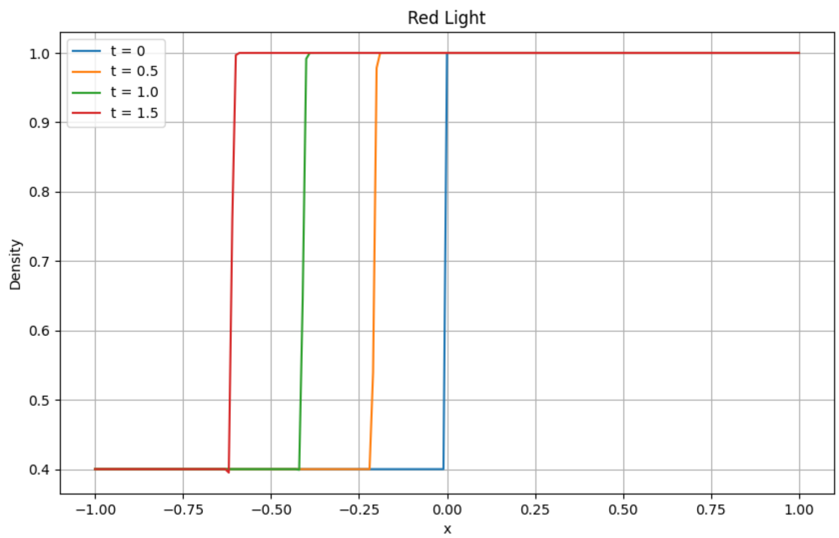

Figure 1: Traffic density evolution for the red light scenario ($q_l = 0.4, q_r = 1.0$) at times $t = 0, 0.5, 1, 1.5$.

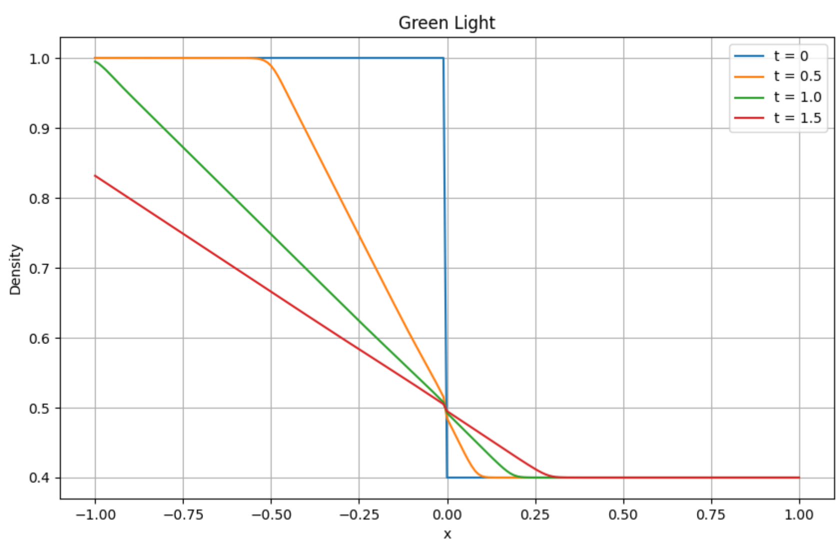

Figure 2: Traffic density evolution for the green light scenario ($q_l = 1.0, q_r = 0.4$) at times $t = 0, 0.5, 1, 1.5$.

1 Solution to Problem 2

(a) On the infinite grid $h\mathbb{Z}$ with $h = 1$, the second derivative operator applied to the constant function $v(x) = 1$ yields zero at all grid points. Consider the standard Fourier-Spectral formula

$$S''(j) = \begin{cases} -\frac{\pi^2}{3} & j = 0 \\ \frac{2(-1)^{j+1}}{j^2} & j \neq 0 \end{cases}$$

Applying this to $v(x) = 1$, where its second derivative is zero, we have

$$0 = S''(0) + \sum_{j\neq 0} S''(j) = -\frac{\pi^2}{3} + 2\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2}.$$

Rearranging gives

$$\frac{\pi^2}{12} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots.$$

(b) See the accompanying code file `Homework5.py` for the implementation of the numerical computation of E_α and the fitting to a power law. I decided to follow the hint and use a library function to numerically integrate. The results yield specific values for C and q. Below is the output from the fitting process:
`Power law fit: E_alpha = 2.341002e+00 * alpha^-1.243544`

(c) The band-limited interpolant $p(x)$ converges to $v(x)$ as $\alpha \to \infty$ because $v(x)$ is band-limited with a maximum frequency of $\frac{\pi}{2}$. The sinc interpolation reconstructs any band-limited function exactly when sampled at the Nyquist rate, which in this case is satisfied by the grid spacing $h = 1$. Thus, as α increases, the approximation $p_\alpha(x)$ approaches $v(x)$.

(d) Considering the first derivative of $v(x) = \sin\frac{\pi x}{2}$, we have

$$v'(x) = \frac{\pi}{2}\cos\frac{\pi x}{2}.$$

Applying the Fourier-Spectral formula for the first derivative, we find that

$$0 = S'(0) + \sum_{j\neq 0} S'(j) = 0 + \sum_{j=1}^{\infty} \frac{2(-1)^j}{j}.$$

Rearranging gives

$$\frac{\pi}{4} = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

(e) By considering the function $v(x) = x^2$ on the infinite grid $h\mathbb{Z}$ with $h = 1$, we can apply the second derivative operator. The second derivative of $v(x)$ is constant, specifically $v''(x) = 2$. Using the Fourier-Spectral formula for the second derivative, we have

$$2 = S''(0) + \sum_{j\neq 0} S''(j) = -\frac{\pi^2}{3} + 2\sum_{j=1}^{\infty} \frac{1}{j^2}.$$

Rearranging gives

$$\frac{\pi^2}{6} = \sum_{j=1}^{\infty} \frac{1}{j^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

as desired.[1]

[1] I am a fan of the Basel problem.

2 Solution to Problem 3

(a) To find f, we compute the derivatives of $u(x) = e^x(x^2 - 1)$:

$$u_x = e^x(x^2 - 1) + e^x(2x) = e^x(x^2 + 2x - 1)$$
$$u_{xx} = e^x(x^2 + 2x - 1) + e^x(2x + 2) = e^x(x^2 + 4x + 1)$$

Also, $u^5 = e^{5x}(x^2 - 1)^5$.

Therefore, the source term is:

$$f(x) = u_{xx} + u^5 = e^x(x^2 + 4x + 1) + e^{5x}(x^2 - 1)^5 \tag{1}$$

(b) Using a Chebyshev spectral method with Newton's method, we compute the numerical solution $p_N(x)$ for various values of N. The following plot shows the semilog plot of $\|p_N - u\|_2$ as a function of N:

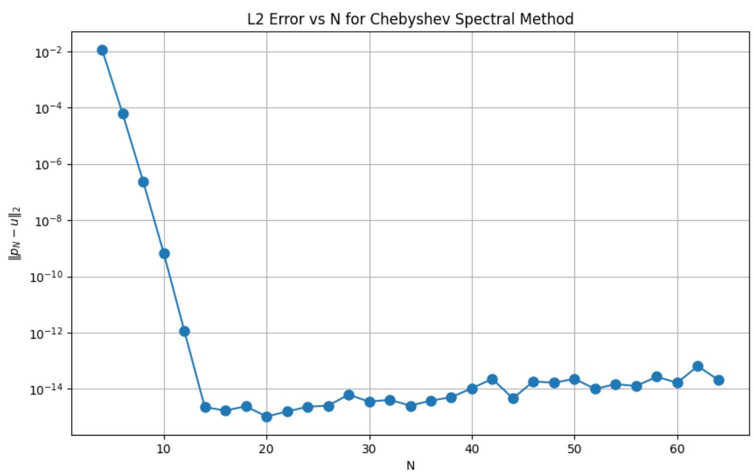

Figure 3: Semilog plot of the error $\|p_N - u\|_2$ as a function of N.