Vasicek Model

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In risk-neutral measure Q, the money market account $\beta(t)$ is the associated numeraire and $W(t)$ is the adapted Brownian motion process. In T-forward measure Q^T, zero coupon bond $P(t,T)$ is the associated numeraire and $W^T(t)$ is the adapted Brownian motion process.

1 Zero Coupoun Bond Price Dynamics

In risk-neutral measure, the deflated zero coupoun bond $P_\beta(t,T) = P(t,T)/\beta(t)$ is a martingale with lognormal distribution. The SDE is given as

$$\mathrm{d}P_\beta(t,T) = -P_\beta(t,T)\sigma_P(t,T)\,\mathrm{d}W(t) \tag{1}$$

where $\sigma_P(t,T)$ is the zero coupon bond volatility.

In risk-neutral measure, zero coupon bond $P(t,T)$ is a geometric brownian motion (GBM) process and it is not a martingale. (Piterbarg Eq. 4.31)

$$\mathrm{d}P(t,T)/P(t,T) = r(t)\,\mathrm{d}t - \sigma_P(t,T)\,\mathrm{d}W(t) \tag{2}$$

In the risk-neutral measure, the forward zero coupon bond $P(t,T_1,T_2) = P(t,T_2)/P(t,T_1)$ is a GBM process, and it is not a martingale(Piterbarg Eq. 4.32). The SDE is given as

$$\begin{aligned}\mathrm{d}P(t,T_1,T_2)/P(t,T_1,T_2) = &-\left[\sigma_P(t,T_2) - \sigma(t,T_1)\right]\sigma_P(t,T_1)\,\mathrm{d}t \\ &-\left[\sigma_P(t,T_2) - \sigma_P(t,T_1)\right]\mathrm{d}W(t)\end{aligned} \tag{3}$$

In the T-forward measure, $P(t,T_1,T_2)$ is a martingale (Piterbarg Eq. 4.33). The SDE is given as

$$\mathrm{d}P(t,T_1,T_2)/P(t,T_1,T_2) = -\left[\sigma_P(t,T_2) - \sigma_P(t,T_1)\right]\mathrm{d}W^T(t) \tag{4}$$

The relation between $W(t)$ and $W^T(t)$ is

$$\mathrm{d}W^T(t) = \mathrm{d}W(t) + \sigma_P(t,T)\,\mathrm{d}t \tag{5}$$

2 Forward Rate Dynamics

By definition, the instantaneous forward rate is $f(t,T) = \frac{dP(t,T)}{dT}$. In the risk-neutral measure, the process for $f(t,T)$ is normally distributed with non-zero drift. The SDE is given as

$$\mathrm{d}f(t,T) = \sigma_f(t,T)\sigma_P(t,T)\,\mathrm{d}t + \sigma_f(t,T)\,\mathrm{d}W(t) \tag{6}$$

$$\mathrm{d}f(t,T) = \sigma_f(t,T)\left(\int_t^T \sigma_f(t,u)\,\mathrm{d}u\right)\mathrm{d}t + \sigma_f(t,T)\,\mathrm{d}W(t) \tag{7}$$

where

$$\sigma_f(t,T) = \frac{\partial \sigma_P(t,T)}{\partial T} \tag{8}$$

In the T-forward measure, The process for $f(t,T)$ is a martingale with normal distribution and the SDE is given as

$$\mathrm{d}f(t,T) = \sigma_f(t,T)\,\mathrm{d}W^T(t) \tag{9}$$

3 Connection to Short-Rate Model

From the instantaneous forward rate $f(t,T)$, the short rate $r(t) = f(t,t)$ is given as

$$r(t) = f(t,t) = f(0,t) + \int_0^t \sigma_f(u,t) \int_u^T \sigma_f(u,s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t \sigma_f(u,t)\,\mathrm{d}W(t) \tag{10}$$

Consider the case where $\sigma_f(t,T)$ is a deterministic function with the special choice

$$\sigma_f(t,T) = g(t)h(T) \tag{11}$$

where $h(u)$ is a positive real function and $g(u)$ can take any sign.

This leads to the general Vasicek model, which gives the SDE of the short rate $r(t)$ as

$$\mathrm{d}r(t) = [\alpha(t) - \kappa(t)r(t)]\,\mathrm{d}t + \sigma_r(t)\,\mathrm{d}W(t) \tag{12}$$

Note that the functions $\alpha(t)$, $\kappa(t)$ and $\sigma_r(t)$ are not arbitrary. They are all linked with Eq.10 through $g(u)$, $h(u)$ and the initial status, which are given as

$$\alpha(t) = \frac{\partial f(0,t)}{\partial t} + \kappa(t)f(0,t) + \int_0^t \sigma_f(u,t)\sigma_f(u,t)\,\mathrm{d}u \tag{13}$$

$$h(t) = e^{-\int_0^t \kappa(u)\,\mathrm{d}u} \tag{14}$$

$$g(t) = e^{\int_0^t \kappa(u)\,\mathrm{d}u}\sigma_r(t) \tag{15}$$

$$\sigma_f(t,T) = e^{-\int_t^T \kappa(u)\,\mathrm{d}u}\sigma_r(t) \tag{16}$$

Following Piterbarg Eq. 10.19, one can derive the zero coupoun bond price $P(t,T)$, which is the main and foundamental result from Vasicek model. One can follow Piterbarg Eq. 10.19 to derive the following results.

First define the a list of quantities.

$$x(t) = h(t)\int_0^t g(u)^2 \int_u^t h(s)\,\mathrm{d}s\,\mathrm{d}u + h(t)\int_0^t g(u)\,\mathrm{d}W(u) \tag{17}$$

$$y(t) = h(t)^2 \int_0^t g(u)^2\,\mathrm{d}u \tag{18}$$

The forward rate $f(t,T)$ is given as

$$f(t,T) = f(0,T) + \frac{h(T)}{h(t)}\left(x(t) + \frac{y(t)}{h(t)}\int_t^T h(s)\,\mathrm{d}s\right) \tag{19}$$

The zero coupoun bond price $P(t,T)$ is given as

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right) \tag{20}$$

$$= \exp\left(-\int_t^T \left[f(0,u) + \frac{h(u)}{h(t)}\left(x(t) + \frac{y(t)}{h(t)}\int_t^u h(s)\,\mathrm{d}S\right)\right]\mathrm{d}u\right) \tag{21}$$

$$= \frac{P(0,T)}{P(0,t)}\exp\left[-\frac{x(t)}{h(t)}B(t,T) - \frac{y(t)}{h(t)^2}\int_t^T h(u)B(t,u)\,\mathrm{d}u\right] \tag{22}$$

$$= \frac{P(0,T)}{P(0,t)}\exp\left[-\int_0^t g(u)^2 B(u,t)\,\mathrm{d}u B(t,T) - V(t)\int_t^T h(u)B(t,u)\,\mathrm{d}u - S(t)B(t,T)\right] \tag{23}$$

$$= \frac{P(0,T)}{P(0,t)}\exp\left[-\int_0^t g(u)^2 B(u,t)\,\mathrm{d}u B(t,T) - \frac{1}{2}V(t)B(t,T)^2 - S(t)B(t,T)\right] \tag{24}$$

$$= \frac{P(0,T)}{P(0,t)}A(t,T)\exp\left[-B(t,T)S(t)\right] \tag{25}$$

where

$$A(t,T) = \exp\left[-\int_0^t g(u)^2 B(u,t)\,\mathrm{d}u B(t,T) - \frac{1}{2}V(t)B(t,T)^2\right] \tag{26}$$

$$B(t,T) = \int_t^T h(u)\,\mathrm{d}u \tag{27}$$

$$S(t) = \int_0^t g(u)\,\mathrm{d}W(u) \tag{28}$$

$$V(t) = \int_0^t g(u)^2\,\mathrm{d}u \tag{29}$$

The equality from Eq.23 to Eq.24 is because $\mathrm{d}B(t,u) = h(u)\,\mathrm{d}u$. $S(t)$ is called the state variable which introduces the randomness. $V(t)$ is the variance of the state variable $S(t)$ up to time t.

Eqs.25 - 29 are the final expressions to be used in the implementation for bond price under Vasicek model framework. Note here, the results are in the risk-neutral measure, rather than the T-forward measure. Using either risk-neutral measure or T-forward measure, the volatilities $\sigma_f(t,T)$, $\sigma_P(t,T)$ are the foundamental quantities from Vasicek model that are eventually expressed from $g(u)$ and $h(u)$ functions and used to derive the bond price $P(t,T)$ and forward bond price $P(t,T_1,T_2)$. Therefore, functions $g(u)$ and $h(u)$ are conceptually served as the Vasicek model parameters for calibration purpose. Below, the relations for these volatilities are summarized.

$$\sigma_f(t,T) = g(t)h(T) \tag{30}$$

$$\sigma_P(t,T) = g(t)\int_t^T h(u)\,\mathrm{d}u = g(t)B(t,T) \tag{31}$$

$$\sigma_P(t,T_1,T_2) = \sigma_P(t,T_1) - \sigma_P(t,T_2) = -g(t)B(T_1,T_2) \tag{32}$$

4 Verification of P(t,T) bond price from two approaches

As an execise, I am verifying that the expression of $P(T,T') = P(T,T,T')$ derived from forward bond SDE (Eq.4) is the same as the one directly given by the Vasicek model for $P(T,T')$ from Eq.24.

Under T-forward measure, the forward bond price $P(t,T,T')$ is a GBM process with SDE shown in Eq.4. Therefore, the expression for $P(t,T,T')$ is given as

$$P(t,T,T') = P(0,T,T')\exp\left(\int_0^t -\frac{1}{2}\sigma_P(u,T,T')^2\,\mathrm{d}u + \int_0^t \sigma_P(u,T,T')\mathrm{d}W^T(u)\right) \tag{33}$$

$$= P(0,T,T')\exp\left(\int_0^t -\frac{1}{2}g(u)^2 B(T,T')^2\,\mathrm{d}u - \int_0^t g(u)B(T,T')\left[\mathrm{d}W(u) + \sigma_P(u,T)\,\mathrm{d}t\right]\right) \tag{34}$$

$$= \frac{P(0,T')}{P(0,T)}\exp\left(-\frac{1}{2}V(t)B(T,T')^2 - S(t)B(T,T') - \int_0^t g(u)^2 B(u,T)\,\mathrm{d}uB(T,T')\right) \tag{35}$$

Therefore, let $t = T$ and the $P(T,T')$ is given as

$$P(T,T,T') = \frac{P(T,T')}{P(T,T)} = P(T,T') \tag{36}$$

$$= \frac{P(0,T')}{P(0,T)}\exp\left(-\frac{1}{2}V(T)B(T,T')^2 - S(T)B(T,T') - \int_0^T g(u)^2 B(u,T)\,\mathrm{d}uB(T,T')\right) \tag{37}$$

This equation is identical to the results from Vasicek model by Eq.24.