Math/CS 714: Assignment 4

Problem 1

Beam–Warming method (4 points). The Beam–Warming method for the linear advection equation $u_t + au_x = 0$ is given by

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{a^2k^2}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n), \tag{1}$$

where U_j^n is the approximation of $u(jh, nk)$.

1. Use Taylor series to show that this method is second-order accurate.
2. For a given plane wave solution $U_j^0 = e^{ijh\xi}$, compute the amplification factor $g(\xi)$, and hence determine the stability restriction for this method.

a) Expand the exact solution in time at (x_j, t_n):

$$u_j^{n+1} = u + ku_t + \frac{k^2}{2}u_{tt} + \frac{k^3}{6}u_{ttt} + O(k^4),$$

with $u_t = -au_x,\ u_{tt} = a^2u_{xx},\ u_{ttt} = -a^3u_{xxx}$, so

$$u_j^{n+1} = u - ak\,u_x + \frac{a^2k^2}{2}u_{xx} - \frac{a^3k^3}{6}u_{xxx} + O(k^4).$$

Expand spatial shifts about x_j:

$$\begin{aligned} U_{j-1} &= u - hu_x + \tfrac{h^2}{2}u_{xx} - \tfrac{h^3}{6}u_{xxx} + \tfrac{h^4}{24}u_{xxxx} + O(h^5), \\ U_{j-2} &= u - 2hu_x + 2h^2u_{xx} - \tfrac{4}{3}h^3u_{xxx} + \tfrac{2}{3}h^4u_{xxxx} + O(h^5). \end{aligned}$$

Form the combinations occurring in the scheme:

$$\begin{aligned} 3U_j - 4U_{j-1} + U_{j-2} &= 2h\,u_x - \tfrac{2}{3}h^3u_{xxx} + \tfrac{1}{2}h^4u_{xxxx} + O(h^5), \\ U_j - 2U_{j-1} + U_{j-2} &= h^2u_{xx} - h^3u_{xxx} + \tfrac{7}{12}h^4u_{xxxx} + O(h^5). \end{aligned}$$

Substitute into the Beam–Warming update

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j - 4U_{j-1} + U_{j-2}) + \frac{a^2k^2}{2h^2}(U_j - 2U_{j-1} + U_{j-2})$$

to obtain

$$U_j^{n+1} = u - ak\,u_x + \frac{a^2k^2}{2}u_{xx} + \left(\tfrac{1}{3}akh^2 - \tfrac{1}{2}a^2k^2h\right)u_{xxx} + O(kh^3, k^2h^2, k^3).$$

Comparing with the exact time–Taylor expansion shows the leading matching terms are $u - aku_x + \frac{a^2k^2}{2}u_{xx}$; the local truncation error is

$$\tau = u^{\text{exact}}(t+k) - U_j^{n+1} = O(k^3) + O(kh^2).$$

Thus if the time step scales with the mesh, e.g. $k = O(h)$ (fixed Courant number), the local error is $O(h^3)$ and the global error is $O(h^2)$ — the scheme is second order accurate in space and time.

Problem 2

(9 points). Dropping the last term in the Beam–Warming method from Eq. (1) gives

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n), \tag{2}$$

which corresponds to forward Euler method in time, and a second-order one-sided derivative in space. Define $\nu = ak/h$.

1. Calculate the amplification factor $g(\xi)$ for a plane wave solution $U_j^0 = e^{ijh\xi}$.
2. Define $A(\xi) = |g(\xi)|^2$ and calculate a Taylor series for A at $\xi = 0$ up to second order. Using the Taylor series, explain why we consider the numerical scheme of Eq. (2) to be unstable regardless of the choice of timestep.
3. Make two plots of $A(\xi)$ for $\nu = 1/100$ using two different axis ranges:
   - $0 \le h\xi \le 2\pi$ and $0.91 \le A \le 1.01$,
   - $0 \le h\xi \le 0.17$ and $1 - 10^{-6} \le A \le 1 + 10^{-6}$.
4. Write a program to simulate Eq. (2) on a periodic interval $[0, 2\pi)$ using $N = 40$ grid points and a grid spacing of $h = 2\pi/N$. Use the initial condition $u = \exp(2\sin x)$ and $\nu = 1/100$. Plot the solution for $n = 0, 1000, 2000, 4000$. Define the root mean squared value of the solution,

   $$R(n) = \sqrt{\frac{1}{N}\sum_{j=0}^{N-1}(U_j^n)^2}. \tag{3}$$

   Make a plot of R over the range from $n = 0$ to $n = 10000$. You should find that R does not grow over time, indicating that the method is stable.
5. Using the discrete Fourier transform, it can be shown that an arbitrary initial condition on the periodic interval can be written as

   $$U_j^0 = \sum_{l=0}^{N-1} \alpha_l e^{ijlh} \tag{4}$$

   for some constants α_l. Write down an expression for the general solution U_j^n. Using your answer, explain why your result in part (d) does not contradict the result in part (b).

Problem 3

Lax–Wendroff method (7 points). Consider the hyperbolic conservation equation

$$q_t + [A(x)q]_x = 0 \tag{5}$$

for a function on $q(x,t)$ on the periodic interval $[0, 2\pi)$. Let $A(x) = 2 + \frac{4}{3}\sin x$. Following the finite volume approach, divide the intervals into m domains $\mathcal{C}_i$ of length $h = \frac{2\pi}{m}$, for $i = \{0, 1, \ldots, m-1\}$. Let $Q_i^n \approx q((i + 1/2)h, n\Delta t)$ be the discretized solution at the center of each $\mathcal{C}_i$. The generalized Lax–Wendroff scheme for this equation is given by

$$Q_i^{n+1} = Q_i - \frac{\Delta t}{h}\left[\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right] \tag{6}$$

where the fluxes are

$$\mathcal{F}_{i-1/2}^n = \frac{A_{i-1}Q_{i-1}^n + A_i Q_i^n}{2} - \frac{A_{i-1/2}\Delta t}{2h}\left[A_i Q_i^n - A_{i-1}Q_{i-1}^n\right]. \tag{7}$$

Here, $A_i = A((i + 1/2)h)$ and $A_{i-1/2} = A(ih)$. It can be shown that the solution to Eq. (5) is time-periodic so that $q(x, t + T) = q(x, t)$ where $T = 3\pi/\sqrt{5}$.

1. The CFL condition requires that $\Delta t \leq \frac{h}{c}$ for stability. What is c in this case?
2. Implement Eq. (7) and set $\Delta t = \frac{h}{3c}$. Use the initial condition

$$q(x, 0) = \exp\left(\sin x + \tfrac{1}{2}\sin 4x\right). \tag{8}$$

   For $m = 512$, plot snapshots of the solution for $t = 0, \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}, T$.[a]
3. By considering a range of m (*e.g.* 256 and upward) with the initial condition in Eq. (8), calculate the L_2 norm between the numerical solution at $t = T$ and the exact answer. Determine the order of convergence.[b]
4. Repeat parts (b) and (c) for the initial condition

$$q(x, 0) = \max\{\tfrac{\pi}{2} - |x - \pi|, 0\}. \tag{9}$$

5. **Optional.** By the considering the characteristics, or otherwise, derive the result that q is time-periodic with period T.

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[a]Since multiples of Δt do not exactly match these snapshot times, you may need to make a small adjustment to the timestep.

[b]When determining the order of convergence, you are interested in the asymptotic properites of error as m gets large. You can ignore initial transients in error.