

Math/CS 714: Assignment 4

Problem 1

Beam-Warming method (4 points). The Beam-Warming method for the linear advection equation $u_t + au_x = 0$ is given by

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{a^2k^2}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n), \quad (1)$$

where U_j^n is the approximation of $u(jh, nk)$.

- a) Use Taylor series to show that this method is second-order accurate.
- b) For a given plane wave solution $U_j^0 = e^{ijh\xi}$, compute the amplification factor $g(\xi)$, and hence determine the stability restriction for this method.

- a) For brevity, define $U_j^n := U$. Then we can Taylor expand U about (j, n) and we have

$$\begin{aligned} U_{j-1}^n &= U - hU_x + \frac{h^2}{2}U_{xx} \\ U_{j-2}^n &= U - 2hU_x + 2h^2U_{xx} \\ U_j^{n+1} &= U + kU_t + \frac{k^2}{2}U_{tt} + O(k^3) \end{aligned}$$

Substituting these into the Beam-Warming method gives

$$\begin{aligned} U + kU_t + \frac{k^2}{2}U_{tt} + O(k^3) &= U - \frac{ak}{2h} \left(3U - 4 \left(U - hU_x + \frac{h^2}{2}U_{xx} \right) + (U - 2hU_x + 2h^2U_{xx}) \right) \\ &\quad + \frac{a^2k^2}{2h^2} \left(U - 2 \left(U - hU_x + \frac{h^2}{2}U_{xx} \right) + (U - 2hU_x + 2h^2U_{xx}) \right) + O(h^3) \\ &= U - \frac{ak}{2h} (3U - 4U + 4hU_x - 2h^2U_{xx} + U - 2hU_x + 2h^2U_{xx}) \\ &\quad + \frac{a^2k^2}{2h^2} (U - 2U + 2hU_x - h^2U_{xx} + U - 2hU_x + 2h^2U_{xx}) + O(h^3) \\ &= U - \frac{ak}{2h} (2hU_x) + \frac{a^2k^2}{2h^2} (h^2U_{xx}) + O(h^3) \\ &= U - akU_x + \frac{a^2k^2}{2}U_{xx} + O(h^3) \end{aligned}$$

To reiterate, we have

$$kU_t + \frac{k^2}{2}U_{tt} + O(k^3) = -akU_x + \frac{a^2k^2}{2}U_{xx} + O(h^3)$$

Dividing by k gives

$$U_t + \frac{k}{2}U_{tt} + O(k^2) = -aU_x + \frac{a^2k}{2}U_{xx} + O(h^2)$$

Then we can use the fact that $u_t = -au_x$ and $u_{tt} = a^2u_{xx}$, and substitute for $U_{tt} = a^2U_{xx}$ which gives

$$U_t + aU_x = O(k^2) + O(h^2)$$

Thus the Beam-Warming method is second-order accurate in both space and time.

- b) Given that $U_j^0 = e^{ijh\xi}$, we can calculate the amplification factor with $g(\xi)e^{ijh\xi} = U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{a^2k^2}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$. Plugging in the plane wave solution and letting $\nu = \frac{ak}{h}$ we have:

$$\begin{aligned} g(\xi)e^{ijh\xi} &= e^{ijh\xi} - \frac{\nu}{2} \left(3e^{ijh\xi} - 4e^{i(j-1)h\xi} + e^{i(j-2)h\xi} \right) + \frac{\nu^2}{2} \left(e^{ijh\xi} - 2e^{i(j-1)h\xi} + e^{i(j-2)h\xi} \right) \\ \implies g(\xi) &= \frac{e^{ijh\xi} - \frac{\nu}{2} (3e^{ijh\xi} - 4e^{i(j-1)h\xi} + e^{i(j-2)h\xi}) + \frac{\nu^2}{2} (e^{ijh\xi} - 2e^{i(j-1)h\xi} + e^{i(j-2)h\xi})}{e^{ijh\xi}} \\ &= 1 - \frac{\nu}{2} (3 - 4e^{-ih\xi} + e^{-2ih\xi}) + \frac{\nu^2}{2} (1 - 2e^{-ih\xi} + e^{-2ih\xi}) \end{aligned}$$

Then, noticing that $e^x = 1 + x + \frac{x^2}{2} + \dots$ we have:

$$\begin{aligned} g(\xi) &= 1 - \frac{\nu}{2} \left(3 - 4 \left(1 - ih\xi + \frac{(ih\xi)^2}{2} \right) + (1 - 2ih\xi + 2(ih\xi)^2) \right) + \frac{\nu^2}{2} \left(1 - 2 \left(1 - ih\xi + \frac{(ih\xi)^2}{2} \right) + (1 - 2ih\xi + 2(ih\xi)^2) \right) \\ &= 1 - \frac{\nu}{2} (0 + 2ih\xi - (ih\xi)^2) + \frac{\nu^2}{2} (0 + 0 + (ih\xi)^2) + O((h\xi)^3) \\ &= 1 - i\nu h\xi + \frac{\nu}{2}(h\xi)^2 + \frac{\nu^2}{2}(h\xi)^2 + O((h\xi)^3) \\ &= 1 - i\nu h\xi + \frac{\nu(1+\nu)}{2}(h\xi)^2 + O((h\xi)^3) \end{aligned}$$

Then we can compute $|g(\xi)|^2$:

$$\begin{aligned} |g(\xi)|^2 &= \left(1 + \frac{\nu(1+\nu)}{2}(h\xi)^2 + O((h\xi)^3) \right)^2 + (-\nu h\xi + O((h\xi)^3))^2 \\ &= 1 + \nu(1+\nu)(h\xi)^2 + \nu^2(h\xi)^2 + O((h\xi)^3) \\ &= 1 + \nu(1+2\nu)(h\xi)^2 + O((h\xi)^3) \end{aligned}$$

For stability, we require that $|g(\xi)|^2 \leq 1$ for all ξ . Therefore we need $\nu(1+2\nu) \leq 0$. This gives the stability restriction of $-\frac{1}{2} \leq \nu \leq 0$.

Problem 2

(9 points). Dropping the last term in the Beam–Warming method from Eq. (1) gives

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n), \quad (2)$$

which corresponds to forward Euler method in time, and a second-order one-sided derivative in space. Define $\nu = ak/h$.

- a) Calculate the amplification factor $g(\xi)$ for a plane wave solution $U_j^0 = e^{ijh\xi}$.
- b) Define $A(\xi) = |g(\xi)|^2$ and calculate a Taylor series for A at $\xi = 0$ up to second order. Using the Taylor series, explain why we consider the numerical scheme of Eq. (2) to be unstable regardless of the choice of timestep.
- c) Make two plots of $A(\xi)$ for $\nu = 1/100$ using two different axis ranges:
 - $0 \leq h\xi \leq 2\pi$ and $0.91 \leq A \leq 1.01$,
 - $0 \leq h\xi \leq 0.17$ and $1 - 10^{-6} \leq A \leq 1 + 10^{-6}$.
- d) Write a program to simulate Eq. (2) on a periodic interval $[0, 2\pi]$ using $N = 40$ grid points and a grid spacing of $h = 2\pi/N$. Use the initial condition $u = \exp(2 \sin x)$ and $\nu = 1/100$. Plot the solution for $n = 0, 1000, 2000, 4000$. Define the root mean squared value of the solution,

$$R(n) = \sqrt{\frac{1}{N} \sum_{j=0}^{N-1} (U_j^n)^2}. \quad (3)$$

Make a plot of R over the range from $n = 0$ to $n = 10000$. You should find that R does not grow over time, indicating that the method is stable.

- e) Using the discrete Fourier transform, it can be shown that an arbitrary initial condition on the periodic interval can be written as

$$U_j^0 = \sum_{l=0}^{N-1} \alpha_l e^{ijlh} \quad (4)$$

for some constants α_l . Write down an expression for the general solution U_j^n . Using your answer, explain why your result in part (d) does not contradict the result in part (b).

- a) We can calculate the amplification factor with $g(\xi)e^{ijh\xi} = U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n)$. Plugging in the plane wave solution and simplifying as in the previous problem gives $g(\xi) = 1 - \frac{\nu}{2}(3 - 4e^{-ih\xi} + e^{-2ih\xi})$.

- b) Taking $A(\xi) = |g(\xi)|^2$ and expanding in a Taylor series about $\xi = 0$, we can use the fact that $e^x = 1 + x + \frac{x^2}{2} + \dots$ to find that

$$\begin{aligned} A(\xi) &= \left| 1 - \frac{\nu}{2} \left(3 - 4(1 - ih\xi + \frac{(ih\xi)^2}{2}) + (1 - 2ih\xi + 2(ih\xi)^2) \right) \right|^2 \\ &= \left| 1 - \frac{\nu}{2} (0 + 2ih\xi - (ih\xi)^2) \right|^2 \end{aligned}$$

Thus we have

$$\begin{aligned} A(\xi) &= \left(1 + \frac{\nu}{2} (h\xi)^2 \right)^2 + (-\nu h\xi)^2 \\ &= 1 + \nu(h\xi)^2 + \frac{\nu^2}{4} (h\xi)^4 + \nu^2 (h\xi)^2 \\ &= 1 + \nu(1 + \nu)(h\xi)^2 + O((h\xi)^4) \end{aligned}$$

Since $\nu(1 + \nu) > 0$ for all $\nu > 0$, we have that $A(\xi) > 1$ for sufficiently small but nonzero ξ . This indicates that the method is unstable regardless of the choice of timestep.

- c) In Figure 1 and Firgure 2 we have the two plots of $A(\xi)$ for $\nu = \frac{1}{100}$.

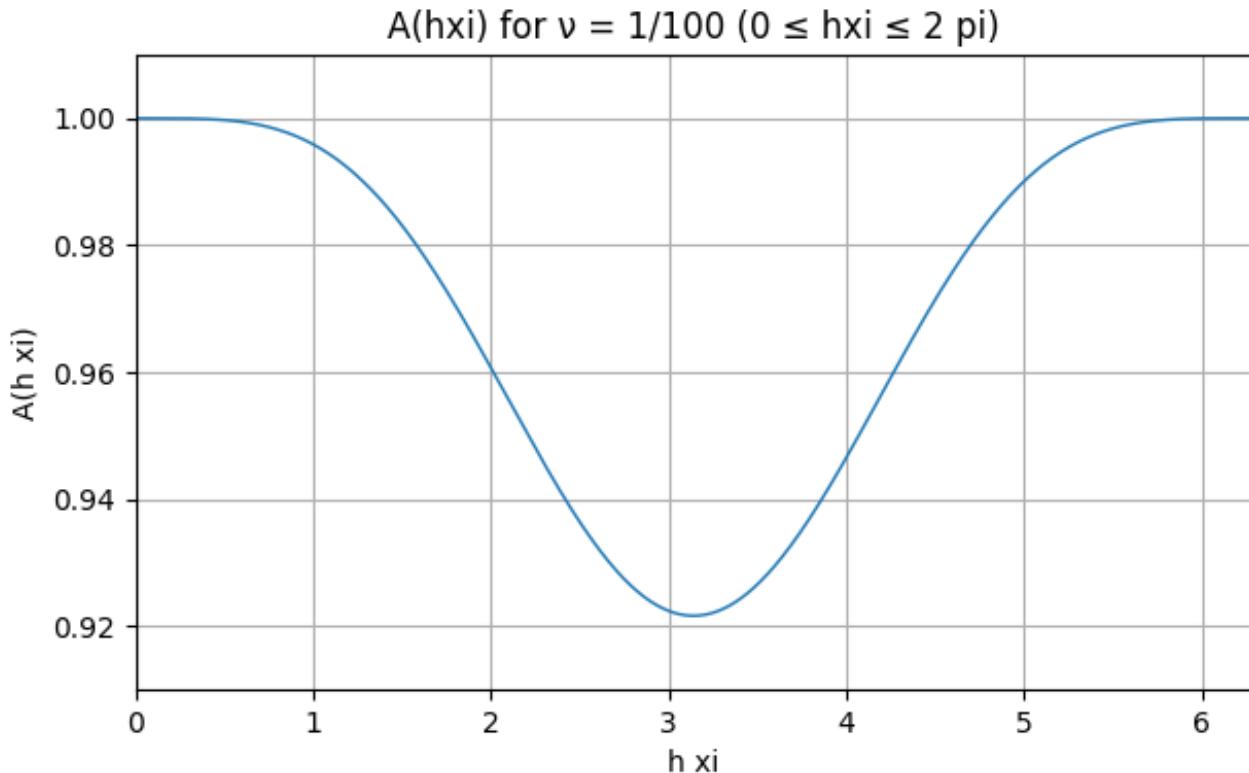


Figure 1: $A(\xi)$ for $0 \leq h\xi \leq 2\pi$ and $0.91 \leq A \leq 1.01$.

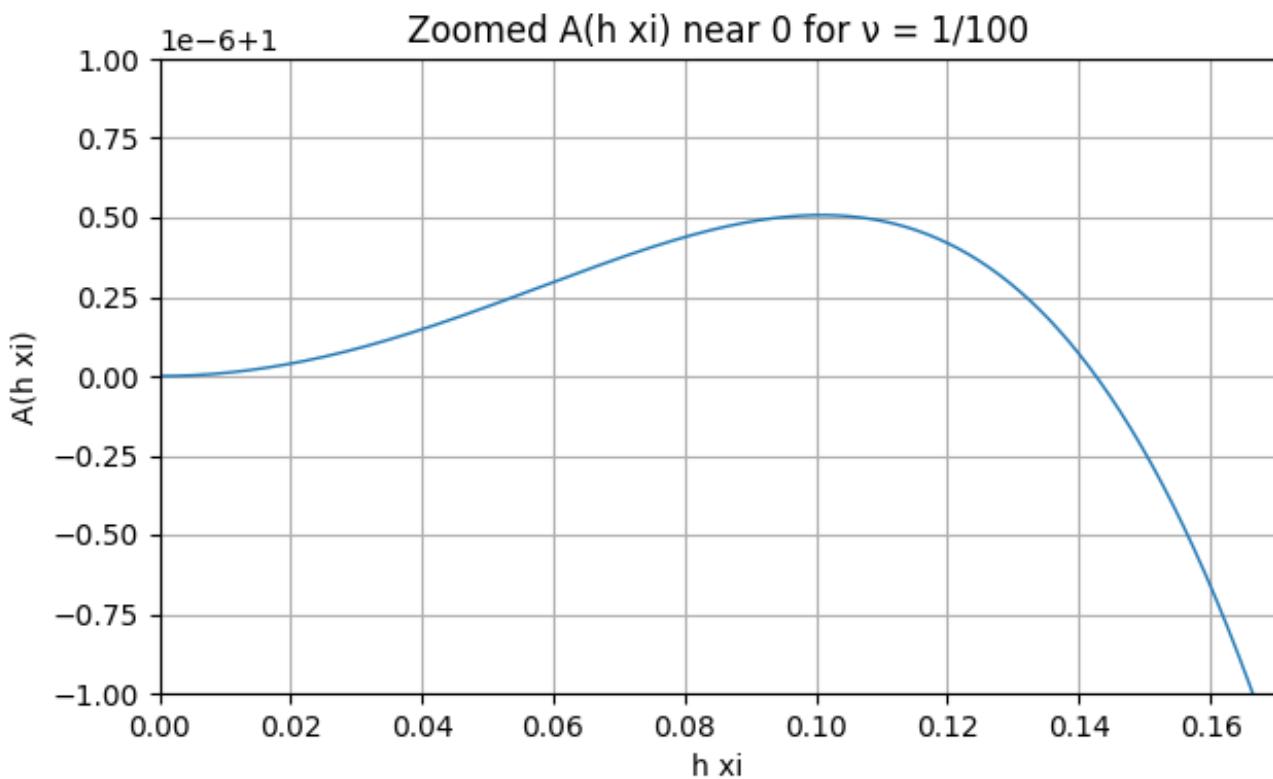


Figure 2: $A(\xi)$ for $0 \leq h\xi \leq 0.17$ and $1 - 10^{-6} \leq A \leq 1 + 10^{-6}$.

- d) See Figure 3 for the plots of the solution at $n = 0, 1000, 2000, 4000$ and Figure 4 for the plot of R over the range from $n = 0$ to $n = 10000$.

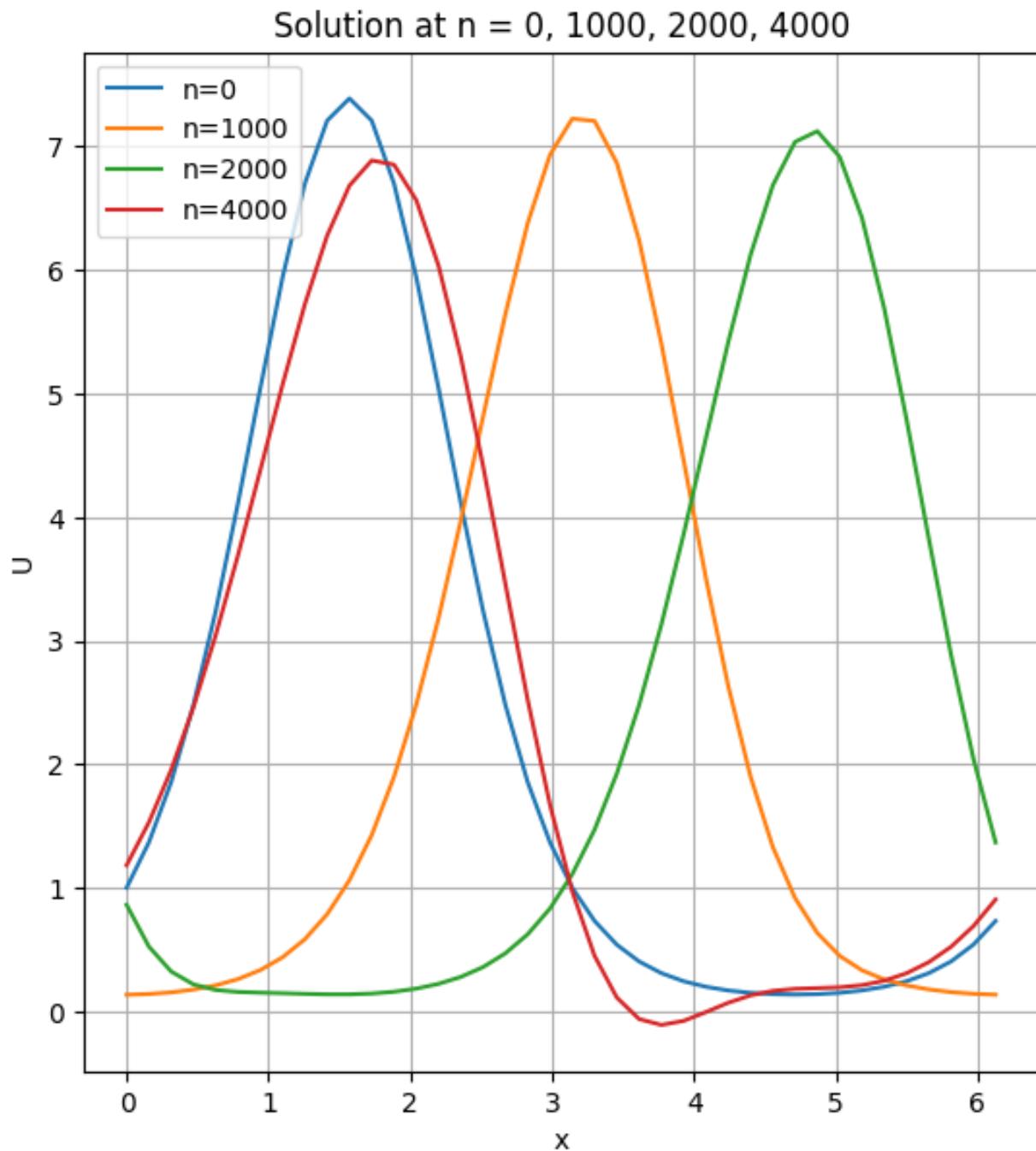


Figure 3: Plots of the solution at $n = 0, 1000, 2000, 4000$.

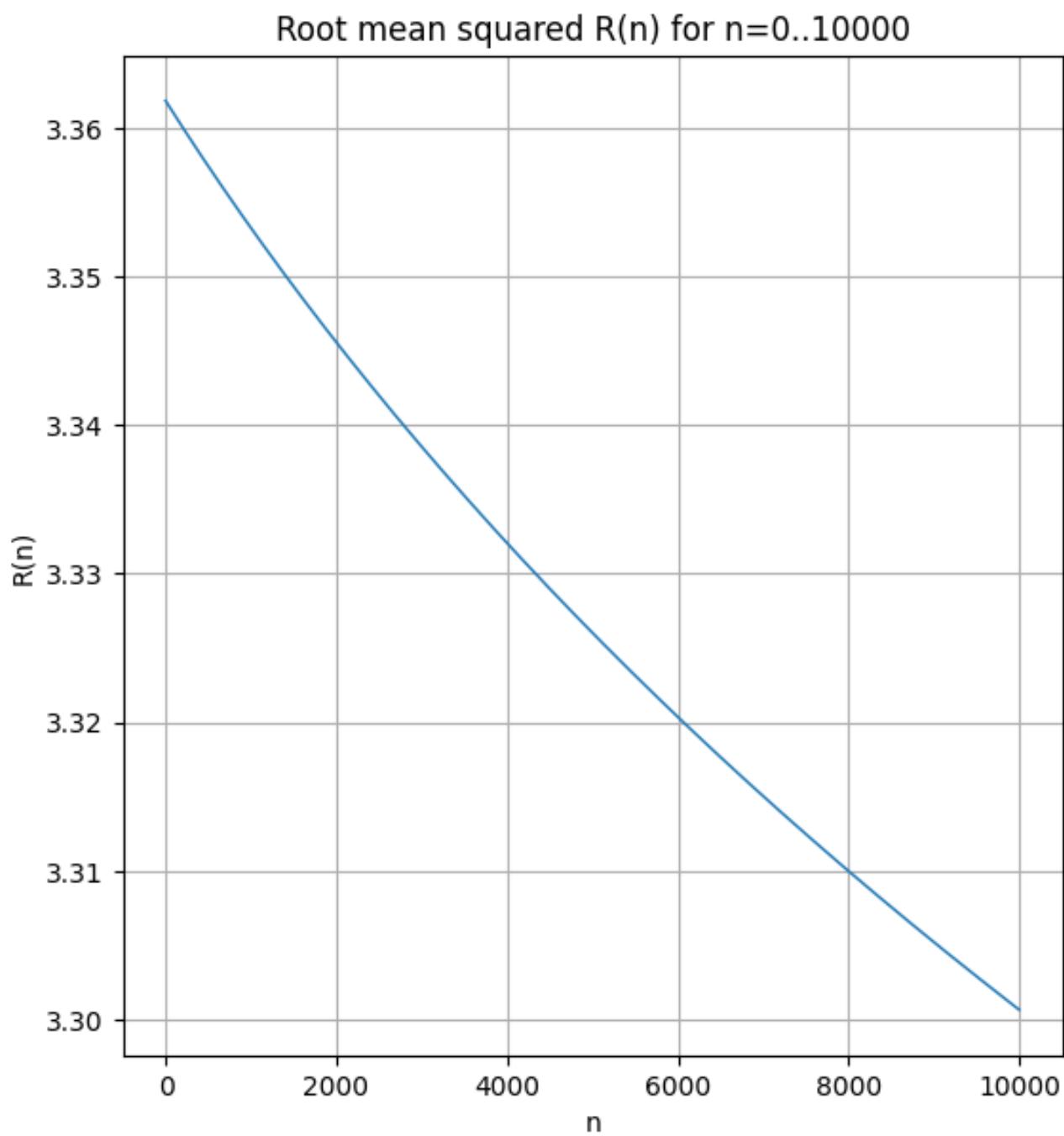


Figure 4: Plot of R over the range from $n = 0$ to $n = 10000$.

Problem 3

Lax–Wendroff method (7 points). Consider the hyperbolic conservation equation

$$q_t + [A(x)q]_x = 0 \quad (5)$$

for a function on $q(x, t)$ on the periodic interval $[0, 2\pi]$. Let $A(x) = 2 + \frac{4}{3} \sin x$. Following the finite volume approach, divide the intervals into m domains \mathcal{C}_i of length $h = \frac{2\pi}{m}$, for $i = \{0, 1, \dots, m-1\}$. Let $Q_i^n \approx q((i+1/2)h, n\Delta t)$ be the discretized solution at the center of each \mathcal{C}_i . The generalized Lax–Wendroff scheme for this equation is given by

$$Q_i^{n+1} = Q_i - \frac{\Delta t}{h} [\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n] \quad (6)$$

where the fluxes are

$$\mathcal{F}_{i-1/2}^n = \frac{A_{i-1}Q_{i-1}^n + A_iQ_i^n}{2} - \frac{A_{i-1/2}\Delta t}{2h} [A_iQ_i^n - A_{i-1}Q_{i-1}^n]. \quad (7)$$

Here, $A_i = A((i+1/2)h)$ and $A_{i-1/2} = A(ih)$. It can be shown that the solution to Eq. (5) is time-periodic so that $q(x, t+T) = q(x, t)$ where $T = 3\pi/\sqrt{5}$.

- a) The CFL condition requires that $\Delta t \leq \frac{h}{c}$ for stability. What is c in this case?
- b) Implement Eq. (7) and set $\Delta t = \frac{h}{3c}$. Use the initial condition

$$q(x, 0) = \exp(\sin x + \frac{1}{2} \sin 4x). \quad (8)$$

For $m = 512$, plot snapshots of the solution for $t = 0, \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}, T$.^a

- c) By considering a range of m (e.g. 256 and upward) with the initial condition in Eq. (8), calculate the L_2 norm between the numerical solution at $t = T$ and the exact answer. Determine the order of convergence.^b
- d) Repeat parts (b) and (c) for the initial condition

$$q(x, 0) = \max\left\{\frac{\pi}{2} - |x - \pi|, 0\right\}. \quad (9)$$

- e) **Optional.** By the considering the characteristics, or otherwise, derive the result that q is time-periodic with period T .

^aSince multiples of Δt do not exactly match these snapshot times, you may need to make a small adjustment to the timestep.

^bWhen determining the order of convergence, you are interested in the asymptotic properites of error as m gets large. You can ignore initial transients in error.

- a) The wave (characteristic) speed is $A(x)$, so

$$c = \max_{x \in [0, 2\pi]} |A(x)| = \max_x \left(2 + \frac{4}{3} \sin x\right) = 2 + \frac{4}{3} = \frac{10}{3}.$$

(Here $A(x) \geq 2 - \frac{4}{3} = \frac{2}{3} > 0$, hence the maximum absolute value is 10/3.) Thus the CFL condition is $\Delta t \leq h/c = \frac{3h}{10}$.