Problem 1

Let V be an n-dimensional vector space, and let $\phi : V \to V$ be a linear map. Show that, for any n-linear antisymmetric form $\beta(v_1, \ldots, v_n)$ on n, we have

$$\beta(\phi(v_1), \ldots, \phi(v_n)) = \det(\phi)\beta(v_1, \ldots, v_n).$$

(This formalizes the following idea: any "unit of volume" on V, given by β, gets scaled by $\det(\phi)$ when we apply ϕ. In fact, this can be used as a definition of $\det(\phi)$: this way, some of its properties, such as independence of basis and multiplicativity become clear.)

Problem 2

Let V be the space of polynomials of degree at most n (over some field K; if you want, you can assume $K = \mathbb{R}$). Fix $a, b \in \mathbb{R}$ and consider the linear map

$$\phi : V \to V, \quad p(t) \mapsto p(at + b).$$

Compute $\det(\phi)$.

Problem 3

Let M be an $n \times n$ matrix (over some field). Let V be the space of $n \times n$ matrices. Consider the linear map $m_M : V \to V$ given by left multiplication by M:

$$A \mapsto MA.$$

Find $\det(m_M)$. (Of course, the answer depends on M.)

Problem 4

Let K be a field and V be a finite-dimensional vector space. Let

$$\gamma : V \times V \to K$$

be an antisymmetric bilinear form. Show that there exists $k \leq \frac{n}{2}$ and a basis

$$\{a1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_{n-2k}\}$$

of V such that

$$\gamma(a_i, b_i) = 1, \quad \gamma(b_i, a_i) = -1, \quad (i = 1, \ldots, k),$$

and γ vanishes on all other pairs of basis vectors. (When $n = 2k$, this is called a symplectic basis.)

Problem 5

Consider $\det(A)$ as a multivariable function of the entries of a real matrix A. Compute the directional lderivative of this function at the point $A = I$ in the direction of some matrix B. (Equivalently, find the linear approximation for $f(t) = \det(I + tB)$ at $t = 0$.)

Problem 6

Let A be a square $n \times n$ matrix whose characteristic polynomial has n roots in K, counting with multiplicity. Consider the Jordan form of A: suppose that it consists of blocks J_{λ_i, n_i}, where λ_i is the eigenvalue and n_i is the size of the block.

Express the following invariants of A in terms of n_i and λ_i: its characteristic polynomial, its minimal polynomial, the dimension of eigenspace for each λ (this is called "the geometric multiplicity of an eigenvalue") and rank. (No explanation is required.)

Problem 7

Following up on the previous problem, let us go in the opposite direction: explain how to find λ_i and n_i from the data of $\mathrm{rk}(A - \lambda I)^k$ for all $\lambda \in K$ and $k > 0$. In particular, this implies that the Jordan form is unique.