Problem 1

The discrete Fourier transform. Let V and W be two n-dimensional vector spaces over a field F, and $T : V \to W$ a linear transformation.

(a) Define when we say that T is invertible.

(b) Suppose V and W are finite dimensional. Show that TFAE:

1. T is invertible.
2. T maps a basis $\mathscr{B}$ of V to a basis $\mathscr{C} = \{T(v); v \in \mathscr{B}\}$ for W.

(c) Let $N > 1$, be an integer, and consider the set $\mathbb{Z}_N = \{0, 1, 2, \dots, N-1\}$, with the addition and multipication is defined modulo N. Inside the vector space $\mathscr{H} = \mathbb{C}(\mathbb{Z}_N)$ of all functions from $\mathbb{Z}_N$ to $\mathbb{C}$, consider the subset

$$\mathscr{D} = \{\delta_t : t \in \mathbb{Z}_N\},$$

where δ_t is the delta function at t, $\delta_t(s) = 1$ if $s = t$, and 0 otherwise, and consider the subset

$$\mathscr{E} = \{e_w : w \in \mathbb{Z}_N\},$$

where e_w is the function given by

$$e_w(s) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} ws}, \quad s \in \mathbb{Z}_N.$$

1. Show that $\mathscr{E}$ is a basis for $\mathscr{H}$. You can do it using the following facts.
   - The dimenson of $\mathscr{H}$ is N.
   - The elements of $\mathscr{E}$ are linearly independent. To show this you can use the fact that on $\mathscr{H}$ we have the so called "inner product"
   $$\langle , \rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C},$$
   given by
   $$\langle f, g \rangle = \sum_{s \in \mathbb{Z}_N} f(s)\overline{g(s)},$$
   where $\overline{g(s)}$ is the complex conjugate of $g(s)$. Then we have,
   Fact. The collection $\mathscr{E}$ satisfies
   $$\langle e_w, e_{w'} \rangle = \begin{cases} 1, & w = w' \\ 0, & w \neq w'. \end{cases}$$
   In particular, using the fact that $\langle , \rangle$ is linear in the first coordinate, i.e., $\langle f + f', g \rangle = \langle f, g \rangle + \langle f', g \rangle$ and $\langle af, g \rangle = a\langle f, g \rangle$ for every $f, f' \in \mathscr{H}, a \in \mathbb{C}$, it is easy to show the linear independency of $\mathscr{E}$.
2. The operator $F_N : \mathscr{H} \to \mathscr{H}$ that is given by
   $$F_N[\delta_t] = e_{-t}$$
   is called the discrete Fourier transform modulo N. For $f \in \mathscr{H}$, denote $\hat{f} = F_N(f)$. Show that
   1. we have the formula
   $$\hat{f}(w) = \frac{1}{\sqrt{N}} \sum_{t \in \mathbb{Z}_N} f(t) e^{-\frac{2\pi i}{N} wt},$$
   for $w \in \mathbb{Z}_N$
3. The operator F_N is invertible.

Problem 2

Diagonalization. Let T be an operator on a vector space V over a field $\mathbb{F}$.

(a) We say that

1. a scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if
2. a vector $v \in V, v \neq 0$ is an eigenvector, with eigenvalue $\lambda \in \mathbb{F}$, if

(b) Show that if $\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis of V, consisting of eigenvectors of T, then there exists scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$ such that

$$[T]_{\mathscr{B}} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Remark. The process (if possible) of finding a basis $\mathscr{B}$ of V consisting of eigenvectors of T, and the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, is called a diagonalization of T.

(c) Consider the space $\mathscr{H} = \mathbb{C}(\mathbb{Z}_3)$.

1. we have an operator called time shift

$$\begin{cases} L : \mathscr{H} \to \mathscr{H}, \\ L[f](t) = f(t-1), \end{cases}$$

for every $f \in \mathscr{H}, t \in \mathbb{Z}_3$. Find a diagonalization of L, and write the corresponding diagonal matrix

$$D = [L]_{\mathscr{B}}.$$

2. in addition, we have an operator called frequency shift

$$\begin{cases} M : \mathscr{H} \to \mathscr{H}, \\ M[f](t) = e^{\frac{2\pi i}{3} t} f(t), \end{cases}$$

for every $f \in \mathscr{H}, t \in \mathbb{Z}_3$. Find a diagonalization of M, and write the corresponding diagonal matrix

$$D = [M]_{\mathscr{B}}.$$

Problem 3

Heisenberg's commutation relations. consider the vector space $\mathscr{H} = \mathbb{C}(\mathbb{Z}_N)$.

(a) For every $\tau \in \mathbb{Z}_N$, we have an operator $L_\tau : \mathscr{H} \to \mathscr{H}$, called time shift, given by

$$L_\tau[f](t) = \text{-----}$$

and, for every $\omega \in \mathbb{Z}_N$, we have an operator $M_\omega : \mathscr{H} \to \mathscr{H}$, called frequency shift, given by

$$M_\omega[f](t) = \text{-----}$$

(b) Show that $M_\omega \circ L_\tau = e^{-\frac{2\pi i}{N}\omega\tau} L_\tau \circ M_\omega$, for every $\tau, \omega \in \mathbb{Z}_N$.

(c) Show that for every $\tau, \omega \in \mathbb{Z}_N$,

$$L_\tau \circ F_N = F_N \circ M_\tau,$$

where F_N is the discrete Fourier transform modulo N described in Problem 1.