1

Logic

1.1 Truth Tables

In the last chapter, we presented a rather informal introduction to some of the basic concepts of mathematical logic. There are times, however, when a more formal approach can be useful. We begin to look at such an approach now.

Let p and q be statements. For us, remember that *statement* means a statement of fact that is either true or false. The compound statements "p or q" and "p and q," which were introduced in Section 0.1, will henceforth be written "$p \vee q$" and "$p \wedge q$," respectively.

$$p \vee q: \quad p \text{ or } q$$
$$p \wedge q: \quad p \text{ and } q.$$

The way in which the truth values of these compound statements depend on those of p and q can be neatly summarized by tables called *truth tables*. Truth tables for $p \vee q$ and $p \wedge q$ are shown in Fig. 1.1.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 1.1 Truth tables for $p \vee q$ (p or q) and $p \wedge q$ (p and q).

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	$\neg p$
T	F
F	T

Figure 1.2 Truth tables for $p \to q$ (p implies q) and $\neg p$ (not p).

In each case, the first two columns show all possible truth values for p and q—each is either true (T) or false (F)—and the third column shows the corresponding truth value for the compound statement.

The truth table for the implication $p \to q$, introduced in Section 0.1, is shown on the left in Fig. 1.2. On the right, we show the particularly simple truth table for "$\neg p$," the negation of p

Truth tables for more complicated compound statements can be constructed using the truth tables we have seen so far. For example, the statement "$p \leftrightarrow q$," defined in Section 0.1 as "$(p \to q)$ and $(q \to p)$," is "$(p \to q) \wedge (q \to p)$."

The truth values for $p \to q$ and $q \to p$ are shown in Fig. 1.3. Focusing on columns 3 and 4 and remembering that $r \wedge s$ is true if and only if both r and s are true—see the truth table for "$\wedge$" shown in Fig. 1.1—we obtain the truth table for $(p \to q) \wedge (q \to p)$, that is, for $p \leftrightarrow q$.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Figure 1.3 The truth table for $p \leftrightarrow q$ (p if and only if q).

The first two columns and the last column are the most important, of course, so in future applications we remember $p \leftrightarrow q$ with the simple truth table shown in Fig. 1.4.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 1.4 The truth table for $p \leftrightarrow q$.

Here is another demonstration of how to analyze complex compound statements with truth tables.

EXAMPLE 1 Suppose we want the truth table for $p \to \neg(q \vee p)$.

p	q	$p \vee q$	$\neg(p \vee q)$	$p \to \neg(p \vee q)$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	T
F	F	F	T	T

Although the answer is presented as a single truth table, the procedure is to construct appropriate columns one by one until the answer is reached. Here, columns 1 and 2 are used to form column 3 ($q \vee p$). Then column 4 follows from column 3 and, finally, columns 1 and 4 are used to construct column 5, using the truth table for an implication in Fig. 1.2. ■

When three statements p, q, and r are involved, eight rows are required in a truth table since it is necessary to consider the two possible truth values for r for each of the four possible truth values of p and q.

PROBLEM 2. Construct a truth table for $(p \vee q) \leftrightarrow [((\neg p) \wedge r) \to (q \wedge r)]$.

Solution. The truth table for $p \leftrightarrow q$ shows false exactly when the values of p, q are

T, F. We use this idea here for the last column.

p	q	r	$\neg p$	$(\neg p) \wedge r$	$q \wedge r$	$((\neg p) \wedge r) \rightarrow (q \wedge r)$	$p \vee q$	$(p \vee q) \leftrightarrow [((\neg p) \wedge r) \rightarrow (q \wedge r)]$
T	T	T	F	F	T	T	T	T
T	F	T	F	F	F	T	T	T
F	T	T	T	T	T	T	T	T
F	F	T	T	T	F	F	F	T
T	T	F	F	F	F	T	T	T
T	F	F	F	F	F	T	T	T
F	T	F	T	F	F	T	T	T
F	F	F	T	F	F	T	F	F

▲

Of course, it is only necessary to construct an entire truth table if a complete analysis of a certain compound statement is desired. We do not need to construct all 32 rows of a truth table to do the next problem.

PROBLEM 3. Find the truth value of

$$[p \rightarrow ((q \wedge (\neg r)) \vee s)] \wedge [(\neg t) \leftrightarrow (s \wedge r)],$$

where p, q, r, and s are all true, while t is false.

Solution. We evaluate the expression step by step, showing just the relevant row of the truth table.

p	q	r	s	t	$\neg r$	$q \wedge (\neg r)$	$(q \wedge (\neg r)) \vee s$
T	T	T	T	F	F	F	T

$p \rightarrow [(q \wedge (\neg r)) \vee s]$	$\neg t$	$s \wedge r$	$(\neg t) \leftrightarrow (s \wedge r)$
T	T	T	T

$[p \rightarrow [((q \wedge (\neg r)) \vee s)] \wedge [(\neg t) \leftrightarrow (s \wedge r)]$
T

The truth value is true. ▲

A notion that will be important in later sections is that of *logical equivalence*. Formally, statements $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent if they have identical truth tables.

EXAMPLE 4 In Section 0.1, we defined the *contrapositive* of the statement "$p \rightarrow q$" as the statement "$(\neg q) \rightarrow (\neg p)$." In Theorem 0.2.1, we proved that these implications are logically equivalent without actually introducing the terminology. Here is how to establish the same result using truth tables.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg q$	$\neg p$	$(\neg q) \to (\neg p)$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

■

PROBLEM 5. Show that "$\mathcal{A}: p \to (\neg q)$" and "$\mathcal{B}: \neg(p \wedge q)$" are logically equivalent.

Solution. We simply observe that the final columns of the two truth tables are identical.

p	q	$\neg q$	$p \to (\neg q)$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

p	q	$p \wedge q$	$\neg (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

▲

Pause 1

Let $\mathcal{A}: p \to (\neg q)$ and $\mathcal{B}: \neg(p \wedge q)$ as in Problem 5. Show that $\mathcal{A} \leftrightarrow \mathcal{B}$ is always true.

1.1.1 DEFINITIONS

A compound statement that is always true, regardless of the truth values assigned to its variables, is a *tautology*. A compound statement that is always false is a *contradiction*. ❖

For example, and as illustrated in PAUSE 1, statements $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent precisely when the statement $\mathcal{A} \leftrightarrow \mathcal{B}$ is a tautology. An example of a contradiction is $p \wedge (\neg p)$: This statement is always false.

p	$\neg p$	$p \wedge (\neg p)$
T	F	F
F	T	F

We shall give many other examples of logical equivalences in Section 1.2.

Answer to Pause

1. The truth table in Fig. 1.3 shows that a double implication $p \leftrightarrow q$ is true precisely when both p and q have the same truth values. Looking at the truth tables for $\mathcal{A}$ and $\mathcal{B}$ in Problem 5 we see that $\mathcal{A}$ and $\mathcal{B}$ always have the same value, so $\mathcal{A} \leftrightarrow \mathcal{B}$ is always true.

True/False Questions

(Answers can be found in the back of the book.)

1. "$p \vee q$" means "p and q."
2. A truth table based on four simple statements p, q, r, and s has 16 rows.
3. If $p \wedge q$ is true, then $p \vee q$ is also true.
4. If p and q are both false, the truth value of $(\neg p \vee \neg q) \to (p \leftrightarrow q)$ is also false.
5. If $p \to q$ is false, the truth value of $(\neg p \vee \neg q) \to (p \leftrightarrow q)$ is also false.
6. "$p \to q$" and "$q \to p$" are logically equivalent.
7. A statement and its contrapositive are logically equivalent.
8. "$(p \vee q) \to (p \to q)$" is a tautology.
9. If $\mathcal{B}$ is a tautology and $\mathcal{A}$ is a contradiction, then $(\neg\mathcal{A}) \vee \mathcal{B}$ is a tautology.
10. If $\mathcal{A}$ and $\mathcal{B}$ are both contradictions, then $\mathcal{A} \to \mathcal{B}$ is a tautology.

Exercises

*The answers to exercises marked [BB] can be found in the **Back** of the **Book**.*

1. Construct a truth table for each of the following compound statements.
   (a) [BB] $p \wedge ((\neg q) \vee p)$
   (b) $(p \wedge q) \vee ((\neg p) \to q)$
   (c) $\neg\,(p \wedge (q \vee p)) \leftrightarrow p$
   (d) [BB] $(\neg\,(p \vee (\neg q))) \wedge ((\neg p) \vee r)$
   (e) $(p \to (q \to r)) \to ((p \wedge q) \vee r)$
2. **(a)** If $p \to q$ is false, determine the truth value of $(p \wedge (\neg q)) \vee ((\neg p) \to q)$.
   (b) [BB] Is it possible to answer 2(a) if $p \to q$ is true instead of false? Why or why not?
3. [BB] Determine the truth value for

$$[p \to (q \wedge (\neg r))] \;\vee\; [r \leftrightarrow ((\neg s) \vee q)]$$

   when p, q, r, and s are all true.
4. Repeat Exercise 3 in the case where p, q, r, and s are all false.
5. **(a)** [BB] Show that $(p \wedge q) \to (p \vee q)$ is a tautology.
   (b) [BB] Show that $((\neg p) \wedge q) \wedge (p \vee (\neg q))$ is a contradiction.
6. **(a)** Show that $q \to (p \to q)$ is a tautology.
   (b) Show that $[p \wedge q] \wedge [(\neg p) \vee (\neg q)]$ is a contradiction.
7. **(a)** [BB] Show that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.
   (b) [BB] Explain in plain English why the answer to 7a makes sense.
8. Show that the statement

$$[p \vee ((\neg r) \to (\neg s))] \;\vee$$
$$[(s \to ((\neg t) \vee p)) \vee ((\neg q) \to r)]$$

   is neither a tautology nor a contradiction.
9. Given that the compound statement $\mathcal{A}$ is a contradiction, establish each of the following.
   (a) [BB] If $\mathcal{B}$ is any statement, $\mathcal{A} \to \mathcal{B}$ is a tautology.
   (b) If $\mathcal{B}$ is a tautology, $\mathcal{B} \to \mathcal{A}$ is a contradiction.
10. **(a)** Show that the statement $p \to (q \to r)$ is not logically equivalent to the statement $(p \to q) \to r$.
    (b) What can you conclude from 10a about the compound statement $[p \to (q \to r)] \leftrightarrow [(p \to q) \to r]$?
11. If p and q are statements, then the compound statement $p \veebar q$ (often called the *exclusive or*) is defined to be true if and only if exactly one of p, q is true; that is, either p is true or q is true, but not both p and q are true.
    (a) [BB] Construct a truth table for $p \veebar q$.
    (b) Construct a truth table for $(p \veebar ((\neg p) \wedge q)) \vee q$.
    (c) [BB] Show that $(p \veebar q) \to (p \vee q)$ is a tautology.
    (d) Show that $p \veebar q$ is logically equivalent to $\neg(p \leftrightarrow q)$.

1.2 The Algebra of Propositions

At the conclusion of Section 1.1, we discussed the notion of logical equivalence and noted that statements $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent precisely when the statement $\mathcal{A} \leftrightarrow \mathcal{B}$ is a tautology.

We use the notation $\mathcal{A} \iff \mathcal{B}$ to denote the fact that $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent. When this is the case, we often think of statement $\mathcal{B}$ as just a rewording of statement $\mathcal{A}$. Clearly then, it is of interest to be able to determine in an efficient manner when two statements are logically equivalent and when they are not. Truth tables will do this job for us, but, as you may already have noticed, they can become cumbersome rather easily. Another approach is first to gather together some of the fundamental examples of logically equivalent statements and then to analyze more complicated situations by showing how they reduce to these basic examples.

The word *proposition* is a synonym for *(mathematical) statement*. Just as there are rules for addition and multiplication of real numbers—commutativity and associativity, for instance—there are properties of $\wedge$ and $\vee$ that are helpful in recognizing that a given compound statement is logically equivalent to another, often more simple, one.

Some Basic Logical Equivalences

1. **Idempotence:** (i) $(p \vee p) \iff p$
   (ii) $(p \wedge p) \iff p$
2. **Commutativity:** (i) $(p \vee q) \iff (q \vee p)$
   (ii) $(p \wedge q) \iff (q \wedge p)$
3. **Associativity:** (i) $((p \vee q) \vee r) \iff (p \vee (q \vee r))$
   (ii) $((p \wedge q) \wedge r) \iff (p \wedge (q \wedge r))$
4. **Distributivity:** (i) $(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$
   (ii) $(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
5. **Double Negation:** $\neg(\neg p) \iff p$
6. **De Morgan's Laws:** (i) $\neg(p \vee q) \iff ((\neg p) \wedge (\neg q))$
   (ii) $\neg(p \wedge q) \iff ((\neg p) \vee (\neg q))$

Property 6 was discussed in a less formal manner in Section 0.1.

It is clear that any two tautologies are logically equivalent and that any two contradictions are logically equivalent. Letting **1** denote a tautology and **0** a contradiction, we can add the following properties to our list.

7. (i) $(p \vee \mathbf{1}) \iff \mathbf{1}$
   (ii) $(p \wedge \mathbf{1}) \iff p$
8. (i) $(p \vee \mathbf{0}) \iff p$
   (ii) $(p \wedge \mathbf{0}) \iff \mathbf{0}$
9. (i) $(p \vee (\neg p)) \iff \mathbf{1}$
   (ii) $(p \wedge (\neg p)) \iff \mathbf{0}$
10. (i) $\neg\mathbf{1} \iff \mathbf{0}$
    (ii) $\neg\mathbf{0} \iff \mathbf{1}$

We add three more properties.

11. $(p \to q) \iff [(\neg q) \to (\neg p)]$
12. $(p \to q) \iff [(\neg p) \vee q]$
13. $(p \leftrightarrow q) \iff [(p \to q) \wedge (q \to p)]$

Property 11 simply restates the fact, proved in Theorem 0.2.1, that an implication and its contrapositive are logically equivalent. Property 12 shows that an implication is logically equivalent to a statement that does not use the symbol $\to$. The definition of "$\leftrightarrow$" gives Property 13 immediately.

Pause 2

Given an implication $p \to q$, explain why its *converse*, $q \to p$, and its *inverse*, $[(\neg p) \to (\neg q)]$, are logically equivalent. ∎

Pause 3 Show that $[\neg(p \leftrightarrow q)] \iff [(p \wedge (\neg q)) \vee (q \wedge (\neg p))]$. ▮

PROBLEM 6. Show that $(\neg p) \to (p \to q)$ is a tautology.

Solution. Using Property 12, we have

$$\begin{aligned}[(\neg p) \to (p \to q)] &\iff [(\neg p) \to ((\neg p) \vee q)] \\ &\iff [(\neg(\neg p)) \vee ((\neg p) \vee q)] \\ &\iff p \vee [(\neg p) \vee q] \\ &\iff [p \vee (\neg p)] \vee q \iff \mathbf{1} \vee q \iff \mathbf{1}.\end{aligned}$$ ▲

In the exercises, we ask you to verify all the properties of logical equivalence that we have stated. Some are very simple. For example, to see $(p \vee p) \iff p$, we need only observe that p and $p \vee p$ have the same truth tables:

p	$p \vee p$
T	T
F	F

Others require more work. To verify the second distributive property, for example, we would construct two truth tables.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	F
F	F	T	T	F
T	T	F	T	T
T	F	F	F	F
F	T	F	T	F
F	F	F	F	F

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	F	T	F	T	T
F	T	T	F	F	F
F	F	T	F	F	F
T	T	F	T	F	T
T	F	F	F	F	F
F	T	F	F	F	F
F	F	F	F	F	F

PROBLEM 7. Simplify the statement $[\neg(p \vee q)] \vee [(\neg p) \wedge q]$.

Solution. We have

$$\begin{aligned}&[[\neg(p \vee q)] \vee [(\neg p) \wedge q]] \\ &\iff [[(\neg p) \wedge (\neg q)] \vee [(\neg p) \wedge q]] && \text{by one of the laws of De Morgan} \\ &\iff (\neg p) \wedge [(\neg q) \vee q] && \text{using a distributive law} \\ &\iff (\neg p) \wedge \mathbf{1} \iff \neg p,\end{aligned}$$

so the given statement is logically equivalent simply to $\neg p$. ▲

In Problems 6 and 7 we used, in a sneaky way, a very important principle of logic, which we now state as a theorem. The fact that we didn't really think about this at the time tells us that the theorem is easily understandable and quite painless to apply in practice.

1.2.1 THEOREM

Suppose $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent statements involving variables $p_1, p_2, \dots, p_n$. Suppose that $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ are statements. If, in $\mathcal{A}$ and $\mathcal{B}$, we replace p_1 by $\mathcal{C}_1$, p_2 by $\mathcal{C}_2$ and so on until we replace p_n by $\mathcal{C}_n$, then the resulting statements will still be logically equivalent.

Pause 4

Explain how Theorem 1.2.1 was used in Problem 7. ▮

PROBLEM 8. Show that $[(p \vee q) \vee ((q \vee (\neg r)) \wedge (p \vee r))] \iff \neg\,[(\neg p) \wedge (\neg q)]$.

Solution. Using one of the distributive laws, the left-hand side is logically equivalent to

$$[(p \vee q) \vee (q \vee (\neg r))] \wedge [(p \vee q) \vee (p \vee r)].$$

Associativity and idempotence say that the first term here, $[(p \vee q) \vee (q \vee (\neg r))]$, is logically equivalent to

$$\begin{aligned}[p \vee (q \vee (q \vee (\neg r)))] &\iff [p \vee ((q \vee q) \vee (\neg r))]\\ &\iff [p \vee (q \vee (\neg r))] \iff [(p \vee q) \vee (\neg r)],\end{aligned}$$

while the second term, $[(p \vee q) \vee (p \vee r)]$, is logically equivalent to

$$\begin{aligned}[(q \vee p) \vee (p \vee r)] &\iff [q \vee (p \vee (p \vee r))]\\ &\iff [q \vee ((p \vee p) \vee r)]\\ &\iff [q \vee (p \vee r)]\\ &\iff [(q \vee p) \vee r] \iff [(p \vee q) \vee r].\end{aligned}$$

Hence the expression on the left-hand side of the statement we are trying to establish is logically equivalent to

$$\begin{aligned}[((p \vee q) \vee (\neg r)) \wedge ((p \vee q) \vee r)] &\iff [(p \vee q) \vee ((\neg r) \wedge r)]\\ &\iff [(p \vee q) \vee \mathbf{0}] \iff p \vee q.\end{aligned}$$

But this is logically equivalent to the right-hand side of the desired statement by double negation and one of the laws of De Morgan. ▲

The next problem illustrates clearly why employing the basic logical equivalences discussed in this section is often more efficient than working simply with truth tables.

PROBLEM 9. Show that $[s \to (((\neg p) \wedge q) \wedge r)] \iff \neg[(p \vee (\neg\,(q \wedge r))) \wedge s]$.

Solution.
$$\begin{aligned}[s \to (((\neg p) \wedge q) \wedge r)] &\iff [(\neg s) \vee (((\neg p) \wedge q) \wedge r)]\\ &\iff [(\neg s) \vee ((\neg p) \wedge (q \wedge r))]\\ &\iff [(\neg s) \vee (\neg(p \vee (\neg(q \wedge r))))]\\ &\iff \neg[s \wedge (p \vee (\neg\,(q \wedge r)))]\\ &\iff \neg[(p \vee (\neg(q \wedge r))) \wedge s].\end{aligned}$$
▲

A primary application of the work in this section is reducing statements to logically equivalent simpler forms. There are times, however, when a different type of logically equivalent statement is required.

1.2.2 DEFINITION Let $n \geq 1$ be an integer and let $x_1, x_2, \ldots, x_n$ be variables. A *minterm* based on these variables is a compound statement of the form $a_1 \wedge a_2 \wedge \cdots \wedge a_n$, where each a_i is x_i or $\neg x_i$. A compound statement in $x_1, x_2, \ldots, x_n$ is said to be in *disjunctive normal form* if it looks like $y_1 \vee y_2 \vee \cdots \vee y_m$ where the statements $y_1, y_2, \ldots, y_m$ are different minterms. ❖

EXAMPLE 10 $x_1 \wedge \neg x_2 \wedge \neg x_3$ is a minterm (on variables x_1, x_2, x_3) and the statement $(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge (\neg x_2) \wedge (\neg x_3))$ is in disjunctive normal form.

$(p \wedge q) \vee (\neg p \wedge \neg q)$ is in disjunctive normal form on variables p, q, but $(p \wedge q) \vee (\neg p \wedge \neg q) \vee (p \wedge q)$ is not (because two minterms are the same). ■

EXAMPLE 11 $p \wedge (q \vee r)$ is not a minterm (because it involves the symbol $\vee$) and the statement $((p \wedge q) \vee r) \wedge ((p \wedge q) \vee (\neg q))$ is not in disjunctive normal form, one reason being that the minterms $(p \wedge q) \vee r$ and $(p \wedge q) \vee (\neg q)$ involve the symbol $\vee$. This statement is logically equivalent to $(p \wedge q) \vee (r \wedge (\neg q))$, which is still not in disjunctive normal form because the minterms, $p \wedge q$ and $r \wedge (\neg q)$, don't contain all the variables. Continuing, however, our statement is logically equivalent to

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge (\neg r)) \vee (p \wedge (\neg q) \wedge r) \vee ((\neg p) \wedge (\neg q) \wedge r),$$

which is in disjunctive normal form (on the variables p, q, r). ■

As shown in Example 11, when writing a statement in disjunctive normal form, it is very useful to note that

$$x \iff [(x \wedge y) \vee (x \wedge (\neg y))] \tag{1}$$

for any statements x and y. This follows from

$$x \iff (x \wedge \mathbf{1}) \iff [x \wedge (y \vee (\neg y))] \iff [(x \wedge y) \vee (x \wedge (\neg y))].$$

PROBLEM 12. Express $p \to (q \wedge r)$ in disjunctive normal form.

Solution. Method 1: We construct a truth table.

p	q	r	$q \wedge r$	$p \to (q \wedge r)$
T	T	T	T	T
T	F	T	F	F
F	T	T	T	T
F	F	T	F	T
T	T	F	F	F
T	F	F	F	F
F	T	F	F	T
F	F	F	F	T

Now focus attention on the rows for which the statement is true—each of these will contribute a minterm to our answer. For example, in row 1, p, q, and r are all T, so $p \wedge q \wedge r$ agrees with the T in the last column. In row 4, p is F, while q and r are both T. This gives the minterm $(\neg p) \wedge q \wedge r$. In this way, we obtain

$$(p \wedge q \wedge r) \vee ((\neg p) \wedge q \wedge r) \vee ((\neg p) \wedge q \wedge (\neg r)) \\ \vee ((\neg p) \wedge (\neg q) \wedge r) \vee ((\neg p) \wedge (\neg q) \wedge (\neg r)).$$

Method 2: We have

$$
\begin{aligned}
&[p \to (q \wedge r)]\\
&\iff [(\neg p) \vee (q \wedge r)]\\
&\iff [((\neg p) \wedge q) \vee ((\neg p) \wedge (\neg q)) \vee (q \wedge r)]\\
&\iff [((\neg p) \wedge q \wedge r) \ \vee \ ((\neg p) \wedge q \wedge (\neg r)) \ \vee \ ((\neg p) \wedge (\neg q) \wedge r)\\
&\qquad \vee ((\neg p) \wedge (\neg q) \wedge (\neg r)) \ \vee \ (p \wedge q \wedge r) \vee ((\neg p) \wedge q \wedge r)]\\
&\iff [((\neg p) \wedge q \wedge r) \ \vee \ ((\neg p) \wedge q \wedge (\neg r)) \ \vee \ ((\neg p) \wedge (\neg q) \wedge r)\\
&\qquad \vee ((\neg p) \wedge (\neg q) \wedge (\neg r)) \ \vee \ (p \wedge q \wedge r)],
\end{aligned}
$$

omitting the second occurrence of $(\neg p) \wedge q \wedge r$ at the last step. We leave it to you to decide for yourself which method you prefer. ▲

Application: Three-Way Switches

Disjunctive normal form is useful in applications of logic to computer science, particularly in the construction *logic circuits*.

Most of us can guess how an ordinary light switch works. Flipping the switch up completes a circuit and sends electricity to the light bulb, which then (if it's not broken) goes on. Flipping the switch down cuts the circuit and the light goes off. But how do three-way switches work, you know, two switches that control the same light? You flip one up and the light goes on; you flip the other up and the light goes off, and so on. Exactly how do these switches work?

We illustrate with a *logic circuit*, the key components of which are *OR-gates*, *AND-gates*, and *NOT-gates*. In diagrams, these are depicted with the standard symbols shown in Fig. 1.5. As you might guess, an AND-gate implements the logical connective $\wedge$, an OR-gate implements $\vee$, and a NOT-gate implements $\neg$.

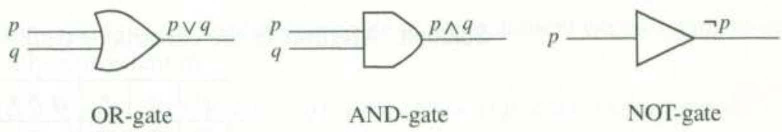

Figure 1.5

In Fig. 1.6, we show how these gates can be put together to make three-way switches work as they should. There are two inputs to the circuit, labeled p and q, corresponding to the two switches. We assign these variables the values 0 and 1 to indicate up and down, respectively. Let v denote the current reaching the light bulb, 0 indicating "no current" (so the light is off) and 1 indicating "current" (so the light is on). The table to the left shows how we would like v to depend on p and q. In particular, notice that when both switches are up, the light should be off. Look at the two rows where $v = 1$. Just as in Problem 12, we see that v is logically equivalent to $[(p \wedge (\neg q)] \vee [(\neg p) \wedge q]$, which is in disjunctive normal form. Thus, assuming we can build devices that implement $\wedge$, $\vee$, and $\neg$, we can see easily how to build three-way switches.

p	q	v
0	0	0
1	0	1
0	1	1
1	1	0

Answers to Pauses

2. This is just a restatement of Property 11, writing p instead of q and q instead of p.

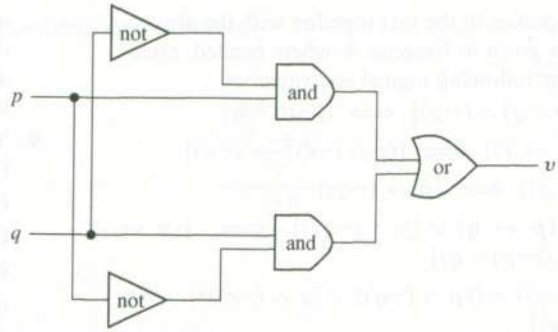

Figure 1.6 Implementation of three-way switches.

3. This could be done with truth tables. Alternatively, we note that

$$\begin{aligned}\neg(p \leftrightarrow q) &\iff [\neg((p \to q) \wedge (q \to p))]\\ &\iff \neg[((\neg p) \vee q) \wedge ((\neg q) \vee p)]\\ &\iff [\neg((\neg p) \vee q) \vee (\neg((\neg q) \vee p))]\\ &\iff [(p \wedge (\neg q)) \vee (q \wedge (\neg p))].\end{aligned}$$

4. In applying the distributive property, we are using $\neg p$, $\neg q$, and q in place of p, q, and r. Also, when applying Property 7, we use $\neg p$ instead of p.

True/False Questions

(Answers can be found in the back of the book.)

1. Two statements $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent precisely when the statement $\mathcal{A} \to \mathcal{B}$ is a tautology.
2. "$\mathcal{A} \iff \mathcal{B}$" and "$\mathcal{A} \leftrightarrow \mathcal{B}$" mean the same thing.
3. $(p \vee p) \iff (p \wedge p)$ for any statement p.
4. $((p \vee q) \wedge r) \iff (p \vee (q \wedge r))$ for any statements p, q, r.
5. $(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$ for any statements p, q, r.
6. $(\neg(p \wedge q)) \iff ((\neg p) \vee (\neg q))$ for any statements p, q.
7. If $\mathcal{A} \iff \mathcal{B}$ and $\mathcal{C}$ is any statement, then $(\mathcal{A} \to \mathcal{C}) \iff (\mathcal{B} \to \mathcal{C})$.
8. $(p \wedge q \wedge (\neg r)) \vee ((\neg p) \wedge (\neg q))$ is in disjunctive normal form.
9. $(p \wedge q \wedge (\neg r)) \vee ((\neg p) \wedge (\neg q) \wedge (\neg r))$ is in disjunctive normal form.
10. Disjunctive normal form is useful in applications of logic to computer science.

Exercises

The answers to exercises marked [BB] can be found in the Back of the Book.

1. Verify each of the 13 properties of logical equivalence that appear in this section [BB; 1,3,5,7,9,11,13].
2. (a) Show that $p \vee [\neg(p \wedge q)]$ is a tautology.
   (b) What is the negation of the statement in (a)? Show that this negation is a contradiction.
3. Simplify each of the following statements.
   (a) [BB] $(p \wedge q) \vee (\neg((\neg p) \vee q))$
   (b) $(p \vee r) \to [(q \vee (\neg r)) \to ((\neg p) \to r)]$
   (c) $[(p \to q) \vee (q \to r)] \wedge (r \to s)$
4. Using truth tables, verify the following *absorption* properties.
   (a) [BB] $(p \vee (p \wedge q)) \iff p$
   (b) $(p \wedge (p \vee q)) \iff p$

5. Using the properties in the text together with the absorption properties given in Exercise 4, where needed, establish each of the following logical equivalences.
   (a) [BB] $[(p \vee q) \wedge (\neg p)] \iff [(\neg p) \wedge q]$
   (b) $[p \to (q \to r)] \iff [(p \wedge (\neg r)) \to (\neg q)]$
   (c) $[\neg(p \leftrightarrow q)] \iff [p \leftrightarrow (\neg q)]$
   (d) [BB] $\neg[(p \leftrightarrow q) \vee (p \wedge (\neg q))] \iff [(p \leftrightarrow (\neg q)) \wedge ((\neg p) \vee q)]$
   (e) $[(p \wedge (\neg q)) \wedge ((p \wedge (\neg q)) \vee (q \wedge (\neg r)))] \iff [p \wedge (\neg q)]$
   (f) $[p \to (q \vee r)] \iff [p \wedge (\neg q)] \to r]$
   (g) $\neg(p \vee q) \vee [(\neg p) \wedge q] \iff \neg p$
6. Prove that the statements $(p \wedge (\neg q)) \to q$ and $(p \wedge (\neg q)) \to \neg p$ are logically equivalent. What simpler statement is logically equivalent to both of them?
7. Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are statements with $\mathcal{A}$ and $\mathcal{B}$ logically equivalent.
   (a) Show that $\mathcal{A} \vee \mathcal{C}$ and $\mathcal{B} \vee \mathcal{C}$ are logically equivalent.
   (b) Show that $\mathcal{A} \wedge \mathcal{C}$ and $\mathcal{B} \wedge \mathcal{C}$ are logically equivalent.
8. In Exercise 11 of Section 1.1 we defined the *exclusive or* "$p \veebar q$" to be true whenever either p or q is true, but not both. For each of the properties discussed in this section (including those of absorption given in Exercise 4) determine whether the property holds with $\veebar$ replacing $\vee$ wherever it occurs [BB; 1,3,7,9,13].
9. Which of the following are in disjunctive normal form (on the appropriate set of variables)?
   (a) $(p \vee q) \wedge ((\neg p) \vee (\neg q))$
   (b) [BB] $(p \wedge q) \vee ((\neg p) \wedge (\neg q))$
   (c) [BB] $p \vee ((\neg p) \wedge q)$
   (d) $(p \wedge q) \vee ((\neg p) \wedge (\neg q) \wedge r)$
   (e) $(p \wedge q \wedge r) \vee ((\neg p) \wedge (\neg q) \wedge (\neg r))$
10. Express each of the following statements in disjunctive normal form.
   (a) [BB] $p \wedge q$
   (b) [BB] $(p \wedge q) \vee (\neg((\neg p) \vee q))$
   (c) $p \to q$
   (d) $(p \vee q) \wedge ((\neg p) \vee (\neg q))$
   (e) $(p \to q) \wedge (q \wedge r)$
   (f) $p \vee [(q \wedge (p \vee (\neg r))]$
11. ✍ Find out what you can about Augustus De Morgan and write a paragraph or two about him, in good English, of course!

1.3 Logical Arguments

Proving a theorem in mathematics involves drawing a conclusion from some given information. The steps required in the proof generally consist of showing that if certain statements are true then the truth of other statements must follow. Taken in its entirety, the proof of a theorem demonstrates that if an initial collection of statements—called *premises* or *hypotheses*—are all true then the conclusion of the theorem is also true.

Different methods of proof were discussed informally in Section 0.2. Now we relate these ideas to some of the more formal concepts introduced in Sections 1.1 and 1.2. First, we define what is meant by a *valid argument*.

1.3.1 DEFINITIONS An *argument* is a finite collection of statements $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ called *premises* (or *hypotheses*) followed by a statement $\mathcal{B}$ called the *conclusion*. Such an argument is *valid* if, whenever $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true, then $\mathcal{B}$ is also true. ❖

It is often convenient to write an elementary argument in column form, like this.

$$\begin{array}{c} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_n \\ \hline \mathcal{B} \end{array}$$

PROBLEM 13. Show that the argument

$$\begin{array}{c} p \to \neg q \\ r \to q \\ r \\ \hline \neg p \end{array}$$

is valid.

Solution. We construct a truth table.

p	q	r	$\neg q$	$p \to \neg q$	$r \to q$	$\neg p$	
T	T	T	F	F	T	F	
T	F	T	T	T	F	F	
F	T	T	F	T	T	T	⋆
F	F	T	T	T	F	T	
T	T	F	F	F	T	F	
T	F	F	T	T	T	F	
F	T	F	F	T	T	T	
F	F	F	T	T	T	T	

Observe that row 3—marked with the star (⋆)—is the only row where the premises $p \to \neg q$, $r \to q$, r are all marked T. In this row, the conclusion $\neg p$ is also T. Thus the argument is valid. ▲

In Problem 13, we were a bit fortunate because there was only one row where all the premises were marked T. In general, to assert that an argument is valid when there are several rows with all premises marked T, it is necessary to check that the conclusion is also T in every such row.

Arguments can be shown to be valid without the construction of a truth table. For example, here is an alternative way to solve Problem 13.

> Assume that all premises are true. In particular, this means that r is true. Since $r \to q$ is also true, q must also be true. Thus $\neg q$ is false and, because $p \to (\neg q)$ is true, p is false. Thus $\neg p$ is true as desired.

PROBLEM 14. Determine whether the following argument is valid.

> If I like biology, then I will study it.
> Either I study biology or I fail the course.
> ―――――――――――――――――――
> If I fail the course, then I do not like biology.

Solution. Let p be "I like biology," q be "I study biology," and r be "I fail the course." In symbols, the argument we are to check becomes

$$\begin{array}{c} p \to q \\ q \vee r \\ \hline r \to (\neg p). \end{array}$$

This can be analyzed by a truth table.

p	q	r	$p \to q$	$q \vee r$	$\neg p$	$r \to (\neg p)$	
T	T	T	T	T	F	F	$\star$
T	F	T	F	T	F	F	
F	T	T	T	T	T	T	$\star$
F	F	T	T	T	T	T	$\star$
T	T	F	T	T	F	T	$\star$
T	F	F	F	F	F	T	
F	T	F	T	T	T	T	$\star$
F	F	F	T	F	T	T	

The rows marked $\star$ are those in which the premises are true. In row 1, the premises are true, but the conclusion is F. The argument is not valid. ▲

The theorem that follows relates the idea of a valid argument to the notions introduced in Sections 1.1 and 1.2.

1.3.2 THEOREM

An argument with premises $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and conclusion $\mathcal{B}$ is valid precisely when the compound statement $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \cdots \wedge \mathcal{A}_n \to \mathcal{B}$ is a tautology.

Surely, this is not hard to understand. For the implication $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \cdots \wedge \mathcal{A}_n \to \mathcal{B}$ to be a tautology, it must be the case that whenever $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \cdots \wedge \mathcal{A}_n$ is true then $\mathcal{B}$ is also true. But $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \cdots \wedge \mathcal{A}_n$ is true precisely when each of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ is true, so the result follows from our definition of a valid argument.

In the same spirit as Theorem 1.2.1, we have the following important *substitution theorem*.

1.3.3 THEOREM

Substitution Assume that an argument with premises $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and conclusion $\mathcal{B}$ is valid and that all these statements involve variables $p_1, p_2, \ldots, p_m$. If $p_1, p_2, \ldots, p_m$ are replaced by statements $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$, the resulting argument is still valid.

Rules of Inference

Because of Theorem 1.3.3, some very simple valid arguments that regularly arise in practice are given special names. Here is a list of some of the most common *rules of inference*.

1. **Modus ponens:**
$$\begin{array}{c} p \\ p \to q \\ \hline q \end{array}$$

2. **Modus tollens:**
$$\begin{array}{c} p \to q \\ \neg q \\ \hline \neg p \end{array}$$

3. **Disjunctive syllogism:**
$$\begin{array}{c} p \vee q \\ \neg p \\ \hline q \end{array}$$

4. Chain rule:

$$\begin{array}{c} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

5. Resolution:

$$\begin{array}{c} p \vee r \\ q \vee (\neg r) \\ \hline p \vee q \end{array}$$

Pause 5 Verify modus tollens. ∎

We illustrate how the rules of inference can be applied.

PROBLEM 15. Show that the following argument is valid.

$$\begin{array}{c} (p \vee q) \to (s \wedge t) \\ [\neg((\neg s) \vee (\neg t))] \to [(\neg r) \vee q] \\ \hline (p \vee q) \to (r \to q) \end{array}$$

Solution. One of the laws of De Morgan and the principle of double negation—see Section 1.2—tell us that

$$[\neg((\neg s) \vee (\neg t))] \iff [(\neg(\neg s)) \wedge (\neg(\neg t))] \iff (s \wedge t).$$

Property 12 of logical equivalence as given in Section 1.2 says that $(\neg r) \vee q \iff (r \to q)$. Thus the given argument can be rewritten as

$$\begin{array}{c} (p \vee q) \to (s \wedge t) \\ (s \wedge t) \to (r \to q) \\ \hline (p \vee q) \to (r \to q) \end{array}$$

The chain rule now tells us that our argument is valid. ▲

Pause 6 If a truth table were used to answer Problem 15, how many rows would be required? ∎

Sometimes, rules of inference need to be combined.

PROBLEM 16. Determine the validity of the following argument.

If I study, then I will pass.
If I do not go to a movie, then I will study.
I failed.

Therefore, I went to a movie.

Solution. Let p, q, and r be the statements

p: "I study."
q: "I pass."
r: "I go to a movie."

The given argument is

$$\begin{array}{c} p \to q \\ (\neg r) \to p \\ \neg q \\ \hline r \end{array}$$

The first two premises imply the truth of $(\neg r) \to q$ by the chain rule. Since $(\neg r) \to q$ and $\neg q$ imply $\neg(\neg r)$ by modus tollens, the validity of the argument follows by the principle of double negation: $\neg(\neg r) \iff r$. ▲

Answers to Pauses

5. While this can be shown with a truth table, we prefer an argument by words. Since $\neg q$ is true, q is false. Since $p \to q$ is true, p must also be false. Hence $\neg p$ is true and we are done.
6. There are five variables, each of which could be T or F, so we would need $2^5 = 32$ rows.

True/False Questions

(Answers can be found in the back of the book.)

1. An argument is valid if, whenever the conclusion is true, then the premises are also true.
2. If the premises of an argument are all contradictions, then the argument is valid.
3. If the premises of an argument are all tautologies and the conclusion is not a tautology, then the argument is not valid.
4. De Morgan's laws are two examples of rules of inference.
5. The chain rule has $p \to q$ and $q \to r$ as its premises.
6. Resolution has $p \wedge q$ as its conclusion.
7. Modus ponens and modus tollens were named after the famous Canadian logician George Modus.
8. To decide whether a given argument is valid, it is always best to use truth tables.
9. To decide whether a given argument is valid, it is never best to use truth tables.
10. We've done enough logic. Let's get on to something different!

Exercises

*The answers to exercises marked [BB] can be found in the **B**ack of the **B**ook.*

You are encouraged to use the result of any exercise in this set to assist with the solution of any other.

1. Determine whether or not each of the following arguments is valid.

(a) [BB] $\begin{array}{c} p \to (q \to r) \\ q \\ \hline p \to r \end{array}$ (b) [BB] $\begin{array}{c} p \to q \\ q \vee r \\ \hline r \to (\neg q) \end{array}$

(c) $\begin{array}{c} p \to q \\ r \to q \\ \hline r \to p \end{array}$ (d) $\begin{array}{c} p \to q \\ (q \vee (\neg r)) \to (p \wedge s) \\ \hline s \to (r \vee q) \end{array}$

(e) $\begin{array}{c} p \to (\neg q) \\ r \to q \\ r \\ \hline \neg p \end{array}$ (f) $\begin{array}{c} p \to (\neg q) \\ (\neg r) \to p \\ q \\ \hline r \end{array}$

2. Verify that each of the five rules of inference given in this section is a valid argument.

3. Verify that each of the following arguments is valid.

(a) [BB] $\begin{array}{c} p \to r \\ q \to r \\ \hline (p \vee q) \to r \end{array}$ (b) $\begin{array}{c} p \to r \\ q \to s \\ \hline (p \wedge q) \to (r \wedge s) \end{array}$

(c)
$$\begin{array}{c} p \vee q \\ (\neg p) \vee r \\ (\neg r) \vee s \\ \hline q \vee s \end{array}$$

(d)
$$\begin{array}{c} p \vee ((\neg q) \wedge r) \\ \neg (p \wedge s) \\ \hline \neg (s \wedge (q \vee (\neg r))) \end{array}$$

4. Test the validity of each of the following arguments. [*Hint*: You may find Exercise 3(a) helpful.]

(a) [BB]
$$\begin{array}{c} p \to q \\ (\neg r) \vee (\neg q) \\ r \\ \hline \neg p \end{array}$$

(b)
$$\begin{array}{c} p \vee (\neg q) \\ (t \vee s) \to (p \vee r) \\ (\neg r) \vee (t \vee s) \\ p \to (t \vee s) \\ \hline (p \vee r) \to (q \vee r) \end{array}$$

(c) [BB]
$$\begin{array}{c} p \vee (\neg q) \\ (t \vee s) \to (p \vee r) \\ (\neg r) \vee (t \vee s) \\ p \leftrightarrow (t \vee s) \\ \hline (q \vee r) \to (p \vee r) \end{array}$$

(d)
$$\begin{array}{c} p \vee (\neg q) \\ (\neg r) \vee (t \vee s) \\ (t \vee s) \to p \\ \hline (q \vee r) \to (p \vee r) \end{array}$$

(e)
$$\begin{array}{c} [(p \wedge q) \vee r] \to (q \wedge r \wedge s) \\ [(\neg p) \wedge (\neg q)] \to (r \vee p) \\ [p \vee (\neg q) \vee r] \to (q \wedge s) \\ \hline (p \wedge q) \leftrightarrow [(q \wedge r) \vee s] \end{array}$$

(f) [BB]
$$\begin{array}{c} p \to q \vee s \\ q \to r \\ \hline p \to (r \vee s) \end{array}$$

(g)
$$\begin{array}{c} p \vee (q \to r) \\ q \vee r \\ r \to p \\ \hline p \end{array}$$

[*Hint*: part (f)]

(h)
$$\begin{array}{c} p \to (q \vee r) \\ q \to s \\ r \to \neg p \\ \hline p \to s \end{array}$$

[*Hint*: part (f)]

5. Determine the validity of each of the following arguments. If the argument is one of those listed in the text, name it.

(a) [BB] If I stay up late at night,
then I will be tired in the morning.
I stayed up late last night.

I am tired this morning.

(b) [BB] If I stay up late at night,
then I will be tired in the morning.
I am tired this morning.

I stayed up late last night.

(c) If I stay up late at night,
then I will be tired in the morning.
I am not tired this morning.

I did not stay up late last night.

(d) If I stay up late at night,
then I will be tired in the morning.
I did not stay up late last night.

I am not tired this morning.

(e) [BB] Either I wear a red tie or I wear blue socks.
I am wearing pink socks.

I am wearing a red tie.

(f) Either I wear a red tie or I wear blue socks.
I am wearing blue socks.

I am not wearing a red tie.

(g) [BB] If I work hard, then I earn lots of money.
If I earn lots of money,
then I pay high taxes.

If I pay high taxes, then I have worked hard.

(h) If I work hard, then I earn lots of money.
If I earn lots of money,
then I pay high taxes.

If I work hard, then I pay high taxes.

(i) If I work hard, then I earn lots of money.
If I earn lots of money, then I pay high taxes.

If I do not work hard, then I do not pay high taxes.

(j) If I like mathematics, then I will study.
I will not study.
Either I like mathematics or I like football.

I like football.

(k) Either I study or I like football.
If I like football, then I like mathematics.

If I don't study, then I like mathematics.

(l) [BB] If I like mathematics, then I will study.
Either I don't study or I pass mathematics.
If I don't graduate,
then I didn't pass mathematics.

If I graduate, then I studied.

(m) If I like mathematics, then I will study.
Either I don't study or I pass mathematics.
If I don't graduate, then I didn't pass mathematics.

If I like mathematics, then I will graduate.

6. [BB] Given the premises $p \to (\neg r)$ and $r \vee q$, either write down a valid conclusion that involves p and q only and is not a tautology or show that no such conclusion is possible.

7. Repeat Exercise 6 with the premises $(\neg p) \to r$ and $r \vee q$.

8. (a) [BB] Explain why two premises p and q can always be replaced by the single premise $p \wedge q$, and vice versa.

   (b) Using 8(a), verify that this argument is valid:

$$\begin{array}{c} p \wedge q \\ p \to r \\ s \to \neg q \\ \hline (\neg s) \wedge r. \end{array}$$

9. Let n be an integer greater than 1. Show that the following argument is valid.

$$\begin{array}{c} p_1 \to (q_1 \to r_1) \\ p_2 \to (q_2 \to r_2) \\ \vdots \\ p_n \to (q_n \to r_n) \\ q_1 \wedge q_2 \wedge \cdots \wedge q_n \\ \hline (p_1 \to r_1) \wedge (p_2 \to r_2) \wedge \cdots \wedge (p_n \to r_n) \end{array}$$

[*Hint*: Exercise 8(a).]

✍10. [BB] What language is being used when we say "modus ponens" or "modus tollens"? Translate these expressions into English and explain.

Key Terms & Ideas

Here are some technical words and phrases that were used in this chapter. Do you know the meaning of each? If you're not sure, check the glossary or index at the back of the book.

argument
conclusion
contradiction
contrapositive
converse
disjunctive normal form
hypothesis
logically equivalent
minterm
negation
premise
tautology
valid argument

Review Exercises for Chapter 1

1. Construct a truth table for the compound statement $[p \wedge (q \to (\neg r))] \to [(\neg q) \vee r]$.

2. Determine the truth value of $[p \vee (q \to ((\neg r) \wedge s))] \leftrightarrow (r \wedge t)$, where p, q, r, s, and t are all true.

3. Determine whether each statement is a tautology, a contradiction, or neither.
   (a) $[p \wedge (\neg q)] \wedge [(\neg p) \vee q]$
   (b) $(p \to q) \to (p \vee q)$
   (c) $p \vee [(p \wedge (\neg q)) \to r]$
   (d) $[p \vee q] \leftrightarrow [(\neg q) \wedge r]$

4. Two compound statements $\mathcal{A}$ and $\mathcal{B}$ have the property that $\mathcal{A} \to \mathcal{B}$ is logically equivalent to $\mathcal{B} \to \mathcal{A}$. What can you conclude about $\mathcal{A}$ and $\mathcal{B}$?

5. (a) Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are compound statements such that $\mathcal{A} \iff \mathcal{B}$ and $\mathcal{B} \iff \mathcal{C}$. Explain why $\mathcal{A} \iff \mathcal{C}$.
   (b) Give a proof of Property 11 that uses the result of Property 12.

6. Establish the logical equivalence of each of the following pairs of statements.
   (a) $[(p \to q) \to r]$ and $[(p \vee r) \wedge (\neg(q \wedge (\neg r)))]$
   (b) $p \to (q \vee s)$ and $[p \wedge (\neg q)] \to r$

7. Express each of the following statements in disjunctive normal form.
   (a) $((p \vee q) \wedge r) \vee ((p \vee q) \wedge (\neg p))$
   (b) $[p \vee (q \wedge (\neg r))] \wedge \neg(q \wedge r)$

8. Determine whether each of the following arguments is valid.

(a)
$$\begin{array}{c} p \to q \\ \neg p \\ \hline \neg q \end{array}$$

(b)
$$\begin{array}{c} \neg((\neg p) \wedge q) \\ \neg(p \wedge r) \\ r \vee s \\ \hline q \to s \end{array}$$

(c)
$$\begin{array}{c} p \vee (\neg q) \\ (t \vee s) \to (p \vee r) \\ (\neg r) \vee (t \vee s) \\ p \leftrightarrow (t \vee s) \\ \hline (p \wedge r) \to (q \wedge r) \end{array}$$

9. Discuss the validity of the argument

$$\begin{array}{c} p \wedge q \\ (\neg p) \wedge r \\ \hline \text{Purple toads live on Mars.} \end{array}$$

10. Determine the validity of each of the following arguments. If the argument is one of those listed in the text, name it.

(a) Either I wear a red tie or I wear blue socks.
Either I wear a green hat or I do not wear blue socks.
———
Either I wear a red tie or I wear a green hat.

(b) If I like mathematics, then I will study.
Either I don't study or I pass mathematics.
If I don't pass mathematics, then I don't graduate.
———
If I graduate, then I like mathematics.