Vasicek Model: Thoery and Implementation Guidance

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Contents

In risk-neutral measure Q, the money market account $\beta(t)$ is the associated numeraire and $W(t)$ is the adapted Brownian motion process. In T-forward measure Q^T, zero coupon bond $P(t,T)$ is the associated numeraire and $W^T(t)$ is the adapted Brownian motion process.

1 Zero Coupoun Bond Price Dynamics

In risk-neutral measure, the deflated zero coupoun bond $P_\beta(t,T) = P(t,T)/\beta(t)$ is a martingale with lognormal distribution. The SDE is given as

$$\mathrm{d}P_\beta(t,T) = -P_\beta(t,T)\sigma_P(t,T)\,\mathrm{d}W(t) \tag{1}$$

where $\sigma_P(t,T)$ is the zero coupon bond volatility.

In risk-neutral measure, zero coupon bond $P(t,T)$ is a geometric brownian motion (GBM) process and it is not a martingale. One can easily prove this by applying Ito (see Appendix A.1 and Piterbarg Eq. 4.31).

$$\mathrm{d}P(t,T)/P(t,T) = r(t)\,\mathrm{d}t - \sigma_P(t,T)\,\mathrm{d}W(t) \tag{2}$$

In the risk-neutral measure, the forward zero coupon bond $P(t,T_1,T_2) = P(t,T_2)/P(t,T_1)$ is a GBM process, and it is not a martingale. The SDE is given as

$$\begin{aligned}\mathrm{d}P(t,T_1,T_2)/P(t,T_1,T_2) = &-\left[\sigma_P(t,T_2) - \sigma_P(t,T_1)\right]\sigma_P(t,T_1)\,\mathrm{d}t \\ &-\left[\sigma_P(t,T_2) - \sigma_P(t,T_1)\right]\mathrm{d}W(t)\end{aligned} \tag{3}$$

See Appendeix A.2 and Piterbarg Eq. 4.32.

In the T_1-forward measure, $P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)}$ by its definition can be easily concluded to be a martingale (Piterbarg Eq. 4.33). Therefore, the SDE under T_1-forward measure is given as

$$\mathrm{d}P(t,T_1,T_2)/P(t,T_1,T_2) = -\left[\sigma_P(t,T_2) - \sigma_P(t,T_1)\right]\mathrm{d}W^{T_1}(t) \tag{4}$$

Based on Eq. 3 and Eq. 4, the relation between risk neutral measure and T-forward measure ($W(t)$ and $W^T(t)$) is

$$\mathrm{d}W^T(t) = \mathrm{d}W(t) + \sigma_P(t,T)\,\mathrm{d}t \tag{5}$$

2 Forward Rate Dynamics

By definition, the instantaneous forward rate is $f(t,T) = -\frac{\mathrm{d}\ln P(t,T)}{\mathrm{d}T}$. By Ito's lemma, in risk neutral measure,

$$\mathrm{d}\ln P(t,T) = \mathcal{O}(\mathrm{d}t) - \sigma_P(t,T)\,\mathrm{d}W(t) \tag{6}$$

Taking derivatives on both side w.r.t. T, we get the SDE for $f(t,T)$ as

$$\mathrm{d}f(t,T) = \mu_f(t,T)\,\mathrm{d}t + \sigma_f(t,T)\,\mathrm{d}W(t) \tag{7}$$

with

$$\sigma_f(t,T) = \frac{\partial}{\partial T}\sigma_P(t,T) \tag{8}$$

To determine the expression of the drift term $\mu_f(t,T)$ under risk neutral measure, it would be easier to first consider the case under T-forward measure, then change the measure back to risk neutral measure to obtain the analytical expression.

We need to use the property that the forward rate $L(t,T,T+\tau)$ is a martingale under $(T+\tau)$-forward measure, that is,

$$L(t,T,T+\tau) = \mathbb{E}_t^{T+\tau}\left(L(u,T,T+\tau)\right), \quad t \le u \tag{9}$$

Take the limit of τ going to zero, we have

$$f(t,T) = \mathbb{E}_t^{T}\left(f(u,T)\right), \quad t \le u \tag{10}$$

Therefore, the forward rate $f(t,T)$ is a martingale under T-forward measure. With the volatility of $\sigma_f(t,T)$ is given as above, we can conclude that the SDE of $f(t,T)$ under T-forward measure must be

$$\mathrm{d}f(t,T) = \sigma_f(t,T)\,\mathrm{d}W^T(t) \tag{11}$$

Applying change of measure, the process of $f(t,T)$ under risk neutral measure is given as

$$\begin{aligned}
\mathrm{d}f(t,T) &= \sigma_f(t,T)\sigma_P(t,T)\,\mathrm{d}t + \sigma_f(t,T)\,\mathrm{d}W(t) && (12)\\
&= \sigma_f(t,T)\left(\int_t^T \sigma_f(t,u)\,\mathrm{d}u\right)\mathrm{d}t + \sigma_f(t,T)\,\mathrm{d}W(t) && (13)
\end{aligned}$$

which indicates

$$\mu_f(t,T) = \sigma_f(t,T)\left(\int_t^T \sigma_f(t,u)\,\mathrm{d}u\right) \tag{14}$$

3 Connection to Short-Rate Model

From the instantaneous forward rate $f(t,T)$, the short rate $r(t) = f(t,t)$ can be obatined by integrating on both side of the Eq. 13

$$r(t) = f(t,t) = f(0,t) + \int_0^t \sigma_f(u,t)\int_u^T \sigma_f(u,s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t \sigma_f(u,t)\,\mathrm{d}W(t) \tag{15}$$

Consider a special case where $\sigma_f(t,T)$ is a deterministic function

$$\sigma_f(t,T) = g(t)h(T) \tag{16}$$

in which $h(u)$ is a positive real function and $g(u)$ can take any sign. I have to emphasize here that Eq. 16 is a very important assumption, because it will eventually lead to the general Vasicek model that we are familiar with (as shown in Piterbarg's book section 4.5.2). The SDE of the short rate $r(t)$ for the general Vasicek model is

$$\mathrm{d}r(t) = [\alpha(t) - \kappa(t)r(t)]\,\mathrm{d}t + \sigma_r(t)\,\mathrm{d}W(t) \tag{17}$$

Note, because of the connection between Eq. 16 and Eq. 17, the functions $\alpha(t)$, $\kappa(t)$ and $\sigma_r(t)$ in Vasicek model are not arbitrary. They are all linked with Eq.15 through $g(u)$, $h(u)$ and the initial status. The analytical expression are given as below

$$\begin{aligned}
\alpha(t) &= \frac{\partial f(0,t)}{\partial t} + \kappa(t)f(0,t) + \int_0^t \sigma_f(u,t)\sigma_f(u,t)\,\mathrm{d}u && (18)\\
h(t) &= e^{-\int_0^t \kappa(u)\,\mathrm{d}u} && (19)\\
g(t) &= e^{\int_0^t \kappa(u)\,\mathrm{d}u}\sigma_r(t) && (20)\\
\sigma_f(t,T) &= e^{-\int_t^T \kappa(u)\,\mathrm{d}u}\sigma_r(t) && (21)
\end{aligned}$$

4 Vasicek Model Application

4.1 Zero Coupoun Bond Price

Following Piterbarg Eq. 10.19, one can derive the zero coupoun bond price $P(t,T)$

$$P(t,T) = \frac{P(0,T)}{P(0,t)} A(t,T) \exp\left[-B(t,T)S(t)\right] \tag{22}$$

where

$$A(t,T) = \exp\left[-\int_0^t g(u)^2 B(u,t)\,\mathrm{d}u B(t,T) - \frac{1}{2}V(t)B(t,T)^2\right] \tag{23}$$

$$B(t,T) = \int_t^T h(u)\,\mathrm{d}u \tag{24}$$

$$S(t) = \int_0^t g(u)\,\mathrm{d}W(u) \tag{25}$$

$$V(t) = \int_0^t g(u)^2\,\mathrm{d}u \tag{26}$$

$S(t)$ is called the state variable which introduces the randomness. $V(t)$ is the variance of the state variable $S(t)$ up to time t. The detailed derivation of Eq. 22 can be found in Appendix A.2. Details for the implementation of evaluating $A(t,T)$ and $B(t,T)$ can be found in Appendix A.4

The bond price (Eqs.22 - 26) are the most foundamental results from Vasicek model. From the bond price $P(t,T)$, one can derive the forward bond price $P(t,T_1,T_2)$ and the forward simple rate $L(t,T_1,T_2)$. Eqs.22 - 26 are the final expressions to be used in the implementation for bond price under Vasicek model framework. Note here, the results are in the risk-neutral measure, NOT in the T-forward measure. If you want to use the bond price in forward measure, remember to change the measure first (See Appendix A.3 for related topic and comment).

No matter of using risk-neutral measure or T-forward measure, the volatilities $\sigma_f(t,T)$, $\sigma_P(t,T)$ do not change and they are the foundamental quantities from Vasicek model which are expressed by $g(u)$ and $h(u)$ functions and used to derive the bond price $P(t,T)$ and forward bond price $P(t,T_1,T_2)$. Therefore, functions $g(u)$ and $h(u)$ are conceptually served as the Vasicek model parameters for calibration purpose. Below, the relations for these volatilities are summarized.

$$\sigma_f(t,T) = g(t)h(T) \tag{27}$$

$$\sigma_P(t,T) = g(t)\int_t^T h(u)\,\mathrm{d}u = g(t)B(t,T)$$

$$\tilde{\sigma}_P(t,T_1,T_2) = \sigma_P(t,T_2) - \sigma_P(t,T_1) = g(t)B(T_1,T_2) \tag{28}$$

4.2 Simple Averaging Caplet Price

For simple averaging caplet, there is only one cash flow paying at the end of the period, but the interest rate is calculated from the mean averaging of the a sequence of small rates. Denote $T_1, T_2, T_3, \cdots, T_N$ as the dates within the period to give the N number of small rates $L(t,T_i,T_{i+1})$ for for each of the interest accrual interval $[T_i,T_{i+1}]$. Depending on when the rate is fixed, there could be two types of averaging caplet. If all the rates are fixed at the begining (meaning all fixed at T_1), it is the forward looking caplet. If all the rates are fixed at the T_i for each interval accordingly, it is the backward looking caplet. The approach to evalulate the price of these two types of averaging caplets are different. The discussion here also applies to the similiar cases for compounding caplet with forward or backward looking types.

Note the payment is occurred at the end of the period T_N. It would be easier to price using T_N-forward measure. Under T_N-forward measure, the price of the backward caplet at current time t is

$$V^b(t) = P(t, T_N)\,\mathbb{E}^{T_N}\left[\frac{1}{N}\sum_i^N L(T_i, T_i, T_{i+1}) - K\right]^+ \tag{29}$$

The price of the forward caplet at current time t is

$$V^f(t) = P(t, T_N)\,\mathbb{E}^{T_N}\left[\frac{1}{N}\sum_i^N L(T_1, T_i, T_{i+1}) - K\right]^+ \tag{30}$$

In both cases, we need the evoluation of the the forward simple rate L under T_N-forward measure. The only difference is what time the forward rate is evulated at. Below we take the backward case as an example, and similiar derviation can be easily applied to the forward case.

For backward caplet, each small rate $L(T_i, T_i, T_{i+1})$ by its definition is connected with the forward band price $P(T_i, T_i, T_{i+1})$ at time T_i accordingly,

$$L(T_i, T_i, T_{i+1}) = \left(\frac{1}{P(T_i, T_i, T_{i+1})} - 1\right)\frac{1}{\tau_i} \tag{31}$$

Therefore, we need to simulate the evolution of the forwad bond $P(t, T_i, T_{i+1})$ under T_N-forward measure. Apply changing of measure, we have

$$\begin{aligned}
\mathrm{d}W^{T_1}(t) &= \mathrm{d}W(t) + \sigma_P(t, T_1)\,\mathrm{d}t && (32)\\
\mathrm{d}W^{T_2}(t) &= \mathrm{d}W(t) + \sigma_P(t, T_2)\,\mathrm{d}t && (33)\\
\mathrm{d}W^{T_1}(t) &= \mathrm{d}W^{T_2}(t) - [\sigma_P(t, T_2) - \sigma_P(t, T_1)]\,\mathrm{d}t && (34)\\
&= \mathrm{d}W^{T_2} - \tilde{\sigma}_P(t, T_1, T_2)\,\mathrm{d}t && (35)
\end{aligned}$$

Switch from T_1-forwad measure to T_N-forward measure, the SDE of $P(t, T_1, T_2)$ is given as

$$\begin{aligned}
\frac{\mathrm{d}P(t, T_1, T_2)}{P(t, T_1, T_2)} &= -\tilde{\sigma}_P(t, T_1, T_2)\,\mathrm{d}W^{T_1} && (36)\\
&= -\tilde{\sigma}_P(t, T_1, T_2)\tilde{\sigma}_P(t, T_1, T_N)\,\mathrm{d}t - \tilde{\sigma}_P(t, T_1, T_2)\,\mathrm{d}W^{T_N} && (37)
\end{aligned}$$

This is a GBM Process and the solution of $P(t, T_1, T_2)$ under T_N-forwad measure is given as

$$\begin{aligned}
P(t, T_1, T_2) &= \frac{P(0, T_2)}{P(0, T_1)}\exp\left\{\int_0^t \left[\tilde{\sigma}_P(s, T_1, T-2)\tilde{\sigma}_P(s, T_1, T_N) - \frac{1}{2}\tilde{\sigma}_P(s, T_1, T_2)^2\right]\mathrm{d}s\right. && (38)\\
&\quad \left. - \int_0^t \tilde{\sigma}_P(s, T_1, T_2)\,\mathrm{d}W^{T_N}(s)\right\} && (39)\\
&= \frac{P(0, T_2)}{P(0, T_1)}\exp\left[Y(t)\right] && (40)
\end{aligned}$$

where $Y(t)$ is a random variable within the big bracket, and it is a quantity under T_N-forward measure,

$$Y(t) = \int_0^t \left[\tilde{\sigma}_P(s, T_1, T-2)\tilde{\sigma}_P(s, T_1, T_N) - \frac{1}{2}\tilde{\sigma}_P(s, T_1, T_2)^2\right]\mathrm{d}s - \int_0^t \tilde{\sigma}_P(s, T_1, T_2)\,\mathrm{d}W^{T_N}(s) \tag{41}$$

with drift $\mu_Y(t)$ and variance $\sigma_Y(t)$. By some arrangement, we can derive the drfit and variance as below,

$$
\begin{aligned}
\mu_Y(t) &= \int_0^t \left[\tilde{\sigma}_P(s,T_1,T_2)\tilde{\sigma}_P(s,T_1,T_N) - \frac{1}{2}\tilde{\sigma}_P(s,T_1,T_2)^2\right] \mathrm{d}s \\
&= \int_0^t g(s)^2 \,\mathrm{d}s \cdot B(T_1,T_2)B(T_1,T_N) - \frac{1}{2}\int_0^t g(s)^2 \,\mathrm{d}s \cdot B(T_1,T_2)^2 \\
&= V(t)B(T_1,T_2)B(T_1,T_N) - \frac{1}{2}V(t)B(T_1,T_2)^2 \\
&= V(t)B(T_1,T_2)\left[B(T_1,T_N) - \frac{1}{2}B(T_1,T_2)\right] \\
&= V(t)B(T_1,T_2)\left(B(T_1,T_N) - \frac{1}{2}\left[B(T_1,T_N) - B(T_2,T_N)\right]\right) \\
&= \frac{1}{2}V(t)B(T_1,T_2)\left[B(T_1,T_N) + B(T_2,T_N)\right] \qquad (42)\\
\sigma_Y(t) &= \sqrt{V(t)}B(T_1,T_2) \qquad (43)\\
V(t) &= \int_0^t g(s)^2 \,\mathrm{d}s \qquad (44)
\end{aligned}
$$

Now we can express each small rate $L(T_i,T_i,T_{i+1})$ as the function of the random variable $Y(T_i)$ as

$$
\begin{aligned}
L(T_i,T_i,T_{i+1}) &= \left(\frac{1}{P(T_i,T_i,T_{i+1})} - 1\right)\frac{1}{\tau_i} \\
&= \left[\frac{P(0,T_i)}{P(0,T_{i+1})}\exp(-Y(T_i)) - 1\right]\frac{1}{\tau_i} \qquad (45)\\
&= h\left(Y(T_i)\right) \qquad (46)
\end{aligned}
$$

where $Y(T_i) \sim \mathcal{N}\left(\mu_Y(T_i), \sigma_Y(T_i)\right)$ under T_N-forward measure. For short notation, we denote L is a function of Y, $L = h(Y)$.

To calculate the expectation of $L(T_i,T_i,T_{i+1})$ under T_N-forward measure, we can use gauss-hermit quadrature. If $Y \sim \mathcal{N}(\mu,\sigma)$, the expectation can be approximate as

$$
\begin{aligned}
\mathbb{E}^{T_N}\left[h(Y)\right] &= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) h(y)\,\mathrm{d}y \qquad (47)\\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \exp(-x^2)h(\sqrt{2}\sigma x + \mu)\,\mathrm{d}x \qquad (48)\\
&\approx \sum_i^n \frac{1}{\sqrt{\pi}}\omega_i h(\sqrt{2}\sigma x_i + \mu) \qquad (49)
\end{aligned}
$$

where $X \sim \mathcal{N}(0,1)$, ω_i is the gauss-hermit coefficients, and Y can be expressed from X as $Y = \sqrt{2}\sigma X + \mu$.

Therefore, in the case of forward rate $L(T_i,T_i,T_{i+1})$, we have the random variable $-Y(T_i) \sim \mathcal{N}(-\mu_Y(T_i), \sigma_Y(T_i))$. The random forward rate is expressed as

$$
\begin{aligned}
L(T_i,T_i,T_{i+1};X) &= \left[\frac{P(0,T_i)}{P(0,T_{i+1})}\exp\left[-Y(T_i)\right] - 1\right]\frac{1}{\tau_i} \\
&= \left[\frac{P(0,T_i)}{P(0,T_{i+1})}\exp\left[\sigma_Y(T_i)\sqrt{2}X - \mu_Y(T_i)\right] - 1\right]\frac{1}{\tau_i} \qquad (50)\\
& \qquad (51)
\end{aligned}
$$

To evaluate the caplet price, we can use numerical integration (like gauss-hermit quadrature) to calculate the expectation. Note the options payoff function is non-linear and there are multiple simple rates used in the expectation. This leads the exact expectation evulation to be a multi-dimensional integration. However, in practice, we usually approximate the multi-dimensional integraion into a single dimensional integraion.

Appendix A Detailed derivations

A.1 Bond price SDE

$$\mathrm{d}P(t,T)/P(t,T) = r(t)\,\mathrm{d}t - \sigma_P(t,T)\,\mathrm{d}W(t) \tag{52}$$

Proof:

$$P(t,T) = P_\beta(t,T)\cdot\beta(t) \tag{53}$$

$$\begin{aligned}\mathrm{d}P(t,T) &= \mathrm{d}P_\beta\cdot\beta + P_\beta\,\mathrm{d}\beta + \mathrm{d}P_\beta\cdot\mathrm{d}\beta\\ &= -\sigma_P P_\beta\beta\,\mathrm{d}W + P_\beta r\beta\,\mathrm{d}t\\ &= -\sigma_P P\,\mathrm{d}W + Pr\,\mathrm{d}t\end{aligned} \tag{54}$$

A.2 Forward bond price SDE

$$\frac{\mathrm{d}P(t,T_1,T_2)}{P(t,T_1,T_2)} = -\left[\sigma_P(t,T_2)-\sigma_P(t,T_1)\right]\sigma_P(t,T_1)\,\mathrm{d}t - \left[\sigma_P(t,T_2)-\sigma_P(t,T_1)\right]\mathrm{d}W(t) \tag{55}$$

There are two ways to prove the above relation.

Approach 1: using GBM property

$P(t,T)$ is a GBM process.

$$P(t,T) = P(0,T)\exp\left[(r(t)-\frac{1}{2}\sigma_P^2)t - \sigma_P W(t)\right] \tag{56}$$

Therefore, we can write down the expression for the forward bond price.

$$\begin{aligned}\tilde{P} &= \frac{P_2}{P_1}\\ &= \frac{P(0,T_2)}{P(0,T_1)}\exp\left[-\frac{1}{2}(\sigma_2^2-\sigma_1^2)t - (\sigma_2-\sigma_1)W\right]\\ &= \tilde{P}(0,T_1,T_2)\exp\left[\left(-(\sigma_2-\sigma_1)\sigma_1 - \frac{1}{2}(\sigma_2-\sigma_1)^2\right)t - (\sigma_2-\sigma_1)W\right]\end{aligned} \tag{57}$$

From the above equation, we can back out the SDE for the GBM process as

$$\frac{\mathrm{d}\tilde{P}}{\tilde{P}} = -(\sigma_2-\sigma_1)\sigma_1\,\mathrm{d}t - (\sigma_2-\sigma_1)\,\mathrm{d}W(t) \tag{58}$$

Approach 2: using Ito

$$\tilde{P} = \frac{P_2}{P_1} \tag{59}$$

$$\mathrm{d}\tilde{P} = \mathrm{d}P_2\frac{1}{P_1} + P_2\,\mathrm{d}\frac{1}{P_1} + \mathrm{d}P_2\,\mathrm{d}\frac{1}{P_1} \tag{60}$$

$$\mathrm{d}P = P(r\,\mathrm{d}t - \sigma\,\mathrm{d}W) \tag{61}$$

$$\begin{aligned}\mathrm{d}\frac{1}{P} &= -\frac{1}{P^2}\,\mathrm{d}P + \frac{1}{P^3}(\mathrm{d}P)^2\\ &= -\frac{1}{P}(r\,\mathrm{d}t - \sigma\,\mathrm{d}W) + \frac{\sigma^2}{P}\,\mathrm{d}t\\ &= \frac{1}{P}\left[(\sigma^2 - r)\,\mathrm{d}t + \sigma\,\mathrm{d}W\right]\end{aligned} \tag{62}$$

Substitute everything into the definition, we can get

$$
\begin{aligned}
\mathrm{d}\tilde{P} &= \frac{P_2}{P_1}(r\,\mathrm{d}t - \sigma_2\,\mathrm{d}W) + \frac{P_2}{P_1}\left[(\sigma_1^2 - r)\,\mathrm{d}t + \sigma_1\,\mathrm{d}W\right] - \frac{P_2}{P_1}\sigma_1\sigma_2\,\mathrm{d}t \\
&= \frac{P_2}{P_1}\left[(\sigma_1 - \sigma_2)\sigma_1\,\mathrm{d}t - (\sigma_2 - \sigma_1)\,\mathrm{d}W\right] \\
&= \tilde{P}\left[(\sigma_1 - \sigma_2)\sigma_1\,\mathrm{d}t - (\sigma_2 - \sigma_1)\,\mathrm{d}W\right] \qquad (63)
\end{aligned}
$$

A.3 Vasicek model bond price analytical expression derivation

Use Eq. 16 and integrate Eq. 13, the forward rate $f(t,T)$ is

$$
\begin{aligned}
f(t,T) &= f(0,T) + \int_0^t g(u)h(T)\int_u^T g(u)h(s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t g(u)h(T)\,\mathrm{d}W(u) \\
&= f(0,T) + h(T)\int_0^t g(u)^2\int_u^T h(s)\,\mathrm{d}s\,\mathrm{d}u + h(T)\int_0^t g(u)\,\mathrm{d}W(u) \qquad (64)
\end{aligned}
$$

Define the following auxiliary quantities

$$
x(t) = h(t)\int_0^t g(u)^2\int_u^t h(s)\,\mathrm{d}s\,\mathrm{d}u + h(t)\int_0^t g(u)\,\mathrm{d}W(u) \qquad (65)
$$

$$
y(t) = h(t)^2\int_0^t g(u)^2\,\mathrm{d}u \qquad (66)
$$

The forward rate $f(t,T)$ can be rewritten as

$$
f(t,T) = f(0,T) + \frac{h(T)}{h(t)}\left(x(t) + \frac{y(t)}{h(t)}\int_t^T h(s)\,\mathrm{d}s\right) \qquad (67)
$$

The zero coupoun bond price $P(t,T)$ is given as

$$
\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right) \qquad (68) \\
&= \exp\left(-\int_t^T\left[f(0,u) + \frac{h(u)}{h(t)}\left(x(t) + \frac{y(t)}{h(t)}\int_t^u h(s)\,\mathrm{d}s\right)\right]\mathrm{d}u\right) \qquad (69) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left[-\frac{x(t)}{h(t)}B(t,T) - \frac{y(t)}{h(t)^2}\int_t^T h(u)B(t,u)\,\mathrm{d}u\right] \qquad (70) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left[-\int_0^t g(u)^2 B(u,t)\,\mathrm{d}u B(t,T) - V(t)\int_t^T h(u)B(t,u)\,\mathrm{d}u - S(t)B(t,T)\right] \qquad (71) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left[-\int_0^t g(u)^2 B(u,t)\,\mathrm{d}u B(t,T) - \frac{1}{2}V(t)B(t,T)^2 - S(t)B(t,T)\right] \qquad (72) \\
&= \frac{P(0,T)}{P(0,t)}A(t,T)\exp\left[-B(t,T)S(t)\right] \qquad (73)
\end{aligned}
$$

The equality from Eq.71 to Eq.72 is because $\mathrm{d}B(t,u) = h(u)\,\mathrm{d}u$.

A.4 Verification of P(T,T') bond price from two approaches

For bond price $P(T,T') = P(T,T,T') = \frac{P(T,T')}{P(T,T)}$, there are two ways to calculate its value. One is directly from Vasicek model by using Eq. 22, and the other is from the forward bond price $P(t,T,T')|_{t=T}$ by using the forward bond price is a GBM process (Eq. 3 and 4). As an execise, I am verifying the two approaches agree with each other.

Based on the GBM process of forwad bond (Eq. 3 and 4), the expression for $P(t,T,T')$ is given as

$$
\begin{aligned}
P(t,T,T') &= P(0,T,T') \exp\left(\int_0^t -\frac{1}{2}\sigma_P(u,T,T')^2 \,\mathrm{d}u + \int_0^t \sigma_P(u,T,T') \mathrm{d}W^T(u)\right) && (74)\\
&= P(0,T,T') \exp\left(\int_0^t -\frac{1}{2} g(u)^2 B(T,T')^2 \,\mathrm{d}u - \int_0^t g(u)B(T,T') \left[\mathrm{d}W(u) + \sigma_P(u,T)\,\mathrm{d}t\right]\right) && (75)\\
&= \frac{P(0,T')}{P(0,T)} \exp\left(-\frac{1}{2}V(t)B(T,T')^2 - S(t)B(T,T') - \int_0^t g(u)^2 B(u,T)\,\mathrm{d}u B(T,T')\right) && (76)
\end{aligned}
$$

Evaulate the forwad bond at time $t = T$, the $P(T,T')$ is given as

$$
\begin{aligned}
P(T,T,T') &= \frac{P(T,T')}{P(T,T)} = P(T,T') && (77)\\
&= \frac{P(0,T')}{P(0,T)} \exp\left(-\frac{1}{2}V(T)B(T,T')^2 - S(T)B(T,T') - \int_0^T g(u)^2 B(u,T)\,\mathrm{d}u B(T,T')\right) && (78)
\end{aligned}
$$

This equation is identical to the results from Vasicek model by Eq.22.

A.5 Implementation of Vasicek A and B term for PWC

In the case of $g(u)$ and $h(u)$ being peice-wise constant (PWC) function, define

$$
g(u) = \begin{cases} g_1, & t_0 < u < t_1 \\ g_2, & t_1 < u < t_2 \\ \vdots & \\ g_N, & t_{N-1} < u < t_N \end{cases} \tag{79}
$$

Similiar definition for $h(u)$ function. Let $t_i < t < t_{i+1}$, define

$$
\begin{aligned}
\Delta t_i &:= t - t_i && (80)\\
\delta t_i &:= t_i - t_{i-1} && (81)\\
G_i &:= \int_0^{t_i} g(u)^2 \,\mathrm{d}u = \sum_0^{i-1} g_k^2 (t_{k+1} - t_k) && (82)\\
H_i &:= \int_0^{t_i} h(u) \,\mathrm{d}u = \sum_0^{i-1} h_k^2 (t_{k+1} - t_k) && (83)\\
\Delta G_i &:= \int_{t_i}^{t} g(u)^2 \,\mathrm{d}u = g_i^2 \Delta t_i && (84)\\
\Delta H_i &:= \int_{t_i}^{t} h(u) \,\mathrm{d}u = h_i^2 \Delta t_i && (85)
\end{aligned}
$$

A.5.1 Expression of A(t, T)

$$A(t,T) = \exp\left[-\int_0^t g(u)^2 B(u,t)\,\mathrm{d}u B(t,T) - \frac{1}{2}V(t)B(t,T)^2\right] \tag{86}$$

We derive the first term in Eq. 86 here. The second term is trivial to implement. Rewrite the integration on $h(u)$ into two parts,

$$\begin{aligned}
\int_0^t g(u)^2 B(u,t)\,\mathrm{d}u &= \int_0^t g(u)^2 \int_u^t h(s)\,\mathrm{d}s\,\mathrm{d}u \\
&= \int_0^t g(u)^2 \int_0^t h(s)\,\mathrm{d}s\,\mathrm{d}u - \int_0^t g(u)^2 \int_0^u h(s)\,\mathrm{d}s\,\mathrm{d}u \\
&= (G_i + \Delta G_i)(H_i + \Delta H_i) - \int_0^t g(u)^2 \int_0^u h(s)\,\mathrm{d}s\,\mathrm{d}u
\end{aligned} \tag{87}$$

Next, we evaluate the second term in Eq. 87. Break the integral into two parts,

$$\begin{aligned}
\int_0^t g(u)^2 \int_0^u h(s)\,\mathrm{d}s\,\mathrm{d}u &= \int_0^{t_i} g(u)^2 \int_0^u h(s)\,\mathrm{d}s\,\mathrm{d}u + \int_{t_i}^t g(u)^2 \int_0^u h(s)\,\mathrm{d}s\,\mathrm{d}u \\
&= M_i + \int_{t_i}^t g_i^2\,[H_i + h_i(u - t_i)]\,\mathrm{d}u \\
&= M_i + g_i^2 H_i \Delta t_i + \frac{1}{2} g_i^2 h_i (t^2 - t_i^2) - g_i^2 h_i t_i \Delta t_i \\
&= M_i + g_i^2 H_i \Delta t_i + \frac{1}{2} g_i^2 h_i \Delta t_i^2 \\
&= M_i + H_i \Delta G_i + \frac{1}{2}\Delta H_i \Delta G_i
\end{aligned} \tag{88}$$

where

$$M_i \coloneqq \int_0^{t_i} g(u)^2 \int_0^u h(s)\,\mathrm{d}s\,\mathrm{d}u \tag{89}$$

Substitute to evaluate $A(t,T)$, we have

$$A(t,T) = (G_i H_i - M_i) + G_i \Delta H_i + \frac{1}{2}\Delta H_i \Delta G_i, \quad t_i < t < t_{i+1} \tag{90}$$

Note G_i, H_i, and M_i are quantities with integration up to the integer point (no fractions).

A.5.2 Expression of B(t, T)

$$B(t,T) = B(0,T) - B(0,t) \tag{91}$$

$$B(0,t) = \int_0^t h(u)\,\mathrm{d}u = H_i + h_i \Delta t_i, \quad t_i < t < t_{i+1} \tag{92}$$