Math 540 -Linear Algebra II
Fall 2025
HW6: Diagonalizability, Projectors, Degree, Homomorphism, Inverses, Subring
Submit via Canvas by Mon. 10/27/25 at 12pm

Remarks:

A) Definition is just a definition, there is no need to justify or explain it.

B) Answers to questions with proofs should be written, as much as you can, in the following format:

i) Statement.

ii) Main points that will appear in your proof.

iii) The actual proof.

C) Answers to questions with computations should be written, as much as possible, in the following format:

i) Statement and Result.

ii) Main points that will appear in your computation.

iii) The actual computation.

1. *Diagonalizability and projectors - general case.* Let V be a vector space, and $V_j < V$, $j = 1, ..., k$, subspaces.

   (a) Define when V is a <u>direct sum</u> of V_j's, denoted $V = V_1 \oplus ... \oplus V_k$.
   Recall that, an operator P on V is called <u>projector</u> onto a subspace $W < V$, if $P(W) = W$, and the restriction $P_{|W} = Id_W$.

   (b) Show that TFAE:

   1. $V = V_1 \oplus ... \oplus V_k$.
   2. there exist projectors P_j, onto the subspaces V_j, $j = 1, ..., k$, such that
      1. $Id_V = P_1 + ... + P_k$.
      2. $P_i \circ P_j = 0 = P_j \circ P_i$, for every $i \neq j$.

   (c) Suppose V is finite dimensional and $T : V \to V$, linear transformation. Show that TFAE:

   1. $V = \bigoplus\limits_{\lambda \in spec(T)} V_\lambda$, direct sum of the eigenspaces of T, i.e., T is diagonalizable;
   2. there are projectors P_λ, for some finite collection Λ of λ's from $\mathbb{F}$, such that,
      1. $Id_V = \sum\limits_{\lambda \in \Lambda} P_\lambda$,
      2. $P_\lambda \circ P_\mu = 0 = P_\mu \circ P_\lambda$, for very $\lambda \neq \mu \in \Lambda$.
      3. $T = \sum\limits_{\lambda \in \Lambda} \lambda \cdot P_\lambda$.

Moreover, show that in this case $\Lambda = spec(T)$, and for each $\mu \in speck(T)$ we have

$$P_\mu = \prod_{\mu \neq \lambda \in spec(T)} \left(\frac{T - \lambda \cdot Id}{\mu - \lambda} \right).$$

2. *Degree of a polynomial***.** Let $\mathbb{F}$ be a field.

   (a) Define the ring $\mathbb{F}[X]$ of polynomials with coefficients in $\mathbb{F}$.

   (b) Show that $\dim \mathbb{F}[X] = \infty$.

   (c) Now, let R be a ring and $R[X]$ the ring of polynomials with coefficients in R. Recall that the degree $\deg(f)$ of a polynomial $f \in R[X]$ is defined to be $\deg(f) = d$ if $f = a_d X^d + ... + a_1 X + a_0$ and $a_d \neq 0$, and $\deg(f) = -\infty$ if $f = 0$. Show that for every $f, g \in R[X]$ we have

      (i) $\deg(fg) \leq \deg(f) + \deg(g)$,

      (ii) $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$.

      Moreover, show that if R is an integral domain than the inequality in (i) is actually an equality.

3. *Kernel of a homomorphism***.** Let $\varphi : R \to S$ be homomorphism of rings.

   (a) Define the <u>kernel</u> of φ, denoted $\ker(\varphi)$, by $\ker(\varphi) = \{a \in R \text{ such that } \varphi(a) = 0\}$.

   (b) Show that φ is one-to-one if and only if $\ker(\varphi) = \{0\}$.

4. *Uniqueness of inverses.* Suppose R is a ring.

   (a) Define when we say that R is a <u>ring with unit</u>.

   (b) Suppose R is a ring with unit. Show that for $a \in R$ invertible, i.e., there exists at most one inverse $b \in R$, i.e., such that $ab = ba = 1_R$.

   (c) Compute all the invertible elements in the ring $\mathbb{Z}_{12}$, and for each find its inverse.

5. *Subrings from ring homomorphisms.* Let $\varphi : R \to S$ be a homomorphism of rings.

   (a) Define when R' is a <u>subring</u> of a ring R.

      Let $\varphi : R \to S$ be a homomorphism of rings. We define the image of φ, denoted $Im(\varphi)$, by $Im(\varphi) = \{\varphi(r) \mid r \in R\}$.

   (b) Show that $Im(\varphi)$ is a subring of S and that $ker(\varphi)$ is a subring of R.

<u>Important remarks:</u>

- Every W-F 8:30-9am we go (only if students show up to the meeting) over HW in canvas's zoom (and record it).

- You are very much encouraged to consult the lecturer and grader on HW during office hours/appointments/discussions/lectures.

- You are very much encouraged to work with other students on the HW.
- You should submit your HW **ALONE in YOUR OWN ORIGINAL DOCUMENT!!!**
- Remember: "ChatGpt" will not be there for you during the quiz on the HW, so make sure you can really solve the HW problems.

Good Luck!