1 Вывод уравнений

Уравнения Непрерывности

$$\begin{aligned} \frac{\partial \psi_l}{\partial t} + \frac{\partial W_l}{\partial x} = 0 \\ \frac{\partial \psi_g}{\partial t} + \frac{\partial W_g}{\partial x} = 0 \end{aligned} \tag{1}$$

Уравнения Дарси

$$\begin{aligned} W_l = -\frac{k_l}{\eta_l}\left(\frac{\partial P}{\partial z} - \rho_l g\right) \\ W_g = -\frac{k_g}{\eta_g}\left(\frac{\partial P}{\partial z} - \rho_g g\right) \end{aligned} \tag{2}$$

Условие гидростатического равновесия

$$\frac{\partial P}{\partial z} = (\rho_l \theta_l + \rho_g \theta_g + \rho_s \theta_s) g \tag{3}$$

Тогда

$$W_l = -\frac{k_g}{\eta_g}\left((\rho_l \theta_l + \rho_g \theta_g + \rho_s \theta_s - \rho_g\right) g \tag{4}$$

С учетом того, что

$$\begin{aligned} \theta_s &= \frac{1}{1 + \psi_l + \psi_g} \\ \theta_l &= \frac{\psi_l}{1 + \psi_l + \psi_g} \\ \theta_g &= \frac{\psi_g}{1 + \psi_l + \psi_g} \\ \psi_l &= \frac{\theta_l}{\theta_s} \\ \psi_g &= \frac{\theta_g}{\theta_s} \\ k_l &= \theta_l^2 \\ k_g &= \theta_g^2 \end{aligned} \tag{5}$$

Получаем

$$\begin{aligned} W_l = -\frac{\psi_l^2}{\eta_l (1 + \psi_l + \psi_g)^3}((\rho_s - \rho_l) + \psi_g(\rho_g - \rho_l))g \\ W_g = -\frac{\psi_g^2}{\eta_g (1 + \psi_l + \psi_g)^3}((\rho_s - \rho_g) + \psi_l(\rho_l - \rho_g))g \end{aligned} \tag{6}$$

Можно переписать систему в виде

$$\frac{\partial \vec{\psi}}{\partial t} = A \frac{\partial \vec{\psi}}{\partial x} \tag{7}$$

где

$$A = \begin{pmatrix} \frac{\partial W_l}{\partial \psi_l} & \frac{\partial W_l}{\partial \psi_g} \\ \frac{\partial W_g}{\partial \psi_l} & \frac{\partial W_g}{\partial \psi_g} \end{pmatrix} \tag{8}$$

$$\begin{aligned} \frac{\partial W_l}{\partial \psi_l} &= -\frac{\psi_l(2\psi_g + 2 - \psi_l)(\psi_g(\rho_g - \rho_l) - \rho_l + \rho_s)}{\eta_l(\psi_g + \psi_l + 1)^4} g \\ \frac{\partial W_l}{\partial \psi_g} &= -\frac{\psi_l^2(\rho_g(\psi_l + 1 - 2\psi_g) + \rho_l(2\psi_g + 2 - \psi_l) - 3\rho_s)}{\eta_l(\psi_g + \psi_l + 1)^4} g \\ \frac{\partial W_g}{\partial \psi_l} &= -\frac{\psi_g^2(\rho_g(2\psi_l + 2 - \psi_g) + \rho_l(\psi_g - 2\psi_l + 1) - 3\rho_s)}{\eta_g(\psi_g + \psi_l + 1)^4} g \\ \frac{\partial W_g}{\partial \psi_g} &= -\frac{\psi_g(2\psi_l + 2 - \psi_g)(\psi_l(\rho_l - \rho_g) - \rho_g + \rho_s)}{\eta_g(\psi_g + \psi_l + 1)^4} g \end{aligned} \tag{9}$$

Ищем собственные значения

$$\lambda^2 - (W_1 + W_4)\lambda + W_1 W_4 - W_3 W_2 = 0 \tag{10}$$

$$D = (W_1 + W_4)^2 - 4(W_1 W_4 - W_3 W_2) \tag{11}$$