Background

Suppose $X \sim \mathrm{Ber}(\theta)$ & suppose observed x

Want to test hypothesis $\{x, \dots, x\} = \{1, \dots, 1\}$

We use $\hat{\theta} = \frac{\sum(x)}{n}$ to estimate θ.

Example

$\mathrm{E}[\hat{\theta}] \overset{\text{CLT}}{\approx} \frac{\theta}{n}$

Question: How do we measure the accuracy of the estimate?

i.e. of $\sum(x)$ or CI of $\hat{\theta}$

① study $\sigma_{\hat{\theta}}$ or $\sqrt{\mathrm{Var}(\hat{\theta})}$ (hard to calculate or estimate)

② Accuracy

Method

Say we have a parameter θ. We estimate it based on a sample $\vec{X}$.

$\hat{\theta} = \frac{\sum(\vec{X})}{n}$; $\vec{X} = \{x_1, \dots, x_n\}$

we want CI of θ (some one is not)

① Jackknife

$\sum_{-i}(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ X_i

$\hat{\theta}_{-i} = \frac{\sum_{-i}(x)}{n-1}$

$\mathrm{SD}_{\text{jack}} = \sqrt{\frac{n-1}{n} \sum_{i=1}^{n} (\hat{\theta}_{-i} - \hat{\theta}_{\cdot})^2}$; $\hat{\theta}_{\cdot} = \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_{-i}$

Example Say $\hat{\theta} = X = \frac{1}{n} \sum X_i$

then we use $\hat{\theta}_{-i}$ is the average excluding X_i.

$\hat{\theta}_{-i} = \frac{\sum_{-i}(x)}{n-1} = \frac{\sum x - x_i}{n-1} = \frac{n\hat{\theta} - x_i}{n-1}$

$\hat{\theta}_{\cdot} = \frac{1}{n} \sum_{i=1}^{n} (\hat{\theta}_{-i})$

$= \frac{n\hat{\theta} - \sum_{i=1}^{n} x_i}{n-1}$

$= \hat{\theta}$

so $\mathrm{SD}_{\text{jack}} = \left(\frac{n-1}{n} \sum_{i=1}^{n} \left(\frac{n\hat{\theta} - x_i}{n-1} - \hat{\theta} \right)^2 \right)^{\frac{1}{2}}$

$= \left(\frac{n-1}{n} \sum_{i=1}^{n} \left(\frac{n\hat{\theta} - x_i - \hat{\theta}(n-1)}{n-1} \right)^2 \right)^{\frac{1}{2}}$

$= \left(\frac{1}{n(n-1)} \sum (x_i - \hat{\theta})^2 \right)^{\frac{1}{2}}$

$= \frac{\sqrt{\mathrm{Var}(X)}}{\sqrt{n}} = \sqrt{\mathrm{Var}(\hat{\theta})}$

to implement Jackknife

what we need to say.

Input : $\vec{X}$ and $S(\cdot)$.

Function : calculate every $\hat{\theta}_i$ (some one out)

calculate $\mathrm{SD}_{\text{jack}}$

Output : set of estimate $\hat{\sigma}(\cdot)$

② to special form of confidence distribution $F(x)$

① statistic pivot

② quotient to pivot cons.

Say we want to estimate correlation between X and Y.

① standard deviation

$\mathrm{SE} = \hat{\sigma}(\vec{X}, \vec{Y})$

$\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, \alpha Y) = \mathrm{Cov}(\alpha X, Y) = \mathrm{Cov}(\alpha X, \alpha Y) = -\rho$

$\frac{\sigma_X \sigma_Y}{\sigma_X \sigma_Y} = \frac{\sigma_X}{\sigma_X}$

② with c $\frac{\sigma_X}{\sigma_Y}$

$\hat{\theta}_{\cdot}$

$\alpha, \beta > 0$ $\mathrm{SE}(\alpha \cdot \mathrm{cos}(X,Y)) \neq \mathrm{SE}(\mathrm{cos}(X,Y))$

$\mathrm{Cov}(\alpha X, \beta Y) = \alpha\beta \mathrm{Cov}(X,Y)$

$\alpha = 1$ $\mathrm{Var}(\alpha X) = \alpha^2 \mathrm{Var}(X)$

$\alpha = 0$ $\mathrm{Var}(\alpha X) = 0$

③ Bootstrap :

Ideally, better way to estimate SE of $\hat{\theta} = S(x)$

is repeatedly draw samples from distribution F.

$(\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n) \longrightarrow (S(\vec{X}_1), \dots, S(\vec{X}_n))$ $\xrightarrow[\text{variance}]{\text{calculate}}$ $\mathrm{SE}(S(\vec{X}))$

estimate $F(x)$ with empirical cdf $F_n(x)$

$\vec{X}^* = (x_1^*, \dots, x_n^*)$ bootstrap sample

x_i^* is drawn uniformly at random with replacement from $\vec{X}$

same probability not always probability.

we independently draw B bootstrap samples from $\vec{X}$

$(\vec{X}_1^*, \dots, \vec{X}_B^*) \xrightarrow{\text{for}} (S(\vec{X}_1^*), \dots, S(\vec{X}_B^*))$

$\hat{\mathrm{SE}}_{\text{boot}} = \sqrt{\frac{1}{B-1} \sum_{b=1}^{B} (\hat{\theta}_b^* - \hat{\theta}_{\cdot}^*)^2} / \sqrt{n-1}$; $\hat{\theta}_{\cdot}^* = \frac{1}{B} \sum_{b=1}^{B} \hat{\theta}_b^*$

plug in estimator

to implement Bootstrap.

what we need to say.

Input : $\vec{X}$ and $S(\cdot)$

Function : draw B bootstrap for $\vec{X}$

calculate $\hat{\theta}^*$ and

Output : set of estimate $\hat{\sigma}(\cdot)$

① to special form of confidence distribution $F(x)$

② statistic

③ $B \to \infty$ is usually sufficient for deriving S.E.

④ other procedures : just plug in one bootstrap sample statistics

e.g. CI for $\hat{\theta}$.

Consider any of sticky sampling

Example

$\vec{w} = (w_1, \dots, w_n)'$ weight factor of sample $\vec{X}(\omega) = 1$

SAE turn $\hat{\theta}$ changes when we use different $\vec{w}$

For Jackknife

$w = (\frac{1}{n-1}, \dots, \frac{1}{n-1}, 0, \frac{1}{n-1}, \dots)$

For bootstrap

$w^* = (\cdot, \cdot, \cdot, \cdot, \cdot, \cdot)$, anything that satisfies $\vec{X}(\omega) = 1$.

Just : $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$

But : $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4$ *