Problem 1

Minimal Polynomials and Diagonalizability. Let $T : V \to V$ be a linear transformation on a finite imensional vcetor space V over a field $\mathbb{F}$.

(a) Define the minimal polynomial of T, denoted $m_T(x)$.

(b) Prove the following:

Theorem 0.1. *T is diagonalizable if and only if $m_T(x)$ is a product of different linear factors of the form $x-\lambda$, for some $\lambda \in \mathbb{F}$.*

(a) The minimal polynomial of T, denoted $m_T(x)$, is the monic polynomial of smallest degree with coefficients in $\mathbb{F}$ such that $m_T(T) = 0$.

(b) *Proof.* ($\Rightarrow$) Suppose T is diagonalizable. Then there exists a basis $\{v_1, \ldots, v_n\}$ of V consisting of eigenvectors of T with corresponding eigenvalues $\lambda_1, \ldots, \lambda_k$ (distinct). Let $p(x) = (x - \lambda_1)\cdots(x - \lambda_k)$. For any v_i, we have $p(T)(v_i) = (T - \lambda_1 I)\cdots(T - \lambda_k I)(v_i) = 0$ since v_i is an eigenvector. Thus $p(T) = 0$, so $m_T(x)$ divides $p(x)$. Since $m_T(x)$ is monic and has no repeated factors, $m_T(x) = (x - \lambda_1)\cdots(x - \lambda_m)$ for distinct $\lambda_i \in \mathbb{F}$.

($\Leftarrow$) Suppose $m_T(x) = (x - \lambda_1)\cdots(x - \lambda_k)$ with distinct λ_i. Then $V = \ker(T - \lambda_1 I) \oplus \cdots \oplus \ker(T - \lambda_k I)$ by the Primary Decomposition Theorem. Each kernel consists of eigenvectors, so V has a basis of eigenvectors of T. Thus T is diagonalizable. $\square$

Problem 2

Simultaneously diagonalizable operators. Let S, T be two operators defined on a finite-dimensional vector space V.

(a) Define when we say that S and T are simultaneously diagonalizable.

(b) Suppose S and T are both diagonalizable. Show that TFAE:

1. S and T commute, i.e., $S \circ T = T \circ S$.
2. S and T are simultaneously diagonalizable.

(c) Can you extend the result above to arbitrary collection of diagonalizable operators on V?

(a) Two operators S and T on a finite-dimensional vector space V are *simultaneously diagonalizable* if there exists a single basis $\{v_1, \ldots, v_n\}$ of V such that both S and T are diagonal with respect to this basis. Equivalently, S and T share the same eigenbasis.

(b) *Proof.* ($1 \Rightarrow 2$) Suppose $S \circ T = T \circ S$. Since S is diagonalizable, there exists a basis $\{v_1, \ldots, v_n\}$ of eigenvectors of S with eigenvalues $\lambda_1, \ldots, \lambda_k$. Let $V_i = \ker(S - \lambda_i I)$. Since S and T commute, $T(V_i) \subseteq V_i$ for each i. Thus T restricted to each V_i is diagonalizable. Therefore, each V_i has a basis of common eigenvectors of both S and T. The union of these bases is a basis of V diagonalizing both S and T.

($2 \Rightarrow 1$) Suppose S and T are simultaneously diagonalizable. Then there exists a basis where both are diagonal. Diagonal matrices commute, so $S \circ T = T \circ S$. $\square$

(c) Yes. The result extends to any finite collection of diagonalizable operators. If $T_1, \ldots, T_m$ are diagonalizable operators on V, then they are simultaneously diagonalizable if and only if they pairwise commute, i.e., $T_i \circ T_j = T_j \circ T_i$ for all i, j.