Two way ANOVA with equal group sizes

Analysis of factor effects.

① Determine if factors interact.

If $\exists$ no interaction

$$\hat{\mu}_{i\cdot} = \bar{y}_{i\cdot\cdot} , \qquad \hat{\mu}_{\cdot j} = \bar{y}_{\cdot j \cdot}$$

$$Var(\hat{\mu}_{i\cdot}) = \frac{\sigma^2}{bn} \qquad Var(\hat{\mu}_{\cdot j}) = \frac{\sigma^2}{an}$$

Contrast

For $L = \sum_i c_i \mu_{i\cdot}$

$$\hat{L} = \sum_i c_i \hat{\mu}_{i\cdot}$$

$$Var(\hat{L}) = \frac{\sigma^2}{bn} \sum_i c_i^2$$

$$\widehat{Var}(\hat{L}) = \frac{MSE}{bn} \sum_i c_i^2 = \frac{SSE \sum c_i^2}{ab^2 n(n-1)}$$

t-interval
t-test
$\Big\}$ df. $ab(n-1)$

· Multiple comparisons of A-factor effects.

Tukey: $\hat{\mu}_{i\cdot} - \hat{\mu}_{i'\cdot} \pm \frac{1}{\sqrt{2}} q(1-\alpha; a, (n-1)ab) \sqrt{\widehat{Var}(\hat{\mu}_{i\cdot} - \hat{\mu}_{i'\cdot})}$

Bonf. Holm - as for 1-way ANOVA

Scheffé $\hat{L} \pm \sqrt{(a-1) F(1-\alpha;\, a-1,\, (n-1)ab)}\, \sqrt{\widehat{Var}(\hat{L})}$

- Multiple comparisons of A-factor and B-factor effects.

  Scheffe + Bonf. = join two sets of contrasts
  (A contr + B contr by Bonf)

  or Bonf. directly

  or Tukey + Bonf

  or Scheffé with $\hat{L} \pm \sqrt{(a+b-2) F(1-\alpha,\, a+b-2,\, (n-1)ab)}\, \sqrt{\widehat{Var}(\hat{L})}$

② If $\exists$ significant interactions
(compare treatment means μ_{ij})

Tukey: for $\mu_{ij} - \mu_{i'j'}$, use $\frac{1}{\sqrt{2}}\, q(1-\alpha;\, ab,\, (n-1)ab)$

Scheffé: for $L = \sum_j \sum_i c_{ij}\mu_{ij}$, use $\sqrt{(ab-1) F(1-\alpha;\, ab-1,\, (n-1)ab)}$

Bonf. use $t(1-\frac{\alpha}{2g},\, (n-1)ab)$

$g \to$ # of contrasts

Chapter 20. Two-way ANOVA with $n_{ij} \equiv 1$.
(1 obs for each treatment)

Problem: $SSE = \sum\sum\sum (y_{ijk} - \bar{y}_{ij\cdot})^2 = 0$
with $(n-1)ab = 0$ d.f.

Two solutions.

① Assume no interaction

Then (under $H_0: (\alpha\beta)_{ij} \equiv 0$). $E(MSAB) = \sigma^2$

$\Rightarrow$ use MSAB instead of MSE.

Say $H_0: \alpha_i \equiv 0$.

$$F = \frac{MSA}{MSAB} \quad \text{with } (a-1, (a-1)(b-1)) \text{ d.f.}$$

② Tukey test: assume $(\alpha\beta)_{ij} = D\alpha_i\beta_j$
for some constant D

$$y_{ij} = \mu + \alpha_i + \beta_j + D\alpha_i\beta_j + \varepsilon_{ij} \qquad i=1,\cdots a \quad j=1,\cdots b$$

Least squares

$$\min \sum_i \sum_j (y_{ij} - \mu - \alpha_i - \beta_j - D\alpha_i\beta_j)^2$$

$$-\frac{1}{2}\frac{\partial}{\partial\mu} = \sum_i \sum_j (y_{ij} - \mu - \alpha_i - \beta_j - D\alpha_i\beta_j) = 0$$

$$-\frac{1}{2}\frac{\partial}{\partial\alpha_i} = \sum_j (y_{ij} - \mu - \alpha_i - \beta_j - D\alpha_i\beta_j)(1 + D\beta_j) = 0$$

$$-\frac{1}{2}\frac{\partial}{\partial\beta_j} = \sum_i (y_{ij} - \mu - \alpha_i - \beta_j - D\alpha_i\beta_j)(1 + D\alpha_i) = 0$$

$$-\frac{1}{2}\frac{\partial}{\partial D} = \sum_i \sum_j (y_{ij} - \mu - \alpha_i - \beta_j - D\alpha_i\beta_j)\alpha_i\beta_j = 0$$

$$\hat{\mu} = \bar{y}_{..}$$

$$\hat{\alpha}_i = \bar{y}_{i.} - \bar{y}_{..}, \quad \hat{\beta}_j = \bar{y}_{.j} - \bar{y}_{..}$$

$$\hat{D} = \frac{\sum_i \sum_j \hat{\alpha}_i \hat{\beta}_j (y_{ij} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j)}{\sum_i \sum_j \hat{\alpha}_i^2 \hat{\beta}_j^2}$$

$$= \frac{\sum_i \sum_j y_{ij} \hat{\alpha}_i \hat{\beta}_j}{\sum_i \hat{\alpha}_i^2 \sum_j \hat{\beta}_j^2}$$

$$= \frac{\sum_i \sum_j y_{ij} (\bar{y}_{i.} - \bar{y}_{..})(\bar{y}_{.j} - \bar{y}_{..})}{\sum_i (\bar{y}_{i.} - \bar{y}_{..})^2 \sum_j (\bar{y}_{.j} - \bar{y}_{..})^2}$$

ANOVA.

$$SSTO = SSA + SSB + SSAB_{Tukey} + SS(\text{Remaider})$$

Under $H_0: D=0$, independent

df. 1 $\qquad$ $(a-1)(b-1)-1$

Tukey test for interaction (of type $D\alpha_i\beta_j$)

$$F = \frac{MSAB_{Tukey}}{MS_{Rem}} \sim F_{1,\, ab-a-b}.$$

If H_0 is rejected, there are interactions.

May use transformation tech.

Three-way and higher order ANOVA (Chapter 24)

factors A, B, C

$$\mu_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk}$$

where

$$\alpha_i = \mu_{i\cdot\cdot} - \mu$$

$$\beta_j = \mu_{\cdot j \cdot} - \mu$$

$$\gamma_k = \mu_{\cdot\cdot k} - \mu$$

$$(\alpha\beta)_{ij} = \mu_{ij\cdot} - \mu_{i\cdot\cdot} - \mu_{\cdot j\cdot} + \mu = \mu_{ij\cdot} - \mu - \alpha_i - \beta_j$$

$\vdots$

$$\begin{aligned}(\alpha\beta\gamma)_{ijk} &= \mu_{ijk} - \mu_{ij\cdot} - \mu_{i\cdot k} - \mu_{\cdot jk} \\ &\quad + \mu_{i\cdot\cdot} + \mu_{\cdot j\cdot} + \mu_{\cdot\cdot k} - \mu \\ &= \mu_{ijk} - \mu - \alpha_i - \beta_j - \gamma_k - (\alpha\beta)_{ij} \\ &\quad - (\alpha\gamma)_{ik} - (\beta\gamma)_{jk}\end{aligned}$$

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_k \gamma_k = 0$$

$$\sum_i (\alpha\beta)_{ij} = 0, \quad \forall j, \quad = \quad =$$

$$\sum_i (\alpha\beta\gamma)_{ijk} = 0, \quad \forall j, k$$

We say factors A, B, C have 3-way interaction if

- AB interaction changes with the levels of C, or
- BC . . . . of A or
- AC . . . . of B

Eg. in a 3-way interaction of level (i_0, i_0') for factor A, level (j_0, j_0') for B and (k_0, k_0') for C is.

$$\mu_{ijk} - \mu_{i'jk} - \mu_{ij'k} - \mu_{ijk'} + \mu_{i'j'k} + \mu_{i'jk'} + \mu_{ij'k'} - \mu_{i'j'k'}$$

$$= \underbrace{\mu_{ijk} - \mu_{i'jk} - \mu_{ij'k} + \mu_{i'j'k}}_{\text{AB interaction at level } k \text{ of C}} - \Big(\underbrace{\mu_{ijk'} - \mu_{i'jk'} - \mu_{ij'k'} + \mu_{i'j'k'}}_{\text{AB interaction at level } k' \text{ of C}}\Big)$$

$$= \underbrace{\mu_{ijk} - \mu_{ij'k} - \mu_{ijk'} + \mu_{ij'k'}}_{\text{BC interaction at level } i \text{ of A}} - \Big(\underbrace{\mu_{i'jk} - \mu_{i'j'k} - \mu_{i'jk'} + \mu_{i'j'k'}}_{\text{BC interaction at level } i' \text{ of A}}\Big)$$

$$= \ldots$$

Three-way ANOVA

$$\mu_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\gamma\beta)_{jk} + (\alpha\beta\gamma)_{ijk}$$

$$\sum \alpha_i = \sum \beta_j = \sum \gamma_k = 0$$

$$\sum_i (\alpha\beta)_{ij} = 0, \ \forall j \qquad \sum_i (\alpha\gamma)_{ik} = 0, \ \forall k \qquad \sum_j (\beta\gamma)_{jk} = 0, \ \forall k$$

$$\sum_i (\alpha\beta\gamma)_{ijk} = 0, \quad \forall j, k$$

$(\alpha\beta\gamma)_{ijk} \equiv 0 \iff$ A and B have the same interaction with any level of C.

$(\alpha\beta)$- interaction with level $C = k_1$: $\mu_{ijk_1} - \mu_{i\cdot k_1} - \mu_{\cdot jk_1} + \mu_{\cdot\cdot k_1}$

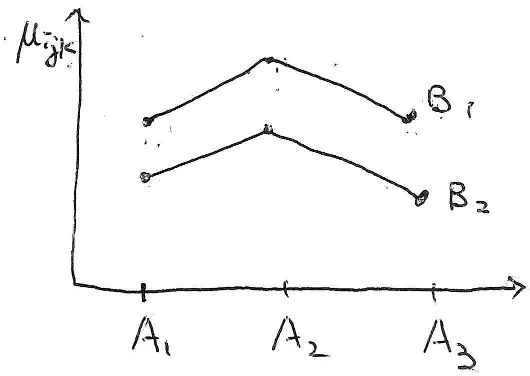
$(\alpha\beta)$ - - - - - - - - - - - - $C = k_2$: $\mu_{ijk_2} - \mu_{i\cdot k_2} - \mu_{\cdot jk_2} + \mu_{\cdot\cdot k_2}$

$$\cancel{\mu} + \cancel{\alpha_i} + \cancel{\beta_j} + \cancel{\gamma_{k_1}} + (\alpha\beta)_{ij} + \cancel{(\alpha\gamma)_{ik_1}} + \cancel{(\beta\gamma)_{jk_1}} + (\alpha\beta\gamma)_{ijk_1} - (\cancel{\mu} + \cancel{\alpha_i} + \cancel{\gamma_{k_1}} + \cancel{(\alpha\gamma)_{ik_1}})$$
$$- (\cancel{\mu} + \cancel{\beta_j} + \cancel{\gamma_{k_1}} + \cancel{(\beta\gamma)_{jk_1}}) + \cancel{\mu} + \cancel{\gamma_{k_1}}$$
$$= \cancel{\mu} + \cancel{\alpha_i} + \cancel{\beta_j} + \cancel{\gamma_{k_2}} + (\alpha\beta)_{ij} + \cancel{(\alpha\gamma)_{ik_2}} + \cancel{(\beta\gamma)_{jk_2}} + (\alpha\beta\gamma)_{ijk_2} - (\cancel{\mu} + \cancel{\alpha_i} + \cancel{\gamma_{k_2}} + \cancel{(\alpha\gamma)_{ik_2}})$$
$$- (\cancel{\mu} + \cancel{\beta_j} + \cancel{\gamma_{k_2}} + \cancel{(\beta\gamma)_{jk_2}}) + \cancel{\mu} + \cancel{\gamma_{k_2}}$$

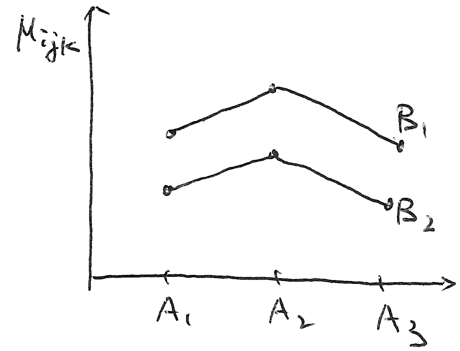
$$\left.\begin{array}{l}(\alpha\beta\gamma)_{ijk_1} = (\alpha\beta\gamma)_{ijk_2} \\ \sum_k (\alpha\beta\gamma)_{ijk} = 0\end{array}\right\} \quad (\alpha\beta\gamma)_{ijk_1} = (\alpha\beta\gamma)_{ijk_2} = 0$$

Treatment means curves

Ex. (C_1)



(C_2)

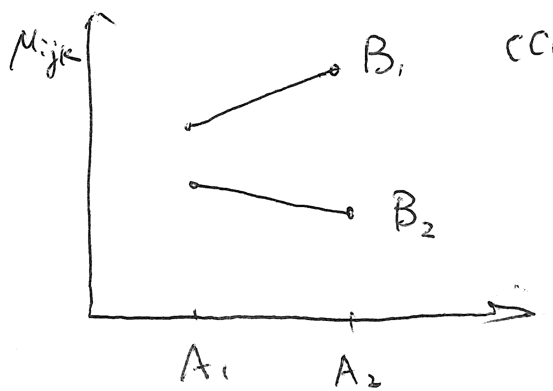


parallel

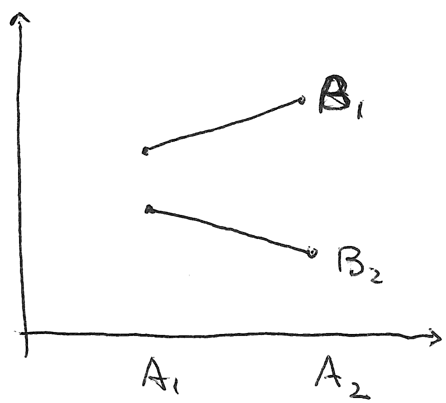
$(\alpha\beta)_{ij} \equiv 0$

$(\alpha\beta\gamma)_{ijk} \equiv 0$

Ex_2 (C_1)

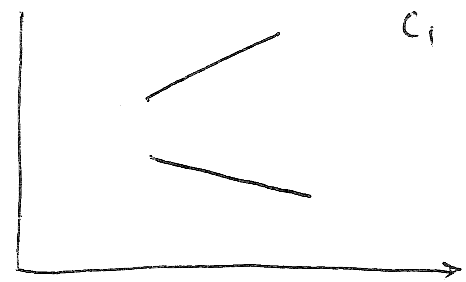


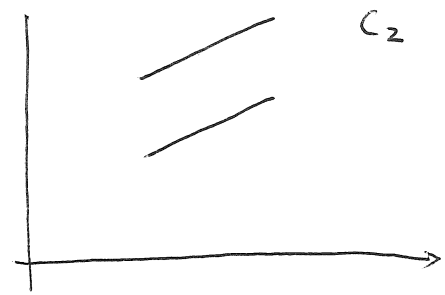
(C_2)



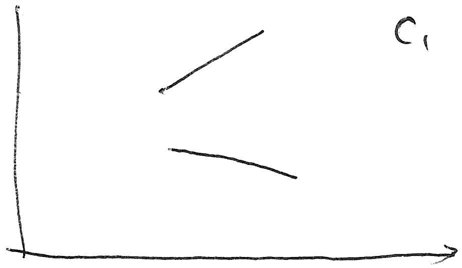
$(\alpha\beta)_{ij} \neq 0$

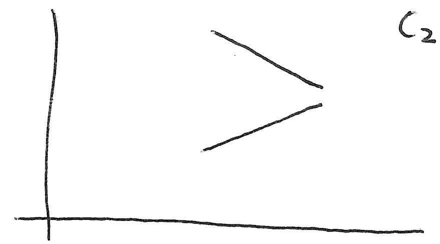
$(\alpha\beta\gamma)_{ijk} = 0$





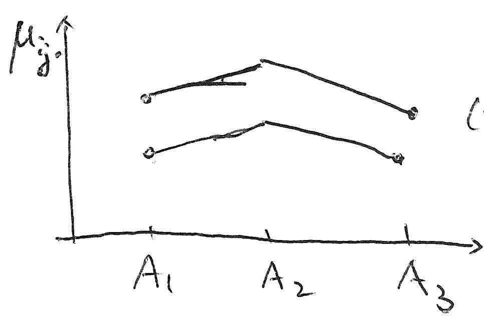
$(\alpha\beta\gamma)_{ijk} \not\equiv 0$



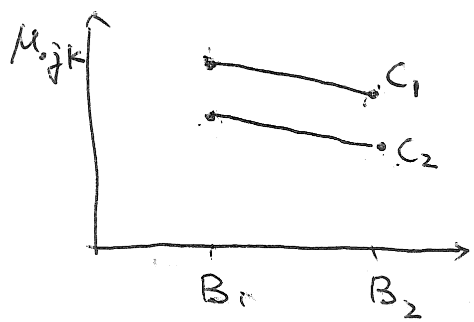


$(\alpha\beta\gamma)_{ijk} \not\equiv 0$

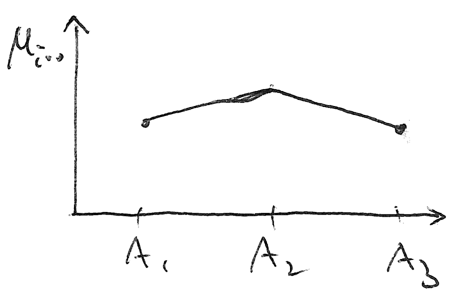
EX1



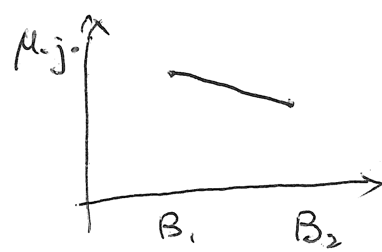
$(\alpha\beta)_{ij} \equiv 0$



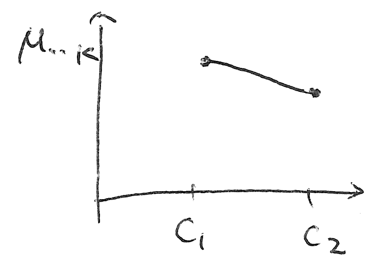
$(\beta\gamma)_{jk} = 0$



$\alpha_i \not\equiv 0$



$\beta_j \not\equiv 0$



$\gamma_k \not\equiv 0$