Problem 1

Let V be an n-dimensional vector space, and let $\phi : V \to V$ be a linear map. Show that, for any n-linear antisymmetric form $\beta(v_1, \ldots, v_n)$ on n, we have

$$\beta(\phi(v_1), \ldots, \phi(v_n)) = \det(\phi)\beta(v_1, \ldots, v_n).$$

(This formalizes the following idea: any "unit of volume" on V, given by β, gets scaled by $\det(\phi)$ when we apply ϕ. In fact, this can be used as a definition of $\det(\phi)$: this way, some of its properties, such as independence of basis and multiplicativity become clear.)

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Then we can write

$$v_i = \sum_{j=1}^{n} a_{ij} e_j, \quad i = 1, \ldots, n,$$

for some scalars a_{ij}. Then we have

$$\begin{aligned}
\beta(\phi(v_1), \ldots, \phi(v_n)) &= \beta\left(\phi\left(\sum_{j=1}^{n} a_{1j} e_j\right), \ldots, \phi\left(\sum_{j=1}^{n} a_{nj} e_j\right)\right) \\
&= \beta\left(\sum_{j=1}^{n} a_{1j}\phi(e_j), \ldots, \sum_{j=1}^{n} a_{nj}\phi(e_j)\right) \\
&= \sum_{j_1, j_2, \ldots, j_n = 1}^{n} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \beta(\phi(e_{j_1}), \phi(e_{j_2}), \ldots, \phi(e_{j_n})) \\
&= \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \beta(\phi(e_{\sigma(1)}), \phi(e_{\sigma(2)}), \ldots, \phi(e_{\sigma(n)})) \\
&= \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \operatorname{sgn}(\sigma) \beta(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)) \\
&= \det(A)\beta(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)),
\end{aligned}$$

where $A = (a_{ij})$ is the matrix whose columns are the coordinates of v_i in the basis $\{e_1, \ldots, e_n\}$. Now, note that

$$\beta(\phi(e_1), \phi(e_2), \ldots, \phi(e_n))$$

is an n-linear antisymmetric form evaluated at the basis vectors $\{e_1, \ldots, e_n\}$ after applying ϕ. By the definition of determinant, we have

$$\beta(\phi(e_1), \phi(e_2), \ldots, \phi(e_n)) = \det(\phi)\beta(e_1, e_2, \ldots, e_n).$$

Therefore, we conclude that

$$\beta(\phi(v_1), \ldots, \phi(v_n)) = \det(A)\det(\phi)\beta(e_1, e_2, \ldots, e_n).$$

Finally, since β is multilinear, we have

$$\beta(v_1, \ldots, v_n) = \det(A)\beta(e_1, e_2, \ldots, e_n).$$

Combining these results, we obtain

$$\beta(\phi(v_1), \ldots, \phi(v_n)) = \det(\phi)\beta(v_1, \ldots, v_n).$$

$\square$

Problem 2

Let V be the space of polynomials of degree at most n (over some field K; if you want, you can assume $K = \mathbb{R}$). Fix $a, b \in \mathbb{R}$ and consider the linear map

$$\phi : V \to V, \quad p(t) \mapsto p(at + b).$$

Compute $\det(\phi)$.

Proof. Let $\{1, t, t^2, \ldots, t^n\}$ be the standard basis of V. Then we have

$$\phi(t^k) = (at + b)^k = \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} t^j.$$

Thus, the matrix representation of ϕ with respect to this basis is given by

$$[\phi] = \begin{pmatrix} 1 & b & b^2 & \cdots & b^n \\ 0 & a & 2ab & \cdots & nab^{n-1} \\ 0 & 0 & a^2 & \cdots & \binom{n}{2}a^2b^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a^n \end{pmatrix}.$$

This is an upper triangular matrix, and the determinant of an upper triangular matrix is the product of its diagonal entries. Therefore, we have

$$\det(\phi) = 1 \cdot a \cdot a^2 \cdot a^3 \cdots a^n = a^{\frac{n(n+1)}{2}}.$$

Hence, the determinant of the linear map ϕ is

$$\det(\phi) = a^{\frac{n(n+1)}{2}}.$$

□

Problem 3

Let M be an $n \times n$ matrix (over some field). Let V be the space of $n \times m$ matrices. Consider the linear map $m_M : V \to V$ given by left multiplication by M:

$$A \mapsto MA.$$

Find $\det(m_M)$. (Of course, the answer depends on M.)

Proof. Let $\{E_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ be the standard basis of V, where E_{ij} is the matrix with a 1 in the (i, j)-th position and 0 elsewhere. Then we have

$$m_M(E_{ij}) = ME_{ij} = \text{the } j\text{-th column of } M \text{ placed in the } i\text{-th column of a zero matrix.}$$

Thus, the matrix representation of m_M with respect to this basis is given by a block matrix where each block corresponds to the action of M on the respective basis element. Specifically, the matrix representation of m_M can be viewed as an $nm \times nm$ matrix composed of m blocks of size $n \times n$, each block being the matrix M. Therefore, the matrix representation of m_M is given by

$$[m_M] = \begin{pmatrix} M & 0 & 0 & \cdots & 0 \\ 0 & M & 0 & \cdots & 0 \\ 0 & 0 & M & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M \end{pmatrix}.$$

This is a block diagonal matrix with m blocks of M. The determinant of a block diagonal matrix is the product of the determinants of its blocks. Therefore, we have

$$\det(m_M) = (\det(M))^m.$$

Hence, the determinant of the linear map m_M is $\det(m_M) = (\det(M))^m$. □

Problem 4

Let K be a field and V be a finite-dimensional vector space. Let

$$\gamma : V \times V \to K$$

be an antisymmetric bilinear form. Show that there exists $k \leq \frac{n}{2}$ and a basis

$$\{a1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_{n-2k}\}$$

of V such that

$$\gamma(a_i, b_i) = 1, \quad \gamma(b_i, a_i) = -1, \quad (i = 1, \ldots, k),$$

and γ vanishes on all other pairs of basis vectors. (When $n = 2k$, this is called a symplectic basis.)

Proof. We proceed by induction on the dimension n of the vector space V.
Base Case: If $n = 0$, the statement is trivially true with $k = 0$ and an empty basis. If $n = 1$, since γ is antisymmetric, we have $\gamma(v, v) = 0$ for any $v \in V$. Thus, we can take $k = 0$ and any basis $\{c_1\}$ of V.
Inductive Step: Assume the statement holds for all vector spaces of dimension less than n. We consider two cases:

i.) If γ is the zero form, then we can take $k = 0$ and any basis $\{c_1, \ldots, c_n\}$ of V.

ii.) If γ is not the zero form, there exist vectors $u, v \in V$ such that $\gamma(u, v) \neq 0$. We can scale u and v such that $\gamma(u, v) = 1$. Since $\gamma(u, v) \neq 0$, we have that u, v are linearly independent so by the building-up lemma we can extend $\{u, v\}$ to a basis of V, say $\{u, v, w_1, \ldots, w_{n-2}\}$. Consider the subspace $W = \text{span}\{w_1, \ldots, w_{n-2}\}$. The restriction of γ to W is still an antisymmetric bilinear form. By the inductive hypothesis, there exists a basis of W of the desired form with some $k' \leq \frac{n-2}{2}$. Adding u and v to this basis, we obtain a basis of V of the desired form with $k = k' + 1 \leq \frac{n}{2}$.

Thus, by induction, the statement holds for all finite-dimensional vector spaces V over the field K. □

Problem 5

Consider $\det(A)$ as a multivariable function of the entries of a real matrix A. Compute the directional derivative of this function at the point $A = I$ in the direction of some matrix B. (Equivalently, find the linear approximation for $f(t) = \det(I + tB)$ at $t = 0$.)

Proof. We want to compute the directional derivative of the determinant function at the identity matrix I in the direction of a matrix B. The directional derivative can be expressed as

$$D_B \det(I) = \lim_{t \to 0} \frac{\det(I + tB) - \det(I)}{t}.$$

Since $\det(I) = 1$, we have

$$D_B \det(I) = \lim_{t \to 0} \frac{\det(I + tB) - 1}{t}.$$

To evaluate this limit, we can use the fact that for small t, the determinant can be approximated using the first-order term in its Taylor expansion. Specifically, we have

$$\det(I + tB) = 1 + t\,\text{tr}(B) + O(t^2),$$

where $\text{tr}(B)$ is the trace of the matrix B. Therefore, we have

$$D_B \det(I) = \lim_{t \to 0} \frac{(1 + t\,\text{tr}(B) + O(t^2)) - 1}{t} = \lim_{t \to 0} \frac{t\,\text{tr}(B) + O(t^2)}{t}.$$

Simplifying this expression, we get

$$D_B \det(I) = \text{tr}(B).$$

Thus, the directional derivative of the determinant function at the identity matrix in the direction of the matrix B is given by the trace of B:

$$D_B \det(I) = \text{tr}(B).$$

□

Problem 6

Let A be a square $n \times n$ matrix whose characteristic polynomial has n roots in K, counting with multiplicity. Consider the Jordan form of A: suppose that it consists of blocks J_{λ_i, n_i}, where λ_i is the eigenvalue and n_i is the size of the block.
Express the following invariants of A in terms of n_i and λ_i: its characteristic polynomial, its minimal polynomial, the dimension of eigenspace for each λ (this is called "the geometric multiplicity of an eigenvalue") and rank. (No explanation is required.)

(a) **Characteristic Polynomial:** The characteristic polynomial of A is given by

$$p_A(x) = \prod_i (x - \lambda_i)^{n_i}.$$

(b) **Minimal Polynomial:** The minimal polynomial of A is given by

$$m_A(x) = \prod_i (x - \lambda_i)^{m_i},$$

where m_i is the size of the largest Jordan block corresponding to the eigenvalue λ_i.

(c) **Geometric Multiplicity:** The dimension of the eigenspace for each eigenvalue λ_i is equal to the number of Jordan blocks associated with λ_i. If there are k_i blocks for eigenvalue λ_i, then the geometric multiplicity is

$$\text{geom. mult.}(\lambda_i) = k_i.$$

(d) **Rank:** The rank of the matrix A can be computed as

$$\text{rk}(A) = n - \sum_i (n_i - 1) = n - \left(\sum_i n_i - \text{number of blocks} \right).$$

Problem 7

Following up on the previous problem, let us go in the opposite direction: explain how to find λ_i and n_i from the data of $\text{rk}(A - \lambda I)^k$ for all $\lambda \in K$ and $k > 0$. In particular, this implies that the Jordan form is unique.

To find the eigenvalues λ_i from the data of $\text{rk}(A - \lambda I)^k$, we can use the fact that the eigenvalues of A are precisely the values of λ for which the matrix $A - \lambda I$ is not invertible. This occurs when $\det(A - \lambda I) = 0$. Thus, by examining the ranks of $(A - \lambda I)^k$ for various λ, we can identify the eigenvalues as those values of λ for which the rank drops below n.
To determine the sizes of the Jordan blocks n_i, we can analyze the ranks of the powers of $(A - \lambda I)$. Specifically, for each eigenvalue λ_i, we consider the sequence of ranks:

$$\text{rk}(A - \lambda_i I), \text{rk}((A - \lambda_i I)^2), \text{rk}((A - \lambda_i I)^3), \ldots$$

The size of the largest Jordan block corresponding to the eigenvalue λ_i can be determined by finding the smallest integer k such that

$$\text{rk}((A - \lambda_i I)^k) = \text{rk}((A - \lambda_i I)^{k+1}).$$

This integer k gives us the size of the largest Jordan block n_i for the eigenvalue λ_i. By repeating this process for each eigenvalue, we can reconstruct the entire Jordan form of the matrix A. This method also shows that the Jordan form is unique, as the sizes of the Jordan blocks are determined solely by the ranks of the powers of $(A - \lambda I)$.
To find the other sizes of Jordan blocks corresponding to the same eigenvalue λ_i, we can use the differences in ranks:

$$\text{number of blocks of size } \geq k = \text{rk}((A - \lambda_i I)^{k-1}) - \text{rk}((A - \lambda_i I)^k).$$