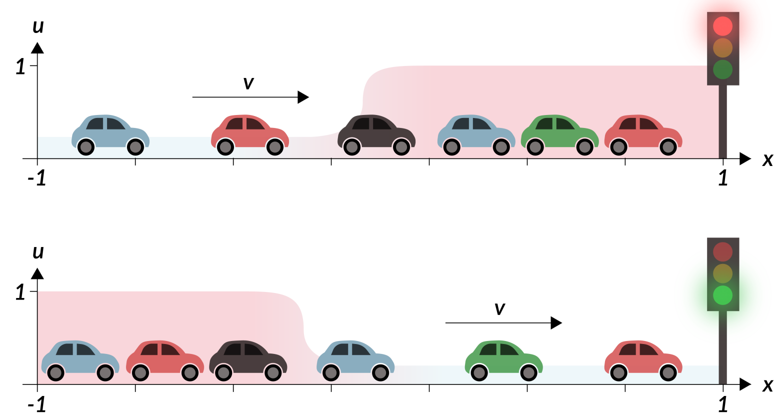
Math/CS 714: Assignment 5

1. **Red light, green light (8 points).** In this problem, we will consider the behavior of traffic on a stretch of road $x \in [-1, 1]$ due to a traffic light at $x = 1$.



   (a) Consider the 1D LWR traffic model with car density $q \in [0, 1]$ and max car speed $u_{\max} = 1$, traffic flow velocity $u(q) = 1 - q \in [0, 1]$, and the flux $f(q) = q(1 - q)$. What is the analytical expression for the characteristic velocity $c(q)$ for this model? For what values of q is the characteristic velocity negative?

   (b) Consider the discretization

$$\frac{Q_j^{n+1} - Q_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0 \tag{1}$$

   with

$$F_{j+\frac{1}{2}}^n = U_{j+\frac{1}{2}}^n Q_{j+\frac{1}{2}}^n, \quad F_{j-\frac{1}{2}}^n = U_{j-\frac{1}{2}}^n Q_{j-\frac{1}{2}}^n.$$

   Here, $F_{j+\frac{1}{2}}^n$, $U_{j+\frac{1}{2}}^n$, and $Q_{j+\frac{1}{2}}^n$ are numerical approximations of the flux $f(x_{j+\frac{1}{2}}, t_n)$, velocity $u(x_{j+\frac{1}{2}}, t_n)$, and density $q(x_{j+\frac{1}{2}}, t_n)$, respectively, with $x_{j+\frac{1}{2}} = -1 + \left(j + \frac{1}{2}\right)\Delta x$ and $t_n = n\Delta t$. Show that Eq. (1) can be rewritten as

$$\frac{Q_j^{n+1} - Q_j^n}{\Delta t} + C_j^n \frac{Q_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n}{\Delta x} = 0 \tag{2}$$

   where the discrete characteristic velocity

$$C_j^n = 1 - Q_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n. \tag{3}$$

   (c) Implement the WENO scheme discussed in class to solve for the density in time using the discretization in Eq. (2). Discretize the domain $x \in [-1, 1]$ by $m = 201$ evenly-spaced points with spacing $\Delta x = 2/(m - 1)$. Use a timestep of

$\Delta t = 0.001$ and integrate to a final time $T = 1.5$, outputting 150 snapshots after the initial state (151 outputs total). As initial conditions, use the step function

$$Q_j^0 = \begin{cases} q_l, & x < 0 \\ q_r, & x \geq 0 \end{cases} \tag{4}$$

for fixed values q_l and q_r. When the interpolant for $Q^n_{j+\frac{1}{2}}$ or $Q^n_{j-\frac{1}{2}}$ relies on points outside the domain, use the ghost node approach and treat the density on these nodes as fixed at q_l on the left and q_r on the right, respectively.

At each grid cell, the sign of the characteristic velocity C_j^n must be determined to select the appropriate upwind condition. At the start of the step, compute the intermediate value

$$\widehat{C}_j^n = 1 - Q_{j+1}^n - Q_{j-1}^n \tag{5}$$

from known values Q_{j+1}^n and Q_{j-1}^n, and use its sign to select the proper set of upwind interpolants and compute $Q^n_{j+\frac{1}{2}}$ and $Q^n_{j-\frac{1}{2}}$. Then compute the final characteristic velocity C_j^n from Eq. (3) and use it in the update rule given by Eq. (2). You can consult this Jupyter notebook for implementing the WENO scheme, but not that the notations for density and velocity is different.

Run your program for the following two cases:

i. **Red light:** $q_l = 0.4, q_r = 1.0$. Physically, we imagine that the traffic light at $x = 1$ is red, and the traffic is initially backed up and standing still from $[0, 1]$ while more vehicles approach from the left.

ii. **Green light:** $q_l = 1.0, q_r = 0.4$. Now, the light at $x = 1$ is green, and traffic on $[0, 1]$ has reduced, while cars on $[-1, 0]$ are initially still at rest.

For each case, plot snapshots of the density at the four times $t = 0, 0.5, 1, 1.5$.

2. **Primers on spectral methods (6 points).**

(a) Consider the infinite grid $h\mathbb{Z}$ with $h = 1$. By considering the second derivative of the constant function $v(x) = 1$ on $h\mathbb{Z}$, show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots. \tag{6}$$

(b) Consider the function $v(x) = \sin\frac{\pi x}{2}$ on the infinite grid $h\mathbb{Z}$ with $h = 1$. Introduce the function

$$p_\alpha(x) = \sum_{m=-\alpha}^{\alpha} v_m S_h(x - x_m) \tag{7}$$

where S_h is the sinc function introduced in the lectures, $x_m = mh$, and $v_m = v(x_m)$. Write a program to compute[1] $E_\alpha = \|p_\alpha - v\|_2$ over the range $[-5, 5]$ for

[1] You will need to use numerical integration to do this. You can use a library function, trapezoid rule, or other quadrature rule.

$\alpha = 1, 2, 4, 8, \ldots, 512$. Fit the data to a power law

$$E_\alpha = C\alpha^q \tag{8}$$

and determine the parameters C and q.

(c) The band-limited interpolant is $p(x) = \lim_{\alpha\to\infty} p_\alpha(x)$. Part (b) shows that $E_\alpha \to 0$ as $\alpha \to \infty$, and hence $p(x) = v(x)$. Explain from the theory of band-limited interpolants why this must be the case.[2]

(d) By considering the first derivative of v from part (b), show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots. \tag{9}$$

(e) By using an appropriate function choice, show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots. \tag{10}$$

3. **Chebyshev spectral method (6 points).** Consider the boundary value problem

$$u_{xx} + u^5 = f \tag{11}$$

for the function $u(x)$ on the non-periodic interval $[-1, 1]$, with a source term $f(x)$. Use the Dirichlet conditions $u(-1) = u(1) = 0$. Write a program that can find u represented on a Chebyshev grid with $N + 1$ grid points. Since Eq. (11) is nonlinear, you will need to solve this using the Newton method.[3]

(a) Use the method of manufactured solutions, with the solution

$$u(x) = e^x(x^2 - 1). \tag{12}$$

Calculate what f will be in order for u to satify Eq. (11).

(b) For a range of N from 4 to 64, calculate the numerical solution $p_N(x)$ to Eq. (11). Start your Newton method from the function $(x + 1)^2(x - 1)$ as an initial guess.[4] Make a semilog plot of $\|p_N - u\|_2$ as a function of N.

[2] It may be helpful to know that the Fourier transform of $e^{i\lambda x}$ is $\delta(k - \lambda)$ where δ is the Dirac delta function.

[3] This is similar to Question 3 on Homework 1.

[4] The nonlinear system has multiple solutions. This function is close enough to Eq. (12) that your Newton method will reliably converge to it.