Problem 1

In this problem, no explanation is required. All parts are worth 2 points.

(a) True or false: In a free abelian group of finite rank, every linearly independent set can be completed to a basis.

(b) How many different (up to isomorphism) abelian groups of order 300 are there?

(c) True or false: For any action of a finite group G on a set X, the cardinality $|X|$ divides $|G|$.

(d) Give an example of an infinite group G such that every element of G has finite order.

(e) Let F_2 be the free group on two generators. True or false: For every n, there exists a normal subgroup $H_n \subset F_2$ such that $F_2/H_n \cong S_n$?

(a) True.

(b) There are 5 different abelian groups of order 300 up to isomorphism. This is because $300 = 2^2 \cdot 3^1 \cdot 5^2$, and the number of abelian groups of order n is given by the product of the number of partitions of the exponents in its prime factorization. The partitions are: for 2^2 (2), for 3^1 (1), and for 5^2 (2). Thus, the total number is $2 \times 1 \times 2 = 4$.

(c) False.

(d) An example of an infinite group where every element has finite order is the group of all roots of unity in the complex numbers, denoted by $\{e^{2\pi ik/n} \mid k \in \mathbb{Z}, n \in \mathbb{N}\}$.

(e) True.

Problem 2

Let $\mathbb{Q}^\times$ be the group of non-zero rational numbers under multiplication.

(a) Show that $\mathbb{Q}^\times$ is isomorphic to the product of $\mathbb{Z}/2\mathbb{Z}$ and a free abelian group.

(b) Describe all group homomorphisms $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}^\times$.

(c) Describe all group homomorphisms $\mathbb{Q}^\times \to \mathbb{Z}/2\mathbb{Z}$.

(a) *Proof.* Every non-zero rational number can be uniquely expressed in the form

$$r = \pm p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where p_i are distinct prime numbers and $a_i \in \mathbb{Z}$. The sign of r corresponds to an element of $\mathbb{Z}/2\mathbb{Z}$, while the exponents a_i correspond to elements of a free abelian group generated by the primes. Thus, we can define an isomorphism

$$\phi : \mathbb{Q}^\times \to \mathbb{Z}/2\mathbb{Z} \times F,$$

where F is the free abelian group generated by the primes. This shows that $\mathbb{Q}^\times$ is isomorphic to the product of $\mathbb{Z}/2\mathbb{Z}$ and a free abelian group. □

(b) The group homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Q}^\times$ are determined by the image of the non-identity element of $\mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Q}^\times$ contains elements of order 2 (specifically, -1), there are two homomorphisms: the trivial homomorphism sending everything to 1, and the homomorphism sending the non-identity element to -1.

(c) The group homomorphisms from $\mathbb{Q}^\times$ to $\mathbb{Z}/2\mathbb{Z}$ are determined by the kernel of the homomorphism. The only non-trivial homomorphism is the one that sends all positive rational numbers to the identity element of $\mathbb{Z}/2\mathbb{Z}$ and all negative rational numbers to the non-identity element. Thus, there are two homomorphisms: the trivial homomorphism and the sign homomorphism.

Problem 3

Let G be a group of order $2017 \times 2027 \times 2029$ (these are all prime numbers). Show that G is cyclic.

Proof. We have that the order of G is the product of three distinct primes: 2017, 2027, and 2029. Then, by the first Sylow theorem, for each prime p dividing the order of G, there exists a Sylow p-subgroup of G. Let n_p denote the number of Sylow p-subgroups of G. By the third Sylow theorem, we have that $n_p \equiv 1 \mod p$ and n_p divides the order of G. Since the primes are distinct and large, the only divisors of the order of G that are congruent to 1 modulo p are 1 itself. Therefore, each Sylow p-subgroup is unique and hence normal in G. Since the Sylow subgroups are normal and their orders are pairwise relatively prime, G is isomorphic to the direct product of its Sylow subgroups, each of which is cyclic of prime order. Thus, we have that $G = \mathbb{Z}/2017\mathbb{Z} \times \mathbb{Z}/2027\mathbb{Z} \times \mathbb{Z}/2029\mathbb{Z}$ is cyclic. □

Problem 4

Let G be a finite group, and let $A = \text{Aut}(G)$ be the group of automorphisms $\phi : G \to G$. Consider the natural action of A on G, and take the quotient G/A.

(a) What is $|G/A|$ if $G = \mathbb{Z}/6\mathbb{Z}$?

(b) Show that if $|G/A| = 2$, then $G \cong (\mathbb{Z}/p\mathbb{Z})^n$ for a prime p and $n > 0$.

(a) For $G = \mathbb{Z}/6\mathbb{Z}$, the automorphism group $\text{Aut}(G)$ consists of all group automorphisms of $\mathbb{Z}/6\mathbb{Z}$. The elements of $\mathbb{Z}/6\mathbb{Z}$ are $\{0, 1, 2, 3, 4, 5\}$. The automorphisms are determined by the images of the generator 1. The possible images are 1 and 5 (since they are coprime to 6). Thus, there are two automorphisms: the identity and the one sending 1 to 5. The orbits under this action are $\{0\}$, $\{1, 5\}$, $\{2, 4\}$, and $\{3\}$. Therefore, there are 4 distinct orbits, so $|G/A| = 4$.

(b) *Proof.* We have that $|G/A| = 2$ implies that there are exactly two orbits under the action of $\text{Aut}(G)$ on G. One orbit must be the identity element $\{e\}$, and the other orbit must contain all other elements of G. This means that for any non-identity element $g \in G$, there exists an automorphism $\phi \in \text{Aut}(G)$ such that $\phi(g) = h$ for any other non-identity element $h \in G$. This property implies that all non-identity elements of G have the same order. Let this common order be p. Since G is finite, p must be a prime number. Thus, every non-identity element of G has order p, and G is a p-group. Furthermore, since all non-identity elements have the same order, G must be isomorphic to a direct product of copies of $\mathbb{Z}/p\mathbb{Z}$. Therefore, we conclude that $G \cong (\mathbb{Z}/p\mathbb{Z})^n$ for some prime p and integer $n > 0$. □

Problem 5

A finite group G acts transitively (that is, with a single orbit) on a finite set X such that $|X| > 1$. Show that there exists an element $g \in G$ which does not fix any element of X.

Problem 6

A map $\phi : \mathbb{R} \to \mathbb{R}$ is said to be an *affine-linear bijection* if it is of the form

$$\phi(x) = ax + b \quad (a, b \in \mathbb{R} : a \neq 0).$$

(a) Show that the set of affine-linear bijections forms a group G under composition.

(b) Show that G is isomorphic to semidirect product of *abelian* groups A and B. Make sure to identify the groups A and B, as well as the action of one on the other used in the semidirect product.