Series Numéricas

Corolario del criterio de Cauchy

$\sum a_n$ convergente $\implies \lim_{n\to\infty} a_n = 0$

Serie de Bertrand $\left(\sum \frac{1}{n^\alpha (\log n)^\beta}\right)$

- $\alpha > 1$ o $\alpha = 1$, $\beta > 1 \implies$ convergente
- $\alpha < 1$ o $\alpha = 1$, $\beta \leq 1 \implies$ divergente

Series positivas

Criterio de Condensación (a_n decreciente, $a_n \geq 0$) $\sum a_n$ convergente $\iff \sum 2^n a_{2^n}$ convergente

Comparación directa ($b_n \geq a_n \geq 0 \quad \forall n \geq n_0$)

- $\sum b_n$ conv. $\implies \sum a_n$ conv.
- $\sum a_n$ divergente $\implies \sum b_n$ divergente

Comparación en el límite $\left(\lim_{n\to\infty} \frac{a_n}{b_n} = L\right)$

- $L < +\infty$, $\sum a_n$ conv. $\implies \sum b_n$ conv.
- $L > 0$, $\sum b_n$ div. $\implies \sum a_n$ div.

Criterio de la raíz y del cociente

$\left(\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L\right)$

- $L < 1 \implies \sum a_n$ convergente
- $L > 1 \implies \sum a_n$ divergente

Criterio de Raabe $\left(\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = L\right)$

- $L > 1 \implies \sum a_n$ convergente
- $L < 1 \implies \sum a_n$ divergente

Criterio logarítmico $\left(\lim_{n\to\infty} \frac{-\log a_n}{\log n} = L\right)$

- $L > 1 \implies \sum a_n$ convergente
- $L < 1 \implies \sum a_n$ divergente

Criterio de Leibnitz $\left(a_n \text{ dec. } \lim_{n\to\infty} a_n = 0\right)$

$\sum(-1)^{n+1} a_n$ convergente

Criterio de la integral ($a_n = f(n)$, f integ.)

- $\int_M^\infty f$ converge $\iff \sum a_k$ converge
- $\sum_M^\infty = \sum_M^{N-1} + \int_N^\infty f + \varepsilon_N$, $\varepsilon_N \in [0, a_N]$

Criterio de Dirichlet ($\lim_{n\to\infty} b_n = 0$, b_n dec. Sumas de $\sum a_n$ acotadas.) $\sum a_n b_n$ convergente

Series de Potencias

Teorema de Cauchy-Hadamard

$\frac{1}{R} = \limsup |a_n|^{1/n}$

Radio de convergencia

$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

Integrales impropias

Criterio de Cauchy ($\forall \varepsilon, \exists c_0$)

$c_1, c_2 > c_0 \implies \left|\int_{c_1}^{c_2} f\right| < \varepsilon. \implies$ convergente

Comparación directa ($g(x) \geq f(x) \geq 0$)

- $\int_a^\infty g$ converge $\implies \int_a^\infty f$ converge
- $\int_a^\infty f$ divergente $\implies \int_a^\infty g$ divergente

Comp. en el límite $\left(g, f \geq 0, \lim_{x\to b} \frac{f(x)}{g(x)} = L\right)$

- $L < \infty$, $\int_a^b g$ conv. $\implies \int_a^b f$ conv.
- $L > 0$, $\int_a^b f$ div. $\implies \int_a^b g$ div.

Criterio de Dirichlet

$\left(\text{g dec., } \lim_{x\to b} g(x) = 0, c < b \implies \left|\int_a^c f\right| < M\right)$

Entonces $\int_a^b f(x)g(x)dx$ converge

Integración múltiple

Conjuntos de medida nula

($Z \subseteq \mathbb{R}^n$ medida nula)

- graph(f) con f unif. cont.
- $f(Z)$ con f lipschitziana $(d(f(x), f(y)) \leq d(x, y))$
- $f(Z)$ con f de clase $\mathcal{C}^1$
- M subvariedad regular de dim $M < n$

Teorema de Lebesgue: f integrable en A sii $disc(f) \cap A$ tiene medida nula

Conjuntos admisibles (A, A' admisibles)

- $A \cap A'$, $A \cup A'$, $A \setminus A'$ son admisibles
- rectángulos acotados y bolas

Medida de Jordan ($C \subseteq \mathbb{R}^n$ admisible)

$\text{vol}(C) = \int_C 1$

Propiedades de la integral (f, g integrables)

- $f + g$ integrable
- fg integrable
- $f \leq g \implies \int_E f \leq \int_E g$
- $m \leq f \leq M \implies m\,\text{vol}(E) \leq \int_E f \leq M\,\text{vol}(E)$
- $\text{vol}(E) = 0 \implies \int_E f = 0$
- E conexo, f continua $\implies \int_E f = f(x_0)\,\text{vol}(E)$
- h continua $\implies h \circ f$ integrable
- $\left|\int_E f\right| \leq \int_E |f|$
- $\int_{A\cup B} f = \int_A f + \int_B f - \int_{A\cap B} f$

Teorema de Fubini (f continua)

$\int_{A\times B} f(x,y)dxdy = \int_A dx \left(\int_B dy f(x,y)\right)$

Región elemental (ψ, ϕ cont. D elemental)

$E = \left\{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}|_{\phi(x)\leq y\leq\psi(x)}^{x\in D}\right\}$

TCV ($V \in \mathbb{R}^n$ abierto, $\varphi\colon V \mapsto \mathbb{R}^n$ inyectiva, de clase C^1, $\det \mathrm{D}\,\varphi \neq 0$), $f\colon U = \varphi(V) \mapsto \mathbb{R}$ integrable). Entonces $\int_U = \int_V (f\circ\varphi)|\det \mathrm{D}\,\varphi|$

- $\int x^n dx = \frac{1}{n+1}x^{n+1}$
- $\int \frac{1}{x}dx = \log(|x|)$
- $\int e^x = e^x$
- $\int a^x dx = \frac{a^x}{\log(a)}$
- $\int \sin(x)dx = -\cos(x)$
- $\int \cos(x)dx = \sin(x)$
- $\int \tan(x)dx = -\log(|\cos(x)|)$
- $\int \arcsin\left(\frac{x}{a}\right)dx =$
  $x\arcsin\left(\frac{x}{a}\right) + \sqrt{a^2-x^2}\ \ a > 0$
- $\int \arccos\left(\frac{x}{a}\right)dx =$
  $x\arccos\left(\frac{x}{a}\right) - \sqrt{a^2-x^2}\ \ a > 0$
- $\int \arctan\left(\frac{x}{a}\right)dx =$
  $x\arctan\left(\frac{x}{a}\right) - \frac{a}{2}\log\left(a^2+x^2\right)\ \ a > 0$
- $\int \sin^2(mx)dx =$
  $\frac{1}{2m}(mx - \sin(mx)\cos(mx))$
- $\int \cos^2(mx)dx =$
  $\frac{1}{2m}(mx + \sin(mx)\cos(mx))$
- $\int \sec^2(x)dx = \tan(x)$
- $\int \csc^2(x)dx = -\cot(x)$
- $\int \sin^n(x)dx =$
  $-\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)dx$
- $\int \cos^n(x)dx =$
  $-\frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)dx$
- $\int \tan^n(x)dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)dx$
- $\int \sinh(x)dx = \cosh(x)$
- $\int \cosh(x)dx = \sinh(x)$
- $\int \tanh(x)dx = \log(|\cosh(x)|)$
- $\int \sinh^2(x)dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x$
- $\int \cosh^2(x)dx = \frac{1}{4}\cosh(2x) + \frac{1}{2}x$
- $\int \frac{1}{\sqrt{a^2+x^2}}dx = \log\left(x + \sqrt{a^2+x^2}\right)$
- $\int \frac{1}{a^2+x^2}dx = \frac{1}{2}\arctan\frac{x}{a}$
- $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$
- $\int (a^2-x^2)^{\frac{3}{2}}dx =$
  $\frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a}$
- $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a}$
- $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{x+a}{x-a}\right|$
- $\int \frac{1}{(a^2-x^2)^{\frac{3}{2}}}dx = \frac{x}{a^2\sqrt{a^2-x^2}}$
- $\int \sqrt{x^2 \pm a^2}dx =$
  $\frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\log\left|x \pm \sqrt{x^2\pm a^2}\right|$
- $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x + \sqrt{x^2-a^2}\right|$
- $\int \frac{1}{x(a+bx)}dx = \frac{1}{a}\log\left|\frac{x}{a+bx}\right|$
- $\int x\sqrt{a+bx}dx = \frac{2(3bx-2a)(a+bx)^{\frac{3}{2}}}{15b^2}$
- $\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx} + a\int \frac{1}{x\sqrt{a+bx}}dx$
- $\int \frac{x}{\sqrt{a+bx}}dx = \frac{2(bx-2a)\sqrt{a+bx}}{3b^2}$
- $\int \frac{1}{x\sqrt{a+bx}}dx = \frac{1}{\sqrt{a}}\log\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|$
- $\int \frac{\sqrt{a^2-x^2}}{x}dx =$
  $\sqrt{a^2-x^2} - a\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$
- $\int x\sqrt{a^2-x^2}dx = -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}$
- $\int x^2\sqrt{a^2-x^2} =$
  $\frac{a}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a}$
- $\int \frac{1}{x\sqrt{a^2-x^2}}dx = -\frac{1}{a}\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$
- $\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2}$
- $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$
- $\int \frac{\sqrt{a^2+x^2}}{x}dx =$
  $\sqrt{a^2+x^2} - a\log\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|$
- $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - \arcsin\frac{x}{a}$
- $\int x\sqrt{x^2\pm a^2}dx = \frac{1}{3}(x^2\pm a^2)^{\frac{3}{2}}$
- $\int \frac{1}{x\sqrt{x^2+a^2}}dx = \frac{1}{a}\log\left|\frac{x}{a+\sqrt{x^2+a^2}}\right|$
- $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\arccos\frac{a}{|x|}$
- $\int \frac{1}{x^2\sqrt{x^2\pm a^2}}dx = \pm\frac{\sqrt{x^2\pm a^2}}{a^2x}$
- $\int \frac{x}{\sqrt{x^2\pm a^2}}dx = \sqrt{x^2\pm a^2}$
- $\int \frac{1}{ax^2+bx+c}dx =$
  $\begin{cases} \frac{1}{\sqrt{b^2-4ac}}\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & (b^2 > 4ac) \\ \frac{2}{\sqrt{b^2-4ac}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} & (b^2 < 4ac) \end{cases}$
- , $\int \frac{x}{ax^2+bx+c}dx =$
  $\frac{1}{2a}\log\left|ax^2+bx+c\right| - \frac{b}{2a}\int \frac{1}{ax^2+bx+c}dx$
- $\int \frac{1}{\sqrt{ax^2+bx+c}}dx =$
  $\begin{cases} \frac{1}{\sqrt{a}}\log\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right| & (a > 0 \\ \frac{1}{\sqrt{-a}}\arcsin\frac{-2ax-b}{\sqrt{b^2-4ac}} & (a < 0 \end{cases}$
- $\int \sqrt{ax^2+bx+c}dx =$
  $\frac{2ax+b}{4a}\sqrt{ax^2+bx+c} -$
  $\frac{4ac-b^2}{8a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx$
- $\int \frac{x}{\sqrt{ax^2+bx+c}}dx =$
  $\frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx$
- $\int x^3\sqrt{x^2+a^2}dx =$
  $\left(\frac{1}{5}x^2 - \frac{2}{15}a^2\right)\sqrt{(a^2+x^2)^3}$
- $\int \frac{\sqrt{x^2\pm a^2}}{x^4}dx = \frac{\pm\sqrt{(x^2\pm a^2)^3}}{3a^2x^3}$
- $\int \sin(ax)\sin(bx)dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$
- $\int \sin(ax)\cos(bx)dx = \frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$
- $\int \cos(ax)\cos(bx)dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$
- $\int x^n\log(ax)dx = x^{n+1}\left(\frac{\log(ax)}{n+1} - \frac{1}{(n+1)^2}\right)$
- $\int e^{ax}\sin bx dx = \frac{e^{ax}(b\sin(bx) - b\cos(bx))}{a^2+b^2}$
- $\int e^{ax}\cos bx dx = \frac{e^{ax}(b\sin(bx) + b\cos(bx))}{a^2+b^2}$