Problem 1

Diagonalizability - geometric definition.
Let V be an n-dimensional vector space over a field $\mathbb{F}$ and let $T : V \to V$ a transformation.

(a) Write down the geometric definition (that we gave in class interms of direct sum decomposition of V) for when T is diagonalizable.

(b) Define what does it mean for $\lambda \in \mathbb{F}$ to be an eigenvalue of T. Denote $\mathrm{Spec}(T)$ the set of eigenvalues of T in $\mathbb{F}$. For each $\lambda \in \mathrm{Spec}(T)$, define the eigenspace V_λ. Show that the following are equivalent:

  (i) T is diagonalizable.

  (ii) $V = \bigoplus_{\lambda \in \mathrm{Spec}(T)} V_\lambda$.

(c) A linear transformation $P : V \to V$ is called a projector of $P^2 = P$. Show that any projector is diagonalizable.

1. We say that T is diagonalizable if there exists $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ distinct and subspaces $V_1, \dots, V_k < V$ such that

$$V = \bigoplus_{i=1}^{k} V_i,$$

and T preserves each V_i, and $T|_{V_i} = \lambda_i Id_{V_i}$ for all $i = 1, \dots, k$.

2. If $V_\lambda \neq 0$ it is called the λ-eigenspace of T, and such λ is called eigenvalue of T, $v \in V_\lambda$ is called an λ-eigenvector of T.

Proof. (i) $\Rightarrow$ (ii):
Suppose T is diagonalizable. Then there exists $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ distinct and subspaces $V_1, \dots, V_k < V$ such that

$$V = \bigoplus_{i=1}^{k} V_i,$$

and T preserves each V_i, and $T|_{V_i} = \lambda_i Id_{V_i}$ for all $i = 1, \dots, k$.
Then for each i, $V_i \subseteq V_{\lambda_i}$, since for any $v \in V_i$, $T(v) = \lambda_i v$. Thus, $V_i \leq V_{\lambda_i}$ for all i.
Now, let $\lambda \in \mathrm{Spec}(T)$. Then there exists $v \in V$ such that $T(v) = \lambda v$. Since $V = \bigoplus_{i=1}^{k} V_i$, we can write $v = v_1 + \dots + v_k$ with $v_i \in V_i$. Then

$$T(v) = T(v_1 + \dots + v_k) = T(v_1) + \dots + T(v_k) = \lambda_1 v_1 + \dots + \lambda_k v_k.$$

But also, $T(v) = \lambda v = \lambda(v_1 + \dots + v_k) = \lambda v_1 + \dots + \lambda v_k$.
Thus, we have

$$\lambda_1 v_1 + \dots + \lambda_k v_k = \lambda v_1 + \dots + \lambda v_k.$$

Since λ_i are distinct, we must have $v_i = 0$ for all $i \neq j$, where j is such that $\lambda_j = \lambda$. Thus, $v = v_j$, and $v_j \in V_\lambda$.
Therefore, $V = \bigoplus_{\lambda \in \mathrm{Spec}(T)} V_\lambda$.
(ii) $\Rightarrow$ (i):
Suppose $V = \bigoplus_{\lambda \in \mathrm{Spec}(T)} V_\lambda$. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of T. Then

$$V = \bigoplus_{i=1}^{k} V_{\lambda_i}.$$

For each i, T preserves V_{λ_i}, and $T|_{V_{\lambda_i}} = \lambda_i Id_{V_{\lambda_i}}$.
Thus, T is diagonalizable. □

3. *Proof.* Let $P : V \to V$ be a projector, i.e., $P^2 = P$. Then for any $v \in V$, we have

$$P(P(v)) = P(v).$$

Thus, $P(v)$ is an eigenvector of P with eigenvalue 1.
Now, consider the subspace $W = \ker(P)$. For any $w \in W$, we have

$$P(w) = 0.$$

Thus, w is an eigenvector of P with eigenvalue 0.
Therefore, we have two eigenspaces: $V_1 = \mathrm{Im}(P)$ with eigenvalue 1 and $V_0 = \ker(P)$ with eigenvalue 0.
Since $V = V_1 \oplus V_0$, we have

$$V = V_1 \oplus V_0,$$

and P preserves each eigenspace. Thus, P is diagonalizable. □

Problem 2

Diagonalizability - computational definition.

(a) Let V be an n-dimensional vector space over a field $\mathbb{F}$ and let $T : V \to V$ a linear transformation. Write down the computational definition (that we gave in class in terms of a basis $\mathscr{B}$ and the corresponding matrix $[T]_{\mathscr{B}}$) for when T is diagonalizable.

(b) For a matrix $A \in M_n(\mathbb{F})$, consider the linear transformation $T_A : \mathbb{F}^n \to \mathbb{F}^n$ given by $v \mapsto Av$. Show that the following are equivalent:

(i) T_A is diagonalizable (in this case we also say that A is diagonalizable).

(ii) There exists a diagonal matrix $D \in M_n(\mathbb{F})$ and an invertible matrix $C \in M_n(\mathbb{F})$ such that $C^{-1}AC = D$.

(c) Consider the operator $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Show that T_A is diagonalizable

(b) find its eigenvalues

(c) find the direct sum decomposition for eigenspaces

(d) find a basis of eigenvectors

(e) find $D, C \in M_2(\mathbb{R})$ with D diagonal and C invertible such that $D = C^{-1}AC$.

(a) We say that T is diagonalizable if there exists a basis $\mathscr{B}$ of V such that the matrix $[T]_{\mathscr{B}}$ is a diagonal matrix.

(b) *Proof.* (i) $\Rightarrow$ (ii):
Suppose T_A is diagonalizable. Then there exists a basis $\mathscr{B} = \{v_1, \ldots, v_n\}$ of $\mathbb{F}^n$ such that $[T_A]_{\mathscr{B}}$ is a diagonal matrix. Let C be the matrix whose columns are the vectors of $\mathscr{B}$. Then C is invertible, and we have

$$[T_A]_{\mathscr{B}} = C^{-1}AC.$$

Since $[T_A]_{\mathscr{B}}$ is diagonal, we can take $D = [T_A]_{\mathscr{B}}$, and we have $C^{-1}AC = D$.
(ii) $\Rightarrow$ (i):
Suppose there exists a diagonal matrix $D \in M_n(\mathbb{F})$ and an invertible matrix $C \in M_n(\mathbb{F})$ such that $C^{-1}AC = D$. Let $\mathscr{B}$ be the basis of $\mathbb{F}^n$ whose columns are the vectors of C. Then we have

$$[T_A]_{\mathscr{B}} = C^{-1}AC = D,$$

which is a diagonal matrix. Thus, T_A is diagonalizable. □

(c) (a) We will show that T_A is diagonalizable by finding a basis of eigenvectors.

(b) To find the eigenvalues, we compute the characteristic polynomial:

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1.$$

The eigenvalues are the roots of the characteristic polynomial, so we have

$$\lambda^2 - 1 = 0 \implies \lambda = \pm 1.$$

(c) Next we find the eigenspaces. For $\lambda = 1$, we solve

$$(A - I)v = 0 \implies \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the system of equations:

$$-x_1 + x_2 = 0,$$

which simplifies to $x_1 = x_2$. Thus, the eigenspace for $\lambda = 1$ is

$$V_1 = \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

For $\lambda = -1$, we solve

$$(A + I)v = 0 \implies \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

This gives us the system of equations:

$$x_1 + x_2 = 0,$$

which simplifies to $x_1 = -x_2$. Thus, the eigenspace for $\lambda = -1$ is

$$V_{-1} = \text{span}\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}.$$

(d) A basis of eigenvectors is given by

$$\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}.$$

(e) We can take $C = \begin{pmatrix}V_1 & V_{-1}\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$, and $D = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$. Then we have

$$C^{-1}AC = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}^{-1}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = D.$$

Thus, we have found D and C such that $D = C^{-1}AC$.