$$\mathrm{Var}(\hat{f}_n(x)) = \frac{1}{nh}\int K^2(v)\,dv\, f(x) + o\left(\frac{1}{nh}\right)$$

MISE (mean integrated square error)

$$\mathrm{MISE}\{\hat{f}_h(x)\} = \int_{-\infty}^{\infty} \mathrm{MSE}\{\hat{f}_h(x)\}\,dx$$

Optimize the MISE over h ?

$$\mathrm{MISE}\{\hat{f}_h(x)\} = \frac{1}{nh} R(K) + \frac{1}{4} h^4 \mu_2(K)^2 R(f'') + o\left(\frac{1}{nh} + h^4\right)$$

here $R(K) = \int_{-\infty}^{\infty} K^2(v)\,dv$

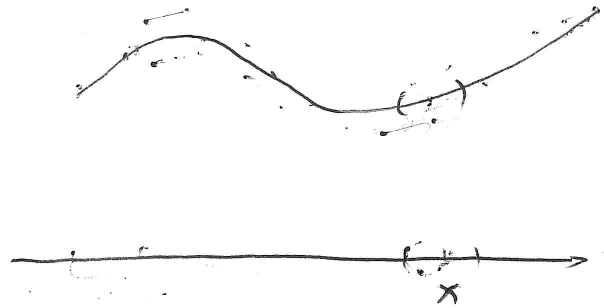
$R(f'') = \int_{-\infty}^{\infty} f''^2(x)\,dx$

$\mu_2(K) = \int x^2 K(x)\,dx$

Regress

Data (X_i, Y_i), $i=1,\dots n$

$Y_i = m(X_i) + \varepsilon_i$



ideal : Fit a weighted polynomial regression of degree p with kernel weights.

polymial : $S_p(x) = \beta_0 + \beta_1(\cdot - x) + \cdots + \beta_p(\cdot - x)^p$

weight : $K_h = \frac{1}{h} K\left(\frac{\cdot - x}{h}\right)$

· Find out $\beta_0, \cdots \beta_p$ that minimize

$$\sum_{j=1}^{n} \left\{ \left(y_i - \beta_0 - \beta_1(x_i - x) - \cdots - \beta_p(x_i - x)^p \right)^2 K_h(x_i - x) \right\}$$

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \vdots \\ \hat{\beta}_p \end{pmatrix} = (X_x' W_x X_x)^{-1} (X_x' W_x Y)$$

$$X_x = \begin{pmatrix} 1 & x_1 - x & \cdots & (x_1 - x)^p \\ \vdots & \vdots & & \vdots \\ 1 & x_n - x & & (x_n - x)^p \end{pmatrix}$$

$$W_x = \mathrm{diag}(K_h(X_1 - x), \cdots K_h(X_n - x))$$

If $p = 0$ Nadaraya-Watson estimator

$$\hat{\beta} = \sum_{j=1}^{n} K_h(X_j - x) y_j \Big/ \sum_{j=1}^{n} K_h(X_j - x)$$

- bandwidth selection (select h)

R kernSmooth package

direct plug-in

cross-validation

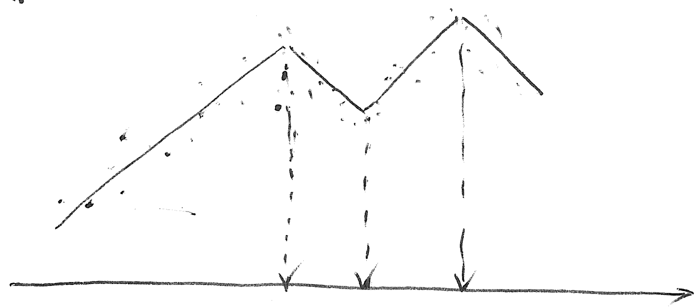
$$CV(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{m}_{(-i)}(x_i))^2$$

↳ the point "i" is left out

pick h that minimize the $CV(\cdot)$.

disadvantage: computationally expensive

Spline Regression

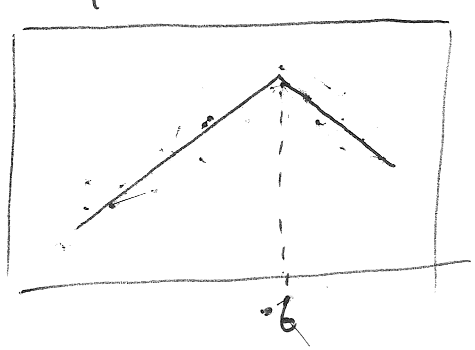


- linear regression

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

basis for the model $\{1, x\}$

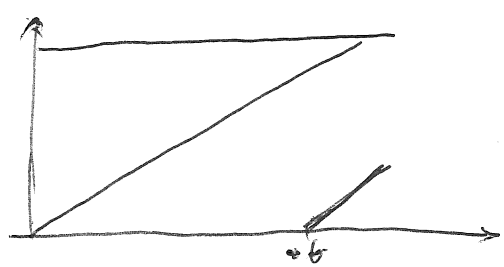
~~For~~ more complicated case, model with more complicated basis

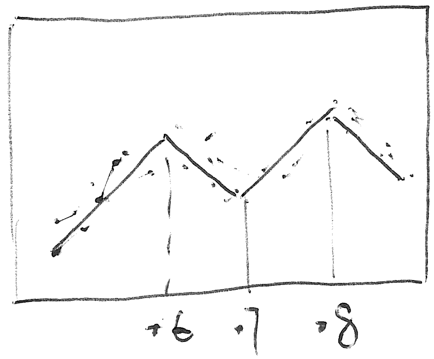


Broken stick basis

$$\{1, x, (x-.6)_+\}$$

$$(x-.6)_+ = \begin{cases} x-.6 & \text{if } x \geq .6 \\ 0 & \text{e.w} \end{cases}$$





$\{1, x, (x-.6)_+, (x-.7)_+, (x-.8)_+\}$

- It is possible to handle any complex type of structure by simply adding more functions of form $(x-k)_+$ to the basis.
- The value of k is usually referred as a knot, where two lines are "tie together"
- For knots $x_1, x_2, \cdots x_N$, the truncated power basis $\{1, x, \cdots x^{m-1}, (x-x_j)^{m-1}, j=1, \cdots N\}$

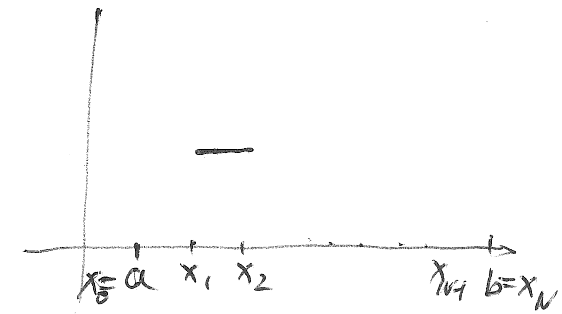
An equivalent basis to truncated power basis is B splines.

B splines.

Recursive formula of B splines order $m-1$ to m.

$$b_{j,1}(x) = \begin{cases} 1, & x \in [x_j, x_{j+1}] \\ 0 & \text{e.w} \end{cases}$$

$$a = x_0 < x_1 < x_2 \cdots x_N = b$$



$$b_{j,m}(x) = \frac{x - x_j}{x_{j+m-1} - x_j} b_{j,m-1}(x) + \frac{x_{j+m} - x}{x_{j+m} - x_{j+1}} b_{j+1,m-1}(x)$$

For equally spaced knots.

$$x_j = a + j\frac{(b-a)}{N+1} = a + jh$$

$$b_{j,m}(x) = \frac{x - x_j}{(m-1)h} b_{j,m-1}(x) + \frac{x_{j+m} - x}{(m-1)h} b_{j+1,m-1}(x)$$

Factorial design (chapter 29)

2^k factorial design.

- k factors, each has 2 levels $(+, -)$
- all interactions

$k = 2$, 2^2

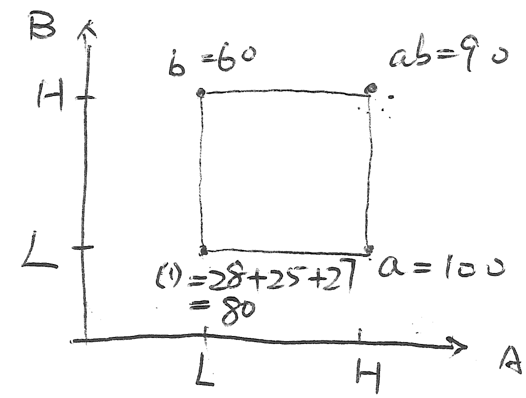
$$\mu_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

4 parameters.

μ, $\alpha_1 = -\alpha_2$, $\beta_1 = -\beta_2$, $(\alpha\beta)_{11} = -(\alpha\beta)_{12} = -(\alpha\beta)_{21}$
$= (\alpha\beta)_{22}$

If we label the factor A and B "+", "−", there are 4 experimental combinations labels.

Symbol	A	B	I	II	III
(1)	−	−	28	25	27
a	+	−	36	32	32
b	−	+	18	19	23
ab	+	+	31	30	29



$ab \quad Ey = \mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22}$

$a \quad Ey = \mu + \alpha_2 - \beta_2 - (\alpha\beta)_{22}$

$b \quad Ey = \mu - \alpha_2 + \beta_2 - (\alpha\beta)_{22}$

$(1) \quad Ey = \mu - \alpha_2 - \beta_2 + (\alpha\beta)_{22}$

$$\hat{\mu} = \frac{ab + a + b + (1)}{4n}$$

n : # of replications
in our ex: $n = 3$

$$\hat{\alpha}_2 = \frac{ab + a - b - (1)}{4n},$$

$$\hat{\beta}_2 = \frac{ab - a + b - (1)}{4n}$$

$$\widehat{(\alpha\beta)}_{22} = \frac{ab - a - b + (1)}{4n}$$

Main effect of A.

The effect of A at the low level of B

$$\frac{1}{n}(a - (1))$$

The effect of A at the high level of B

$$\frac{1}{n}(ab - b)$$

$$A = \frac{1}{2n}(ab - b + a - (1)) = 2\hat{\alpha}_2$$

$$B = 2\hat{\beta}_2$$

interaction effect: $AB = 2\widehat{(\alpha\beta)}_{22}$

In our example: $A = \frac{1}{2\times 3}(90-60+100-80) = 8.33$

$$B = \frac{1}{2\times 3}(90+60-100-80)$$

$$AB = \frac{1}{2\times 3}(90-60-100+80)$$

- Sum of squares

effect	Contrast (1)	a	b	coefficients ab
A	-1	1	-1	1
B	-1	-1	1	1
AB	1	-1	-1	1

$$SS_{\text{Contrast}} = \frac{(\sum c_i y_i)^2}{n\sum_{i=1}^{4} c_i^2}, \quad i = 1, 2, 3, 4$$

$$SS_A = \frac{(ab + a - b - (1))^2}{n\,4} = 4n\,\hat{\alpha}_2^2$$

$$SS_B = 4n\,\hat{\beta}_2^2$$

$$SS_{AB} = 4n\,\widehat{(\alpha\beta)}_{22}^2$$

in our ex. $SS_A = \frac{(50)^2}{3 \times 4} = 208.33$

$SS_B = 75$

$SS_{AB} = 8.33$

$SS_T = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{y_{...}^2}{2 \times 2 \times 3} = 323$

$SS_E = 323 - 208.33 - 75 - 8.33 = 31.34$

ANOVA table followed.

2^3 factorial design

8 combinations

A	B	C	symbol
−	−	−	(1)
+	−	−	a
−	+	−	b
+	+	−	ab
−	−	+	c
+	−	+	ac
−	+	+	bc
+	+	+	abc

A main effect is $2\hat{\alpha}_2$

$$= \frac{(a-(1)) + (ab-b) + (ac-c) + (abc-bc)}{4n}$$

$$= \frac{(a-(1)) + b(a-(1)) + c(a-(1)) + bc(a-(1))}{4n}$$

$$= \frac{(a-(1))(b+(1))(c+(1))}{4n}$$

AB effect: $\dfrac{(a-(1))(b-(1))(c+(1))}{4n}$

Fundamental principles for factorial effects.

hierarchical ordering
- lower order effects are more likely to be more important than higher order effects
- effects of the same order are equally likely to be important.

- Sparsity principle

$$2^2 - 2^3 - 2^4 - 2^5 - \cdots 2^{10}$$

One observation in each combination

- often assume 3 or higher interaction do not occur.
- Pool these interactions as error.

Blocking in 2^k factorial design.

2^k design with 2 blocks (Blocks assumed to allow 2^{k-1} combinations)

- 2^2 factorial (2 combinations in each block)
  - possible parings
    1. (1) and b ~~together~~ (a and ab together)
    2. (1) and a together (b ab)
    3. → (1) and ab together (a b)

Effect of AB $\dfrac{ab + (1) - a - b}{2}$ is block difference

Blocking in 2^k factorial designs.

- cannot run all combinations within one block
- Certain effects confounded, which means 2 effects are indistinguishable.

  (AB effect and block effect are confounded in our 2^2 example)
- "Sacrifice" certain effects.

  2^2 factorial with 2 blocks

1	(1) & b	a & ab	Effect of A is block difference $\frac{ab + a - (1) - b}{2}$ A confounded.
2.	(1) & a	b & ab	B confounded
3.	(1) & ab	a & b	AB confounded.

- Generally we confound highest order interaction

  can use + − table to determine blocks.

A	B	AB	Symbol	
−	−	+	(1)	1st block
+	−	−	a	
−	+	−	b	2nd block
+	+	+	ab	

2^3 factorial

A	B	C	AB	AC	BC	ABC	symbol
−	−	−	+	+	+	−	(1)
+	−	−	−	−	+	+	a
−	+	−	−	+	−	+	b
+	+	−	+	−	−	−	ab
−	−	+	+	−	−	+	c
+	−	+	−	+	−	−	ac
−	+	+	−	−	+	−	bc
+	+	+	+	+	+	+	abc

Confound ABC assign (1), ab, ac, bc

a b c abc

block with (1) is called "principle block"

(1), ab, ac, bc

a (1) a·ab a ac abc

→ a b c abc

$a^2 = b^2 = c^2 = I$

– 2^k factorial in four blocks ($k \geq 4$ usually)

- each block~~s~~ contains 2^{k-2} e.c.s.
- Must select two ~~effects~~ to confound.
- will result in a third confounded ~~factor~~ effect.

2^6 Factorial in 4 blocks (A B C D E F.)

		ABC		DEF
Block 1	use	ABC	and	DEF −
2		ABC −		DEF +
3		ABC +		DEF −
4		ABC +		DEF +

Result in (ABC)(DEF) = ABCDEF confounded.