27/09/17

Recordatori

grup simètric   permutacions

$f \in S_p(E) \iff \forall s \in \mathcal{S}_p \quad sf = f$

$f \in A_p(E) \iff \forall s \in \mathcal{S}_p \quad sf = -f$

$\boxed{\text{car} K \neq 2}$

$\underline{\text{Ej}}^{\text{ex}}$ $f(\ldots, v_i, \ldots, v_j, \ldots) = -f(\ldots, v_j, \ldots, v_i, \ldots) \iff f(\ldots, w, \ldots, w, \ldots) = 0$

$\Rightarrow f(\ldots, w, \ldots, w, \ldots) \underset{\text{hip}}{=} -f(\ldots, w, \ldots, w, \ldots)$

$\Rightarrow 2f(\ldots, w, \ldots, w, \ldots) = 0 \underset{\text{car}K\neq 2}{\Longrightarrow} f(\ldots, w, \ldots, w, \ldots) = 0$

$\Leftarrow (\ldots, v_i + v_j, \ldots, v_i + v_j, \ldots) \underset{\text{hip}}{=} 0$

$\Rightarrow \underbrace{f(\ldots, v_i, \ldots, v_i)}_{=0} + \underbrace{f(\ldots, v_j, \ldots, v_j, \ldots)}_{=0} + f(\ldots, v_i, \ldots, v_j, \ldots) + f(\ldots, v_j, \ldots, v_i, \ldots) = 0$

hip

$\Downarrow$

$f(\ldots, v_i, \ldots, v_j, \ldots) = -f(\ldots, v_j, \ldots, v_i, \ldots)$

Ejemplos:

(1) $T_2(E) \ni f$ (forma bilineal) $\longrightarrow M_B(f) = A = (f(e_i, e_j))$

$B = \{e_1, \ldots, e_n\}$ base

$\mathcal{S}_2 = \{I, s = (1,2)\}$

$\varepsilon(I) = 1 \quad \varepsilon(s) = -1$

$If = f$

$sf \longrightarrow M_B(sf) = (sf) =$

$= (sf(e_i, e_j)) =$

$= (f(e_j, e_i))$

$= A^t$

► En formas bilineales:

f simétrica $\iff A = A^t \iff A$ simétrica.

f antisimétrica $\iff A^t = -A \iff A$ antisimétrica

$\hookrightarrow$ diagonal $= 0$.

(2) $\dim E = n$

$B = \{e_1, \dots, e_n\}$

$E \times \overset{n)}{\dots} \times E \xrightarrow{f} K$

$(u_1, \dots, u_n) \longmapsto \det(u_1, \dots, u_n)_B$

- f multilineal ($\Rightarrow f \in T_n(E)$)
- f antisimétrico porque $\det_B(\dots, u_i, \dots, u_j, \dots) = -\det_B(\dots, u_j, \dots, u_i, \dots)$

(Podemos cambiar "un poco" un tensor para que sea simétrico o antisimétrico)

Def Sea E un K-e.v.; sea $f \in T_p(E)$ (sup car K = 0)

$$\boxed{S(f) = \frac{1}{p!} \sum_{s \in S_p} sf}$$

(simetrizado de f)

$$\boxed{A(f) = \frac{1}{p!} \sum \varepsilon(s)\, sf}$$

Antisimetrizado de f.

Ejemplos. $E = \mathbb{R}^3$ $B = \{e_1, e_2, e_3\}$

(A) $f = e_1^* \otimes e_2^* \in T_2(E)$

$S(f) = \frac{1}{2}(e_1^* \otimes e_2^* + e_2^* \otimes e_1^*)$

$S_2 = \{I, s = (1,2)\}$

$A(f) = \frac{1}{2}(e_1^* \otimes e_2^* - e_2^* \otimes e_1^*)$

(2) $g = e_1^* \otimes e_2^* \otimes e_3^*$

$S(g) = \frac{1}{6}(s_1 g + s_2 g + s_3 g + s_4 g + s_5 g, s_6 g) = (\#)$

$S_3 = \{\underset{s_1}{I}, \underset{s_2}{(1,2)}, \underset{s_3}{(1,3)}, \underset{s_4}{(2,3)}, \underset{s_5}{(1,2,3)}, \underset{s_6}{(1,3,2)}\}$

$\varepsilon(s) =$ 1 -1 -1 -1 1 1

$s_1^{-1} = s_1$ $\qquad s_3^{-1} = s_3$ $\qquad s_5^{-1} = s_6$

$s_2^{-1} = s_2$ $\qquad s_4^{-1} = s_4$ $\qquad s_6^{-1} = s_5$

$$(\#) = \frac{1}{6}\Big(e_1^* \otimes e_2^* \otimes e_3^* + e_2^* \otimes e_1^* \otimes e_3^* + e_3^* \otimes e_2^* \otimes e_1^* + e_1^* \otimes e_3^* \otimes e_2^* + e_3^* \otimes e_1^* \otimes e_2^* + e_2^* \otimes e_3^* \otimes e_1^*\Big)$$

(3) E $\qquad f \in T_2(E)$ (forma bilineal) ← tensor de orden 2

B base $\qquad A = M_B(f) = (f(e_i, e_j))$

$M_B(S(f)) = \frac{1}{2}(A + A^t)$ (hacer el promedio con A^t)

$M_B(A(f)) = \frac{1}{2} = \frac{1}{2}(A - A^t)$

$\boxed{\text{Obs } f = S(f) + A(f)}$ ← Solo para tensores de orden 2

(+ detalles en Problemas)

Prop. E K-ev

$$T_p(E) \xrightarrow[A]{S} T_p(E)$$

S → simetrizar, A → Antisimetrizar.

(0) S, A son lineales

(1) $f \in S_p(E) \Rightarrow S(f) = f$

$f \in A_p(E) \Rightarrow S(f) = f$

(2) $\operatorname{Im} S = S_p(E)$

$\operatorname{Im} A = A_p(E)$

Dem (0) Trivial, por def de S y A.

(1) $S_p(E) \to$ ej

$\boxed{A_p(E)}$ $\qquad A(f) = \frac{1}{p!}\left(\sum_{s \in S_p} \varepsilon(s) \cdot sf\right) \overset{f \in A_p(E)}{=} \frac{1}{p!} \sum_{s \in S_p} \varepsilon(s)(\varepsilon(s) f)$

$$= \frac{1}{p!} \sum_{s \in S_p} \underbrace{\varepsilon(s)^2}_{=1} f = f$$

(2) $\operatorname{Im} S = S_p(E)$ Ejercicio

$\boxed{\mathrm{Im}\, A = A_p(E)}$ (1) f es la imagen de él mismo

$\boxed{\supseteq}$ Si $f \in A_p(E) \overset{(1)}{\Longrightarrow} f = A(f) \Rightarrow f \in \mathrm{Im}A$.

$\boxed{\subseteq}$ Sea $g = A(h) \in \mathrm{Im}\, A$

Hemos que ver que $g \in A_p(E)$

Sea $s \in S_p$

$$\underline{s}\, g = \underline{s}\left(\frac{1}{p!}\sum_{r \in S_p} \varepsilon(r)\, \underline{r} h\right) \overset{\underline{s} \text{ es lineal}}{=}$$

$$= \frac{1}{p!}\left(\sum_{r \in S_p} \varepsilon(r)\, \underline{s}\,\underline{r} h\right) = \varepsilon(s) \cdot \frac{1}{p!}\left(\sum_{r \in S_p} \varepsilon(sr) \cdot \underline{sr}\, h\right) =$$

$\boxed{\underline{s}\,\underline{r} = \underline{sr}}$

$\boxed{\varepsilon(s)^2 = 1}$

$\boxed{\varepsilon(sr) = \varepsilon(s) \cdot \varepsilon(r)}$

$$= \varepsilon(s)\left(\frac{1}{p!}\sum_{t \in S_p} \varepsilon(t) \cdot \underline{t} h\right) = \varepsilon(s)\, g \Longrightarrow g \in A_p(E)$$

reordenació d'índexos

Obs $S_p \xrightarrow{\Phi} S_p$ es biyectiva (s fijado)
$r \longmapsto sr$

$\{r\}_{r \in S_p} = \{sr\}_{r \in S_p} = \{t\}_{t \in S_p}$

Recorda: En lloc de calcular les transposicions de una permutació per a obtenir el signe de la permutació, podem fer el següent:

$s = (1, 2, 3) = (\ \ )(\ \ )$ transposicions

fem: $\Longrightarrow$

$$\det \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \hline \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array} = \varepsilon(s).$$

$\varepsilon(s) = (-1)^{2}$ # transposicions