Problem 1

Verify that the following construction is a contravariant functor F from the category of sets to itself: for a set $A, F(A)$ is the power set of A (that is, the set of all the subsets of A), while for a map $f : A \to B$, the induced map $F(f) : F(B) \to F(A)$ sends $X \subset B$ to $f^{-1}(X) \subset A$.

Proof. Let $X \subset C$. Then
$$F(g \circ f)(X) = (g \circ f)^{-1}(X) = f^{-1}(g^{-1}(X)) = F(f)(F(g)(X)).$$
Thus, $F(g \circ f) = F(f) \circ F(g)$.
Finally, we check the identity property. Let $X \subset A$. Then
$$F(\mathrm{id}_A)(X) = (\mathrm{id}_A)^{-1}(X) = X.$$
Thus, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
Since both properties of **Definition X.1.2** are satisfied, F is indeed a contravariant functor from the category of sets to itself. □

Problem 2

Let $\mathcal{C}$ be any category. Fix an object $a \in \mathcal{C}$ and suppose that for any object $x \in \mathcal{C}$, the product $a \times x$ exists. Show that the correspondence
$$x \mapsto a \times x$$
is a functor. (Technically, there is a subtlety here: we know that if a product exists, it is unique up to isomorphism, however, in order to construct a functor, we need to make a specific choice of $a \times x$ for all x. It would be better to say that the functor is defined up to a cononical isomorphism, but this detail is usually ignored.)

Proof. We need to verify two properties of a functor:

1. For any objects $x, y, z \in \mathcal{C}$ and any morphisms $f : x \to y$ and $g : y \to z$, we have
$$F(g \circ f) = F(g) \circ F(f).$$

2. For any object $x \in \mathcal{C}$, we have
$$F(\mathrm{id}_x) = \mathrm{id}_{F(x)}\,.$$

Let $f : x \to y$ and $g : y \to z$ be morphisms in $\mathcal{C}$. By the universal property of products, there exist unique morphisms $F(f) : a \times x \to a \times y$ and $F(g) : a \times y \to a \times z$ such that the following diagrams commute:

$$\begin{array}{ccccccc} a \times x & \xrightarrow{F(f)} & a \times y & \quad & a \times y & \xrightarrow{F(g)} & a \times z \\ \downarrow \pi_a & & \downarrow \pi_a & & \downarrow \pi_a & & \downarrow \pi_a \\ a & = & a & & a & = & a \end{array}$$

Now, consider the composition $g \circ f : x \to z$. By the universal property of products again, there exists a unique morphism $F(g \circ f) : a \times x \to a \times z$ such that the following diagram commutes:

$$\begin{array}{ccc} a \times x & \xrightarrow{F(g \circ f)} & a \times z \\ \downarrow \pi_a & & \downarrow \pi_a \\ a & = & a \end{array}$$

Since both $F(g) \circ F(f)$ and $F(g \circ f)$ satisfy the same universal property, by the uniqueness part of the universal property of products, we have
$$F(g \circ f) = F(g) \circ F(f).$$

Finally, we check the identity property. For any object $x \in \mathcal{C}$, by the universal property of products, there exists a unique morphism $F(\mathrm{id}_x) : a \times x \to a \times x$ such that the following diagram commutes:

$$\begin{array}{ccc} a \times x & \xrightarrow{F(\mathrm{id}_x)} & a \times x \\ \downarrow \pi_a & & \downarrow \pi_a \\ a & = & a \end{array}$$

Since both $\mathrm{id}_{a\times x}$ and $F(\mathrm{id}_x)$ satisfy the same universal property, by the uniqueness part of the universal property of products, we have

$$F(\mathrm{id}_x) = \mathrm{id}_{a\times x}\,.$$

Therefore we have that $x \mapsto a \times x$ is indeed a functor. □

Problem 3

Let V be a linear space and $P : V \to V$ be a linear operator such that $P^2 = P$. (Operators having this property are called *projectors*.) Show that V is the internal direct sum of subspaces $\mathrm{Im}(P)$ and $\mathrm{Ker}(P)$.

Proof. Notice that for any $x \in V$ we have that $x = x - P(x) + P(x)$. Then $P(x - P(x)) = P(x) - P^2(x) = 0$. So we have that $x - P(x) \in \ker(P)$ and clearly $P(x) \in \mathrm{im}(P)$. Now suppose $x \in \ker(P) \cap \mathrm{im}(P)$. Fix y with $P(x) = y$ then we have that $0 = P(x) = P^2(x) = P(y)$. So $0 = P(x) = P(y) \iff x = y = 0$. Therefore we have that $V = \ker(P) \oplus \mathrm{im}(P)$. □

Problem 4

Continuing with the previous problem, suppose that $\dim(V) = n$ Prove that there exists a basis of V such that the matrix of P is of the form $\mathrm{diag}(1, 1, \ldots, 1, 0, 0, \ldots, 0)$. (diag denotes the diagonal matrix with given entries.)

Proof. If we fix bases $\mathscr{B}$ and $\mathscr{C}$ for $\mathrm{im}(P)$ and $\ker(P)$ respectively, then from question 3 we have that $\mathscr{B} \cup \mathscr{C}$ is a basis for V. Then for any $x \in \mathscr{B}$, fix $y_x \in V$ such that $x = P(y_x)$ so we have $P(x) = P^2(y_x) = P(y_x) = x$. Then for any $y \in \mathscr{C}$, we have $P(y) = 0$ by definition. Therefore the matrix representation of P with respect to the basis $\mathscr{B} \cup \mathscr{C}$ is of the form $\mathrm{diag}(1, 1, \ldots, 1, 0, 0, \ldots, 0)$. □

Problem 5

Fix n, and consider the vector space:

$$V = \{p(t) \in \mathbb{R}[t] : \deg(p) \leq n\}$$

over $\mathbb{R}$. Fix $n + 1$ numbers $a_0, \ldots, a_n \in \mathbb{R}$ and consider the map

$$\phi : V \to \mathbb{R}^{n+1}; \quad \phi(p) = (p(a_0), \ldots, p(a_n)).$$

Show that ϕ is invertible (and therefore an isomorphism of vector spaces) if and only if a_i's are all distinct.

Proof. ($\Leftarrow$) Suppose the a_i's are all distinct. We first show that ϕ is injective. Suppose $\phi(p) = \phi(q)$ for some $p, q \in V$. Then we have that $p(a_i) = q(a_i)$ for all $0 \leq i \leq n$. Thus, the polynomial $r(t) = p(t) - q(t)$ has $n + 1$ distinct roots $a_0, a_1, \ldots, a_n$. Since $\deg(r) \leq n$, we must have $r(t) \equiv 0$, which implies that $p(t) = q(t)$. Therefore, ϕ is injective.
To show that ϕ is surjective, let $(b_0, b_1, \ldots, b_n) \in \mathbb{R}^{n+1}$ be any vector. We need to find a polynomial $p(t) \in V$ such that $\phi(p) = (b_0, b_1, \ldots, b_n)$. This is equivalent to solving the system of equations:

$$p(a_i) = b_i \quad \text{for } i = 0, 1, \ldots, n.$$

Since the a_i's are distinct, we have:

$$p(t) = \sum_{i=0}^{n} b_i \prod_{\substack{0\leq j\leq n \\ j\neq i}} \frac{t - a_j}{a_i - a_j}$$

is a polynomial of degree at most n that satisfies the above equations. Thus, ϕ is surjective. Since ϕ is both injective and surjective, it is invertible.

($\Rightarrow$) Now suppose that the a_i's are not all distinct. Without loss of generality, assume that $a_0 = a_1$. We will show that ϕ is not injective. Consider the polynomial $p(t) = t - a_0$. Then we have:

$$\phi(p) = (p(a_0), p(a_1), \ldots, p(a_n)) = (0, 0, p(a_2), \ldots, p(a_n)).$$

Now consider the zero polynomial $q(t) = 0$. We have:

$$\phi(q) = (q(a_0), q(a_1), \ldots, q(a_n)) = (0, 0, 0, \ldots, 0).$$

Since $\phi(p) \neq \phi(q)$, we have found two distinct polynomials p and q such that $\phi(p) = \phi(q)$. Therefore, ϕ is not injective, and hence not invertible.

□

Problem 6

Let V be a vector space over a field K. A (linear) functional on V is a linear operator $\phi : V \to K$. Show that if two functionals $\phi, \psi : V \to K$ satisfy $\ker(\phi) = \ker(\psi)$, then there exists $a \in K$ such that $\psi = a\phi$. (If you need to, you can assume that V is finite-dimensional, but it should not be necessary.)

Problem 7

Let K be a field and $L \subset K$ be a smaller field (e.g., $L = \mathbb{R}$ and $K = \mathbb{C}$). Given a K-vector space V, we can also consider it as a L-vector space, so we get two notions of dimension: as a vector space over K and as a vector space over L. Denote them by $\dim_K V$ and $\dim_L V$, respectively. Show that

$$\dim_L V = (\dim_K V) \cdot (\dim_L K).$$