Exercise 1.5.14

If $N_1 \triangleleft G_1$, $N_2 \triangleleft G_2$, then $(N_1 \times N_2) \triangleleft (G_1 \times G_2)$ and $(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$.

Proof. Let $(n_1, n_2) \in N_1 \times N_2$ and $(g_1, g_2) \in G_1 \times G_2$. Then

$$(g_1, g_2)(n_1, n_2)(g_1, g_2)^{-1} = (g_1 n_1 g_1^{-1}, g_2 n_2 g_2^{-1}) \in N_1 \times N_2$$

since $N_i \triangleleft G_i$ for $i = 1, 2$. Thus $N_1 \times N_2 \triangleleft G_1 \times G_2$.

Now define $\varphi : G_1 \times G_2 \to (G_1/N_1) \times (G_2/N_2)$ by $\varphi(g_1, g_2) = (g_1 N_1, g_2 N_2)$. This is a homomorphism since

$$\begin{aligned}\varphi((g_1, g_2)(h_1, h_2)) &= \varphi(g_1 h_1, g_2 h_2) = (g_1 h_1 N_1, g_2 h_2 N_2)\\ &= (g_1 N_1, g_2 N_2)(h_1 N_1, h_2 N_2) = \varphi(g_1, g_2)\varphi(h_1, h_2)\end{aligned}$$

for all $(g_i, h_i) \in G_i$, $i = 1, 2$. It is surjective since for any $(g_1' N_1, g_2' N_2) \in (G/N_i)$ we have $\varphi(g_1', g_2') = (g_1' N_i, g_2' N_i)$.

Finally,

$$\ker(\varphi) = \{(g_1, g_2) : (g_1 N_i, g_2 N_i) = (N_i, N_i)\} = \{(g_1, g_2) : g_1 \in N_1, g_2 \in N_2\} = N_1 \times N_2$$

Thus by the First Isomorphism Theorem,

$$(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$$

as desired. □

1.6.11

Find all normal subgroups of D_n.

For notation, let a be a rotation of order n and b be a reflection of order 2. Then $D_n = \langle a, b : a^n = e, b^2 = e, bab = a^{-1}\rangle$. If n is odd then we have that $\langle a^i \rangle \triangleleft D_n$ for all i dividing n, and these are the only normal subgroups. If n is even then we have that $\langle a^i \rangle \triangleleft D_n$ for all i dividing n, as well as $\langle a^2, b\rangle \triangleleft D_n$ and $\langle a^2, ab\rangle \triangleleft D_n$, and these are the only normal subgroups. This is because the rotations form a cyclic subgroup which is normal, and the conjugacy classes of reflections depend on the parity of n.

1.8.2

Give an example of groups H_i, K_j such that $H_1 \times H_2 \cong K_1 \times K_2$ and no H_i is isomorphic to any K_j.

Consider $H_1 = \mathbb{Z}_4, H_2 = \mathbb{Z}_3, K_1 = \mathbb{Z}_6, K_2 = \mathbb{Z}_2$. Then $H_1 \times H_2 \cong \mathbb{Z}_{12} \cong K_1 \times K_2$, but no H_i is isomorphic to any K_j.

1.8.3

Let G be an (additive) abelian group with subgroups H and K. Show that $G \cong H \oplus K$ if and only if there are homomorphisms $H \underset{\iota_1}{\overset{\pi_1}{\leftrightarrows}} G \underset{\iota_2}{\overset{\pi_2}{\leftrightarrows}} K$ such that $\pi_1\iota_1 = 1_H, \pi_2\iota_2 = 1_K, \pi_1\iota_2 = 0$, and $\pi_2\iota_1 = 0$, where 0 is the map sending every element onto the zero (identity) element, and $\iota_1\pi_1(x) + \iota_2\pi_2(x) = x$ for all $x \in G$.

Proof. $(\Rightarrow)$ Suppose $G \cong H \oplus K$. Then every $g \in G$ can be uniquely written as $g = h + k$ for some $h \in H, k \in K$. Define $\pi_1 : G \to H$ by $\pi_1(g) = h$ and $\pi_2 : G \to K$ by $\pi_2(g) = k$. Also define $\iota_1 : H \to G$ by $\iota_1(h) = h + 0_K$ and $\iota_2 : K \to G$ by $\iota_2(k) = 0_H + k$. Then for any $h \in H, k \in K, g \in G$ we have

$$\begin{aligned}\pi_1\iota_1(h) &= \pi_1(h + 0_K) = h, & \pi_2\iota_2(k) &= \pi_2(0_H + k) = k,\\ \pi_1\iota_2(k) &= \pi_1(0_H + k) = 0_H, & \pi_2\iota_1(h) &= \pi_2(h + 0_K) = 0_K,\\ \iota_1\pi_1(g) + \iota_2\pi_2(g) &= (h + 0_K) + (0_H + k) = h + k = g.\end{aligned}$$

Thus the desired homomorphisms exist.

$(\Leftarrow)$ Suppose the homomorphisms π_i, ι_i exist as described. Then for any $g \in G$, we have

$$g = \iota_1\pi_1(g) + \iota_2\pi_2(g)$$

where $\iota_1\pi_1(g) \in H$ and $\iota_2\pi_2(g) \in K$. Thus every element of G can be written as a sum of an element of H and an element of K. Now suppose $h + k = h' + k'$ for some $h, h' \in H$ and $k, k' \in K$. Then

$$\begin{aligned}
h + k &= h' + k' \\
\iota_1\pi_1(h + k) + \iota_2\pi_2(h + k) &= \iota_1\pi_1(h' + k') + \iota_2\pi_2(h' + k') \\
\iota_1(\pi_1(h) + \pi_1(k)) + \iota_2(\pi_2(h) + \pi_2(k)) &= \iota_1(\pi_1(h') + \pi_1(k')) + \iota_2(\pi_2(h') + \pi_2(k')) \\
\iota_1(\pi_1(h) + 0_H) + \iota_2(0_K + \pi_2(k)) &= \iota_1(\pi_1(h') + 0_H) + \iota_2(0_K + \pi_2(k')) \\
\iota_1\pi_1(h) + \iota_2\pi_2(k) &= \iota_1\pi_1(h') + \iota_2\pi_2(k') \\
h + k &= h' + k'
\end{aligned}$$

Thus the representation of elements in G as sums of elements from H and K is unique, and $G \cong H \oplus K$.

□

1.8.5

Let G, H be finite cyclic groups. Then $G \times H$ is cyclic if and only if $(|G|, |H|) = 1$.

Proof. ($\Rightarrow$) Suppose $G \times H$ is cyclic. Then there exists some $(g, h) \in G \times H$ such that $\langle (g, h) \rangle = G \times H$. Thus $|\langle (g, h) \rangle| = |G \times H| = |G||H|$. But $|\langle (g, h) \rangle| = \mathrm{lcm}(|g|, |h|)$, so $\mathrm{lcm}(|g|, |h|) = |G||H|$. Since $|g|$ divides $|G|$ and $|h|$ divides $|H|$, we have that $\mathrm{lcm}(|g|, |h|)$ divides $\mathrm{lcm}(|G|, |H|)$. Thus $\mathrm{lcm}(|G|, |H|)$ must be equal to $|G||H|$, which implies that $(|G|, |H|) = 1$.

($\Leftarrow$) Suppose $(|G|, |H|) = 1$. Let g be a generator of G and h be a generator of H. Then consider the element $(g, h) \in G \times H$. We have that $|(g, h)| = \mathrm{lcm}(|g|, |h|) = \mathrm{lcm}(|G|, |H|) = |G||H|$ since $(|G|, |H|) = 1$. Thus $|(g, h)| = |G \times H|$, so $\langle (g, h) \rangle = G \times H$ and $G \times H$ is cyclic. □

1.8.9

If a group G is the (internal) direct product of its subgroups H, K, then $H \cong G/K$ and $G/H \cong K$.

Proof. □

1.9.1

Every nonidentity element in a free group F has infinite order.

1.9.4

Let F be the free group on the set X, and let $Y \subset X$. If H is the smallest normal subgroup of F containing Y, then F/H is a free group.