Problem 1

In this problem, no explanation is required. All parts are worth 2 points.

(a) True or false: In a free abelian group of finite rank, every linearly independent set can be completed to a basis.

(b) How many different (up to isomorphism) abelian groups of order 300 are there?

(c) True or false: For any action of a finite group G on a set X, the cardinality $|X|$ divides $|G|$.

(d) Give an example of an infinite group G such that every element of G has finite order.

(e) Let F_2 be the free group on two generators. True or false: For every n, there exists a normal subgroup $H_n \subset F_2$ such that $F_2/H_n \cong S_n$?

Problem 2

Let $\mathbb{Q}^\times$ be the group of non-zero rational numbers under multiplication.

(a) Show that $\mathbb{Q}^\times$ is isomorphic to the product of $\mathbb{Z}/2\mathbb{Z}$ and a free abelian group.

(b) Describe all group homomorphisms $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}^\times$.

(c) Describe all group homomorphisms $\mathbb{Q}^\times \to \mathbb{Z}/2\mathbb{Z}$.

Problem 3

Let G be a group of order $2017 \times 2027 \times 2029$ (these are all prime numbers). Show that G is cyclic.

Problem 4

Let G be a finite group, and let $A = \mathrm{Aut}(G)$ be the group of automorphisms $\phi : G \to G$. Consider the natural action of A on G, and take the quotient G/A.

(a) What is $|G/A|$ if $G = \mathbb{Z}/6\mathbb{Z}$?

(b) Show that if $|G/A| = 2$, then $G \cong (\mathbb{Z}/p\mathbb{Z})^n$ for a prime p and $n > 0$.

Problem 5

A finite group G acts transitively (that is, with a single orbit) on a finite set X such that $|X| > 1$. Show that there exists an element $g \in G$ which does not fix any element of X.

Problem 6

A map $\phi : \mathbb{R} \to \mathbb{R}$ is said to be an *affine-linear bijection* if it is of the form

$$\phi(x) = ax + b \quad (a, b \in \mathbb{R} : a \neq 0).$$

(a) Show that the set of affine-linear bijections forms a group G under composition.

(b) Show that G is isomorphic to semidirect product of *abelian* groups A and B. Make sure to identify the groups A and B, as well as the action of one on the other used in the semidirect product.