Main effect	F	R	R
	a	b	n
	i	j	k
τ_i	0	b	n
β_j	a	1	n
γ_{ij}	0	1	n

← prepare the table

1. put 0 if at least one row index matches column index and Fixed
2. put 1 if - - - - - - - - - - - - - - - - - - - and Random

in the remaining put # of levels.

2. get $E(MS)$
   - Delete columns with index and rows without index
   - multiply Fixed $\rightarrow Q(\tau) = \frac{\sum \tau_i^2}{a-1}$

     $+$ Random $\rightarrow \sigma_\beta^2$

     $+ \sigma^2$

$$E(MS_\tau) = bn\,Q(\tau) + n\sigma_\gamma^2 + \sigma^2 = bn\frac{\sum \tau_i^2}{a-1} + n\sigma_\gamma^2 + \sigma^2$$

$$E(MS_\beta) = an\sigma_\beta^2 + 0\cdot n\sigma_\gamma^2 + \sigma^2 = an\sigma_\beta^2 + \sigma^2$$

$$E(MS_\gamma) = n\sigma_\gamma^2 + \sigma^2$$

$$E(MSE) = \sigma^2$$

Example : 2 - factor fixed model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}, \quad \begin{array}{l} i = 1, \cdots a \\ j = 1, \cdots b \\ k = 1, \cdots n \end{array}$$

Factor	F	F	R
	a	b	n
	i	j	k
τ_i	0	b	n
$\cancel{\beta_j}$	$\cancel{a}$	$\cancel{0}$	$\cancel{n}$
$(\tau\beta)_{ij}$	0	0	n

$$E(MS_\tau) = bn\,Q(\tau) + \sigma^2$$

$$E(MS_\beta) = an\,Q(\beta) + \sigma^2$$

$$E(MS_{\tau\beta}) = n\,Q(\tau\beta) + \sigma^2$$

$$Q(\tau) = \frac{\sum \tau_i^2}{a-1}$$

$$Q(\beta) = \frac{\sum \beta_j^2}{b-1}$$

$$Q(\tau\beta) = \frac{\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$$

Example 3-way ANOVA

$$y_{ijkl} = \mu + \tau_i + \beta_j + \delta_k + (\tau\beta)_{ij} + (\tau\delta)_{ik} + (\beta\delta)_{jk} + (\tau\beta\delta)_{ijk} + \varepsilon_{ijkl}$$

τ_i: fixed; β_j, δ_k: random

$i = 1, \dots a$
$j = 1, \dots b$
$k = 1, \dots c$
$l = 1, \dots n$

Factors	F	R	R	R
	a	b	c	n
	i	j	k	l
τ_i	0	b	c	n
β_j	a	1	c	n
δ_k	a	b	1	n
$(\tau\beta)_{ij}$	0	1	c	n
$(\tau\delta)_{ik}$	0	b	1	n
$(\beta\delta)_{jk}$	a	1	1	n
$(\tau\beta\delta)_{ijk}$	0	1	1	n

$$E(MS_\tau) = \sigma^2 + bcn\,Q(\tau) + cn\,\sigma^2_{\tau\beta} + bn\,\sigma^2_{\tau\delta} + n\,\sigma^2_{\tau\beta\delta}$$

$$E(MS_\beta) = \sigma^2 + acn\,\sigma^2_\beta + an\,\sigma^2_{\beta\delta}$$

$$E(MS_\delta) = \sigma^2 + abn\,\sigma^2_\delta + an\,\sigma^2_{\beta\delta}$$

$$E(MS_{\tau\beta}) = \sigma^2 + cn\,\sigma^2_{\tau\beta} + n\,\sigma^2_{\tau\beta\delta}$$

$$E(MS_{\tau\delta}) = \sigma^2 + bn\,\sigma^2_{\tau\delta} + n\,\sigma^2_{\tau\beta\delta}$$

$$E(MS_{\beta\delta}) = \sigma^2 + an\,\sigma^2_{\beta\delta}$$

$$E(MS_{\tau\beta\delta}) = \sigma^2 + n\,\sigma^2_{\tau\beta\delta}$$

$$E(MSE) = \sigma^2$$

Eg: $H_0: \sigma^2_\beta = 0, \quad H_A: \sigma^2_\beta > 0$

$$\frac{MS_\beta}{MS_{\beta\delta}} \sim F_{(b-1),\,(b-1)(c-1)}$$

$H_0: \sigma^2_{\tau\beta} = 0, \quad H_A: \sigma^2_{\tau\beta} > 0$

$$\frac{MS_{\tau\beta}}{MS_{\tau\beta\delta}} \sim F_{(a-1)(b-1),\,(a-1)(b-1)(c-1)}$$

$H_0: \tau_1 = \tau_2 = \cdots \tau_a = 0$, H_A : H_0 not holds

bcn $Q(\tau) + E(MS_{\tau\beta}) + E(MS_{\tau\delta}) = E(MS_\tau + MS_{\tau\beta\delta})$.

$$F = \frac{MS_\tau + MS_{\tau\beta\delta}}{MS_{\tau\beta} + MS_{\tau\delta}} \sim \text{approximate } F \text{ test } F_{p,q}$$

$$p = \frac{(MS_\tau + MS_{\tau\beta\delta})^2}{\frac{MS_\tau^2}{df\,\tau} + \frac{MS_{\tau\beta\delta}^2}{df(\tau\beta\delta)}}$$

$$q = \frac{(MS_{\tau\beta} + MS_{\tau\delta})^2}{\frac{MS_{\tau\beta}^2}{df(\tau\beta)} + \frac{MS_{\tau\delta}^2}{df(\tau\delta)}}$$

In general, approx df. of $Q = \sum c_i MS_i$ is

$$\frac{Q^2}{\sum \frac{(c_i MS_i)^2}{df_i}}$$

Satterthwaith procedure

Test: $H_0: \sigma_\beta^2 = 0 \qquad H_A: \sigma_\beta^2 > 0$

1. Method, approximate F-test (balanced, nearly balanced)
2. Method, LRT (Likelihood ratio ~~test~~)

(Normal dist. assumption)

$$\lambda = \frac{\max_{\Theta \in \Theta_0} L(\theta)}{\max_{\Theta \in \Theta} L(\theta)}$$

$$-2 \log \lambda \sim \chi^2(1)$$

· Parameter estimation.

Likelihood-based Methods.

estimates cannot be written in closed form

Numerical solution. (consistent, asymptotic property)

R: nlme (package (nlme))

lmer (package (lme4))

Restricted ML: maximize L under the constraints

$\sigma^2 \geq 0, \quad \sigma_\mu^2 \geq 0$

- Mixed Model (ANOVA III)

Two-way mixed model (Restricted)

One fixed, one random, the interaction is considered to be random.

$$Y_{ijk} = \mu + \underset{\downarrow \atop f}{\alpha_i} + \underset{\downarrow \atop r}{\beta_j} + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

$$\sum \alpha_i = 0$$

$$\beta_j \sim N(0, \sigma_\beta^2)$$

$$(\alpha\beta)_{ij} \sim N\left(0, \frac{a-1}{a}\sigma_{\alpha\beta}^2\right) \quad \text{subject to} \quad \sum_i (\alpha\beta)_{ij} = 0$$

for each j

$$\mathrm{Cov}(\alpha\beta_{ij}, \alpha\beta_{i'j}) = -\frac{\sigma_{\alpha\beta}^2}{a}$$

Chapter 21. Randomized Block Designs

In any experiment, variability arising from ~~a~~ nuisance factor can affect the results.

nuisance factor → not interested

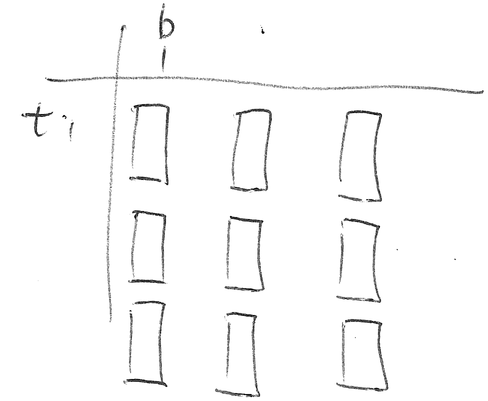
nuisance factor: { known not controlled (Randomization)
known and controllable (blocking design)

Ideal: grouping of experimental units into homogeneous groups (Blocks)

- If all units are homogenous – need only 1 Block
  exp units are assigned to trts at random
  Completely Randomized Design
- If more than 1 groups requred: within each block units are more homogeneous than between blocks
  Some blocked design
  - Randomized complete block design (RCBD)

For "a" trts — "b" blocks with "a" exp units each. Any trt assigned to the "ar" same number of exp units with any block.

trt	b 1	2	3	4
	~~A~~	B	A	B
	B	~~C~~	C	A
	C	A	B	C



· Incomplete block design.

Assumption: No interaction between the components of trt structure and the components of design structure, and among the comp. of design structure.

· Randomized Block Designed.

· Each trt is included once in each Block.

$$y_{ij} = \mu + \rho_i + \tau_j + \varepsilon_{ij}$$

block (row) $\sum \rho_i = 0$, Block levels $i=1, \cdots n_b$

$\sum \tau_j = 0$, trt levels, $j=1, \cdots a$

no interaction.

· 2-way ANOVA with no interaction

$$SSBL = a \sum_i (\bar{y}_{i.} - \bar{y}_{..})^2 \qquad n_b - 1$$

$$SSTR = n_b \sum_j (\bar{y}_{.j} - \bar{y}_{..})^2 \qquad a-1$$

$$SSBL.TR = \sum_i \sum_j (y_{ij} - \hat{y}_{ij})^2$$

$$= \sum_i \sum_j (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2 \qquad (a-1)(n_b-1)$$

· If Block effect is fixed.

	F	F	R
	a	n_b	1
	τ_j	ρ_i	ε_{ik}
τ_j	0	n_b	1
ρ_i	a	0	1

$$E(MS(\tau)) = \sigma^2 + n_b \frac{\sum \tau_j^2}{a-1}$$

$$E(MS(\rho)) = \sigma^2 + a \frac{\sum \rho_i^2}{n_b - 1} \rightarrow a\sigma_\rho^2$$

If Block is random

$$E(MSE) = \sigma^2$$

- If τ_j, ρ_i interact (Restricted version)

  ρ_i random

  $\sum_{j=1}^{a} (\tau\rho)_{ij} = 0$.

- n replications in each cell.

$$E(MS(\tau)) = \sigma^2 + n\, n_b\, Q(\tau) + n\,\sigma^2_{\tau\rho}$$

$$E(MS(\rho)) = \sigma^2 + an\,\sigma_\rho^2$$

$$E(MS(\tau\rho)) = \sigma^2 + n\,\sigma^2_{\tau\rho}$$

$$E\ MSE = \sigma^2$$

- If unrestricted version.

$$E(MS(\rho)) = \sigma^2 + an\,\sigma_\rho^2 + n\,\sigma^2_{\tau\rho}$$

- Efficiency of blocking.

$$E = \frac{\sigma_r^2}{\sigma_b^2}$$

σ_r^2 = variance in a completely random design.

σ_b^2 = variance in a block design

$$\hat{\sigma}_b^2 = MSBL.TR$$

$$\hat{\sigma}_r^2 = \frac{SSBL + SSBL.TR}{df}$$

$$= \frac{(n_b-1)\,MSBL + (n_b-1)(a-1)\,MSBL.TR}{(n_b-1)\,a}$$

In text book an unbiased est of σ_r^2 is

$$\frac{(n_b-1)\,MSBL + n_b(a-1)\,MSBL.TR}{n_b a - 1}$$

$$\hat{E} = \frac{\hat{\sigma}_r^2}{\hat{\sigma}_b^2} = \frac{(n_b-1)\,MSBL + (n_b-1)(a-1)\,MSBL.TR}{(n_b-1)\,a\,MSBL.TR}$$

$$= \frac{1}{a}\frac{MSBL}{MSBL.TR} + 1 - \frac{1}{a}$$

$$\hat{E} > 1 \iff \frac{MSBL}{MSBL.TR} > 1$$