

Problem 1

Let G be a finite group. Prove the following 'inverse Schur Lemma': if every G -equivariant map $\phi : V \rightarrow V$ is scalar, the V is irreducible. You may need to make additional assumptions on the field for the statement to hold, make sure to clearly state this!

Proof. To prove the 'inverse Schur Lemma', we need to show that if every G -equivariant map $\phi : V \rightarrow V$ is scalar, then V is irreducible. We will assume that the field is algebraically closed ($K = \overline{K}$).

Suppose V is reducible. Then there exists a non-trivial proper subrepresentation $W \subset V$ such that W is invariant under the action of G . This means that for any $v \in W$ and $g \in G$, we have $g \cdot v \in W$.

Consider the projection map $\pi : V \rightarrow W$ defined by $\pi(v) = v$ for $v \in W$ and $\pi(v) = 0$ for $v \in V \setminus W$. This map is G -equivariant because for any $g \in G$ and $v \in W$,

$$\pi(g \cdot v) = g \cdot v \in W \quad \text{and} \quad g \cdot \pi(v) = g \cdot v.$$

Since W is a subrepresentation, π is well-defined and G -equivariant.

Now, consider the map $\phi : V \rightarrow V$ defined by $\phi(v) = v - \pi(v)$. This map is also G -equivariant because for any $g \in G$ and $v \in V$,

$$\phi(g \cdot v) = g \cdot v - \pi(g \cdot v) = g \cdot v - g \cdot \pi(v) = g \cdot (v - \pi(v)) = g \cdot \phi(v).$$

However, ϕ is not scalar because it maps elements of W to elements of $V \setminus W$ and vice versa. This contradicts the assumption that every G -equivariant map $\phi : V \rightarrow V$ is scalar.

Therefore, our assumption that V is reducible must be false. Hence, V must be irreducible. \square

Problem 2

Suppose V is a completely reducible representation of a group G . Show that for every subrepresentation $W \subset V$, the quotient V/W is completely reducible. (Obviously, we are assuming that Maschke's Theorem does not apply, otherwise the question is trivial.)

Proof. To show that for every subrepresentation $W \subset V$, the quotient V/W is completely reducible, we will use the fact that V is completely reducible. This means that V can be written as a direct sum of irreducible subrepresentations.

Let $W \subset V$ be a subrepresentation. We need to show that V/W is completely reducible. Since V is completely reducible, we can write

$$V = \bigoplus_{i=1}^m V_i,$$

where each V_i is an irreducible subrepresentation of V .

Consider the quotient representation V/W . We can write

$$V/W = \left(\bigoplus_{i=1}^m V_i \right) / W.$$

We need to show that V/W can be written as a direct sum of irreducible subrepresentations. To do this, we will show that each $V_i/(V_i \cap W)$ is an irreducible subrepresentation of V/W .

Let $V_i/(V_i \cap W)$ be a subrepresentation of V/W . We need to show that it is irreducible. Suppose there exists a non-trivial proper subrepresentation $U \subset V_i/(V_i \cap W)$. Then U corresponds to a subrepresentation $U' \subset V_i$ such that $U' \cap (V_i \cap W) = \{0\}$. Since V_i is irreducible, the only subrepresentations of V_i are $\{0\}$ and V_i itself. Therefore, $U' = V_i$ or $U' = \{0\}$.

If $U' = V_i$, then $U = V_i/(V_i \cap W) = V_i/W_i$, which is not a proper subrepresentation. If $U' = \{0\}$, then $U = \{0\}$, which is not a non-trivial subrepresentation. Therefore, $V_i/(V_i \cap W)$ must be irreducible.

Since each $V_i/(V_i \cap W)$ is irreducible, we can write

$$V/W = \bigoplus_{i=1}^m V_i/(V_i \cap W),$$

where each $V_i/(V_i \cap W)$ is an irreducible subrepresentation of V/W . Therefore, V/W is completely reducible. \square

Problem 3

Let V be an irreducible finite-dimensional representation of a group G . Show that its dual V^\times is irreducible as well.

Proof. Suppose V is an irreducible representation of G . We need to show that V^\times is irreducible. Assume for contradiction that V^\times is reducible. Then there exists a non-trivial proper subrepresentation $W \subset V^\times$.

Consider the natural pairing $\langle \cdot, \cdot \rangle : V \times V^\times \rightarrow \mathbb{C}$ defined by

$$\langle v, f \rangle = f(v)$$

for $v \in V$ and $f \in V^\times$. This pairing is G -equivariant because for any $g \in G$, $v \in V$, and $f \in V^\times$,

$$\langle g \cdot v, f \rangle = f(g \cdot v) = \langle v, g^{-1} \cdot f \rangle.$$

Since W is a subrepresentation of V^\times , the map $\pi : V^\times \rightarrow W$ defined by $\pi(f) = f$ for $f \in W$ and $\pi(f) = 0$ for $f \in V^\times \setminus W$ is a G -equivariant map.

Now, consider the map $\phi : V \rightarrow W$ defined by $\phi(v) = \pi(\langle v, \cdot \rangle)$. This map is G -equivariant because for any $g \in G$ and $v \in V$,

$$\phi(g \cdot v) = \pi(\langle g \cdot v, \cdot \rangle) = \pi(\langle v, g^{-1} \cdot \cdot \rangle) = g^{-1} \cdot \pi(\langle v, \cdot \rangle) = g^{-1} \cdot \phi(v).$$

However, ϕ is not scalar because it maps elements of V to elements of W and vice versa. This contradicts the assumption that V is irreducible.

Therefore, our assumption that V^\times is reducible must be false. Hence, V^\times must be irreducible. \square

Problem 4

Suppose that a finite group G is a product $G = G_1 \times G_2$. Let V be an irreducible representation of G over \mathbb{C} . Denote by $\text{res}_{G_1}^G V$ its restriction to G_1 : it is given by the composition

$$G_1 \rightarrow G \rightarrow GL(V).$$

Show that $\text{res}_{G_1}^G V \cong (V_1)^k$ for some irreducible representation V_1 of G_1 and $k \geq 1$. (In fact, a stronger statement holds: $V \cong V_1 \otimes V_2$, where V_1 is a representation of G_1 and V_2 is a representation of G_2 , but you do not need to prove this.)

Proof. Let $G = G_1 \times G_2$ and V be an irreducible representation of G over \mathbb{C} . We need to show that $\text{res}_{G_1}^G V$ is a direct sum of irreducible representations of G_1 .

Consider the restriction map $\text{res}_{G_1}^G : GL(V) \rightarrow GL(\text{res}_{G_1}^G V)$. This map is G_1 -equivariant because for any $g_1 \in G_1$ and $v \in V$,

$$\text{res}_{G_1}^G(g_1 \cdot v) = g_1 \cdot (\text{res}_{G_1}^G v).$$

Since V is irreducible, the only subrepresentations of V are $\{0\}$ and V itself. Therefore, the only subrepresentations of $\text{res}_{G_1}^G V$ are $\{0\}$ and $\text{res}_{G_1}^G V$ itself.

Now, consider the map $\phi : \text{res}_{G_1}^G V \rightarrow V$ defined by $\phi(v) = v$ for $v \in \text{res}_{G_1}^G V$. This map is G_1 -equivariant because for any $g_1 \in G_1$ and $v \in \text{res}_{G_1}^G V$,

$$\phi(g_1 \cdot v) = g_1 \cdot v.$$

Since $\text{res}_{G_1}^G V$ is a subrepresentation of V , ϕ is well-defined and G_1 -equivariant. However, ϕ is not scalar because it maps elements of $\text{res}_{G_1}^G V$ to elements of V and vice versa. This contradicts the assumption that V is irreducible. Therefore, $\text{res}_{G_1}^G V$ must be a direct sum of irreducible representations of G_1 . Hence, $\text{res}_{G_1}^G V \cong (V_1)^k$ for some irreducible representation V_1 of G_1 and $k \geq 1$. \square

Problem 5

Fix n , and denote X_k the set of all k -element subsets of $\{1, \dots, n\}$ ($k \leq n$). It carries an action of S_n , and we can consider the corresponding representation V_k of S_n , where V_k is the space of \mathbb{C} -valued functions on X_k . Show that $V_k \cong V_{n-k}$.

Proof. We will show that there is an isomorphism of representations between V_k and V_{n-k} . Let $A \in X_k$ be a k -element subset of $\{1, \dots, n\}$. Define a map $\phi : V_k \rightarrow V_{n-k}$ by sending a function $f \in V_k$ to a function $\phi(f) \in V_{n-k}$ defined as follows:

$$\phi(f)(B) = f(\{1, \dots, n\} \setminus B)$$

for every $(n-k)$ -element subset $B \in X_{n-k}$. Here, $\{1, \dots, n\} \setminus B$ is the complement of B in $\{1, \dots, n\}$, which is a k -element subset.

To show that ϕ is a representation isomorphism, we need to verify two things that ϕ is linear and ϕ commutes with the action of S_n .

First we show that ϕ is linear. For any $f_1, f_2 \in V_k$ and scalars $a, b \in \mathbb{C}$, we have

$$\phi(af_1 + bf_2)(B) = (af_1 + bf_2)(\{1, \dots, n\} \setminus B) = af_1(\{1, \dots, n\} \setminus B) + bf_2(\{1, \dots, n\} \setminus B) = a\phi(f_1)(B) + b\phi(f_2)(B).$$

Thus, ϕ is linear.

Next, we show that ϕ commutes with the action of S_n . For any $\sigma \in S_n$, we need to show that

$$\phi(\sigma \cdot f) = \sigma \cdot (\phi(f)).$$

By definition of the action on functions,

$$(\sigma \cdot f)(A) = f(\sigma^{-1}(A)).$$

Therefore,

$$\phi(\sigma \cdot f)(B) = (\sigma \cdot f)(\{1, \dots, n\} \setminus B) = f(\sigma^{-1}(\{1, \dots, n\} \setminus B)).$$

On the other hand,

$$(\sigma \cdot (\phi(f)))(B) = \phi(f)(\sigma^{-1}(B)) = f(\{1, \dots, n\} \setminus \sigma^{-1}(B)).$$

Since σ is a bijection, we have

$$\sigma^{-1}(\{1, \dots, n\} \setminus B) = \{1, \dots, n\} \setminus \sigma^{-1}(B).$$

Thus,

$$\phi(\sigma \cdot f)(B) = f(\{1, \dots, n\} \setminus \sigma^{-1}(B)) = (\sigma \cdot (\phi(f)))(B).$$

This shows that ϕ commutes with the action of S_n . Since ϕ is a linear bijection that commutes with the action of S_n , it is an isomorphism of representations. Therefore, we conclude that $V_k \cong V_{n-k}$. \square

Problem 6

(Continuation of the previous problem) Show that S_n has irreducible representations $W_0, W_1, W_2, \dots, W_{\lfloor n/2 \rfloor}$ such that

$$V_k \cong \bigoplus_{i=0}^k W_i$$

for all $k \leq \frac{n}{2}$.

Proof. We will prove this by induction on k .

Base case: For $k = 0$, we have V_0 is the space of functions on the empty set, which is one-dimensional. Thus $V_0 = W_0$ where W_0 is the trivial representation of S_n .

Inductive step: Assume that for some $k < \lfloor n/2 \rfloor$, we have

$$V_k \cong \bigoplus_{i=0}^k W_i$$

where W_0, W_1, \dots, W_k are irreducible representations of S_n . We need to show that

$$V_{k+1} \cong \bigoplus_{i=0}^{k+1} W_i.$$

Consider the restriction of V_{k+1} as a representation of S_n . We can write V_{k+1} as a sum of irreducible representations:

$$V_{k+1} = \bigoplus_j U_j^{(k+1)}$$

where the $U_j^{(k+1)}$ are irreducible.

By the previous problem, we know that $V_{k+1} \cong V_{n-(k+1)}$. Therefore, we can also write

$$V_{n-(k+1)} = \bigoplus_j U_j^{(n-(k+1))}$$

where the $U_j^{(n-(k+1))}$ are irreducible. Since $V_{k+1} \cong V_{n-(k+1)}$, we have

$$\bigoplus_j U_j^{(k+1)} \cong \bigoplus_j U_j^{(n-(k+1))}.$$

By the induction hypothesis, we know that $V_k \cong \bigoplus_{i=0}^k W_i$. Therefore, we can write

$$V_{k+1} = V_k \oplus U^{(k+1)}$$

where $U^{(k+1)}$ is an irreducible representation of S_n . This is because V_{k+1} is a direct sum of irreducible representations, and V_k is already a direct sum of irreducible representations. Thus, the remaining part must be an irreducible representation. Therefore, we have

$$V_{k+1} \cong \bigoplus_{i=0}^k W_i \oplus U^{(k+1)}.$$

Since $U^{(k+1)}$ is an irreducible representation, we can set $W_{k+1} = U^{(k+1)}$. Therefore, we have

$$V_{k+1} \cong \bigoplus_{i=0}^{k+1} W_i.$$

This completes the inductive step. □