Latin Squares Designs (Chap 28)

- Block on two nuisance factors
- Must have same number of block levels and treatment levels
- Incomplete design.
  only a single treatment is applied with each combination of blocking variables.

Ex: A, B, C = trts

3 trts, 3 block levels with 2 blocking variables.

each row contains all trts / one trt per row

each column contains all trts. / one trt per column

operators

	1	2	3
M	A	B	C
T	C	A	B
W	B	C	A

	1	2	3	4
M	A	B	C	D
T	D	A	B	C
W	C	D	A	B
T	B	C	D	A

For $r=3$, there are 12 arrangement

$3 \times 2 \times 1 \times 2$

For $r=5$, there are 161280 ways.

- R code

```
result ← matrix ("A", 4, 4)
okay ← F
while ( ! okay) {
    result [1, ] ← sample (c("A", "B", "C", "D"), 4, replace=F)
    result [2, ] ←    =
    result [3, ] ←    =
    result [4, ] ←    =
    if ( (all ( table ( result[ ,1]==1)) &
        (all ( table ( result [ ,2]==1) &
        (all ( table ( result[ ,3]== 1) &
        (all ( table ( result[ ,4]==1) )
        okay ← T }
result
```

A
B
A
C

```
table (result[,1])
A B C
2 1 1
```

advantage: Small, balanced

impossible to use each treatment levels for the same comb. of blocking levels.

· Disadv: equal # of rows, cols, trts.

no interaction between trts & blocks

Model: $$y_{ijk} = \mu + \underset{\substack{\downarrow \\ \text{row}}}{\rho_i} + \underset{\substack{\downarrow \\ \text{col}}}{\chi_j} + \underset{\substack{\downarrow \\ \text{trt}}}{\alpha_k} + \varepsilon_{ijk} \qquad i, j, k = 1, \ldots r$$

$$\Sigma \rho_i = 0, \quad \Sigma \chi_j = 0, \quad \Sigma \alpha_k = 0$$

$$\varepsilon_{ijk} \sim N(0, \sigma^2)$$

ANOVA:

α	$r-1$
ρ	$r-1$
χ	$r-1$
Error	$r^2-1-3(r-1) = r^2-3r+2 = (r-1)(r-2) < (r-1)(r-1)$
SSTO	r^2-1

$$\hat{\mu} = \bar{Y}_{\cdot\cdot\cdot}$$

$$\hat{\rho}_i = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot}$$

$$\hat{\kappa}_j = \bar{Y}_{\cdot j \cdot} - \bar{Y}_{\cdot\cdot\cdot}$$

$$\hat{\alpha}_k = \bar{Y}_{\cdot\cdot k} - \bar{Y}_{\cdot\cdot\cdot}$$

$$SSTO = \sum_{i=1}^{r}\sum_{j=1}^{r} (Y_{ijk} - \bar{Y}_{\cdot\cdot\cdot})^2$$

$$SS_P = r \sum_i (\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2$$

$$SS_\kappa = r \sum_j (\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2$$

$$SS_{TR} = r \sum_k (\bar{Y}_{\cdot\cdot k} - \bar{Y}_{\cdot\cdot\cdot})^2$$

$$SS_{REM} = \sum_i \sum_j (Y_{ijk} - \hat{Y}_{ijk})^2$$

$$= \sum_i \sum_j (Y_{ijk} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot\cdot k} + 2\bar{Y}_{\cdot\cdot\cdot})^2$$

$$SSTO = \sum_i \sum_j (\underline{Y_{ijk} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot\cdot k} + 2\bar{Y}_{\cdot\cdot\cdot}} + \underline{\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot}} + \underline{\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot\cdot\cdot}} + \underline{\bar{Y}_{\cdot\cdot k} - \bar{Y}_{\cdot\cdot\cdot}})^2$$

$$= \cdots$$

$$= SS_{REM} + SS_P + SS_\kappa + SS_{TR}$$

$$E(MS_\rho) = \sigma^2 + r\frac{\sum \rho_i^2}{r-1} \qquad E(MS_{REM}) = \sigma^2$$

$$E(MS_\chi) = \sigma^2 + r\frac{\sum \chi_j^2}{r-1}$$

$$E(MS_{TR}) = \sigma^2 + r\frac{\sum \alpha_k^2}{r-1}$$

F - test $\quad \dfrac{MS_\rho}{MS_{REM}}, \quad \dfrac{MS_\chi}{MS_{REM}}, \quad \dfrac{MS_{TR}}{MS_{REM}}.$

- Planning a Latin square.

  compute power.

  Test. $H_0: \alpha_i \equiv 0$,

  F test.

  $$\varphi = \frac{1}{\sigma}\sqrt{\frac{\sum \alpha_i^2}{r}}$$

  DF: $r-1$, $(r-1)(r-2)$

- Efficency of Blocking variables

$$E_1 = \frac{\sigma^2_{CRD}}{\sigma_L^2}, \quad E_2 = \frac{\sigma^2_{BD\,(row)}}{\sigma_L^2}, \quad E_3 = \frac{\sigma^2_{BD\,(col)}}{\sigma_L^2}$$

Estimate variances

Text: $$\hat{\sigma}^2_{CRD} = \frac{MS\,Row + MS\,Col + (r-1)\,MSREM}{r+1}$$

$$\hat{\sigma}^2_{BD(row)} = \frac{MS\,Col + (r-1)\,MSREM}{r}$$

$$\hat{\sigma}^2_{BD(col)} = \frac{MS\,Row + (r-1)\,MSREM}{r}$$

$$\hat{\sigma}^2_{CRD} = \frac{SS_{row} + SS_{col} + SS_{REM}}{nr^2 - r}$$

$$\hat{\sigma}^2_{BD(row)} = \frac{SS_{col} + SSREM}{nr^2 - r - (r-1)}$$

$$\hat{\sigma}^2_{BD(col)} = \frac{SS_{row} + SSREM}{nr^2 - r - (r-1)}$$

SSTR	$r-1$
SScol	$r-1$
SSrow	$r-1$
SSREE	$nr^2-1-3(r-1)$
SSTO	nr^2-1

$n=1$ if regular

Graeco - Latin square design

Ex: An experiment is conducted to compare 4 gasoline additives by testing them on 4 cars with 4 drivers over 4 days. Only 4 runs can be conducted in each day. The response is the amount of automobile emission.

- Treatment factor : gasoline additive A B C D.
- Block factor 1. car, 1 2 3 4
- " " 2 driver 1, 2, 3, 4
- " " 3 day α. β. γ. δ

Graeco - Latin square design.

= control three soures of variation, or blocking in 3 dimensions.

<=> a combination of 2 Latin Squre designs, where any Greek letter occurs with any Latin letter exactly once.

Example : $r = 2$

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \longleftrightarrow \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} A\alpha & B\beta \\ B\beta & A\alpha \end{bmatrix}$$

not right

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \longleftrightarrow \begin{bmatrix} \beta & \alpha \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} A\beta & B\alpha \\ B\alpha & A\beta \end{bmatrix}$$

not right

doesn't exist for $r = 2$

$r = 3$

$$\begin{bmatrix} A\alpha & B\beta & C\gamma \\ B\gamma & C\alpha & A\beta \\ C\beta & A\gamma & B\alpha \end{bmatrix}$$

2×2 Latin sq : 2 arrangements

Graeco-Latin doesn't

3×3 Latin sq : 12 arrangements

Graeco-Latin 72 : 12×6

4×4 Latin square : 576

Graeco-Latin $6912 = 576\times 12$

in 1901, Gaston $\Rightarrow$ ($\nexists$) no Graeco-Latin for $r=6$

X conjecture $\nexists$ Graeco-Latin for $r=4k+2$

Graeco-Latin exists for $r=10$.

$$y_{ijkl} = \mu + \underset{\text{trt}}{\tau_i} + \underset{\text{Greek}}{\alpha_j} + \underset{\text{row}}{\beta_k} + \underset{\text{column}}{\delta_l} + \varepsilon_{ijkl}. \qquad i,j,k\,l = 1,\dots r$$

ANOVA:

SS_{Latin}	$r-1$
SS_{Greek}	$r-1$
SS_{row}	$r-1$
SS_{column}	$r-1$
SS_E	$r^2-1-4(r-1) = (r-1)(r-3)$
SS_{Tot}	r^2-1

- Unequal design (A, B)

$$SS(A \mid 1) = SS(1) - SS(1, A)$$

$$SS(A \mid 1, B) = SS(1, B) - SS(1, A, B)$$

order matters

For a balanced design

$$SSTO = SSTR + SSE$$

$$\overset{\text{if balance}}{=} SSA + SSB + SSAB + SSE \qquad (n_i \equiv n)$$

$$SS(A \mid 1) = SS(1) - SS(1, A)$$

$$= SSTO - (SSB + SSAB + SSE) = SSA$$

$$SS(A \mid 1, B) = SSA + SSAB + SSE - (SSAB + SSE)$$

$$= SSA$$

- Nested Design (chapter 26)
  - Factors A & B are considered crossed if every level of B occurs with every level of A.

		A			
		1	2	3	4
B	1	xx	xx	xx	xx
	2	xx	xx	xx	xx
	3	xx	xx	xx	xx

  - Factors A & B are considered nested if level of B occur with only one level of A.

	A											
	1			2			3			4		
B	1	2	3	4	5	6	7	8	9	10	11	12
	x	x	x	x	x	x	x	x	x	x	x	x
	x	x	x	x	x	x	x	x	x	x	x	x

Example:

	state 1			State 2			state 3		
city	1	2	3	1	2	3	1	2	3
household	1	1	1						
	2	2	2						
	3	3	3						
	4	4	4						

Factor A: state: 3 levels

B: city 3 levels

$$y_{ijk} = \mu + \alpha_i + \beta_{j(i)} + \varepsilon_{ijk}$$

- bracket notation represents nested factor.
- Cannot include interaction.
  - Not all levels of B apprear with all levels of A.
  - Cannot seperate main effect of B and interaction AB algebraically.

$$\alpha_i = \mu_{i\cdot} - \mu_{\cdot\cdot}$$

$$\beta_{j(i)} = \mu_{ij} - \mu_{i\cdot}$$

$$\sum_{i=1}^{a} \alpha_i = 0 \qquad \sum_{j=1}^{b} \beta_{j(i)} = 0 \quad \text{for } i = 1, \ldots a$$