Remarks:

A) Definition is just a definition, there is no need to jjustify or explain it.

B) Answers to questions with proofs should be written, as much as you can, in the following format:

   i) Statement
   ii) Main points that will appear in your proof
   iii) The actual proof

   Answers to questions with computations should be written, as much as possible, in the following format:

   i) Statement and Result
   ii) Main points that will appear in your computation.
   iii) The actual computation

Problem 1

Vector Spaces. Suppose $\mathbb{F}$ is a field.

a) Define when we say that a vector space V over a field $\mathbb{F}$ is *finite dimensional.*

b) Consider the vector space
$$V = \mathbb{F}[x]$$
of all polynomials with coefficients in $\mathbb{F}$. Show that V is not finite dimensional.

c) Suppose X is a finite set. Consider the vector space V, of all functions from X to $\mathbb{F}$,
$$V = \mathbb{F}(X) := \{\text{all } f : X \to \mathbb{F}; \text{s.t. } f \text{ is a function}\},$$
with the standard addition and multiplication by scalars from $\mathbb{F}$. Show that V is finite dimensional.

Problem 2

Short exact sequences. Suppose U, V, W are three vector spaces over $\mathbb{F}$. Consider the following seequence of spaces and linear transformations between them:
$$0 \to U \xrightarrow{\iota} V \xrightarrow{\epsilon} W \to 0, \tag{1}$$
where $0 \to U$, are the obvious maps from the zero space into U, and from the space W onto the zero space, respectively.

a) Define when we say that the sequence (1) is <u>short exact sequence</u> (s.e.s.).

b) Given two subspaces $U, V < V$, such that $V = U \oplus W$, Show that there is a natural s.e.s.associated with the spaces of functions $U = \mathbb{F}(U), V = \mathbb{F}(V)$ and $W = \mathbb{F}(Y \backslash X)$, where $Y \backslash X$ denotes set-minus, i.e., the set of elements which are in Y and are not in X.

Problem 3

Dimension. Denote by Vect $\mathbb{F}^{fd}$ the collection of finite-dimensional vector spaces over $\mathbb{F}$, with linear transformations between them.

a) State the fact about uniqueness and existence of unique dimenstion function

$$\dim : \mathrm{Vect}_{\mathbb{F}}^{fd} \to \mathbb{N},$$

that satisfies certain desired properties.
Def. For V finite dimensional, the integer $\dim(V)$ is called the <u>dimension</u> of V.

b) Show that $\dim(M_n(\mathbb{F})) = n^2$.

c) Suppose $1 + 1 \neq 0$ in $\mathbb{F}$. Consider the spaces $U = A_n(\mathbb{F})$, $V = M_n(\mathbb{F})$, $W = S_n(\mathbb{F})$, of anti-symmetric matrices ($A^T = -A$), all matrices, and symmetric matrices (sastisfy $A^T = A$), respectively.

   i) Show that, they form in a natural why a s.e.s.

   ii) Deduce that $\dim(A_n(\mathbb{F})) = \frac{n(n-1)}{2}$ and $\dim(S_n(\mathbb{F})) = \frac{n(n+1)}{2}$.