Remarks:

A) Definition is just a definition, there is no need to jjustify or explain it.

B) Answers to questions with proofs should be written, as much as you can, in the following format:

   i) Statement
   ii) Main points that will appear in your proof
   iii) The actual proof

   Answers to questions with computations should be written, as much as possible, in the following format:

   i) Statement and Result
   ii) Main points that will appear in your computation.
   iii) The actual computation

Problem 1

Vector Spaces. Suppose $\mathbb{F}$ is a field.

a) Define when we say that a vector space V over a field $\mathbb{F}$ is *finite dimensional.*

b) Consider the vector space
$$V = \mathbb{F}[x]$$
of all polynomials with coefficients in $\mathbb{F}$. Show that V is not finite dimensional.

c) Suppose X is a finite set. Consider the vector space V, of all functions from X to $\mathbb{F}$,
$$V = \mathbb{F}(X) := \{\text{all } f : X \to \mathbb{F}; \text{s.t. } f \text{ is a function}\},$$
with the standard addition and multiplication by scalars from $\mathbb{F}$. Show that V is finite dimensional.

a) We say that a vector space V over a field $\mathbb{F}$ is finite dimensional if there exists $S \subseteq V$ such that $\#(S) < \infty$ and $\text{span}(S) = V$.

b) i) Statement: Show that $V = \mathbb{F}[x]$ is not finite dimensional.

   ii) Main Points:
   - Suppose V is finite dimensional with a finite spanning set S.
   - Let m be the maximum degree of the polynomials in S and find a polynomial $q(x)$ with degree greater than m.
   - Show that $q(x)$ cannot be written as a linear combination of the polynomials in S, leading to a contradiction.

   iii) Proof:

   Proof. Suppose V is finite dimensional. Then there exists $S \subseteq V$ such that $\#(S) < \infty$ and $\text{span}(S) = V$. Let $S = \{p_1(x), p_2(x), \ldots, p_n(x)\}$. Let $m = \max(\deg(p_i(x)))$ for $1 \leq i \leq n$. Then consider the polynomial $q(x) = x^{m+1}$. Since $\deg(q(x)) > m$, $q(x)$ cannot be written as a linear combination of the polynomials in S. This contradicts the fact that $\text{span}(S) = V$. Thus, V is not finite dimensional. □

c) i) Statement: Show that $V = \mathbb{F}(X)$ is finite dimensional.

   ii) Main Points:
   - Since X is finite, say $X = \{x_1, x_2, \ldots, x_n\}$, consider the set of functions $\{\delta_{x_i}\}_{i=1}^n$, where
   $$\delta_{x_i}(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
   - Show that $\{\delta_{x_i}\}_{i=1}^n$ forms a basis for V.

iii) Proof:

Proof. Since X is finite, say $X = \{x_1, x_2, \dots, x_n\}$, consider the set of functions $\{\delta_{x_i}\}_{i=1}^n$, where

$$\delta_{x_i}(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We will show that $\{\delta_{x_i}\}_{i=1}^n$ forms a basis for V. First, we show that they span V. Let $f \in V$. Then we can write

$$f(x) = \sum_{i=1}^{n} f(x_i)\delta_{x_i}(x)$$

for all $x \in X$. Thus, $\{\delta_{x_i}\}_{i=1}^n$ spans V. Next, we show that they are linearly independent. Suppose

$$\sum_{i=1}^{n} a_i \delta_{x_i}(x) = 0$$

for some scalars $a_i \in \mathbb{F}$. Evaluating at $x = x_j$, we get

$$a_j = 0$$

for all $j = 1, 2, \dots, n$. Thus, all $a_i = 0$, and $\{\delta_{x_i}\}_{i=1}^n$ are linearly independent. Therefore, $\{\delta_{x_i}\}_{i=1}^n$ forms a basis for V, and hence V is finite dimensional with $\dim(V) = n$. □

Problem 2

Short exact sequences. Suppose U, V, W are three vector spaces over $\mathbb{F}$. Consider the following sequence of spaces and linear transformations between them:

$$0 \to U \xrightarrow{\iota} V \xrightarrow{\epsilon} W \to 0, \tag{1}$$

where $0 \to U$, are the obvious maps from the zero space into U, and from the space W onto the zero space, respectively.

a) Define when we say that the sequence (1) is short exact sequence (s.e.s.).

b) Given two subspaces $U, V < V$, such that $V = U \oplus W$, Show that there is a natural s.e.s.associated with the spaces of functions $U = \mathbb{F}(U), V = \mathbb{F}(V)$ and $W = \mathbb{F}(Y \backslash X)$, where $Y \backslash X$ denotes set-minus, i.e., the set of elements which are in Y and are not in X.

a) A sequence (1) is a short exact sequence (s.e.s.) if the image of each map is equal to the kernel of the next map, i.e.,

$$\operatorname{im}(0 \to U) = \ker(\iota), \quad \operatorname{im}(\iota) = \ker(\epsilon), \quad \operatorname{im}(\epsilon) = \ker(0 \to W).$$

Since the maps from and to the zero space are trivial, this reduces to

$$\iota \text{ is injective}, \quad \epsilon \text{ is surjective}, \quad \operatorname{im}(\iota) = \ker(\epsilon).$$

b) i) Statement: There is a natural s.e.s. associated with the spaces of functions $U = \mathbb{F}(U), V = \mathbb{F}(V)$ and $W = \mathbb{F}(Y \backslash X)$.

ii) Main Points:

- Define the inclusion map $\iota : U \to V$ by extending functions by zero outside U.
- Define the projection map $\epsilon : V \to W$ by restricting functions to W.
- Show that ι is injective.
- Show that ϵ is surjective.
- Show that $\operatorname{im}(\iota) = \ker(\epsilon)$.

iii) Proof:

Proof. Since $V = U \oplus W$, every element $v \in V$ can be uniquely written as $v = u + w$ for some $u \in U$ and $w \in W$. Define the inclusion map $\iota : U \to V$ by

$$\iota(f)(x) = \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

for all $f \in U$. This map is injective because if $\iota(f) = 0$, then $f(x) = 0$ for all $x \in U$, which implies $f = 0$. Next, define the projection map $\epsilon : V \to W$ by

$$\epsilon(g)(x) = g(x)$$

for all $g \in V$ and $x \in W$. This map is surjective because for any $h \in W$, we can define a function $g \in V$ by

$$g(x) = \begin{cases} h(x) & \text{if } x \in W \\ 0 & \text{if } x \notin W \end{cases}$$

such that $\epsilon(g) = h$. Finally, we need to show that $\mathrm{im}(\iota) = \ker(\epsilon)$. If $f \in U$, then $\epsilon(\iota(f)) = 0$ since $\iota(f)$ is zero outside of U. Conversely, if $g \in V$ and $\epsilon(g) = 0$, then $g(x) = 0$ for all $x \in W$, which means that g must be in the image of ι. Therefore, we have shown that the sequence

$$0 \to U \xrightarrow{\iota} V \xrightarrow{\epsilon} W \to 0$$

is a short exact sequence. □

Remark(s): Since the problem stated that $U = \mathbb{F}(U)$, $V = \mathbb{F}(V)$ and $W = \mathbb{F}(Y \backslash X)$, I assumed that ι and ϵ were defined in terms of functions instead of elements of the vector spaces. If this is not the case, please let me know and I can adjust the proof accordingly.

Problem 3

Dimension. Denote by $\mathrm{Vect}_{\mathbb{F}}^{fd}$ the collection of finite-dimensional vector spaces over $\mathbb{F}$, with linear transformations between them.

a) State the fact about uniqueness and existence of unique dimenstion function

$$\dim : \mathrm{Vect}_{\mathbb{F}}^{fd} \to \mathbb{N},$$

that satisfies certain desired properties.
Def. For V finite dimensional, the integer $\dim(V)$ is called the <u>dimension</u> of V.

b) Show that $\dim(M_n(\mathbb{F})) = n^2$.

c) Suppose $1 + 1 \neq 0$ in $\mathbb{F}$. Consider the spaces $U = A_n(\mathbb{F})$, $V = M_n(\mathbb{F})$, $W = S_n(\mathbb{F})$, of anti-symmetric matrices ($A^T = -A$), all matrices, and symmetric matrices (sastisfy $A^T = A$), respectively.

   i) Show that, they form in a natural why a s.e.s.

   ii) Deduce that $\dim(A_n(\mathbb{F})) = \frac{n(n-1)}{2}$ and $\dim(S_n(\mathbb{F})) = \frac{n(n+1)}{2}$.

a)