Stochastic Processes October 30, 2022
Sharif University of Technology CE 40-695
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Time: 25 mins

Name: Std. Number:

Quiz 2 (Ergodicity + LTI)

Questions

1. We have a discrete time continuous state random process that represents i.i.d sampling with the following definition:

$$X[n] = A_n \quad \text{s.t.} \quad A_n \overset{i.i.d}{\sim} \text{uniform}(0,2)$$

For each of the following statements, specify whether the statement is true or false and give a brief explanation.

(a) This process is strict sense stationary.

(b) This process is mean ergodic.

Solution:

(a) **True:** Intuitively, you can conclude that the process $X[n]$ is strict sense stationary because there is no way to tell where on the time axis you are by looking at any set of samples, no matter at what times they are taken. Mathematically, this intuition is captured by saying that PDF's of all orders are shift invariant, or $f_{X[n_1]X[n_2]X[n_3]\dots X[N]} = f_{X[n_1+l]X[n_2+l]X[n_3+l]\dots X[N+l]}$ for all N and l.

(b) **True:** Because the law of large numbers we know that:

$$\lim_{N\to\infty} \sum_{<N>} = \mathbb{E}[A] = \frac{0+2}{2} = 1$$

Alternatively because of the independence we know that $C_{XX}[l] = 0$ then:

$$\lim_{L\to\infty} \frac{1}{L} \sum_{l=0}^{L} C_{XX}[l] = 0 \implies \text{mean ergodic}$$

2. Suppose the input of an LTI system, as $X[n]$, is a white noise with zero mean and variance σ^2. Let the impulse response of the system be defined as below:

$$h_L[n] = \frac{1}{2(L+1)} \sum_{i=-L-1}^{L} \delta_{\frac{1}{2}+i}[n]$$

(a) For L = 0, find autocorrelation function for the response of $X[n]$ into the system.

(b) Calculate response autocorrelation function for arbitrary L. (Hint: Draw h_L)

Solution:
We have:

$$R_{YY}[k] = h_L[k] * h_L[-k] * R_{XX}[k] = (h_L[k] * h_L[-k]) * (\sigma^2 \delta[k]) = \sigma^2 (h_L[k] * h_L[-k])$$

So we have to compute $h_L[k] * h_L[-k]$

$$h_L[k] * h_L[-k] = \frac{1}{4(L+1)^2} \left(\sum_{l=-2(L+1)}^{2(L+1)} (2(L+1) - |l|)\delta_l[k] \right)$$

$$= \frac{1}{2(L+1)} \left(\sum_{l=-2(L+1)}^{2(L+1)} \left(1 - \frac{|l|}{2(L+1)}\right) \delta_l[k] \right)$$

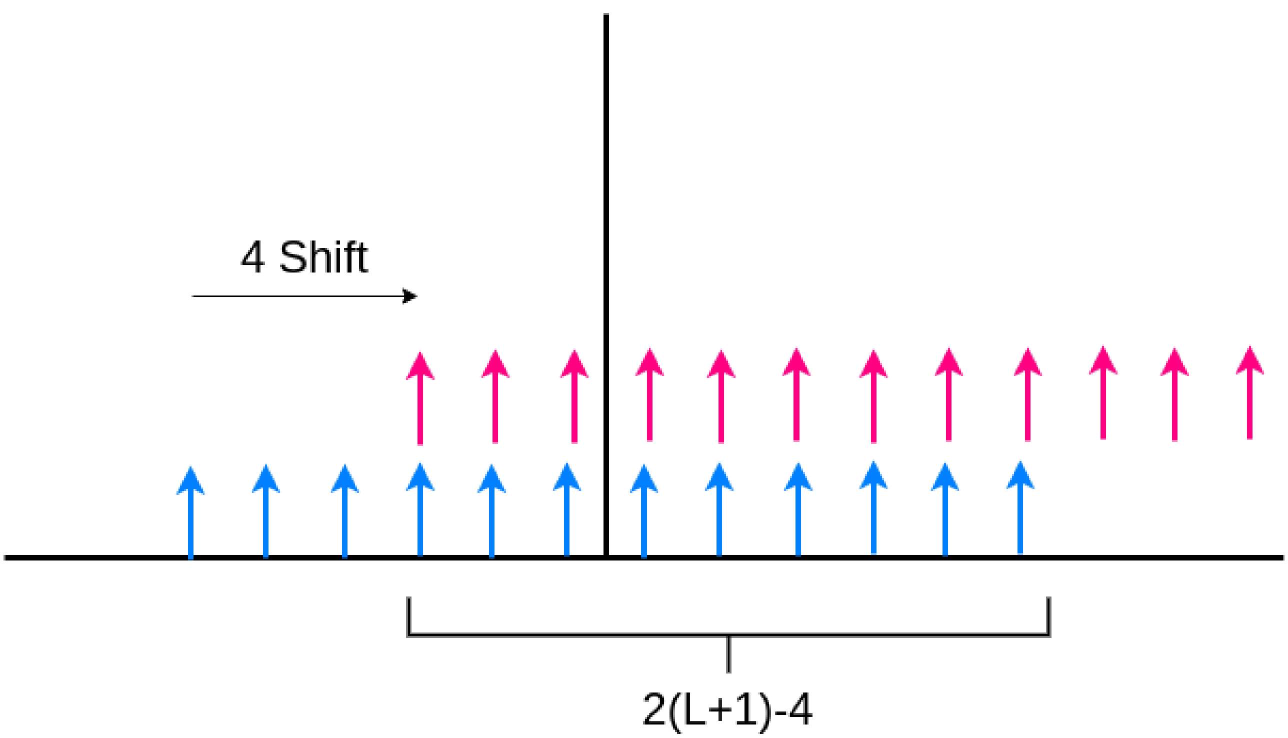

Figure 1: Illustration of $h_L[k] * h_L[-k]$ for $k = 4$