2

Sets and Relations

2.1 Sets

Any branch of science, like a foreign language, has its own terminology. *Isomorphism*, *cyclotomic*, and *coset* aren't words used except in a mathematical context. On the other hand, quite a number of common English words—field, complex, function—have precise mathematical meanings quite different from their usual ones. Students of French or Spanish know that memory work is a fundamental part of their studies; it is perfectly obvious to them that if they don't know what the words mean their ability to learn grammar and to communicate will be severely hindered. It is, however, not always understood by science students that they too must memorize the terminology of their discipline. Without constant review of the meanings of words, one's understanding of a paragraph of text or the words of a teacher is very limited. We advise readers of this book to maintain and constantly review a mathematical vocabulary list. The authors have included their own such list in a glossary at the back of this book.

What would it be like to delve into a dictionary if you didn't already know the meanings of some of the words in it? Most people, at one time or another, have gone to a dictionary in search of a word only to discover that the definition uses another unfamiliar word. Some reflection indicates that a dictionary can be of no use unless there are some words that are so basic that we can understand them without definitions. Mathematics is the same way. There are a few basic terms that we accept without definitions.

Most of mathematics is based on the single undefined concept of *set*, which we think of as just a collection of things called *elements* or *members*. Primitive humans discovered the set of *natural numbers* with which they learned to count. The set of natural numbers, which is denoted with a capital boldface $\mathbf{N}$ or, in handwriting, with this symbol, $\mathbb{N}$, consists of the numbers $1, 2, 3, \ldots$ (the three dots meaning "and so on").[1] The elements of $\mathbf{N}$ are, of course, just the positive integers. The full set of *integers*, denoted $\mathbf{Z}$ or $\mathbb{Z}$, consists of the natural numbers, their negatives, and 0. We might describe this set by $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. Our convention, which is not universal, is that 0 is an integer, but **not** a natural number.

[1] Since the manufacture of boldface symbols such as $\mathbf{N}$ is a luxury not afforded users of chalk or pencil, it has long been traditional to use $\mathbb{N}$ on blackboards or in handwritten work as the symbol for the natural numbers and to call $\mathbb{N}$ a *blackboard bold* symbol.

There are various ways to describe sets. Sometimes it is possible to list the elements of a set within braces.

- $\{\text{egg1}, \text{egg2}\}$ is a set containing two elements, egg1 and egg2.
- $\{x\}$ is a set containing one element, x.
- $\mathsf{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers.
- $\mathsf{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of integers.

On other occasions, it is convenient to describe a set with *set builder* notation. This has the format

$$\{x \mid x \text{ has certain properties}\},$$

which is read "the set of x such that x has certain properties." We read "such that" at the vertical line, $|$.

More generally, we see

$$\{\text{some expression} \mid \text{the expression has certain properties}\}.$$

Thus, the set of odd natural numbers could be described as

$$\{n \mid n \text{ is an odd integer}, n > 0\}$$

or as

$$\{2k - 1 \mid k = 1, 2, 3, \dots\}$$

or as

$$\{2k - 1 \mid k \in \mathsf{N}\}.$$

The expression "$k \in \mathsf{N}$" is read "k belongs to N," the symbol $\in$ denoting set membership. Thus, "$m \in \mathsf{Z}$" simply says that m is an integer. Recall that a slash (/) written over any mathematical symbol negates the meaning of that symbol. So, in the same way that $\pi \neq 3.14$, we have $0 \notin \mathsf{N}$.

The set of common fractions—numbers like $\frac{3}{4}$, $\frac{-2}{17}$, and 5 $(= \frac{5}{1})$, which are ratios of integers with nonzero denominators—is more properly called the set of *rational numbers* and is denoted Q or $\mathbb{Q}$. Formally,

$$\mathsf{Q} = \left\{\tfrac{m}{n} \mid m, n \in \mathsf{Z}, n \neq 0\right\}.$$

The set of all real numbers is denoted R or $\mathbb{R}$. To define the real numbers properly requires considerable mathematical maturity. For our purposes, we think of real numbers as numbers that have decimal expansions of the form $a.a_1a_2\dots$, where a is an integer and $a_1, a_2, \dots$ are integers between 0 and 9 inclusive. In addition to the rational numbers, whose decimal expansions terminate or repeat, the real numbers include numbers like $\sqrt{2}$, $\sqrt[3]{17}$, e, π, $\ln 5$, and $\cos\frac{\pi}{6}$ whose decimal expansions neither terminate nor repeat. Such numbers are called *irrational*. An irrational number is a number that cannot be written in the form $\frac{m}{n}$ with m and n both integers. Incidentally, it can be very difficult to decide whether a given real number is irrational. For example, it is unknown whether such numbers as $e + \pi$ or $\frac{e}{\pi}$ are irrational.

The *complex numbers*, denoted C or $\mathbb{C}$, have the form $a + bi$, where a and b are real numbers and $i^2 = -1$; that is,

$$\mathsf{C} = \{a + bi \mid a, b \in \mathsf{R}, i^2 = -1\}.$$

Sometimes people are surprised to discover that a set can be an element of another set. For example, $\{\{a, b\}, c\}$ is a set with two elements, one of which is $\{a, b\}$ and the other c.

Pause 1 Let S denote the set $\{\{a\}, b, c\}$. True or false?

(a) $a \notin S$.
(b) $\{a\} \in S$.

Equality of Sets

2.1.1 DEFINITION Sets A and B are *equal*, and we write $A = B$, if and only if A and B contain the same elements or neither set contains any element. ❖

EXAMPLE 1

- $\{1, 2, 1\} = \{1, 2\} = \{2, 1\}$;
- $\left\{\frac{1}{2}, \frac{2}{4}, \frac{-3}{-6}, \frac{\pi}{2\pi}\right\} = \left\{\frac{1}{2}\right\}$;
- $\{t \mid t = r - s,\ r, s \in \{0, 1, 2\}\} = \{-2, -1, 0, 1, 2\}$.

The Empty Set

One set that arises in a variety of different guises is the set that contains no elements. Consider, for example, the set SMALL of people less than 1 millimeter in height, the set LARGE of people taller than the Eiffel Tower, the set

$$\mathfrak{PECULIAR} = \{n \in \mathsf{N} \mid 5n = 2\},$$

and the set

$$S = \{n \in \mathsf{N} \mid n^2 + 1 = 0\}.$$

These sets are all equal since none of them contains any elements. The unique set that contains no elements is called the *empty set*. Set theorists originally used 0 (zero) to denote this set, but now it is customary to use a 0 with a slash through it, $\emptyset$, to avoid confusion between zero and a capital "Oh."

Pause 2 True or false? $\{\emptyset\} = \emptyset$.

Subsets

2.1.2 DEFINITION A set A is a *subset* of a set B, and we write $A \subseteq B$, if and only if every element of A is an element of B. If $A \subseteq B$ but $A \neq B$, then A is called a *proper subset* of B and we write $A \subsetneqq B$. ❖

When $A \subseteq B$, it is common to say "A is contained in B" as well as "A is a subset of B." The notation $A \subset B$, which is common, unfortunately means $A \subsetneqq B$ to some people and $A \subseteq B$ to others. For this reason, we avoid it, while reiterating that it is present in a lot of mathematical writing. When you see it, make an effort to discover what the intended meaning is.

We occasionally see "$B \supseteq A$," read "B is a *superset* of A." This is an alternative way to express "$A \subseteq B$," A is a subset of B, just as "$y \geq x$" is an alternative way to express "$x \leq y$." We generally prefer the subset notation.

EXAMPLE 2

- $\{a, b\} \subseteq \{a, b, c\}$
- $\{a, b\} \subsetneqq \{a, b, c\}$
- $\{a, b\} \subseteq \{a, b, \{a, b\}\}$
- $\{a, b\} \in \{a, b, \{a, b\}\}$
- $\mathsf{N} \subsetneqq \mathsf{Z} \subsetneqq \mathsf{Q} \subsetneqq \mathsf{R} \subsetneqq \mathsf{C}$

Note the distinction between $A \subsetneqq B$ and $A \nsubseteq B$, the latter expressing the negation of $A \subseteq B$; for example,

$$\{a, b\} \subsetneqq \{a, b, c\} \nsubseteq \{a, b, x\}.$$

2.1.3 PROPOSITION For any set A, $A \subseteq A$ and $\emptyset \subseteq A$.

Proof If $a \in A$, then $a \in A$, so $A \subseteq A$. The proof that $\emptyset \subseteq A$ is a classic model of proof by contradiction. If $\emptyset \subseteq A$ is false, then there must exist some $x \in \emptyset$ such that $x \notin A$. This is an absurdity since there is no $x \in \emptyset$. ●

Pause 3 True or false?

(a) $\{\emptyset\} \in \{\{\emptyset\}\}$
(b) $\emptyset \subseteq \{\{\emptyset\}\}$
(c) $\{\emptyset\} \subseteq \{\{\emptyset\}\}$

(As Shakespeare once wrote, "Much ado about nothing.") ▮

The following proposition is an immediate consequence of the definitions of "subset" and "equal sets," and it illustrates the way we prove two sets are equal in practice.

2.1.4 PROPOSITION If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Two assertions are being made here.

($\longrightarrow$) If $A = B$, then A is a subset of B and B is a subset of A.

($\longleftarrow$) If A is a subset of B and B is a subset of A, then $A = B$.

Remember that another way to state Proposition 2.1.4 is to say that, for two sets to be equal, it is **necessary and sufficient** that each be a subset of the other.

Note the distinction between **membership**, $a \in b$, and **subset**, $a \subseteq b$. By the former statement, we understand that a is an element of the set b; by the latter, that a is a set each of whose elements is also in the set b.[2]

EXAMPLE 3 Each of the following assertions is true.

- $\{a\} \in \{x, y, \{a\}\}$
- $\{a\} \subsetneqq \{x, y, a\}$
- $\{a\} \nsubseteq \{x, y, \{a\}\}$
- $\{a, b\} \subseteq \{a, b\}$
- $\emptyset \in \{x, y, \emptyset\}$
- $\emptyset \subseteq \{x, y, \emptyset\}$
- $\{\emptyset\} \notin \{x, y, \emptyset\}$ ■

The Power Set

An important example of a set, **all** of whose elements are themselves sets, is the *power set* of a set.

2.1.5 DEFINITION The *power set* of a set A, denoted $\mathcal{P}(A)$, is the set of all subsets of A:

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}.$$ ❖

[2] Note the use of lowercase letters for sets, which is not common but certainly permissible.

EXAMPLE 4

- If $A = \{a\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}\}$.
- If $A = \{a, b\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
- $\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Answers to Pauses

1. Both statements are true. The set S contains the **set** $\{a\}$ as one of its elements, but not the element a.
2. This statement is false: $\{\emptyset\}$ is not the empty set for it contains one element, the set $\emptyset$.
3. (a) True: $\{\{\emptyset\}\}$ is a set that contains the single element $\{\emptyset\}$.
   (b) True: The empty set is a subset of any set.
   (c) False: There is just one element in the set $\{\emptyset\}$, (that is, $\emptyset$), and this is not an **element** of the set $\{\{\emptyset\}\}$, whose only element is $\{\emptyset\}$.

True/False Questions

(Answers can be found in the back of the book.)

1. $5 \in \{x + 2y \mid x \in \{0, 1, 2\}, y \in \{-2, 0, 2\}\}$
2. $-5 \in \{x + 2y \mid x \in \{0, 1, 2\}, y \in \{-2, 0, 2\}\}$
3. If $A = \{a, b\}$, then $b \subseteq A$.
4. If $A = \{a, b\}$, then $\{a\} \in A$.
5. $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$
6. $\emptyset \in \{\emptyset, \{\emptyset\}\}$
7. $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$
8. (Assume A, B, C are sets.) $A \in B, B \in C \rightarrow A \subseteq C$.
9. (Assume A, B are sets.) $A \subsetneq B \rightarrow B \not\subseteq A$.
10. If A has two elements, then $\mathcal{P}(\mathcal{P}(A))$ has eight elements.

Exercises

*The answers to exercises marked [BB] can be found in the **B**ack of the **B**ook.*

1. List the (distinct) elements in each of the following sets:
   (a) [BB] $\{x \in \mathsf{R} \mid x^2 = 5\}$
   (b) $\{x \in \mathsf{Z} \mid xy = 15 \text{ for some } y \in \mathsf{Z}\}$
   (c) [BB] $\{x \in \mathsf{Q} \mid x(x^2 - 2)(2x + 3) = 0\}$
   (d) $\{x + y \mid x \in \{-1, 0, 1\}, y \in \{0, 1, 2\}\}$
   (e) $\{a \in \mathsf{N} \mid a < -4 \text{ and } a > 4\}$
2. List five elements in each of the following sets:
   (a) [BB] $\{a + bi \mid a, b \in \mathsf{Z}, i^2 = -1\}$
   (b) $\{a + b\sqrt{2} \mid a \in \mathsf{N}, -b \in \{2, 5, 7\}\}$
   (c) $\left\{\frac{x}{y} \mid x, y \in \mathsf{R}, x^2 + y^2 = 25\right\}$
   (d) $\{n \in \mathsf{N} \mid n^2 + n \text{ is a multiple of } 3\}$
3. Let $A = \{1, 2, 3, 4\}$. List all the subsets B of A such that
   (a) [BB] $\{1, 2\} \subseteq B$;
   (b) $B \subseteq \{1, 2\}$;
   (c) $\{1, 2\} \not\subseteq B$;
   (d) $B \not\subseteq \{1, 2\}$;
   (e) $\{1, 2\} \subsetneq B$;
   (f) $B \subsetneq \{1, 2\}$.
4. [BB] Let $A = \{\{a, b\}\}$. Are the following statements true or false? Explain your answer.
   (a) $a \in A$.
   (b) $A \in A$.
   (c) $\{a, b\} \in A$.
   (d) There are two elements in A.
5. Determine which of the following are true and which are false. Justify your answers.
   (a) [BB] $3 \in \{1, 3, 5\}$
   (b) $\{3\} \in \{1, 3, 5\}$
   (c) $\{3\} \subsetneq \{1, 3, 5\}$
   (d) [BB] $\{3, 5\} \not\subseteq \{1, 3, 5\}$
   (e) $\{1, 3, 5\} \subsetneq \{1, 3, 5\}$
   (f) $1 \in \{a + 2b \mid a, b \text{ even integers}\}$
   (g) $0 \in \{a + b\sqrt{2} \mid a, b \in \mathsf{Q}, b \neq 0\}$
6. Find the power sets of each of the following sets:
   (a) [BB] $\emptyset$
   (b) $\{\emptyset\}$
   (c) $\{\emptyset, \{\emptyset\}\}$
7. Determine whether each of the following statements is true or false. Justify your answers.
   (a) [BB] $\emptyset \subseteq \emptyset$
   (b) $\emptyset \subseteq \{\emptyset\}$

(c) $\emptyset \in \emptyset$ (d) $\emptyset \in \{\emptyset\}$

(e) [BB] $\{1,2\} \not\subseteq \{1,2,3,\{1,2,3\}\}$

(f) $\{1,2\} \in \{1,2,3,\{1,2,3\}\}$

(g) $\{1,2\} \subsetneq \{1,2,\{\{1,2\}\}\}$

(h) [BB] $\{1,2\} \in \{1,2,\{\{1,2\}\}\}$

(i) $\{\{1,2\}\} \subseteq \{1,2,\{1,2\}\}$

8. [BB] Let A be a set and suppose $x \in A$. Is $x \subseteq A$ also possible? Explain.

9. (a) List all the subsets of the set $\{a,b,c,d\}$ that contain
   i. four elements;
   ii. [BB] three elements;
   iii. two elements;
   iv. one element;
   v. no elements;

   (b) How many subsets of $\{a,b,c,d\}$ are there altogether?

10. (a) How many elements are in the power set of the power set of the empty set?

   (b) Suppose A is a set containing one element. How many elements are in $\mathcal{P}(\mathcal{P}(A))$?

11. (a) [BB] If A contains two elements, how many elements are there in the power set of A?

   (b) [BB] If A contains three elements, how many elements are there in the power set of A?

   (c) [BB] If a set A contains $n \geq 0$ elements, guess how many elements are in the power set of A.

12. Suppose A, B, and C are sets. For each of the following statements, either prove it is true or give a counterexample to show that it is false.

   (a) [BB] $A \in B, B \in C \to A \in C$

   (b) $A \subseteq B, B \subseteq C \to A \subseteq C$

   (c) $A \subsetneq B, B \subsetneq C \to A \subsetneq C$

   (d) [BB] $A \in B, B \subseteq C \to A \in C$

   (e) $A \in B, B \subseteq C \to A \subseteq C$

   (f) $A \subseteq B, B \in C \to A \in C$

   (g) $A \subseteq B, B \in C \to A \subseteq C$

13. Suppose A and B are sets.

   (a) Answer true or false and explain: $A \not\subseteq B \to B \subsetneq A$.

   (b) Is the converse of the implication in (a) true or false? Explain.

14. Suppose A, B, and C are sets. Prove or give a counterexample that disproves each of the following assertions.

   (a) [BB] $C \in \mathcal{P}(A) \leftrightarrow C \subseteq A$

   (b) $A \subseteq B \leftrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$

   (c) $A = \emptyset \leftrightarrow \mathcal{P}(A) = \emptyset$

2.2 Operations on Sets

In this section, we discuss ways in which two or more sets can be combined to form a new set.

Union and Intersection

2.2.1 DEFINITIONS

The *union* of sets A and B, written $A \cup B$, is the set of elements in A or in B (or in both). The *intersection* of A and B, written $A \cap B$, is the set of elements that belong to both A and B. ❖

EXAMPLE 5

- If $A = \{a,b,c\}$ and $B = \{a,x,y,b\}$, then

$$A \cup B = \{a,b,c,x,y\}, \quad A \cap B = \{a,b\},$$

$$A \cup \{\emptyset\} = \{a,b,c,\emptyset\} \quad \text{and} \quad B \cap \{\emptyset\} = \emptyset.$$

- For any set A, $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$. ■

As with addition and multiplication of real numbers, the union and intersection of sets are *associative* operations. To say that set union is associative is to say that

$$(A_1 \cup A_2) \cup A_3 = A_1 \cup (A_2 \cup A_3)$$

for any three sets A_1, A_2, A_3. It follows that the expression

$$A_1 \cup A_2 \cup A_3$$

is unambiguous. The two different interpretations (corresponding to different insertions of parentheses) agree. The union of n sets $A_1, A_2, \ldots, A_n$ is written

$$A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \quad \text{or} \quad \bigcup_{i=1}^{n} A_i \tag{1}$$

and represents the set of elements that belong to one or more of the sets A_i. The intersection of $A_1, A_2, \ldots, A_n$ is written

$$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \quad \text{or} \quad \bigcap_{i=1}^{n} A_i \tag{2}$$

and denotes the set of elements which belong to all of the sets.

Do not assume from the expression $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$ that n is actually greater than 3 since the first part of this expression—$A_1 \cup A_2 \cup A_3$—is present only to make the general pattern clear; a union of sets is being formed. The last term—A_n—indicates that the last set in the union is A_n. If $n = 2$, then $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$ means $A_1 \cup A_2$. Similarly, if $n = 1$, the expression $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$ simply means A_1.

While parentheses are not required in expressions like (1) or (2), they are mandatory when both union and intersection are involved. For example, $A \cap (B \cup C)$ and $(A \cap B) \cup C$ are, in general, different sets. This is probably most easily seen by the use of the *Venn diagram* shown in Fig. 2.1.

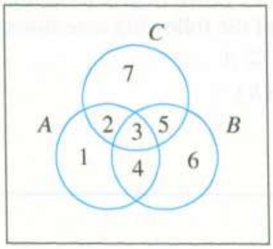


Figure 2.1 A Venn diagram.

The diagram indicates that A consists of the points in the regions labeled 1, 2, 3, and 4; B consists of those points in regions 3, 4, 5, and 6 and C of those in 2, 3, 5, and 7. The set $B \cup C$ consists of points in the regions labeled 3, 4, 5, 6, 2, and 7. Notice that $A \cap (B \cup C)$ consists of the points in regions 2, 3, and 4. The region $A \cap B$ consists of the points in regions 3 and 4; thus, $(A \cap B) \cup C$ is the set of points in the regions labeled 3, 4, 2, 5, and 7. The diagram enables us to see that, in general, $A \cap (B \cup C) \neq (A \cap B) \cup C$ and it shows how we could construct a specific counterexample: We could let A, B, and C be the sets

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6\}, \quad C = \{2, 3, 5, 7\}$$

as suggested by the diagram and then calculate

$$A \cap (B \cup C) = \{2, 3, 4\} \neq \{2, 3, 4, 5, 7\} = (A \cap B) \cup C.$$

There is a way to rewrite $A \cap (B \cup C)$. In Fig. 2.1, we see that $A \cap B$ consists of the points in the regions labeled 3 and 4 and that $A \cap C$ consists of the points in 2 and 3. Thus, the points of $(A \cap B) \cup (A \cap C)$ are those of 2, 3, and 4. These are just the points of $A \cap (B \cup C)$ (as observed previously), so the Venn diagram makes it easy to believe that, in general,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{3}$$

While pictures can be helpful in making certain statements seem plausible, they should not be relied on because they can also mislead. For this reason, and because there are situations in which Venn diagrams are difficult or impossible to create, it is important to be able to establish relationships among sets without resorting to a picture.

PROBLEM 6. Let A, B, and C be sets. Verify equation (3) without the aid of a Venn diagram.

Solution. As observed in Proposition 2.1.4, to show that two sets are equal it is sufficient to show that each is a subset of the other. Here this just amounts to expressing the meaning of $\cup$ and $\cap$ in words.

To show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, let $x \in A \cap (B \cup C)$. Then x is in A and also in $B \cup C$. Since $x \in B \cup C$, either $x \in B$ or $x \in C$. This suggests cases.

Case 1: $x \in B$.
In this case, x is in A as well as in B, so it's in $A \cap B$.

Case 2: $x \in C$.
Here x is in A as well as in C, so it's in $A \cap C$.

We have shown that either $x \in A \cap B$ or $x \in A \cap C$. By definition of union, $x \in (A \cap B) \cup (A \cap C)$, completing this half of our proof.

Conversely, we must show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. For this, let $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$. Thus, x is in both A and B or in both A and C. In either case, $x \in A$. Also, x is in either B or C; thus, $x \in B \cup C$. So x is both in A and in $B \cup C$; that is, $x \in A \cap (B \cup C)$. This completes the proof. ▲

PROBLEM 7. For sets A and B, prove that $A \cap B = A$ if and only if $A \subseteq B$.

Solution. Remember that there are two implications to establish and that we use the symbolism $(\longrightarrow)$ and $(\longleftarrow)$ to mark the start of the proof of each implication.

$(\longrightarrow)$ Here we assume $A \cap B = A$ and must prove $A \subseteq B$. For this, suppose $x \in A$. Then, $x \in A \cap B$ (because we are assuming $A = A \cap B$). Therefore, x is in A and in B, in particular, x is in B. This proves $A \subseteq B$.

$(\longleftarrow)$ Now we assume $A \subseteq B$ and prove $A \cap B = A$. To prove the equality of $A \cap B$ and A, we must prove that each set is a subset of the other. By definition of intersection, $A \cap B$ is a subset of A, so $A \cap B \subseteq A$. On the other hand, suppose $x \in A$. Since $A \subseteq B$, x is in B too; thus, x is in both A and B. Therefore, $A \subseteq A \cap B$. Therefore, $A = A \cap B$. ▲

Pause 4 For sets A and B, prove that $A \cup B = B$ if and only if $A \subseteq B$.

Set Difference

2.2.2 DEFINITIONS The *set difference* of sets A and B, written $A \setminus B$, is the set of those elements of A that are not in B. The *complement* of a set A is the set $A^c = U \setminus A$, where U is some universal set made clear by the context. ❖

EXAMPLE 8

- $\{a, b, c\} \setminus \{a, b\} = \{c\}$
- $\{a, b, c\} \setminus \{a, x\} = \{b, c\}$
- $\{a, b, \emptyset\} \setminus \emptyset = \{a, b, \emptyset\}$
- $\{a, b, \emptyset\} \setminus \{\emptyset\} = \{a, b\}$
- If A is the set {Monday, Tuesday, Wednesday, Thursday, Friday}, the context suggests that the universal set is the days of the week, so $A^c =$ {Saturday, Sunday}. ■

Notice that $A \setminus B = A \cap B^c$ and also that $(A^c)^c = A$. For example, if $A = \{x \in \mathsf{Z} \mid x^2 > 0\}$, then $A^c = \{0\}$ (it being understood that $U = \mathsf{Z}$) and so

$$(A^c)^c = \{0\}^c = \{x \in \mathsf{Z} \mid x \neq 0\} = A.$$

You may have previously encountered standard notation to describe various types of intervals of real numbers.

2.2.3 DEFINITION

Interval Notation If a and b are real numbers with $a < b$, then

$$[a, b] = \{x \in \mathsf{R} \mid a \le x \le b\} \quad \textbf{closed}$$
$$(a, b) = \{x \in \mathsf{R} \mid a < x < b\} \quad \textbf{open}$$
$$(a, b] = \{x \in \mathsf{R} \mid a < x \le b\} \quad \textbf{half open}$$
$$[a, b) = \{x \in \mathsf{R} \mid a \le x < b\} \quad \textbf{half open.}$$

As indicated, a closed interval is one that includes both endpoints, an open interval includes neither, and a half-open interval includes just one endpoint. A square bracket indicates that the adjacent endpoint is in the interval. To describe infinite intervals, we use the symbol ∞ (which is just a symbol) and make obvious adjustments to our notation. For example,

$$(-\infty, b] = \{x \in \mathsf{R} \mid x \le b\},$$
$$(a, \infty) = \{x \in \mathsf{R} \mid x > a\}.$$

The first interval here is half open; the second is open. ❖

Pause 5

If $A = [-4, 4]$ and $B = [0, 5]$, then $A \setminus B = [-4, 0)$. What is $B \setminus A$? What is A^c? ▮

2.2.4 THE LAWS OF DE MORGAN

The following two laws, of wide applicability, are attributed to Augustus De Morgan (1806–1871), who, together with George Boole (1815–1864), helped to make England a leading center of logic in the nineteenth century.[3]

$$(A \cup B)^c = A^c \cap B^c; \qquad (A \cap B)^c = A^c \cup B^c.$$

Readers should be struck by the obvious connection between these laws and the rules for negating *and* and *or* compound sentences described in Section 0.1. We illustrate by showing the equivalence of the first law of De Morgan and the rule for negating "$\mathcal{A}$ or $\mathcal{B}$."

PROBLEM 9. Prove that $(A \cup B)^c = A^c \cap B^c$ for any sets A, B, and C.

Solution. Let $\mathcal{A}$ be the statement "$x \in A$" and $\mathcal{B}$ be the statement "$x \in B$." Then

$$\begin{array}{rlll} x \in (A \cup B)^c & \leftrightarrow & \neg(x \in A \cup B) & \\ & \leftrightarrow & \neg(\mathcal{A} \text{ or } \mathcal{B}) & \text{definition of union} \\ & \leftrightarrow & \neg\mathcal{A} \text{ and } \neg\mathcal{B} & \text{rule for negating "or"} \\ & \leftrightarrow & x \in A^c \text{ and } x \in B^c & \\ & \leftrightarrow & x \in A^c \cap B^c & \text{definition of intersection.} \end{array}$$

The sets $(A \cup B)^c$ and $A^c \cap B^c$ contain the same elements, so they are the same. ▲

Symmetric Difference

2.2.5 DEFINITION

The *symmetric difference* of two sets A and B is the set $A \oplus B$ of elements that are in A or in B, but not in both. ❖

[3]As pointed out by Rudolf and Gerda Fritsch (*Der Vierfarbensatz*, B. I. Wissenschaftsverlag, Mannheim, 1994 and English translation, *The Four-Color Theorem*, by J. Peschke, Springer-Verlag, 1998), it was in a letter from De Morgan to Sir William Rowan Hamilton that the question giving birth to the famous Four-Color Theorem was first posed. See Section 13.2 for a detailed account of this theorem, whose proof was found relatively recently after over 100 years of effort!

Readers should note that the symbol $\triangle$, as in $A \triangle B$, is also used to denote symmetric difference.

Notice that the symmetric difference of sets can be expressed in terms of previously defined operations. For example,

$$A \oplus B = (A \cup B) \setminus (A \cap B)$$

and

$$A \oplus B = (A \setminus B) \cup (B \setminus A).$$

EXAMPLE 10

- $\{a, b, c\} \oplus \{x, y, a\} = \{b, c, x, y\}$
- $\{a, b, c\} \oplus \emptyset = \{a, b, c\}$
- $\{a, b, c\} \oplus \{\emptyset\} = \{a, b, c, \emptyset\}$ ■

PROBLEM 11. Use a Venn diagram to illustrate the plausibility of the fact that $\oplus$ is an associative operation; that is, use a Venn diagram to illustrate that for any three sets A, B, and C,

$$(A \oplus B) \oplus C = A \oplus (B \oplus C). \tag{4}$$

Solution. With reference to Fig. 2.1 again, $A \oplus B$ consists of the points in the regions labeled 1, 2, 5, and 6 while C consists of the points in the regions 2, 3, 5, and 7. Thus, $(A \oplus B) \oplus C$ is the set of points in the regions 1, 3, 6, and 7. On the other hand, $B \oplus C$ consists of the regions 2, 7, 4, and 6 and A, of regions 1, 2, 3, 4. Thus, $A \oplus (B \oplus C)$ also consists of the points in regions 1, 3, 6, and 7. ▲

As a consequence of (4), the expression $A \oplus B \oplus C$, which conceivably could be interpreted in two ways, is in fact unambiguous. Notice that $A \oplus B \oplus C$ is the set of points in an odd number of the sets A, B, C: Regions 1, 6, and 7 contain the points of just one of the sets while region 3 consists of points in all three. More generally, the symmetric difference $A_1 \oplus A_2 \oplus A_3 \oplus \cdots \oplus A_n$ of n sets $A_1, A_2, A_3, \ldots, A_n$ is well defined and, as it turns out, is the set of those elements which are members of an odd number of the sets A_i. (See Exercise 20 of Section 5.1.)

The Cartesian Product of Sets

There is yet another way in which two sets can be combined to obtain another.

2.2.6 DEFINITIONS

If A and B are sets, the *Cartesian product* (sometimes also called the *direct product*) of A and B is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

(We say "A cross B" for "$A \times B$.") More generally, the Cartesian product of $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

When all the sets are equal to the same set A, $\underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$ is written A^n. ❖

The elements of $A \times B$ are called *ordered pairs* because their order is important: $(a, b) \neq (b, a)$ (unless $a = b$). The elements a and b are the *coordinates* of the ordered pair (a, b); the first coordinate is a and the second is b. The elements of A^n are called *n-tuples*.

Elements of $A \times B$ are equal if and only if they have the same first coordinates and the same second coordinates:

$$(a_1, b_1) = (a_2, b_2) \text{ if and only if } a_1 = a_2 \text{ and } b_1 = b_2.$$

EXAMPLE 12 Let $A = \{a, b\}$ and $B = \{x, y, z\}$. Then

$$A \times B = \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$$

and

$$B \times A = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}.$$

This example illustrates that, in general, the sets $A \times B$ and $B \times A$ are different. ■

EXAMPLE 13 The Cartesian plane, in which calculus students sketch curves, is a picture of $\mathsf{R} \times \mathsf{R} = \mathsf{R}^2 = \{(x, y) \mid x, y \in \mathsf{R}\}$. The adjective Cartesian is derived from Descartes,[4] as Cartesius was Descartes's name in Latin. ■

PROBLEM 14. Let A, B, and C be sets. Prove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Solution. We must prove that any element in $A \times (B \cup C)$ is in $(A \times B) \cup (A \times C)$. Since the elements in $A \times (B \cup C)$ are ordered pairs, we begin by letting $(x, y) \in A \times (B \cup C)$ (this is more helpful than starting with "$x \in A \times B$") and ask ourselves what this means. It means that x, the first coordinate, is in A and y, the second coordinate, is in $B \cup C$. Therefore, y is in either B or C. If y is in B, then, since x is in A, $(x, y) \in A \times B$. If y is in C, then, since x is in A, $(x, y) \in A \times C$. Thus, (x, y) is either in $A \times B$ or in $A \times C$; thus, (x, y) is in $(A \times B) \cup (A \times C)$, which is what we wanted to show. ▲

Pause 6 Let A, B, and C be three sets. Prove that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. What can you conclude about the sets $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$? Why? ▮

Pause 7 Let A and B be nonempty sets. Prove that $A \times B = B \times A$ if and only if $A = B$. Is this true if $A = \emptyset$? ▮

Answers to Pauses

4. ($\longrightarrow$) Suppose the first statement, $A \cup B = B$, is true. We show $A \subseteq B$. So let $x \in A$. Then x is certainly in $A \cup B$, by the definition of $\cup$. But $A \cup B = B$, so $x \in B$. Thus, $A \subseteq B$.

   ($\longleftarrow$) Conversely, suppose the second statement, $A \subseteq B$, is true. We have to show $A \cup B = B$. To prove the sets $A \cup B$ and B are equal, we have to show each is a subset of the other. First, let $x \in A \cup B$. Then x is either in A or in B. If the latter, $x \in B$, and if the former, $x \in B$ because A is a subset of B. In either case, $x \in B$. Thus, $A \cup B \subseteq B$. Second, assume $x \in B$. Then x is in $A \cup B$ by definition of $\cup$. So $B \subseteq A \cup B$ and we have equality, as required.

5. $B \setminus A = (4, 5]$; $A^c = (-\infty, -4) \cup (4, \infty)$.

6. An element of $(A \times B) \cup (A \times C)$ is either in $A \times B$ or in $A \times C$; in either case, it's an ordered pair. So we begin by letting $(x, y) \in (A \times B) \cup (A \times C)$ and noting that either $(x, y) \in A \times B$ or $(x, y) \in A \times C$. In the first case, x is in A and y is in B; in the second case, x is in A and y is in C. In either case, x is in A and y is either in B or in C; so $x \in A$ and $y \in B \cup C$. Therefore, $(x, y) \in A \times (B \cup C)$, establishing the required subset relation. The reverse subset relation was established in Problem 14. We conclude that the two sets in question are equal; that is, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

[4] René Descartes (1596–1650), together with Pierre de Fermat, the inventor of analytic geometry, introduced the method of plotting points and graphing functions in R^2 with which we are so familiar today.

7. ($\longrightarrow$) Suppose that the statement $A \times B = B \times A$ is true. We prove $A = B$. So suppose $x \in A$. Since $B \neq \emptyset$, we can find some $y \in B$. Thus, $(x, y) \in A \times B$. Since $A \times B = B \times A$, $(x, y) \in B \times A$. So $x \in B$, giving us $A \subseteq B$. Similarly, we show that $B \subseteq A$ and conclude $A = B$.

($\longleftarrow$) On the other hand, if $A = B$ is a true statement, then $A \times B = A \times A = B \times A$.

Finally, if $A = \emptyset$ and B is any nonempty set, then $A \times B = \emptyset = B \times A$, but $A \neq B$. So $A \times B = B \times A$ does not mean $A = B$ in the case $A = \emptyset$.

True/False Questions

(Answers can be found in the back of the book.)

1. If A and B are sets and $A \neq B$, then $A \cap B \subsetneq A \cup B$.
2. If A and B are sets, then $(A \setminus B) \cap (B \setminus A) = \emptyset$.
3. If A and B are sets, then $(A^c \cup B)^c = A \cap B^c$.
4. If A and B are sets, then $A \setminus B \subseteq A \oplus B$.
5. If A and B are sets and $A \neq B$, then $A \oplus B \neq \emptyset$.
6. If A and B are nonempty sets, then $A \times B$ is a nonempty set.
7. The name of Augustus De Morgan appears in both Chapter 1 and Chapter 2 of this text.
8. $(A \subseteq B) \to (B^c \subseteq A^c)$.
9. $(B^c \subseteq A^c) \to (A \subseteq B)$.
10. $((A \setminus B) \subseteq (B \setminus A)) \to (A \subseteq B)$.

Exercises

The answers to exercises marked [BB] can be found in the Back of the Book.

1. Let $A = \{x \in \mathsf{N} \mid x < 7\}$, $B = \{x \in \mathsf{Z} \mid |x - 2| < 4\}$, and $C = \{x \in \mathsf{R} \mid x^3 - 4x = 0\}$.
   (a) [BB] List the elements in each of these sets.
   (b) Find $A \cup C$, $B \cap C$, $B \setminus C$, $A \oplus B$, $C \times (B \cap C)$, $(A \setminus B) \setminus C$, $A \setminus (B \setminus C)$, and $(B \cup \emptyset) \cap \{\emptyset\}$.
   (c) List the elements in $S = \{(a, b) \in A \times B \mid a = b+2\}$ and in $T = \{(a, c) \in A \times C \mid a \leq c\}$.
2. Let $S = \{2, 5, \sqrt{2}, 25, \pi, \frac{5}{2}\}$ and $T = \{4, 25, \sqrt{2}, 6, \frac{3}{2}\}$.
   (a) [BB] Find $S \cap T$, $S \cup T$, and $T \times (S \cap T)$.
   (b) [BB] Find $\mathsf{Z} \cup S$, $\mathsf{Z} \cap S$, $\mathsf{Z} \cup T$, and $\mathsf{Z} \cap T$.
   (c) List the elements in each of the sets $\mathsf{Z} \cap (S \cup T)$ and $(\mathsf{Z} \cap S) \cup (\mathsf{Z} \cap T)$. What do you notice?
   (d) List the elements of $\mathsf{Z} \cup (S \cap T)$ and list the elements of $(\mathsf{Z} \cup S) \cap (\mathsf{Z} \cup T)$. What do you notice?
3. Let $A = \{(-1, 2), (4, 5), (0, 0), (6, -5), (5, 1), (4, 3)\}$. List the elements in each of the following sets.
   (a) [BB] $\{a + b \mid (a, b) \in A\}$
   (b) $\{a \mid a > 0 \text{ and } (a, b) \in A \text{ for some } b\}$
   (c) $\{b \mid b = k^2 \text{ for some } k \in \mathsf{Z} \text{ and } (a, b) \in A \text{ for some } a\}$
4. List the elements in the sets $A = \{(a, b) \in \mathsf{N} \times \mathsf{N} \mid a \leq b, b \leq 3\}$ and $B = \{\frac{a}{b} \mid a, b \in \{-1, 1, 2\}\}$.
5. For $A = \{a, b, c, \{a, b\}\}$, find
   (a) [BB] $A \setminus \{a, b\}$ (b) $\{\emptyset\} \setminus \mathcal{P}(A)$
   (c) $A \setminus \emptyset$ (d) $\emptyset \setminus A$
   (e) [BB] $\{a, b, c\} \setminus A$ (f) $(\{a, b, c\} \cup \{A\}) \setminus A$
6. Find A^c (with respect to $U = \mathsf{R}$) in each of the following cases.
   (a) [BB] $A = (1, \infty) \cup (-\infty, -2]$
   (b) $A = (-3, \infty) \cap (-\infty, 4]$
   (c) $A = \{x \in \mathsf{R} \mid x^2 \leq -1\}$
7. Let $X = \{1, 2, 3, 4\}$, $Y = \{2, 3, 4, 5\}$, and $Z = \{3, 4, 5, 6\}$. List the elements in the indicated sets. (The universal set is the set of integers.)
   (a) $X \oplus (Y \cap Z)$ (b) $(X^c \cup Y)^c$
8. Let $n > 3$ and $A = \{1, 2, 3, \ldots, n\}$.
   (a) [BB] How many subsets of A contain $\{1, 2\}$?
   (b) How many subsets B of A have the property that $B \cap \{1, 2\} = \emptyset$?
   (c) How many subsets B of A have the property that $B \cup \{1, 2\} = A$?

   Explain your answers.
9. [BB] Let a and b be real numbers with $a < b$. Find $(a, b)^c$, $[a, b)^c$, $(a, \infty)^c$, and $(-\infty, b]^c$.

10. The universal set for this problem is the set of students attending Miskatonic University. Let
    - M denote the set of math majors
    - CS denote the set of computer science majors
    - T denote the set of students who had a test on Friday
    - P denote those students who ate pizza last Thursday

    Using only the set theoretical notation we have introduced in this chapter, rewrite each of the following assertions.
    (a) [BB] Computer science majors had a test on Friday.
    (b) [BB] No math major ate pizza last Thursday.
    (c) Some math majors did not eat pizza last Thursday.
    (d) Those computer science majors who did not have a test on Friday ate pizza on Thursday.
    (e) Math or computer science majors who ate pizza on Thursday did not have a test on Friday.
11. Use the set theoretical notation introduced in this chapter to express the negation each of statements (a)–(e) in Exercise 10. Do the same for the converse of any statement that is an implication.
12. Let P denote the set of primes and E the set of even integers. As always, Z and N denote the integers and natural numbers, respectively. Find equivalent formulations of each of the following statements using the notation of set theory that has been introduced in this section.
    (a) [BB] There exists an even prime.
    (b) 0 is an integer but not a natural number.
    (c) Every natural number is an integer.
    (d) Not every integer is a natural number.
    (e) Every prime except 2 is odd.
    (f) 2 is an even prime.
    (g) 2 is the only even prime.
13. For $n \in \mathsf{Z}$, let $A_n = \{a \in \mathsf{Z} \mid a \leq n\}$. Find each of the following sets.
    (a) [BB] $A_3 \cup A_{-3}$ (b) $A_3 \cap A_{-3}$
    (c) $A_3 \cap (A_{-3})^c$ (d) $\bigcap_{i=0}^{4} A_i$
14. [BB] In Fig. 2.1, the region labeled 7 represents the set $C \setminus (A \cup B)$. What set is represented by the region labeled 2? By that labeled 3? By that labeled 4?
15. Let $A = \{1, 2, 4, 5, 6, 9\}$, $B = \{1, 2, 3, 4\}$, and $C = \{5, 6, 7, 8\}$.
    (a) Draw a Venn diagram showing the relationship between these sets. Show which elements are in which region.
    (b) What are the elements in each of the following sets?
    i. $(A \cup B) \cap C$ ii. $A \setminus (B \setminus A)$
    iii. $(A \cup B) \setminus (A \cap C)$ iv. $A \oplus C$
    v. $(A \cap C) \times (A \cap B)$
16. (a) [BB] Suppose A and B are sets such that $A \cap B = A$. What can you conclude? Why?
    (b) Repeat (a) assuming $A \cup B = A$.
17. [BB] Let $n \geq 1$ be a natural number. How many elements are in the set $\{(a, b) \in \mathsf{N} \times \mathsf{N} \mid a \leq b \leq n\}$? Explain.
18. Suppose A is a subset of $\mathsf{N} \times \mathsf{N}$ with the properties
    - $(1, 1) \in A$ and
    - if $(a, b) \in A$, then both $(a + 1, b)$ and $(a + 1, b + 1)$ are also in A.

    Do you think that $\{(m, n) \in \mathsf{N} \times \mathsf{N} \mid m \geq n\}$ is a subset of A? Explain. [*Hint*: A picture of A in the xy-plane might help.]
19. Let A, B, and C be subsets of some universal set U.
    (a) If $A \cap B \subseteq C$ and $A^c \cap B \subseteq C$, prove that $B \subseteq C$.
    (b) [BB] Given that $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, does it follow that $B = C$? Justify your answer.
20. Let A, B, and C be sets.
    (a) Find a counterexample to the statement $A \cup (B \cap C) = (A \cup B) \cap C$.
    (b) Without using Venn diagrams, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
21. Use the first law of De Morgan to prove the second: $(A \cap B)^c = A^c \cup B^c$.
22. [BB] Use the laws of De Morgan and any other set theoretic identities discussed in the text to prove that $(A \setminus B) \setminus C = A \setminus (B \cup C)$ for any sets A, B, and C.
23. Let A, B, C, and D be subsets of a universal set U. Use set theoretic identities discussed in the text to simplify the expression $[(A \cup B)^c \cap (A^c \cup C)^c]^c \setminus D^c$.
24. Let A, B, and C be subsets of some universal set U. Use set theoretic identities discussed in the text to prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \setminus C^c)$.
25. Suppose A, B, and C are subsets of some universal set U.
    (a) [BB] Generalize the laws of De Morgan by finding equivalent ways to describe the sets $(A \cup B \cup C)^c$ and $(A \cap B \cap C)^c$..
    (b) Find a way to describe the set $(A \cap (B \setminus C))^c \cap A$ without using the symbol ^{c} for set complement.
26. Let A and B be sets.
    (a) [BB] Find a necessary and sufficient condition for $A \oplus B = A$.
    (b) Find a necessary and sufficient condition for $A \cap B = A \cup B$.

    Explain your answers (with Venn diagrams if you wish).
27. Which of the following conditions imply that $B = C$? In each case, either prove or give a counterexample.
    (a) [BB] $A \cup B = A \cup C$
    (b) $A \cap B = A \cap C$
    (c) $A \oplus B = A \oplus C$
    (d) $A \times B = A \times C$
28. True or false? In each case, provide a proof or a counterexample.
    (a) $A \subseteq C, B \subseteq D \rightarrow A \times B \subseteq C \times D$.
    (b) $A \not\subseteq B, B \subseteq C \rightarrow A \not\subseteq C$.
    (c) $A \times B \subseteq C \times D \rightarrow A \subseteq C$ and $B \subseteq D$.
    (d) $A \subseteq C$ and $B \subseteq D$ if and only if $A \times B \subseteq C \times D$.
    (e) [BB] $A \cup B \subseteq A \cap B \rightarrow A = B$.

29. Show that $(A \cap B) \times C = (A \times C) \cap (B \times C)$ for any sets A, B, and C.

30. Let A, B, and C be arbitrary sets. For each of the following, either prove the given statement is true or exhibit a counterexample to prove it is false.
(a) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$
(b) $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$
(c) [BB] $(A \oplus B) \times C = (A \times C) \oplus (B \times C)$
(d) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$
(e) $(A \setminus B) \times (C \setminus D) = (A \times C) \setminus (B \times D)$

31. Find out what you can about George Boole and write a paragraph or two about him (in good English, of course).

2.3 Binary Relations

If A and B are sets, remember that the Cartesian product of A and B is the set $A \times B = \{(a, b) \mid a \in A, b \in B\}$. There are occasions when we are interested in a certain subset of $A \times B$. For example, if A is the set of former major league baseball players and $B = \mathsf{N} \cup \{0\}$ is the set of nonnegative integers, then we might naturally be interested in

$$\mathcal{R} = \{(a, b) \mid a \in A, b \in B, \text{player } a \text{ had } b \text{ career home runs}\}.$$

For example, (Hank Aaron, 755) and (Mickey Mantle, 536) are elements of $\mathcal{R}$.

2.3.1 DEFINITIONS Let A and B denote sets. A *binary relation from A to B* is a subset of $A \times B$. A *binary relation on A* is a subset of $A \times A$. ❖

2.3.2 REMARK When $\mathcal{R}$ is a binary relation from A to B and the pair (a, b) is in $\mathcal{R}$, we naturally write $(a, b) \in \mathcal{R}$, though the reader should be aware that other authors prefer the notation $a\mathcal{R}b$. ❖

The empty set and the entire Cartesian product $A \times B$ are always binary relations from A to B, although these are generally not as interesting as certain nonempty proper subsets of $A \times B$.

EXAMPLE 15

- If A is the set of students who were registered at the University of Toronto during the Fall 2001 semester and B is the set {History, Mathematics, English, Biology}, then $\mathcal{R} = \{(a, b) \mid a \in A \text{ is enrolled in a course in subject } b\}$ is a binary relation from A to B.
- Let A be the set of surnames of people listed in the Seattle telephone directory. Then $\mathcal{R} = \{(a, n) \mid a \text{ appears on page } n\}$ is a binary relation from A to the set N of natural numbers.
- $\{(a, b) \mid a, b \in \mathsf{N}, \frac{a}{b} \text{ is an integer}\}$ and $\{(a, b) \mid a, b \in \mathsf{N}, a - b = 2\}$ are binary relations on N.
- $\{(x, y) \mid y = x^2\}$ is a binary relation on R whose graph the reader should recognize. ■

Pause 8 What is the common name for this graph? ▮

Our primary intent in this section is to identify special properties of binary relations on a set, so, henceforth, all binary relations will be subsets of $A \times A$ for some set A.

2.3.3 DEFINITION A binary relation $\mathcal{R}$ on a set A is *reflexive* if and only if $(a, a) \in \mathcal{R}$ for all $a \in A$. ❖

EXAMPLE 16

- $\{(x, y) \in \mathsf{R}^2 \mid x \leq y\}$ is a reflexive relation on R since $x \leq x$ for any $x \in \mathsf{R}$.
- $\{(a, b) \in \mathsf{N}^2 \mid \frac{a}{b} \in \mathsf{N}\}$ is a reflexive relation on N since $\frac{a}{a}$ is an integer, 1, for any $a \in \mathsf{N}$.
- $\mathcal{R} = \{(x, y) \in \mathsf{R}^2 \mid x^2 + y^2 > 0\}$ is not a reflexive relation on R since $(0, 0) \notin \mathcal{R}$. [This example reminds us that a reflexive relation must contain **all** pairs of the form (a, a): **Most** is not enough.] ■

PROBLEM 17. Suppose $\mathcal{R} = \{(a, b) \in \mathsf{Z} \times \mathsf{Z} \mid a^2 = b^2\}$. Criticize and then correct the following "proof" that $\mathcal{R}$ is reflexive:

$(a, a) \in \mathcal{R}$ if $a^2 = a^2$.

Solution. The statement "$(a, a) \in \mathcal{R}$ if $a^2 = a^2$" is the implication "$a^2 = a^2 \rightarrow (a, a) \in \mathcal{R}$," which has almost nothing to do with what is required. To prove that $\mathcal{R}$ is reflexive, we must establish an implication of the form "something $\rightarrow \mathcal{R}$ is reflexive." Here is a good argument, in this case.

For any integer a, we have $a^2 = a^2$ and, hence, $(a, a) \in \mathcal{R}$. Therefore, $\mathcal{R}$ is reflexive. ▲

2.3.4 DEFINITION A binary relation $\mathcal{R}$ on a set A is *symmetric* if and only if

if $a, b \in A$ and $(a, b) \in \mathcal{R}$, then $(b, a) \in \mathcal{R}$. ❖

EXAMPLE 18

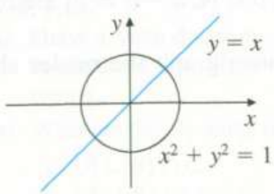
- $\mathcal{R} = \{(x, y) \in \mathsf{R}^2 \mid x^2 + y^2 = 1\}$ is a symmetric relation on R since if $x^2 + y^2 = 1$ then $y^2 + x^2 = 1$ too: If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$.
- $\{(x, y) \in \mathsf{Z} \times \mathsf{Z} \mid x - y \text{ is even}\}$ is a symmetric relation on Z since if $x - y$ is even so is $y - x$.
- $\mathcal{R} = \{(x, y) \in \mathsf{R}^2 \mid x^2 \geq y\}$ is not a symmetric relation on R. For example, $(2, 1) \in \mathcal{R}$ because $2^2 \geq 1$, but $(1, 2) \notin \mathcal{R}$ because $1^2 \not\geq 2$. ■

Suppose $\mathcal{R}$ is a binary relation on $A = \mathsf{R}^2$. In this case, the elements of $\mathcal{R}$, being ordered pairs of elements of A, are ordered pairs of elements each of which is an ordered pair of real numbers. Consider, for example,

$$\mathcal{R} = \{(\,(x, y), (u, v)\,) \in \mathsf{R}^2 \times \mathsf{R}^2 \mid x^2 + y^2 = u^2 + v^2\}.$$

This is a symmetric relation on R^2 since if $(\,(x, y), (u, v)\,) \in \mathcal{R}$, then $x^2 + y^2 = u^2 + v^2$, so $u^2 + v^2 = x^2 + y^2$, so $(\,(u, v), (x, y)\,) \in \mathcal{R}$.

Pause 9 Is this relation reflexive? ▮



A binary relation on the real numbers (or on any subset of R) is symmetric if, when its points are plotted as usual in the Cartesian plane, the figure is symmetric about the line with equation $y = x$. The set $\{(x, y) \in \mathsf{R}^2 \mid x^2 + y^2 = 1\}$ is a symmetric relation because its points are those on the graph of the unit circle centered at the origin, and this circle is certainly symmetric about the line with equation $y = x$.

If a set A has n elements and n is reasonably small, a binary relation on A can be conveniently described by labelling with the elements of A the rows and the columns of an $n \times n$ grid and then inserting some symbol in row a and column b to indicate that (a, b) is in the relation.

EXAMPLE 19 The picture in Fig. 2.2 describes the relation

$$\mathcal{R} = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 2), (3, 3), (4, 4)\}$$

on the set $A = \{1, 2, 3, 4\}$. This relation is reflexive (all points on the *main diagonal*—top left corner to lower right—are present), but not symmetric (the ×'s are not symmetrically located with respect to the main diagonal). For example, there is a × in row 1, column 4, but not in row 4, column 1. ■

	1	2	3	4
1	×	×		×
2	×	×		
3		×	×	
4				×

Figure 2.2

2.3.5 DEFINITION A binary relation $\mathcal{R}$ on a set A is *antisymmetric* if and only if

$$\text{if } a, b \in A \text{ and both } (a, b) \text{ and } (b, a) \text{ are in } \mathcal{R}, \text{ then } a = b.$$ ❖

EXAMPLE 20 $\mathcal{R} = \{(x, y) \in \mathsf{R}^2 \mid x \leq y\}$ is an antisymmetric relation on R since $x \leq y$ and $y \leq x$ implies $x = y$; thus, $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ implies $x = y$. ■

EXAMPLE 21 If S is a set and $A = \mathcal{P}(S)$ is the power set of S, then $\{(X, Y) \mid X, Y \in \mathcal{P}(S), X \subseteq Y\}$ is antisymmetric since $X \subseteq Y$ and $Y \subseteq X$ implies $X = Y$. ■

EXAMPLE 22 $\mathcal{R} = \{(1, 2), (2, 3), (3, 3), (2, 1)\}$ is not antisymmetric on $A = \{1, 2, 3\}$ because $(1, 2) \in \mathcal{R}$ and $(2, 1) \in \mathcal{R}$ but $1 \neq 2$. ■

Note that "antisymmetric" is not the same as "not symmetric." The relation in Example 22 is not symmetric but neither is it antisymmetric.

Pause 10 Why is this relation not symmetric? ▮

Pause 11 Is the relation $\mathcal{R} = \{(\,(x, y), (u, v)\,) \in \mathsf{R}^2 \times \mathsf{R}^2 \mid x^2 + y^2 = u^2 + v^2\}$ antisymmetric? ▮

2.3.6 DEFINITION A binary relation $\mathcal{R}$ on a set A is *transitive* if and only if

$$\text{if } a, b, c \in A, \text{ and both } (a, b) \text{ and } (b, c) \text{ are in } \mathcal{R}, \text{ then } (a, c) \in \mathcal{R}.$$ ❖

EXAMPLE 23 $\mathcal{R} = \{(x, y) \in \mathsf{R}^2 \mid x \leq y\}$ is a transitive relation on R since, if $x \leq y$ and $y \leq z$, then $x \leq z$: if (x, y) and (y, z) are in $\mathcal{R}$, then $(x, z) \in \mathcal{R}$. ■

EXAMPLE 24 $\{(a, b) \in \mathsf{Z} \times \mathsf{Z} \mid \frac{a}{b} \text{ is an integer}\}$ is a transitive relation on Z since, if $\frac{a}{b}$ and $\frac{b}{c}$ are integers, then so is $\frac{a}{c}$ because $\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c}$. ■

3. Let $\mathcal{R}$ be a binary relation on a set A and let $a \in A$. If $\mathcal{R}$ is not reflexive, then we can conclude that $(a, a) \notin \mathcal{R}$.
4. Let $\mathcal{R}$ be a binary relation on a set A and let $a \in A$. If $(a, a) \notin \mathcal{R}$, then $\mathcal{R}$ is not reflexive.
5. Let $\mathcal{R}$ be a binary relation on a set A. If $\mathcal{R}$ is not symmetric, then there exist $a, b \in A$ such that $(a, b) \in \mathcal{R}$ but $(b, a) \notin \mathcal{R}$.
6. Let $\mathcal{R}$ be a binary relation on a set A. If $\mathcal{R}$ is antisymmetric, then we can conclude that there exist $a, b \in A$ such that $(a, b) \in \mathcal{R}$ but $(b, a) \notin \mathcal{R}$.
7. Let $\mathcal{R}$ be a binary relation on a set A. If there exist $a, b, c \in A$ such that $(a, b) \in \mathcal{R}$, $(b, c) \in \mathcal{R}$, and $(a, c) \in \mathcal{R}$, then $\mathcal{R}$ must be transitive.
8. If a binary relation $\mathcal{R}$ is antisymmetric, then $\mathcal{R}$ is not symmetric.
9. If a binary relation $\mathcal{R}$ is not symmetric, then it is antisymmetric.
10. Let $\mathcal{R}$ be a binary relation on a set A containing two elements. If $\mathcal{R}$ is reflexive, then $\mathcal{R}$ is transitive.

Exercises

*The answers to exercises marked [BB] can be found in the **Back** of the **Book**.*

1. [BB] Let B denote the set of books in a college library and S denote the set of students attending that college. Interpret the Cartesian product $S \times B$. Give a sensible example of a binary relation from S to B.
2. Let A denote the set of names of streets in St. John's, Newfoundland, and B denote the names of all the residents of St. John's. Interpret the Cartesian product $A \times B$. Give a sensible example of a binary relation from A to B.
3. Determine which of the properties reflexive, symmetric, transitive apply to the following relations on the set of people.
   (a) [BB] is a father of
   (b) is a friend of
   (c) [BB] is a descendant of
   (d) have the same parents
   (e) is an uncle of
4. With a table like that in Fig. 2.2, illustrate a relation on the set $\{a, b, c, d\}$ that is
   (a) [BB] reflexive and symmetric
   (b) not symmetric and not antisymmetric
   (c) not symmetric but antisymmetric
   (d) transitive

   Include at least six elements in each relation.
5. Let $A = \{1, 2, 3\}$. List the ordered pairs in a relation on A that is
   (a) [BB] not reflexive, not symmetric, and not transitive
   (b) reflexive, but neither symmetric nor transitive
   (c) symmetric, but neither reflexive nor transitive
   (d) transitive, but neither reflexive nor symmetric
   (e) reflexive and symmetric, but not transitive
   (f) reflexive and transitive, but not symmetric
   (g) [BB] symmetric and transitive, but not reflexive
   (h) reflexive, symmetric, and transitive
6. Is it possible for a binary relation to be both symmetric and antisymmetric? If the answer is no, why not? If it is yes, find all such binary relations.
7. [BB] What is wrong with the following argument, which purports to prove that a binary relation that is symmetric and transitive must necessarily be reflexive as well?

   Suppose $\mathcal{R}$ is a symmetric and transitive relation on a set A and let $a \in A$. Then, for any b with $(a, b) \in \mathcal{R}$, we have also $(b, a) \in \mathcal{R}$ by symmetry. Since now we have both (a, b) and (b, a) in $\mathcal{R}$, we have $(a, a) \in \mathcal{R}$ as well, by transitivity. Thus, $(a, a) \in \mathcal{R}$, so $\mathcal{R}$ is reflexive.
8. Determine whether each of the binary relations $\mathcal{R}$ defined on the given sets A is reflexive, symmetric, antisymmetric, or transitive. If a relation has a certain property, prove this is so; otherwise, provide a counterexample to show that it does not.
   (a) [BB] A is the set of all English words; $(a, b) \in \mathcal{R}$ if and only if a and b have at least one letter in common.
   (b) A is the set of all people. $(a, b) \in \mathcal{R}$ if and only if neither a nor b is currently enrolled at Miskatonic University or else both are enrolled at MU and are taking at least one course together.
9. Answer Exercise 8 for each of the following relations:
   (a) $A = \{1, 2\}$; $\mathcal{R} = \{(1, 2)\}$.
   (b) [BB] $A = \{1, 2, 3, 4\}$; $\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (3, 4)\}$.
   (c) [BB] $A = \mathsf{Z}$; $(a, b) \in \mathcal{R}$ if and only if $ab \geq 0$.
   (d) $A = \mathsf{R}$; $(a, b) \in \mathcal{R}$ if and only if $a^2 = b^2$.
   (e) $A = \mathsf{R}$; $(a, b) \in \mathcal{R}$ if and only if $a - b \leq 3$.
   (f) $A = \mathsf{Z} \times \mathsf{Z}$; $((a, b), (c, d)) \in \mathcal{R}$ if and only if $a - c = b - d$.
   (g) $A = \mathsf{N}$; $(a, b) \in \mathcal{R}$ if and only if $a \neq b$.
   (h) $A = \mathsf{Z}$; $\mathcal{R} = \{(x, y) \mid x + y = 10\}$.

(i) [BB] $A = \mathsf{R}^2$; $\mathcal{R} = \{((x, y), (u, v)) \mid x+y \le u+v\}$.

(j) $A = \mathsf{N}$; $(a, b) \in \mathcal{R}$ if and only if $\frac{a}{b}$ is an integer.

(k) $A = \mathsf{Z}$; $(a, b) \in \mathcal{R}$ if and only if $\frac{a}{b}$ is an integer.

10. Define $\mathcal{R}$ on R by $(x, y) \in \mathcal{R}$ if and only if $1 \le |x| + |y| \le 2$.

(a) Make a sketch in the Cartesian plane showing the region of R^2 defined by $\mathcal{R}$.

(b) Show that $\mathcal{R}$ is neither reflexive nor transitive.

(c) Is $\mathcal{R}$ symmetric? Is it antisymmetric? Explain.

11. Let S be a set that contains at least two elements a and b. Let A be the power set of S. Determine which of the properties—reflexivity, symmetry, antisymmetry, transitivity—each of the following binary relations $\mathcal{R}$ on A possesses. Give a proof or counterexample as appropriate.

(a) [BB] $(X, Y) \in \mathcal{R}$ if and only if $X \subseteq Y$.

(b) $(X, Y) \in \mathcal{R}$ if and only if $X \subsetneq Y$.

(c) $(X, Y) \in \mathcal{R}$ if and only if $X \cap Y = \emptyset$.

12. Let A be the set of books for sale in a certain university bookstore and assume that among these are books with the following properties.

Book	Price	Length
U	\$10	100 pages
W	\$25	125 pages
X	\$20	150 pages
Y	\$10	200 pages
Z	\$5	100 pages

(a) [BB] Suppose $(a, b) \in \mathcal{R}$ if and only if the price of book a is greater than or equal to the price of book b **and** the length of a is greater than or equal to the length of b. Is $\mathcal{R}$ reflexive? Symmetric? Antisymmetric? Transitive?

(b) Suppose $(a, b) \in \mathcal{R}$ if and only if the price of a is greater than or equal to the price of b **or** the length of a is greater than or equal to the length of b. Is $\mathcal{R}$ reflexive? Symmetric? Antisymmetric? Transitive?

13. Returning to Example 26, suppose that Mike shot 120 instead of 74 at Pippy Park (and all other scores remained unchanged). Is it now possible to retrieve all the information from the three binary relations?

2.4 Equivalence Relations

It is useful to think of a binary relation on a set A as establishing relationships between elements of A, the assertion "$(a, b) \in \mathcal{R}$" relating the elements a and b. Such relationships occur everywhere. Two people may be of the same sex, have the same color eyes, live on the same street. These three particular relationships are reflexive, symmetric, and transitive and hence *equivalence relations*.

2.4.1 DEFINITION

An *equivalence relation* on a set A is a binary relation $\mathcal{R}$ on A that is reflexive, symmetric, and transitive. ❖

Suppose A is the set of all people in the world and

$$\mathcal{R} = \{(a, b) \in A \times A \mid a \text{ and } b \text{ have the same parents}\}.$$

This relation is reflexive (every person has the same set of parents as himself/herself), symmetric (if a and b have the same parents, then so do b and a), and transitive (if a and b have the same parents, and b and c have the same parents, then a and c have the same parents) and so $\mathcal{R}$ is an equivalence relation. It may be because of examples like this that it is common to say "a is related to b," rather than "$(a, b) \in \mathcal{R}$," even for an abstract binary relation $\mathcal{R}$.

If $\mathcal{R}$ is a binary relation on a set A and $a, b \in A$, some authors use the notation $a\mathcal{R}b$ to indicate that $(a, b) \in \mathcal{R}$. In this section, we will usually write $a \sim b$ and, in the case of an equivalence relation, say "a is equivalent to b." Thus, to prove that $\mathcal{R}$ is an equivalence relation, we must prove that $\mathcal{R}$ is

reflexive: $a \sim a$ for all $a \in A$,

symmetric: if $a \in A$ and $b \in A$ and $a \sim b$, then $b \sim a$, and

transitive: if $a, b, c \in A$ and both $a \sim b$ and $b \sim c$, then $a \sim c$.

EXAMPLE 27 Let A be the set of students currently registered at the University of Southern California. For $a, b \in A$, call a and b equivalent if their student numbers have the same first two digits. Certainly, $a \sim a$ for every student a because any number has the same first two digits as itself. If $a \sim b$, the student numbers of a and b have the same first two digits, so the student numbers of b and a have the same first two digits; therefore, $b \sim a$. Finally, if $a \sim b$ and $b \sim c$, then the student numbers of a and b have the same first two digits, and the student numbers of b and c have the same first two digits, so the student numbers of a and c have the same first two digits. Since $\mathcal{R}$ is reflexive, symmetric, and transitive, $\mathcal{R}$ is an equivalence relation on A. ■

EXAMPLE 28 Let A be the set of all residents of the United States. Call a and b equivalent if a and b are residents of the same state. The student should mentally confirm that $\sim$ defines an equivalence relation. ■

EXAMPLE 29 (Congruence mod 3)[5] Define $\sim$ on the set Z of integers by $a \sim b$ if $a - b$ is divisible (evenly) by 3.[6] For any $a \in \mathsf{Z}$, $a - a = 0$ is divisible by 3 and so $a \sim a$. If $a, b \in \mathsf{Z}$ and $a \sim b$, then $a - b$ is divisible by 3, so $b - a$ (the negative of $a - b$) is also divisible by 3. Hence, $b \sim a$. Finally, if $a, b, c \in \mathsf{Z}$ with $a \sim b$ and $b \sim c$, then both $a - b$ and $b - c$ are divisible by 3, so $a - c$, being the sum of $a - b$ and $b - c$, is also divisible by 3. Thus, $\sim$ is an equivalence relation. ■

EXAMPLE 30 The relation $\leq$ on the real numbers—$a \sim b$ if and only if $a \leq b$—is not an equivalence relation on R. While it is reflexive and transitive, it is not symmetric: $4 \leq 5$ but $5 \not\leq 4$. ■

Pause 13 Let A be the set of all people. For $a, b \in A$, define $a \sim b$ if either (i) both a and b are residents of the same state of the United States or (ii) neither a nor b is a resident of any state of the United States. Does $\sim$ define an equivalence relation? ▮

Surely the three most fundamental properties of equality are

reflexivity: $a = a$ for all a;

symmetry: if $a = b$, then $b = a$; and

transitivity: if $a = b$ and $b = c$, then $a = c$.

Thus, equality is an equivalence relation on any set. For this reason, we think of equivalence as a weakening of equality. We have in mind a certain characteristic or property of elements and wish only to consider as different elements that differ with respect to this characteristic. Little children may think of their brothers and sisters as the same and other children as "different." A statistician trying to estimate the percentages of people in the world with different eye colors is only interested in eye color; for her, two people are only "different" if they have different colored eyes. All drop-off points in a given neighborhood of town may be the "same" to the driver of a newspaper truck. An equivalence relation changes our view of the universe (the underlying set A); instead of viewing it as individual elements, attention is directed to certain groups or subsets. The equivalence relation "same parents" groups people into families; "same color eyes" groups people by eye color; "same neighborhood" groups newspaper drop-off points by neighborhood.

The groups into which an equivalence relation divides the underlying set are called *equivalence classes*. The equivalence class of an element is the collection of all things related to it.

[5]This is an example of an important equivalence relation called *congruence* to which we later devote an entire section, Section 4.4.

[6]Within the context of integers, *divisible* always means *divisible evenly*, that is, with remainder 0.

2.4.2 DEFINITION If $\sim$ denotes an equivalence relation on a set A, the *equivalence class* of an element $a \in A$ is the set $\overline{a} = \{x \in A \mid x \sim a\}$ of all elements equivalent to a. The set of all equivalence classes is called the *quotient set* of A *mod* $\sim$ and denoted $A/\sim$. ❖

Since an equivalence relation is symmetric, it does not matter whether we write $x \sim a$ or $a \sim x$ in the definition of $\overline{a}$. The set of things related to a is the same as the set of things to which a is related.

For the equivalence relation in Example 27, the students who are related to a particular student x are those whose student numbers have the same first two digits as x's student number. For this equivalence relation, an equivalence class is the set of all students whose student numbers begin with the same first two digits. The set of all students has been grouped into smaller sets—the class of 99, for instance (all students whose numbers begin 99), the class of 02 (all students whose numbers begin 02), and so forth. The quotient set is the set of all equivalence classes, so it's

$$A/\sim = \{\text{class of } n \mid n = 05, 04, 03, 02, 01, 00, 99, 98, \ldots\}.$$

In Example 28, if x is a resident of some state of the United States, then the people to whom x is related are those people who reside in the same state. The residents of Colorado, for example, form one equivalence class, as do the residents of Rhode Island, the residents of Florida, and so on. The quotient set is the set of all states in the United States.

What are the equivalence classes for the equivalence relation that is congruence mod 3? What is $\overline{0}$, the equivalence class of 0, for instance? If $a \sim 0$, then $a - 0$ is divisible by 3; in other words, a is divisible by 3. Thus, $\overline{0}$ is the set of all integers that are divisible by 3. We shall denote this set $3\mathsf{Z}$. So $\overline{0} = 3\mathsf{Z}$. What is $\overline{1}$? If $a \sim 1$, then $a - 1 = 3k$ for some integer k, so $a = 3k + 1$. Thus, $\overline{1} = \{3k + 1 \mid k \in \mathsf{Z}\}$, a set we denote $3\mathsf{Z} + 1$. Similarly, $\overline{2} = \{3k + 2 \mid k \in \mathsf{Z}\} = 3\mathsf{Z} + 2$ and we have found all the equivalence classes for congruence mod 3.

Pause 14 Why? ▮

Thus the quotient set for congruence mod 3 is

$$\mathsf{Z}/\sim = \{3\mathsf{Z}, 3\mathsf{Z} + 1, 3\mathsf{Z} + 2\}.$$

In general, for given natural numbers n and r, $n\mathsf{Z} + r$ is the set of integers of the form $na + r$ for some $a \in \mathsf{Z}$:

$$n\mathsf{Z} + r = \{na + r \mid a \in \mathsf{Z}\}. \tag{5}$$

Also, we write $n\mathsf{Z}$ instead of $n\mathsf{Z} + 0$:

$$n\mathsf{Z} = \{na \mid a \in \mathsf{Z}\}. \tag{6}$$

The even integers, for instance, can be denoted $2\mathsf{Z}$.

Pause 15 What are the equivalence classes for the equivalence relation described in Pause 13? How many elements does the quotient set contain? ▮

PROBLEM 31. For (x, y) and (u, v) in R^2, define $(x, y) \sim (u, v)$ if $x^2+y^2 = u^2+v^2$. Prove that $\sim$ defines an equivalence relation on R^2 and interpret the equivalence classes geometrically.

Solution. If $(x, y) \in \mathsf{R}^2$, then $x^2 + y^2 = x^2 + y^2$, so $(x, y) \sim (x, y)$: The relation is reflexive. If $(x, y) \sim (u, v)$, then $x^2 + y^2 = u^2 + v^2$, so $u^2 + v^2 = x^2 + y^2$ and $(u, v) \sim (x, y)$: The relation is symmetric. Finally, if $(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$, then $x^2 + y^2 = u^2 + v^2$ and $u^2 + v^2 = w^2 + z^2$. Thus, $x^2 + y^2 = u^2 + v^2 = w^2 + z^2$. Since $x^2 + y^2 = w^2 + z^2$, $(x, y) \sim (w, z)$, so the relation is transitive.

The equivalence class of (a, b) is

$$\overline{(a, b)} = \{(x, y) \mid (x, y) \sim (a, b)\} = \{(x, y) \mid x^2 + y^2 = a^2 + b^2\}.$$

For example, $\overline{(1, 0)} = \{(x, y) \mid x^2 + y^2 = 1^2 + 0^2 = 1\}$, which we recognize as the graph of a circle in the Cartesian plane with center $(0, 0)$ and radius 1. For general (a, b), letting $c = a^2 + b^2$, the equivalence class $\overline{(a, b)}$ is the set of points (x, y) satisfying $x^2 + y^2 = c$. So this equivalence class is the circle with center $(0, 0)$ and radius $\sqrt{c}$. With one exception, the equivalence classes are circles with center $(0, 0)$. ▲

Pause 16 What is the exception? ▮

2.4.3 PROPOSITION Let $\sim$ denote an equivalence relation on a set A. Let $a \in A$. Then, for any $x \in A$, $x \sim a$ if and only if $\overline{x} = \overline{a}$.

Proof ($\longleftarrow$) Suppose $\overline{x} = \overline{a}$. We know $x \in \overline{x}$ because $x \sim x$, so $x \in \overline{a}$; thus, $x \sim a$. It is the implication $\rightarrow$ that is the substance of this proposition.

($\longrightarrow$) Suppose that $x \sim a$. We must prove that the two sets $\overline{x}$ and $\overline{a}$ are equal. As always, we do this by proving that each set is a subset of the other. First suppose $y \in \overline{x}$. Then $y \sim x$ and $x \sim a$, so $y \sim a$ by transitivity. Therefore, $y \in \overline{a}$, so $\overline{x} \subseteq \overline{a}$. On the other hand, suppose $y \in \overline{a}$. Then $y \sim a$. Since we also have $a \sim x$, we have both $y \sim a$ and $a \sim x$; therefore, by transitivity, $y \sim x$. Thus, $y \in \overline{x}$ and $\overline{a} \subseteq \overline{x}$. Therefore, $\overline{a} = \overline{x}$. ●

In each of the examples of equivalence relations that we have discussed in this section, different equivalence classes never overlapped. If a person is a resident of one state, he or she is not a resident of another. A student number cannot begin with 79 and also with 84. An integer that is a multiple of 3 is not of the form $3k + 1$. These examples are suggestive of a result that is true in general.

2.4.4 PROPOSITION Suppose $\sim$ denotes an equivalence relation on a set A and $a, b \in A$. Then the equivalence classes $\overline{a}$ and $\overline{b}$ are either the same or *disjoint*; that is, $\overline{a} \cap \overline{b} = \emptyset$.

Proof We prove that if $\overline{a} \neq \overline{b}$, then $\overline{a}$ and $\overline{b}$ are disjoint, and we do so by contradiction. Suppose that $\overline{a} \cap \overline{b} \neq \emptyset$. Then there is an element $x \in \overline{a} \cap \overline{b}$. Since $x \in \overline{a}$, $\overline{x} = \overline{a}$ by Proposition 2.4.3. Similarly, since $x \in \overline{b}$, we also have $\overline{x} = \overline{b}$. Thus, $\overline{a} = \overline{b}$, which is a contradiction. ●

If $\sim$ denotes an equivalence relation on A, reflexivity says that every element a in A belongs to some equivalence class, specifically to $\overline{a}$. In conjunction with Proposition 2.4.4, this observation says that the equivalence classes of any equivalence relation divide A into disjoint (that is, nonoverlapping) subsets that cover the entire set, just like the pieces of a jigsaw puzzle. We say that the equivalence classes "partition" A or "form a partition of" A. (The word *partition* is used as both verb and noun.)

2.4.5 DEFINITION

A *partition* of a set A is a collection of disjoint nonempty subsets of A whose union is A. These disjoint sets are called *cells* (or *blocks*). The cells are said to *partition* A. ❖

EXAMPLE 32

- Canada is partitioned into ten provinces and three territories.[7]
- Students are partitioned into groups according to the first two digits of their student numbers.
- The human race is partitioned into groups by eye color.
- A deck of playing cards is partitioned into four suits.
- If $A = \{a, b, c, d, e, f, x\}$, then $\{\{a, b\}, \{c, d, e\}, \{f\}, \{x\}\}$ is a partition of A. So is $\{\{a, x\}, \{b, d, e, f\}, \{c\}\}$. ■

We have seen that the equivalence classes of an equivalence relation on a set A are disjoint sets whose union is A; each element $a \in A$ is in precisely one equivalence class, $\overline{a}$. Thus, we have the following basic theorem about equivalence relations.

2.4.6 THEOREM

The equivalence classes associated with an equivalence relation on a set A form a partition of A.

Not only does an equivalence relation determine a partition, but, conversely, any partition of a set A determines an equivalence relation, specifically, that equivalence relation whose equivalence classes are the cells of the partition. The partition of the integers into "evens" and "odds" corresponds to the equivalence relation that says two integers are equivalent if and only if they are both even or both odd. The partition

$$\{\{a, g\}, \{b, d, e, f\}, \{c\}\}$$

of the set $\{a, b, c, d, e, f, g\}$ corresponds to the equivalence relation whose equivalence classes are $\{a, g\}$, $\{b, d, e, f\}$ and $\{c\}$, that is, to the equivalence relation described in the figure, where a cross in row x and column y is used to indicate $x \sim y$.

	a	g	b	d	e	f	c
a	×	×					
g	×	×					
b			×	×	×	×	
d			×	×	×	×	
e			×	×	×	×	
f			×	×	×	×	
c							×

Pause 17

The suits "heart," "diamond," "club," "spade" partition a standard deck of playing cards. Describe the corresponding equivalence relation on a deck of cards. ▮

The correspondence between equivalence relations and partitions provides a simple way to exhibit equivalence relations on small sets. For example, the equivalence relation defined in Fig. 2.3 can also be described by listing its equivalence classes: $\{a, b\}$ and $\{c\}$.

[7]Nunavut, created from the eastern half of the former Northwest Territories, joined Canada as a third territory on April 1, 1999.

	a	b	c
a	×	×	
b	×	×	
c			×

Figure 2.3 An equivalence relation with two equivalence classes, $\{a, b\}$ and $\{c\}$.

Answers to Pauses

13. It sure does. First, every person is either a resident of the same state in the United States as himself/herself or not a resident of any U.S. state, so $a \sim a$ for all $a \in A$: $\sim$ is reflexive. Second, if $a, b \in A$ and $a \sim b$, then either a and b are residents of the same U.S. state (in which case, so are b and a) or else neither a nor b is a resident of any state in the United States (in which case, neither is b or a). Thus, $b \sim a$: $\sim$ is symmetric. Finally, suppose $a \sim b$ and $b \sim c$. Then either a and b are residents of the same U.S. state or neither is a resident of any U.S. state, and the same holds true for b and c. It follows that either all three of a, b, and c live in the same U.S. state, or none is a resident of a U.S. state. Thus, $a \sim c$: $\sim$ is transitive as well.

14. The equivalence class of $\underline{3}$ is $\{3k + 3 \mid k \in \mathsf{Z}\}$, but this is just the set $3\mathsf{Z}$ of multiples of 3. Thus $\overline{3} = \overline{0}$. The equivalence class of 4 is $\{3k + 4 \mid k \in \mathsf{Z}\}$, but $\underline{3k + 4} = 3(k + 1) + 1$, so this set is just $3\mathsf{Z} + 1$, the equivalence class of 1: $\overline{4} = \overline{1}$. In general, the equivalence class of an integer r is $3\mathsf{Z}$ if r is a multiple of 3, $3\mathsf{Z} + 1$ if r is of the form $3a + 1$, and $3\mathsf{Z} + 2$ if r is of the form $3a + 2$. Since every integer r is either a multiple of 3, or $3a + 1$, or $3a + 2$ for some a, the only equivalence classes are $3\mathsf{Z}$, $3\mathsf{Z} + 1$, and $3\mathsf{Z} + 2$.

15. The equivalence class of a consists of those people equivalent to a in the sense of $\sim$. If a does not live in any state of the United States, the equivalence class of a consists of all those people who also live outside any U.S. state. If a does live in a U.S. state, the equivalence class of a consists of those people who live in the same state. The quotient set has 51 elements, consisting of the residents of the 50 U.S. states and the set of people who do not live in any U.S. state.

16. The equivalence class of $(0, 0)$ is the set $\{(0, 0)\}$ whose only element is the single point $(0, 0)$.

17. Two cards are equivalent if and only if they have the same suit. ∎

True/False Questions

(Answers can be found in the back of the book.)

1. "$\iff$" defines an equivalence relation.
2. An equivalence relation on a set A is a binary relation $\mathcal{R}$ on A that is reflexive, symmetric, and transitive.
3. Define a relation $\sim$ on the set of all people in the world by $a \sim b$ if a and b were born in the same year. Then $\sim$ is an equivalence relation.
4. Define a relation $\sim$ on the set of all people in the world by $a \sim b$ if a and b were born within a year of each other. Then $\sim$ is an equivalence relation.
5. If $\sim$ is not an equivalence relation on a set A, then there must exist $a, b \in A$ with $a \sim b$ and $b \not\sim a$.

6. If $\sim$ is an equivalence relation on a set A, the equivalence class of an element $a \in A$ is the set $\overline{a} = \{x \in A \mid a \sim x\}$.
7. $3\mathsf{Z}$ is the set of odd integers.
8. $2\mathsf{Z} + 3$ is the set of odd integers.
9. If $\sim$ is an equivalence relation on a set A and $\overline{x} \neq \overline{a}$, then $x \not\sim a$.
10. If $\sim$ is an equivalence relation on a set A and $a \not\sim b$, then $\overline{a} \cap \overline{b} \neq \emptyset$.
11. The natural numbers can be partitioned into even numbers and odd numbers.

Exercises

The answers to exercises marked [BB] can be found in the ***Back of the Book.***

1. Let A be the set of all citizens of New York City. For $a, b \in A$, define $a \sim b$ if and only if
   - neither a nor b have a cell phone, or
   - both a and b have cell phones in the same exchange (that is, the first three digits of each phone number are the same).

   Show that $\sim$ defines an equivalence relation on A and find the corresponding equivalence classes.
2. Explain why each of the following binary relations on $S = \{1, 2, 3\}$ is not an equivalence relation on S.
   (a) [BB] $\mathcal{R} = \{(1, 1), (1, 2), (3, 2), (3, 3), (2, 3), (2, 1)\}$
   (b) $\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 1), (1, 3)\}$
   (c)

	1	2	3
1	×	×	×
2	×	×	
3			×

3. [BB] The sets $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$ are the equivalence classes for a well-known equivalence relation on the set $S = \{1, 2, 3, 4, 5\}$. What is the usual name for this equivalence relation?
4. Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $S = \mathcal{P}(A)$, the power set of A.
   (a) For $a, b \in S$, define $a \sim b$ if a and b have the same number of elements. Prove that $\sim$ defines an equivalence relation on S.
   (b) How many equivalence classes are there? List one element from each equivalence class.
5. [BB] For $a, b \in \mathsf{R} \setminus \{0\}$, define $a \sim b$ if and only if $\frac{a}{b} \in \mathsf{Q}$.
   (a) Prove that $\sim$ is an equivalence relation.
   (b) Find the equivalence class of 1.
   (c) Show that $\overline{\sqrt{3}} = \overline{\sqrt{12}}$.
6. For natural numbers a and b, define $a \sim b$ if and only if $a^2 + b$ is even. Prove that $\sim$ defines an equivalence relation on N and find the quotient set determined by $\sim$.
7. [BB] For $a, b \in \mathsf{R}$, define $a \sim b$ if and only if $a - b \in \mathsf{Z}$.
   (a) Prove that $\sim$ defines an equivalence relation on Z.
   (b) What is the equivalence class of 5? What is the equivalence class of $5\frac{1}{2}$?
   (c) What is the quotient set determined by this equivalence relation?
8. [BB] For integers a, b, define $a \sim b$ if and only if $2a + 3b = 5n$ for some integer n. Show that $\sim$ defines an equivalence relation on Z.
9. Define $\sim$ on Z by $a \sim b$ if and only if $3a + b$ is a multiple of 4.
   (a) Prove that $\sim$ defines an equivalence relation.
   (b) Find the equivalence class of 0.
   (c) Find the equivalence class of 2.
   (d) Make a guess about the quotient set.
10. For integers a and b, define $a \sim b$ if $3a + 4b = 7n$ for some integer n.
    (a) Prove that $\sim$ defines an equivalence relation.
    (b) Find the equivalence class of 0.
11. [BB] For $a, b \in \mathsf{Z} \setminus \{0\}$, define $a \sim b$ if and only if $ab > 0$.
    (a) Prove that $\sim$ defines an equivalence relation on Z.
    (b) What is the equivalence class of 5? What's the equivalence class of -5?
    (c) What is the partition of $\mathsf{Z} \setminus \{0\}$ determined by this equivalence relation?
12. For $a, b \in \mathsf{Z}$, define $a \sim b$ if and only if $a^2 - b^2$ is divisible by 3.
    (a) [BB] Prove that $\sim$ defines an equivalence relation on Z.
    (b) What is $\overline{0}$? What is $\overline{1}$?
    (c) What is the partition of Z determined by this equivalence relation?
13. Determine, with reasons, whether or not each of the following defines an equivalence relation on the set A.
    (a) [BB] A is the set of all triangles in the plane; $a \sim b$ if and only if a and b are congruent.
    (b) A is the set of all circles in the plane; $a \sim b$ if and only if a and b have the same center.
    (c) A is the set of all straight lines in the plane; $a \sim b$ if and only if a is parallel to b.
    (d) A is the set of all lines in the plane; $a \sim b$ if and only if a is perpendicular to b.
14. List the pairs in the equivalence relation associated with each of the following partitions of $A = \{1, 2, 3, 4, 5\}$.

(a) [BB] $\{\{1, 2\}, \{3, 4, 5\}\}$
(b) $\{\{1\}, \{2\}, \{3, 4\}, \{5\}\}$ (c) $\{\{1, 2, 3, 4, 5\}\}$

15. (a) List all the equivalence relations on the set $\{a\}$. How many are there altogether?
(b) Repeat (a) for the set $\{a, b\}$.
(c) [BB] Repeat (a) for the set $\{a, b, c\}$.
(d) Repeat (a) for the set $\{a, b, c, d\}$.
(Remark: The number of partitions of a set of n elements grows rather rapidly. There are 52 partitions of a set of five elements, 203 partitions of a set of six elements, and 877 partitions of a set of seven elements.)

16. Define $\sim$ on R^2 by $(x, y) \sim (u, v)$ if and only if $x - y = u - v$.
(a) [BB] Criticize and then correct the following "proof" that $\sim$ is reflexive.
"If $(x, y) \sim (x, y)$, then $x - y = x - y$, which is true."
(b) What is wrong with the following interpretation of symmetry in this situation?
"If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$."
Write a correct statement of the symmetric property (as it applies to the relation $\sim$ in this exercise).
(c) Criticize and then correct the following "proof" that $\sim$ is symmetric.
"$(x, y) \sim (u, v)$ if $x - y = u - v$. Then $u - v = x - y$. So $(u, v) \sim (x, y)$."
(d) Criticize and correct the following "proof" of transitivity.
"$(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$. Then $u - v = w - z$, so if $x - y = u - v$, then $x - y = w - z$. So $(x, y) \sim (w, z)$."
(e) Why does $\sim$ define an equivalence relation on R^2?
(f) Determine the equivalence classes of $(0, 0)$ and $(2, 3)$ and describe these geometrically.

17. [BB] For (x, y) and $(u, v) \in \mathsf{R}^2$ define $(x, y) \sim (u, v)$ if and only if $x^2 - y^2 = u^2 - v^2$. Prove that $\sim$ defines an equivalence relation on R^2. Describe geometrically the equivalence class of $(0, 0)$. Describe geometrically the equivalence class of $(1, 0)$.

18. Determine which of the following define equivalence relations in R^2. For those that do, give a geometrical interpretation of the quotient set.
(a) $(a, b) \sim (c, d)$ if and only if $a + 2b = c + 2d$.
(b) $(a, b) \sim (c, d)$ if and only if $ab = cd$.
(c) $(a, b) \sim (c, d)$ if and only if $a^2 + b = c + d^2$.
(d) $(a, b) \sim (c, d)$ if and only if $a = c$.
(e) $(a, b) \sim (c, d)$ if and only if $ab = c^2$.

19. Let $\overline{a}$ and $\overline{b}$ be two equivalence classes of an equivalence relation. According to Proposition 2.4.4, "if $\overline{a} \neq \overline{b}$, then $\overline{a}$ and $\overline{b}$ are disjoint."
(a) State the converse of the quoted assertion.
(b) Is the converse true? Justify your answer.

20. Let $\sim$ denote an equivalence relation on a set A. Assume $a, b, c, d \in A$ are such that $a \in \overline{b}$, $c \in \overline{d}$, and $d \in \overline{b}$. Prove that $\overline{a} = \overline{c}$.

21. [BB] Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For $a, b \in A$, define $a \sim b$ if and only if ab is a perfect square (that is, the square of an integer).
(a) What are the ordered pairs in this relation?
(b) For each $a \in A$, find $\overline{a} = \{x \in A \mid x \sim a\}$.
(c) Explain why $\sim$ defines an equivalence relation on A.

22. [BB] Let A be the set of all natural numbers and $\sim$ be as in Exercise 21. Show that $\sim$ defines an equivalence relation on A.

23. Repeat Exercise 21 for $A = \{1, 2, 3, 4, 5, 6, 7\}$ and the relation on A defined by $a \sim b$ if and only if $\frac{a}{b}$ is a power of 2, that is, $\frac{a}{b} = 2^t$ for some integer t, positive, negative, or zero.

24. Let A be the set of all natural numbers and $\sim$ be as in Exercise 23. Show that $\sim$ defines an equivalence relation on A.

25. Let $\mathcal{R}$ be an equivalence relation on a set S and let $\{S_1, S_2, \ldots, S_t\}$ be a collection of subsets of S with the property that $(a, b) \in \mathcal{R}$ if and only if a and b are elements of the same set S_i, for some i. Suppose that, for each i, $S_i \not\subseteq \bigcup_{j \neq i} S_j$. Prove that $\{S_1, S_2, \ldots, S_t\}$ is a partition of S.

2.5 Partial Orders

In the previous section, we defined an equivalence relation as a binary relation that possesses the three fundamental properties of equality—reflexivity, symmetry, transitivity. We mentioned that we view equivalence as a weak form of equality and employed a symbol, $\sim$, suggesting "equals." In an analogous manner, in this section we focus on three fundamental properties of the order relation $\leq$ on the real numbers—reflexivity, antisymmetry, transitivity—and define a binary relation called *partial order*, which can be viewed as a weak form of $\leq$. We shall use the symbol $\preceq$ for a partial order to remind us of its connection with $\leq$ and, for the same reason, say that "a is less than or equal to b" whenever $a \preceq b$.

2.5.1 DEFINITIONS A *partial order* on a set A is a binary relation that is reflexive, antisymmetric, and transitive. A *partially ordered set, poset* for short, is a pair $(A, \preceq)$, where $\preceq$ is a partial order on the set A. ❖

Writing $a \preceq b$ to mean that (a, b) is in the relation, a partial order on A is a binary relation that is

reflexive: $a \preceq a$ for all $a \in A$,

antisymmetric: If $a, b \in A$, $a \preceq b$ and $b \preceq a$, then $a = b$, and

transitive: If $a, b, c \in A$, $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

It is convenient to use the notation $a \prec b$ (and to say "a is less than b") to signify $a \preceq b$, $a \neq b$, just as we use $a < b$ to mean $a \leq b$, $a \neq b$. Similarly, the meanings of $a \succeq b$ and $a \succ b$ should be apparent.

There is little purpose in making a definition unless there is at hand a variety of examples that fit the definition. Here then are a few examples of partial orders.

EXAMPLE 33

- The binary relation $\leq$ on the real numbers (or on any subset of the real numbers) is a partial order because $a \leq a$ for all $a \in \mathsf{R}$ (reflexivity), $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry), and $a \leq b$, $b \leq c$ implies $a \leq c$ (transitivity).
- For any set S, the binary relation $\subseteq$ on the power set $\mathcal{P}(S)$ of S is a partial order because $X \subseteq X$ for any $X \in \mathcal{P}(S)$ (reflexivity), $X \subseteq Y$, $Y \subseteq X$ for $X, Y \in \mathcal{P}(S)$ implies $X = Y$ (antisymmetry), and $X \subseteq Y$, $Y \subseteq Z$ for $X, Y, Z \in \mathcal{P}(S)$ implies $X \subseteq Z$ (transitivity). ■

EXAMPLE 34 **(Lexicographic Ordering)** Suppose we have some alphabet of symbols (perhaps the English alphabet) that is partially ordered by some relation $\preceq$. By *word* in this context, we mean any string of letters from this alphabet, not necessarily real words. For "words" $a = a_1a_2 \cdots a_n$ and $b = b_1b_2 \cdots b_m$, define $a \preceq b$ if

- a and b are identical, or
- $a_i \preceq b_i$ in the alphabet at the first position i where the words differ, or
- $n < m$ and $a_i = b_i$ for $i = 1, \ldots, n$. (This is the situation where word a, which is shorter than b, forms the initial sequence of letters in b.)

This ordering of words is called *lexicographic* because, when the basic alphabet is the English alphabet, it is precisely how words are ordered in a dictionary; car $\preceq$ cat $\preceq$ catalog. ■

The adjective *partial*, as in "partial order," draws our attention to the fact that the definition does not require that every pair of elements be *comparable*, in the following sense.

2.5.2 DEFINITION If $(A, \preceq)$ is a partially ordered set, elements a and b of A are said to be *comparable* if and only if either $a \preceq b$ or $b \preceq a$. ❖

If X and Y are subsets of a set S, it need not be the case that $X \subseteq Y$ or $Y \subseteq X$; for example, $\{a\}$ and $\{b, c\}$ are not comparable.

2.5.3 DEFINITION If $\preceq$ is a partial order on a set A and every two elements of A are comparable, then $\preceq$ is called a *total order* and the pair $(A, \preceq)$ is called a *totally ordered set.* ❖

The real numbers are totally ordered by $\leq$ because, for every pair a, b of real numbers, either $a \leq b$ or $b \leq a$. On the other hand, the set of sets, $\{\{a\}, \{b\}, \{c\}, \{a, c\}\}$ is not totally ordered by $\subseteq$ since neither $\{a\} \subseteq \{b\}$ nor $\{b\} \subseteq \{a\}$.

Pause 18 Is lexicographic order on a set of words (in the usual sense) a total order? ∎

Partial orders are often pictured by means of a diagram named after Helmut Hasse (1898–1979), for many years professor of mathematics at Göttingen.[8] In the *Hasse diagram* of a partially ordered set $(A, \preceq)$,

- there is a dot (or vertex) associated with each element of A;
- if $a \preceq b$, then the dot for b is positioned higher than the dot for a; and
- if $a \prec b$ and there is no intermediate c with $a \prec c \prec b$, then a line is drawn from a to b. (In this case, we say that the element b *covers* a.)

The effect of the last property here is to remove redundant lines. Two Hasse diagrams are shown in Fig. 2.4. The reader should appreciate that these would be unnecessarily complicated were we to draw all lines from a to b whenever $a \preceq b$ instead of just those lines where b covers a. No knowledge of the partial order is lost by this convention: After all, if $a \preceq b$ and $b \preceq c$, then (by transitivity) $a \preceq c$; so if there is a line from a to b and a line from b to c, then we can correctly infer that $a \preceq c$, from the diagram. For example, in the diagram on the left, we can infer that $1 \preceq 3$ since $1 \preceq 2$ and $2 \preceq 3$. In the diagram on the right, we similarly infer that $\{b\} \preceq \{a, b, c\}$.

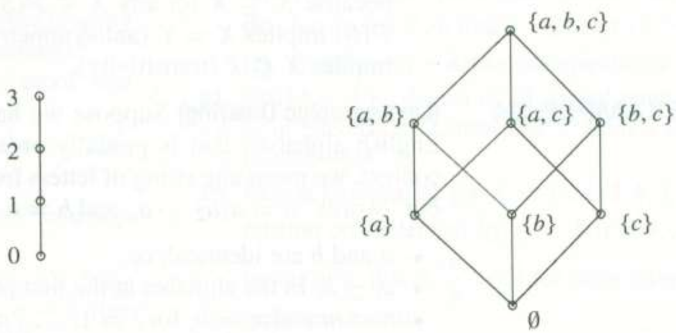


Figure 2.4 The Hasse diagrams for $(\{0, 1, 2, 3\}, \leq)$ and $(\mathcal{P}(\{a, b, c\}), \subseteq)$.

Pause 19 Suppose that in some Hasse diagram a vertex c is "above" another vertex a, but there is no line from a to c. Is it the case that $a \preceq c$? Explain. ∎

2.5.4 DEFINITIONS An element a of a poset $(A, \preceq)$ is *maximum* if and only if $b \preceq a$ for every $b \in A$ and *minimum* if and only if $a \preceq b$ for every $b \in A$. ❖

In the poset $(\mathcal{P}(\{a, b, c\}), \subseteq)$, $\emptyset$ is a minimum element and the set $\{a, b, c\}$ a maximum element. In the poset $\{\{a\}, \{b\}, \{c\}, \{a, c\}\}$ (with respect to $\subseteq$), there is neither a maximum nor a minimum because, for each of the elements $\{a\}$, $\{b\}$, $\{c\}$, and $\{a, c\}$, there is another of these with which it is not comparable.

If a poset has a maximum element, then this element is unique; similarly, a poset can have at most one minimum. (See Exercise 11.)

[8] There is a fascinating account by S. L. Segal of the ambiguous position in which Hasse found himself during the Nazi period. The article, entitled "Helmut Hasse in 1934," appears in *Historia Mathematica* 7 (1980), 45–56.

One must be careful to distinguish between maximum and **maximal** elements and between minimum and **minimal** elements.

2.5.5 DEFINITIONS

An element a of a poset A is *maximal* if and only if,

$$\text{if } b \in A \text{ and } a \preceq b, \text{ then } b = a$$

and *minimal* if and only if,

$$\text{if } b \in A \text{ and } b \preceq a, \text{ then } b = a.$$ ❖

Thus, a **maximum** element is "bigger" (in the sense of $\preceq$) than every other element in the set, while a **maximal** element is one that is not less than any other. Considering again the poset $\{\{a\}, \{b\}, \{c\}, \{a, c\}\}$, while there is neither a maximum nor a minimum, each of $\{a\}$, $\{b\}$, and $\{c\}$ is minimal, while both $\{b\}$ and $\{a, c\}$ are maximal.

Pause 20

What, if any, are the maximum, minimum, maximal, and minimal elements in the poset whose Hasse diagram appears in Fig. 2.5? ▮

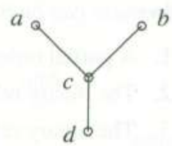


Figure 2.5

2.5.6 DEFINITIONS

Let $(A, \preceq)$ be a poset. An element g is a *greatest lower bound* (abbreviated *glb*) of elements $a, b \in A$ if and only if

1. $g \preceq a$, $g \preceq b$, and
2. if $c \preceq a$ and $c \preceq b$ for some $c \in A$, then $c \preceq g$.

Elements a and b can have at most one glb (see Exercise 14). When this element exists, it is denoted $a \wedge b$, pronounced "a meet b."

An element ℓ is a *least upper bound* (abbreviated *lub*) of a and b if

1. $a \preceq \ell$, $b \preceq \ell$, and
2. if $a \preceq c$, $b \preceq c$ for some $c \in A$, then $\ell \preceq c$.

As with greatest lower bounds, a least upper bound is unique if it exists. The lub of a and b is denoted $a \vee b$, "a join b," if there is such an element. ❖

EXAMPLE 35

- In the poset $(\mathbf{R}, \leq)$, the glb of two real numbers is the smaller of the two and the lub the larger.
- In the poset $(\mathcal{P}(S), \subseteq)$ of subsets of a set S, $A \wedge B = A \cap B$ and $A \vee B = A \cup B$. (See Exercise 12.) Remembering that $\vee$ means $\cup$ and $\wedge$ means $\cap$ in a poset of sets provides a good way to avoid confusing the symbols $\vee$ and $\wedge$ in a general poset. ■

Pause 21

With reference to Fig. 2.5, find $a \vee b$, $a \wedge b$, $b \vee d$, and $b \wedge d$, if these exist. ▮

2.5.7 DEFINITION

A poset $(A, \preceq)$ in which every two elements have a greatest lower bound in A and a least upper bound in A is called a *lattice*. ❖

EXAMPLE 36 The posets described in Examples 35 are both lattices.

Answers to Pauses

18. Sure it is; otherwise, it would be awfully hard to use a dictionary.

19. The answer is "not necessarily." We can conclude $a \preceq c$ **only** if there is a sequence of intermediate vertices between a and c with lines between each adjacent pair. Look at the Hasse diagram $(\mathcal{P}(\{a, b, c\}), \subseteq)$ in Fig. 2.4. Here we have $\{b, c\}$ above $\{a\}$, but $\{a\} \not\preceq \{b, c\}$ because these elements are incomparable. On the other hand, $\{a, b, c\}$ is above $\{a\}$ and we can infer that $\{a\} \preceq \{a, b, c\}$ because, for the intermediate vertex $\{a, c\}$, we have upward directed lines from $\{a\}$ to $\{a, c\}$ and from $\{a, c\}$ to $\{a, b, c\}$.

20. There is no maximum, but a and b are maximal; d is both minimal and a minimum.

21. $a \vee b$ does not exist; $a \wedge b = c$; $b \vee d = b$; $b \wedge d = d$.

True/False Questions

(Answers can be found in the back of the book.)

1. A partial order on a set A is a reflexive, antisymmetric, transitive relation on A.
2. The binary relation "$<$" on the set of real numbers is a partial order.
3. The binary relation "$\geq$" on the set of real numbers is a total order.
4. If a set A has more than one element, a total order on A cannot be an equivalence relation.
5. Hasse diagrams are used to identify the equivalence class of a partial order.
6. In a totally ordered set, every maximal element is maximum.
7. If, in a partially ordered set A, every minimal element is minimum, then any two elements of A must be comparable.
8. If a and b are distinct elements of a poset A, then $a \vee b \neq a \wedge b$ (assuming both elements exist).
9. If $a \vee b = a \wedge b$ for elements a, b in a poset A, then $a = b$.
10. The statement in Question 9 is the contrapositive of the statement in Question 8.

Exercises

*The answers to exercises marked [BB] can be found in the **B**ack of the **B**ook.*

1. Determine whether each of the following relations is a partial order and state whether each partial order is a total order.

(a) [BB] For $a, b \in \mathsf{R}$, $a \preceq b$ means $a \geq b$.

(b) [BB] For $a, b \in \mathsf{R}$, $a \preceq b$ means $a < b$.

(c) $(\mathsf{R}, \preceq)$, where $a \preceq b$ means $a^2 \leq b^2$.

(d) $(\mathsf{N} \times \mathsf{N}, \preceq)$, where $(a, b) \preceq (c, d)$ if and only if $a \leq c$.

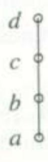
(e) $(\mathsf{N} \times \mathsf{N}, \preceq)$, where $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $b \geq d$.

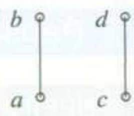
(f) $(\mathcal{W}, \preceq)$, where $\mathcal{W}$ is the set of all strings of letters from the alphabet ("words" real or imaginary), and $w_1 \preceq w_2$ if and only if w_1 has length not exceeding the length of w_2. (Length means number of letters.)

2. (a) [BB] List the elements of the set $\{11, 1010, 100, 1, 101, 111, 110, 1001, 10, 1000\}$ in lexicographic order, given $0 \preceq 1$.

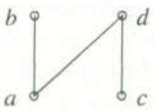
(b) Repeat part (a) assuming $1 \preceq 0$.

3. [BB; (a), (b)] List all pairs (x, y) with $x \prec y$ in the partial orders described by each of the following Hasse diagrams.

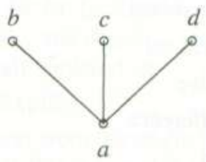
(a) 

(b) 

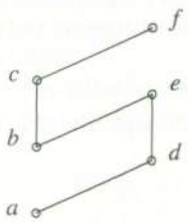
(c)



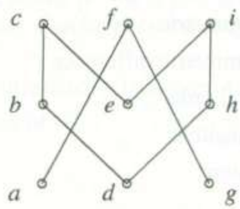
(d)



(e)



(f)



4. [BB; (a), (b)] List all minimal, maximal, minimum, and maximum elements for each of the partial orders described in Exercise 3.

5. Draw the Hasse diagrams for each of the following partial orders.
   (a) $(\{1, 2, 3, 4, 5, 6\}, \le)$
   (b) $(\{\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, c\}, \{c, d\}\}, \subseteq)$

6. List all minimal, minimum, maximal, and maximum elements for each of the partial orders in Exercise 5.

7. [BB] In the poset $(\mathcal{P}(S), \subseteq)$ of subsets of a set S, under what conditions does one set B cover another set A?

✍ 8. Learn what you can about Helmut Hasse and write a short biographical note about this person, in good clear English of course!

9. (a) [BB] Prove that any finite (nonempty) poset must contain maximal and minimal elements.
   (b) Is the result of (a) true in general for posets of arbitrary size? Explain.

10. (a) Let $A = \mathbf{Z}^2$ and, for $\mathsf{a} = (a_1, a_2)$ and $\mathsf{b} = (b_1, b_2)$ in A, define $\mathsf{a} \preceq \mathsf{b}$ if and only if $a_1 \le b_1$ and $a_1 + a_2 \le b_1 + b_2$. Prove that $\preceq$ is a partial order on A. Is this partial order a total order? Justify your answer with a proof or a counterexample.
    (b) Generalize the result of part (a) by defining a partial order on the set Z^n of n-tuples of integers. (No proof is required.)

11. (a) [BB] Prove that a poset has at most one maximum element.
    (b) Prove that a poset has at most one minimum element.

12. Let S be a nonempty set and let A and B be elements of the power set of S. In the partially ordered set $(\mathcal{P}(S), \subseteq)$,
    (a) [BB] prove that $A \wedge B = A \cap B$;
    (b) prove that $A \vee B = A \cup B$.

13. Let a and b be two elements of a poset $(A, \preceq)$ with $a \preceq b$.
    (a) [BB] Show that $a \vee b$ exists, find this element, and explain your answer.
    (b) Show that $a \wedge b$ exists, find this element, and explain your answer.

14. (a) [BB] Prove that a glb of two elements in a poset $(A, \preceq)$ is unique whenever it exists.
    (b) Prove that a lub of two elements in a poset $(A, \preceq)$ is unique whenever it exists.

15. (a) If a and b are two elements of a partially ordered set $(A, \preceq)$, the concepts

$$\max(a, b) = \begin{cases} a & \text{if } b \preceq a \\ b & \text{if } a \preceq b \end{cases}$$

and

$$\min(a, b) = \begin{cases} a & \text{if } a \preceq b \\ b & \text{if } b \preceq a \end{cases}$$

do not make sense unless the poset is totally ordered. Explain.
    (b) Show that any totally ordered set is a lattice.

16. (a) [BB] Give an example of a partially ordered set that has a maximum and a minimum element but is **not** totally ordered.
    (b) Give an example of a totally ordered set that has no maximum or minimum elements.

17. Prove that in a totally ordered set any maximal element is a maximum.

18. Suppose $(A, \preceq)$ is a poset containing a minimum element a.
    (a) [BB] Prove that a is minimal.
    (b) Prove that a is the only minimal element.

Key Terms & Ideas

Here are some technical words and phrases that were used in this chapter. Do you know the meaning of each? If you're not sure, check the glossary or index at the back of the book.

antisymmetric
binary relation
Cartesian product
cell (of a partition)
comparable
complement
direct product
disjoint
equivalence class
equivalence relation
greatest lower bound
intersection