

Math/CS 714: Assignment 4

Problem 1

Beam-Warming method (4 points). The Beam-Warming method for the linear advection equation $u_t + au_x = 0$ is given by

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{a^2k^2}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n), \quad (1)$$

where U_j^n is the approximation of $u(jh, nk)$.

1. Use Taylor series to show that this method is second-order accurate.
2. For a given plane wave solution $U_j^0 = e^{ijh\xi}$, compute the amplification factor $g(\xi)$, and hence determine the stability restriction for this method.

a) Taking the Taylor series expansion of U_j^{n+1} , U_{j-1}^n , and U_{j-2}^n about the point (jh, nk) gives

$$\begin{aligned} U_j^{n+1} &= U_j^n + kU_t + \frac{k^2}{2}U_{tt} + O(k^3), \\ U_{j-1}^n &= U_j^n - hU_x + \frac{h^2}{2}U_{xx} - \frac{h^3}{6}U_{xxx} + O(h^4), \\ U_{j-2}^n &= U_j^n - 2hU_x + 2h^2U_{xx} - \frac{4h^3}{3}U_{xxx} + O(h^4). \end{aligned}$$

Substituting these into Eq. (1) and simplifying gives

$$U_t + aU_x = \frac{k}{2}U_{tt} - \frac{ah^2}{3}U_{xxx} + O(k^2, h^3).$$

Using the PDE to substitute for U_{tt} gives

$$U_t + aU_x = -\frac{a^2k}{2}U_{xx} - \frac{ah^2}{3}U_{xxx} + O(k^2, h^3),$$

which shows that the method is second-order accurate in both space and time.

Problem 2

(9 points). Dropping the last term in the Beam–Warming method from Eq. (1) gives

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n), \quad (2)$$

which corresponds to forward Euler method in time, and a second-order one-sided derivative in space. Define $\nu = ak/h$.

1. Calculate the amplification factor $g(\xi)$ for a plane wave solution $U_j^0 = e^{ijh\xi}$.
2. Define $A(\xi) = |g(\xi)|^2$ and calculate a Taylor series for A at $\xi = 0$ up to second order. Using the Taylor series, explain why we consider the numerical scheme of Eq. (2) to be unstable regardless of the choice of timestep.
3. Make two plots of $A(\xi)$ for $\nu = 1/100$ using two different axis ranges:
 - $0 \leq h\xi \leq 2\pi$ and $0.91 \leq A \leq 1.01$,
 - $0 \leq h\xi \leq 0.17$ and $1 - 10^{-6} \leq A \leq 1 + 10^{-6}$.
4. Write a program to simulate Eq. (2) on a periodic interval $[0, 2\pi)$ using $N = 40$ grid points and a grid spacing of $h = 2\pi/N$. Use the initial condition $u = \exp(2 \sin x)$ and $\nu = 1/100$. Plot the solution for $n = 0, 1000, 2000, 4000$. Define the root mean squared value of the solution,

$$R(n) = \sqrt{\frac{1}{N} \sum_{j=0}^{N-1} (U_j^n)^2}. \quad (3)$$

Make a plot of R over the range from $n = 0$ to $n = 10000$. You should find that R does not grow over time, indicating that the method is stable.

5. Using the discrete Fourier transform, it can be shown that an arbitrary initial condition on the periodic interval can be written as

$$U_j^0 = \sum_{l=0}^{N-1} \alpha_l e^{ijlh} \quad (4)$$

for some constants α_l . Write down an expression for the general solution U_j^n . Using your answer, explain why your result in part (d) does not contradict the result in part (b).

Problem 3

Lax–Wendroff method (7 points). Consider the hyperbolic conservation equation

$$q_t + [A(x)q]_x = 0 \quad (5)$$

for a function on $q(x, t)$ on the periodic interval $[0, 2\pi)$. Let $A(x) = 2 + \frac{4}{3} \sin x$. Following the finite volume approach, divide the intervals into m domains \mathcal{C}_i of length $h = \frac{2\pi}{m}$, for $i = \{0, 1, \dots, m-1\}$. Let $Q_i^n \approx q((i + 1/2)h, n\Delta t)$ be the discretized solution at the center of each \mathcal{C}_i . The generalized Lax–Wendroff scheme for this equation is given by

$$Q_i^{n+1} = Q_i - \frac{\Delta t}{h} \left[\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right] \quad (6)$$

where the fluxes are

$$\mathcal{F}_{i-1/2}^n = \frac{A_{i-1}Q_{i-1}^n + A_iQ_i^n}{2} - \frac{A_{i-1/2}\Delta t}{2h} [A_iQ_i^n - A_{i-1}Q_{i-1}^n]. \quad (7)$$

Here, $A_i = A((i + 1/2)h)$ and $A_{i-1/2} = A(ih)$. It can be shown that the solution to Eq. (5) is time-periodic so that $q(x, t + T) = q(x, t)$ where $T = 3\pi/\sqrt{5}$.

1. The CFL condition requires that $\Delta t \leq \frac{h}{c}$ for stability. What is c in this case?
2. Implement Eq. (7) and set $\Delta t = \frac{h}{3c}$. Use the initial condition

$$q(x, 0) = \exp\left(\sin x + \frac{1}{2} \sin 4x\right). \quad (8)$$

For $m = 512$, plot snapshots of the solution for $t = 0, \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}, T$.^a

3. By considering a range of m (e.g. 256 and upward) with the initial condition in Eq. (8), calculate the L_2 norm between the numerical solution at $t = T$ and the exact answer. Determine the order of convergence.^b
4. Repeat parts (b) and (c) for the initial condition

$$q(x, 0) = \max\left\{\frac{\pi}{2} - |x - \pi|, 0\right\}. \quad (9)$$

5. **Optional.** By the considering the characteristics, or otherwise, derive the result that q is time-periodic with period T .

^aSince multiples of Δt do not exactly match these snapshot times, you may need to make a small adjustment to the timestep.

^bWhen determining the order of convergence, you are interested in the asymptotic properties of error as m gets large. You can ignore initial transients in error.