Sharif University of Technology

CE Department

Course: Stochastic Processes

PS. 6

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1.1 a

We can model the problem as an HMM. The hidden states are the state of weather and the observed states are if her mother has recieved any flower today. Hidden state transition probabilites are given as:

$$P_H = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

The first row corresponds to the weather being good and the second row is weather being bad.
Also for the observation probabilites:

$$P_O = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}$$

The first is the weather being good and the First column is the probability of her mother getting an origami.

1.2 b

Now we can calculate the stationary probability for hidden states:

$$\pi = P_H\pi \to (P_H - I)\pi = 0 \to \pi_H = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Then for any given day $\mathbb{E}[O_i] = \mathbb{E}[\mathbb{E}[O_i|H_i]]$.

$$\mathbb{E}[\mathbb{E}[O_i|H_i]] = 0.5 \times 0.75 + 0.5 \times 0.5 = 0.625$$

On a course of 3650 days (10 years), the expected number of flowers can be obtained by:

$$3650 \times 0.625$$

1.3 c

First condition on the next three days. We need to calculate the following probabilites.

$\mathbb{P}$[Two origamis|3 good days]+$\mathbb{P}$[Two origamis|2 good days and 1 bad day]+$\mathbb{P}$[Two origamis|1 good day and 2 bad days]+$\mathbb{P}$[T

$$= 0.75 \times 0.75 \times 0.25$$

$$+3 \times (0.75 \times 0.75 \times 0.5 + 0.75 \times 0.25 \times 0.5 + 0.25 \times 0.75 \times 0.5)$$

$$+3 \times (0.75 \times 0.5 + 0.25 \times 0.5 \times 0.5)$$

$$+0.5^3$$

1.4 d

$$\mathbb{P}[GGG] = 0.7 \times 0.7 \times 0.75 \times 0.25 \times 0.25$$

$$\mathbb{P}[GGB] = 0.7 \times 0.3 \times 0.75 \times 0.25 \times 0.5$$

$$\mathbb{P}[GBG] = 0.3 \times 0.3 \times 0.75 \times 0.5 \times 0.25$$

$$\mathbb{P}[GBB] = 0.3 \times 0.7 \times 0.75 \times 0.5 \times 0.5$$

$$\mathbb{P}[BGG] = 0.3 \times 0.7 \times 0.5 \times 0.25 \times 0.25$$

$$\mathbb{P}[BGB] = 0.3 \times 0.3 \times 0.5 \times 0.25 \times 0.5$$

$$\mathbb{P}[BBG] = 0.7 \times 0.3 \times 0.5 \times 0.5 \times 0.25$$

$$\mathbb{P}[BBB] = 0.7 \times 0.7 \times 0.5 \times 0.5 \times 0.5$$

So the most probable event is BBB.

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The only possible values for the travels are: (Ba is Bangs, S is Speed, F is Fries, Bl is Bluffs and C is Cool)

$$BSBSB, BSBFB, BFBSB, BFBFB$$

$$\mathbb{P}[BSBSB] = 0.1 \times 0.2 \times 0.1 \times 0.2 = 4 \times 10^{-4}$$

$$\mathbb{P}[BSBFB] = 0.1 \times 0.2 \times 0.3 \times 0.2 = 12 \times 10^{-4}$$

$$\mathbb{P}[BFBSB] = 0.3 \times 0.2 \times 0.1 \times 0.2 = 12 \times 10^{-4}$$

$$\mathbb{P}[BFBFB] = 0.3 \times 0.2 \times 0.3 \times 0.2 = 36 \times 10^{-4}$$

$$\mathbb{P}[\text{Twice in Bands}] = 64 \times 10^{-4}$$

3

First starting from T:

$$\mathbb{P}[TTAC] = \mathbb{P}[X_1 = T|X_0 = T] \times \mathbb{P}[X_2 = T|X_1 = T] \times \mathbb{P}[X_3 = A|X_2 = T] \times \mathbb{P}[X_4 = C|X_3 = A]$$

$$= 0.25 \times 0.25 \times 0.25 \times 0 = 0$$

$$\mathbb{P}[GGCGG] = \mathbb{P}[X_1 = G|X_0 = T] \times \mathbb{P}[X_2 = G|X_1 = G] \times \mathbb{P}[X_3 = C|X_2 = G] \times \mathbb{P}[X_4 = G|X_3 = C] \times \mathbb{P}[X_5 = G|X_4 = G]$$

$$= 0$$

Starting randomly:

$P = 0.25 \times \mathbb{P}[\text{Starting from T}] + 0.25 \times \mathbb{P}[\text{Starting from A}] + 0.25 \times \mathbb{P}[\text{Starting from G}] + 0.25 \times \mathbb{P}[\text{Starting from C}]$

Note that $\mathbb{P}[TTAC]$ is always zero because of $\mathbb{P}[X_{i+1} = C | X_i = A] = 0$.

$$\mathbb{P}[GGCGG] = 0.25 \times 0$$

$$+0.25 \times 0.125 \times 0.125 \times 0.5 \times 0.25 \times 0.125$$

$$+0.25 \times 0.25 \times 0.125 \times 0.5 \times 0.25 \times 0.125$$

$$+0.25 \times 0.5 \times 0.125 \times 0.5 \times 0.25 \times 0.125$$

4

First part: if it's not bipartite then it's aperiodic.

If the graph is not bopartite then there must exist at least one odd cycle $v_{a_1}, \ldots, v_{a_n}$. Take the walk from path from an arbitrary node v_s to v_{a_1}. And then take the odd cycle and path from v_{a_1} back to v_s. We know that this walk has k nodes and k is odd. So $\mathbb{P}[X_k = v_s | X_0 = v_s] > 0$.

Also take the path from v_s to v_n and back for $v_n \in \text{neighbors}(v_s)$. So $\mathbb{P}[X_2 = v_s | X_0 = v_s] > 0$.

Take the gcd of the first k and 2. Then $\text{Period}(v_s) = 1$.

Second part: if its aperiodic then it's not bipartite. Suppose it's bipartite.

Then all cycles must have an even number of vertices. Then all paths that start from v_s and end in v_s must have an even number of nodes. Then only for odd k's $\mathbb{P}[X_k = v_s | X_0 = v_s] > 0$.

$$\text{Period}(v_s) > 1$$

By contradiction the given graph is not bipartite.

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Proof by contradiction: Take $X_1, \ldots, X_M$ to be the number of times being in state $1, \ldots, M$. Also the take N to be the total number of steps. Also suppose that none of the states are recurrent. Note that $\sum_i X_i = N$.

$$\lim_{n \to \infty} \mathbb{E}[X_1 + \cdots + X_M | N = n] = \lim_{n \to \infty} \mathbb{E}[N | N = n] = \infty$$

$$\lim_{n \to \infty} \mathbb{E}[X_1 + \cdots + X_M | N = n] = \lim_{n \to \infty} \sum_{i=1}^{M} \mathbb{E}[X_i | N = n]$$

Also $\lim_{n \to \infty} \mathbb{E}[X_i | N = n] < \infty$.

$$\lim_{n \to \infty} \mathbb{E}[X_1 + \cdots + X_M | N = n] < \infty.$$

Which is a contradiction. So there must exist at least one recurrent state.

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6.1 a

Suppose that no set of nodes exist that starting from arbitrary v_s we can get back to v_s. (no cycles exist on the MDP)

Then v_s can't be positive recurrent because there is no path back to v_s. Also note that there can't be only one cycle with length M. Because this causes the Markov Decision to be not aperiodic. So the graph contains a cycle with length equal to τ such that $< M$.

6.2 b

Every state that can be reached with path which has length m then it can also be reached with moving on the cycle and then taking the path. Which means it is reachable with length $m + \tau$.