Problem 1

Let V be a finite-dimensional vector space. For any subspace $W \subset V$, put

$$W^\perp = \{\phi \in V^* \mid \phi|_W = 0\}.$$

Construct natural isomorphisms $W^* \cong (V^*/W^\perp)$ and $W^\perp \cong (V/W)^*$. (The assumption that V is finite-dimensional is not actually required, but it makes the problem easier.)

Proof. We give explicit constructions and check they are linear bijections.

(1) Constructing $W^* \cong V^*/W^\perp$. Define

$$R : V^* \to W^*, \qquad R(\phi) = \phi|_W.$$

Then R is linear and $R(\phi) = 0$ iff $\phi \in W^\perp$, so $\ker R = W^\perp$. Hence R induces a linear map

$$\overline{R} : V^*/W^\perp \to W^*, \qquad \overline{R}(\phi + W^\perp) = \phi|_W,$$

which is well-defined. To see $\overline{R}$ is bijective directly: injectivity holds because if $\overline{R}(\phi+W^\perp) = 0$ then $\phi|_W = 0$ so $\phi \in W^\perp$ and $\phi + W^\perp = 0$. For surjectivity, let $\psi \in W^*$. Choose a basis $w_1, \ldots, w_k$ of W and extend it to a basis $w_1, \ldots, w_k, v_{k+1}, \ldots, v_n$ of V. Define $\phi \in V^*$ by $\phi(w_i) = \psi(w_i)$ and $\phi(v_j) = 0$ for the extra basis vectors. Then $\phi|_W = \psi$, so $\overline{R}(\phi + W^\perp) = \psi$. Thus $\overline{R}$ is an isomorphism.

(2) Constructing $W^\perp \cong (V/W)^*$. Let $\pi : V \to V/W$ be the quotient map. For $\alpha \in W^\perp$ define

$$S(\alpha) : V/W \to K, \qquad S(\alpha)(v + W) = \alpha(v).$$

This is well-defined because $\alpha|_W = 0$. Clearly $S(\alpha)$ is linear, and $S : W^\perp \to (V/W)^*$ is linear. If $S(\alpha) = 0$ then $\alpha(v) = 0$ for all $v \in V$, so $\alpha = 0$, hence S is injective. For surjectivity, given $f \in (V/W)^*$ set $\alpha = f \circ \pi \in V^*$. Then $\alpha|_W = 0$, so $\alpha \in W^\perp$ and $S(\alpha) = f$. Thus S is an isomorphism. □

Problem 2

Let V be a finite-dimesional vector space, $\dim(V) = n$. For $k \geq 0$, let $G(V, k)$ be the set of all the k-dimensional subspaces of V (it is called the Grassmannian of V). Show that the correspondence

$$W \mapsto W^\perp$$

is a bijection

$$G(V, k) \cong G(V^*, n - k).$$

Proof. To show that the map $W \mapsto W^\perp$ is a bijection from $G(V, k)$ to $G(V^*, n-k)$, we need to prove that it is both injective and surjective.

Injectivity: Suppose $W_1, W_2 \in G(V, k)$ such that $W_1^\perp = W_2^\perp$. We want to show that $W_1 = W_2$. Since $W_1^\perp = W_2^\perp$, any linear functional that vanishes on W_1 also vanishes on W_2, and vice versa. This implies that the annihilators of W_1 and W_2 are the same, which means that the subspaces themselves must be equal. Therefore, $W_1 = W_2$, proving injectivity.

Surjectivity: Let $U \in G(V^*, n - k)$. We want to find a subspace $W \in G(V, k)$ such that $W^\perp = U$. Consider the annihilator of U, denoted by $U^\perp$. By the properties of dual spaces, we have $\dim(U^\perp) = n - \dim(U) = n - (n - k) = k$. Thus, $U^\perp$ is a k-dimensional subspace of V. Now, we can set $W = U^\perp$. Then, by definition, we have

$$W^\perp = (U^\perp)^\perp = U,$$

completing the proof of surjectivity.

Since the map is both injective and surjective, we conclude that it is a bijection:

$$G(V, k) \cong G(V^*, n - k).$$

□

Problem 3

Let V and W be two vector spaces, not neccessarily finite-dimensional. Consider the map

$$V^* \otimes W \to \mathrm{Hom}_K(V, W) \quad \phi \otimes w \mapsto (v \mapsto \phi(v)w).$$

(the maps were considered in class). Prove that the map is an isomorphism if either V or W is finite-dimensional. (In fact, this is an "if and only if"; in general, the map is only injective.)

Proof. First we show injectivity. Suppose that

$$\sum_{i=1}^{m} \phi_i \otimes w_i \mapsto 0,$$

meaning that for all $v \in V$,

$$\sum_{i=1}^{m} \phi_i(v) w_i = 0.$$

We want to show that $\sum_{i=1}^{m} \phi_i \otimes w_i = 0$ in $V^* \otimes W$. To do this, we can use the fact that if a linear combination of tensors maps to zero under the given map, then the coefficients must be zero. Then, without loss of generality, we have that the w_i are linearly independent so it follows that $\phi_i(v) = 0$ for all $v \in V$ and for all i. Thus, each $\phi_i = 0$, and hence $\sum_{i=1}^{m} \phi_i \otimes w_i = 0$. This shows that the map is injective.

Now suppose that V is finite-dimensional. Let $\{v_1, \ldots, v_m\}$ be a basis for V, and let $\{\phi_1, \ldots, \phi_m\}$ be the dual basis for V^*. For any linear transformation $T : V \to W$, we can express it as

$$T(v) = \sum_{i=1}^{m} \phi_i(v) T(v_i).$$

We can do this since for any $v \in V$, we can write $v = \sum_{i=1}^{m} a_i v_i$, and thus

$$T(v) = T\left(\sum_{i=1}^{m} a_i v_i\right) = \sum_{i=1}^{m} a_i T(v_i) = \sum_{i=1}^{m} \phi_i(v) T(v_i).$$

Thus, we can write

$$T = \sum_{i=1}^{m} \phi_i \otimes T(v_i) = \psi \otimes w, \quad \text{where } \psi = \sum \phi_i, w = \sum T(v_i),$$

showing that the map is surjective when V is finite-dimensional.

Now suppose that W is finite-dimensional. Let $\{w_1, \ldots, w_n\}$ be a basis for W. For any linear transformation $T : V \to W$, we can express it as

$$T(v) = \sum_{j=1}^{n} \psi_j(v) w_j,$$

where $\psi_j : V \to K$ are linear functionals. Thus, we can write

$$T = \sum_{j=1}^{n} \psi_j \otimes w_j,$$

showing that the map is surjective when W is finite-dimensional.

Therefore, the map is an isomorphism if either V or W is finite-dimensional. □

Problem 4

Let V and W be finite-dimensional vector spaces, and let $\phi : V \to V$ and $\psi : W \to W$ be linear transformations. Consider the linear transformation

$$\phi \otimes \psi : V \otimes W \to V \otimes W, \quad v \otimes w \mapsto \phi(v) \otimes \psi(w).$$

Find a formula for $\det(\phi \otimes \psi)$. If you want, you can assume that you work over an algebraically closed field.

Proof. Let $\dim(V) = m$ and $\dim(W) = n$. Let $\{\lambda_1, \ldots, \lambda_m\}$ be the eigenvalues of ϕ and $\{\mu_1, \ldots, \mu_n\}$ be the eigenvalues of ψ. The eigenvalues of the tensor product transformation $\phi \otimes \psi$ are given by the products of the eigenvalues of ϕ and ψ. Specifically, the eigenvalues of $\phi \otimes \psi$ are

$$\{\lambda_i \mu_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Therefore, the determinant of $\phi \otimes \psi$ is the product of all these eigenvalues:

$$\det(\phi \otimes \psi) = \prod_{i=1}^{m} \prod_{j=1}^{n} (\lambda_i \mu_j).$$

This can be rewritten as

$$\det(\phi \otimes \psi) = \left(\prod_{i=1}^{m} \lambda_i^n \right) \left(\prod_{j=1}^{n} \mu_j^m \right) = (\det(\phi))^n (\det(\psi))^m.$$

Thus, we have the formula

$$\det(\phi \otimes \psi) = (\det(\phi))^{\dim(W)} (\det(\psi))^{\dim(V)}.$$

□

Problem 5

Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $\phi, \psi : V \to V$ be two linear operators. Suppose that the operators commute: $\phi \circ \psi = \psi \circ \phi$. Prove that there exists a vector $v \in V, v \neq 0$, that is an eigenvector for both ϕ and ψ simultaneously. (Consider the restriction of ψ to the eigenspaces of ϕ.)

Proof. Since we work over $\mathbb{C}$, the operator ϕ has at least one eigenvalue. Let $\lambda \in \mathbb{C}$ be an eigenvalue of ϕ and set

$$E_\lambda = \ker(\phi - \lambda I),$$

the corresponding eigenspace. By choice of λ we have $E_\lambda \neq \{0\}$.

Since ϕ and ψ commute, for any $v \in E_\lambda$ we have

$$(\phi - \lambda I)(\psi v) = \phi(\psi v) - \lambda \psi v = \psi(\phi v) - \lambda \psi v = \psi(\lambda v) - \lambda \psi v = 0,$$

so $\psi v \in E_\lambda$. Thus E_λ is a nonzero ψ-invariant subspace, and $\psi|_{E_\lambda}$ is a linear endomorphism of the finite-dimensional space E_λ.

Working over $\mathbb{C}$, the restriction $\psi|_{E_\lambda}$ has an eigenvector $v \neq 0$ with eigenvalue μ. Since $v \in E_\lambda$ we also have $\phi v = \lambda v$. Hence v is a nonzero vector that is simultaneously an eigenvector for ϕ and ψ:

$$\phi v = \lambda v, \qquad \psi v = \mu v.$$

This completes the proof. □

Problem 6

Let G be an abelian group, and V be a finite-dimensional irreducible representation of G over $\mathbb{C}$. Show that $\dim(V) = 1$. (This is closely related to the previous problem.)

Proof. We can use Schur's lemma to prove this. Since V is an irreducible representation of the abelian group G, any linear operator that commutes with all $\rho(g)$ for $g \in G$ must be a scalar multiple of the identity operator.

Now, consider the action of G on V. Since G is abelian, for any $g, h \in G$, we have $\rho(g)\rho(h) = \rho(h)\rho(g)$. Thus, all the linear operators $\rho(g)$ commute with each other. By the previous problem, there exists a nonzero vector $v \in V$ that is an eigenvector for all $\rho(g)$ simultaneously. That is, for each $g \in G$, there exists a scalar $\lambda_g \in \mathbb{C}$ such that

$$\rho(g)(v) = \lambda_g v.$$

Consider the subspace $W = \text{span}\{v\}$. Since V is irreducible, the only subspaces invariant under the action of G are $\{0\}$ and V itself. Since W is nonzero and invariant under the action of G, we must have $W = V$. Therefore, $\dim(V) = 1$. □

Problem 7

Let $V_{d,n}$ be the vector space of homogeneous polynomials of degree d in the variables $x_1, \dots, x_n$. The symmetric group S_n acts on $V_{d,n}$ by permuting the variables. Write this representation as a direct sum of irreducibles of $(d,n) = (2,2), (3,1), (3,2)$.

- For $(d,n) = (2,2)$, the space $V_{2,2}$ consists of homogeneous polynomials of degree 2 in two variables x_1 and x_2. A basis for this space is given by $\{x_1^2, x_1x_2, x_2^2\}$. For the basis element x_1x_2, the action of S_2 in effect, does nothing. So we have that $\langle x_1x_2 \rangle$ is a one-dimensional invariant subspace of $V_{2,2}$. The action of S_2 on the other basis elements x_1^2 and x_2^2 swaps them. So we have that the subspace spanned by $\{x_1^2, x_2^2\}$ is also invariant under the action of S_2. This subspace can be further decomposed into two one-dimensional irreducible representations: the symmetric part spanned by $x_1^2 + x_2^2$ and the antisymmetric part spanned by $x_1^2 - x_2^2$. Therefore, we have

$$V_{2,2} \cong \langle x_1x_2 \rangle \oplus \langle x_1^2 + x_2^2 \rangle \oplus \langle x_1^2 - x_2^2 \rangle.$$

- For $(d,n) = (3,1)$, the space $V_{3,1}$ consists of homogeneous polynomials of degree 3 in one variable x_1. A basis for this space is given by $\{x_1^3\}$. Since there is only one variable, the action of S_1 (which is trivial) leaves the polynomial unchanged. Therefore, $V_{3,1}$ is already irreducible and we have

$$V_{3,1} \cong \langle x_1^3 \rangle.$$

- For $(d,n) = (3,2)$, the space $V_{3,2}$ consists of homogeneous polynomials of degree 3 in two variables x_1 and x_2. A basis for this space is given by $\{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$. The action of S_2 on these basis elements swaps x_1 and x_2. We can identify invariant subspaces under this action. The polynomial $x_1^2x_2 + x_1x_2^2$ is symmetric under the action of S_2, while the polynomial $x_1^3 + x_2^3$ is also symmetric. The antisymmetric part can be represented by $x_1^3 - x_2^3$. Therefore, we can decompose $V_{3,2}$ as follows:

$$V_{3,2} \cong \langle x_1^2x_2 + x_1x_2^2 \rangle \oplus \langle x_1^3 + x_2^3 \rangle \oplus \langle x_1^3 - x_2^3 \rangle.$$