Sharif University of Technology

CE Department

Course: Stochastic Processes

PS. 1

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First start by writing each X_i in terms of Y_i's:

$$X_0 = Y_0$$

$$X_1 = Y_1 + \lambda X_0 = Y_1 + \lambda Y_0$$

$$X_2 = Y_2 + \lambda X_1 = Y_2 + \lambda Y_1 + \lambda^2 Y_0$$

$$\vdots$$

So if X_n can be written as $\sum_{i=0}^{n} \lambda^{n-i} Y_i$ which is true for X_0, X_{n+1} can be written as:

$$X_{n+1} = \lambda^{n+1-(n+1)} Y_{n+1} + \lambda \sum_{i=0}^{n} \lambda^{n-i} Y_i = \lambda^{n+1-(n+1)} Y_{n+1} + \sum_{i=0}^{n} \lambda^{n+1-i} Y_i = \sum_{i=0}^{n+1} \lambda^{n+1-i} Y_i$$

Thus, by induction we can write X_n as :

$$X_n = \sum_{i=0}^{n} \lambda^{n-i} Y_i = \lambda^n \sum_{i=0}^{n} \frac{Y_i}{\lambda^i}$$

Then we can calculate $\mu_X(n)$:

$$\mu_X(n) = \mathbb{E}[X(n)] = \lambda^n \sum_{i=0}^{n} \frac{\mathbb{E}[Y_i]}{\lambda^i} = 0$$

$$\rightarrow \mu_X(n) = \mu_X = 0$$

Then we calculate $R_x(n, m+n)$:(suppose for now $m \geq 0$)

$$R_x(n, m+n) = \mathbb{E}[(\lambda^n \sum_{i=0}^{n} \frac{Y_i}{\lambda^i})(\lambda^{m+n} \sum_{j=0}^{n+m} \frac{Y_j}{\lambda^j})]$$

$$= \lambda^{2n+m} \sum_{i=0}^{n} \sum_{j=0}^{n+m} \frac{\mathbb{E}[Y_i Y_j]}{\lambda^{i+j}}$$

Note that:

$$\mathbb{E}[Y_i Y_j] = \begin{cases} 0 & i \neq j \\ \frac{\sigma^2}{1-\lambda^2} & i = j = 0 \\ \sigma^2 & \text{O.W} \end{cases}$$

We can simplify $R_x(n, m+n)$ more:

Lets first assume $n > 0$:

$$R_x(n, m+n) = \lambda^{2n+m} \sum_{i=0}^{n} \sum_{j=0}^{n+m} \frac{\mathbb{E}[Y_i Y_j]}{\lambda^{i+j}} \quad \text{terms that } i \neq j \text{ are } 0$$

$$\rightarrow R_X(n, n+m) = \lambda^{2n+m} [\frac{\sigma^2}{1-\lambda^2} + \sum_{i=1}^{n} \frac{\sigma^2}{\lambda^{2i}}]$$

$$= \lambda^{2n+m} \times \frac{\sigma^2}{1-\lambda^2} \times [1 + \frac{1-\lambda^{2n}}{\lambda^{2n}}]$$

$$= \lambda^{2n+m} \times \frac{\sigma^2}{1-\lambda^2} \times [\frac{1}{\lambda^{2n}}] = \lambda^m \frac{\sigma^2}{1-\lambda^2}$$

here we used the fact that:

$$\sum_{i=1}^{n} \frac{1}{\lambda^{2i}} = \frac{1-\lambda^{2n}}{\lambda^{2n}(1-\lambda^2)}$$

for the case which $n = 0$:

$$R_x(0, m) = \lambda^m \sum_{i=0}^{0} \sum_{j=0}^{m} \frac{\mathbb{E}[Y_i Y_j]}{\lambda^{i+j}} = \lambda^m \mathbb{E}[Y_0^2] = \lambda^m \frac{\sigma^2}{1-\lambda^2}$$

We can conclude that for any $n \geq 0$ and any $m \geq 0$:

$$R_x(n, m+n) = \lambda^m \frac{\sigma^2}{1-\lambda^2}$$

In case the first argument is greater:

$$R_x(m+n, n) = R_x(n, m+n) = \lambda^{-(n-(n+m))} \frac{\sigma^2}{1-\lambda^2}$$

So in general:

$$R_x(m, n) = \lambda^{|m-n|} \frac{\sigma^2}{1-\lambda^2}$$

So $X(n)$ is WSS. now we can calculate C_x:

$$C_x(m, n) = R_x(m, n) - \eta_x^2 = R_x(m, n) = \lambda^{|m-n|} \frac{\sigma^2}{1-\lambda^2}$$

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From the definition of Covariance we know that:

$$\mathrm{Var}(X) = \mathrm{Cov}(X, X)$$

And also from Covariace properties we know that:

$$\mathrm{Cov}(X+Y, W+Z) = \mathrm{Cov}(X, W) + \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, W) + \mathrm{Cov}(Y, Z)$$

$$\mathrm{Cov}(aX, Z) = a\mathrm{Cov}(X, Z)$$

Also because $X(t)$ is WSS:

$$\mu_x(t) = [X(t)] = \mu_x$$

$$R(t_1, t_2) = R(t_1 - t_2)$$

It follows that:

$$\mathrm{Var}(X(t+s) - X(t)) = \mathrm{Cov}(X(t+s) - X(t), X(t+s) - X(t))$$

$$= \mathrm{Cov}(X(t+s), X(t+s)) - 2\mathrm{Cov}(X(t+s), X(t)) + \mathrm{Cov}(X(t), X(t))$$

$$= R_x(0) - \mu_x^2 - 2(R_x(s) - \mu_x^2) + R_x(0) - \mu_x^2$$

$$= 2R_x(0) - 2R_x(s)$$

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Let $m_1, m_2, \dots, m_n \quad (\forall_i \quad m_i \in \mathbb{Z})$ be the indeces. n^{th} order density function can be written as:

$$f(x_1, \dots, x_n, m_1, \dots, m_n)$$

Because they are independent this can be simplified to:

$$= f(x_1, m_1) f(x_2, m_2) \dots f(x_n, m_n) = \prod_{i=1}^{n} f(x_i, m_i)$$

And because they are identical, index can be omitted.

$$= \prod_{i=1}^{n} f(x_i)$$

Similarly for the shifted n^{th} order density function we have: ($c \in \mathbb{Z}$)

$$f(x_1, \dots, x_n, m_1 + c, \dots, m_n + c) = \prod_{i=1}^{n} f(x_i)$$

Thus,

$$f(x_1, \dots, x_n, m_1 + c, \dots, m_n + c) = f(x_1, \dots, x_n, m_1, \dots, m_n) \quad \forall c \in \mathbb{Z}$$

Equality holds for any n. So $X(t)$ is n^{th} order stationary for any n. Therefore $X(t)$ is SSS.

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First consider the following lemmas:

Lemma 1

If $f : \mathbb{R} \to \mathbb{R}$ is a periodic function with period T and $X(t) = f(t + \tau), \quad \tau \sim \text{Unif}(0, T)$, then:

$$F(x_1, \dots, x_n, t_1, \dots, t_n) = F(x_1, \dots, x_n, t'_1, \dots, t'_n)$$

Where

$$\forall_i \quad t'_i = kT + t_i \quad \text{for some } k \in \mathbb{Z} \quad \text{and} \quad 0 \le t'_i < T$$

Proof:

$$F(x_1, \dots, x_n, t'_1, \dots, t'_n) = \mathbb{P}[X(t'_1) \le x_1, \dots, X(t'_n) \le x_n] = \mathbb{P}[f(t'_1 + \tau) \le x_1, \dots, f(t'_n + \tau) \le x_n]$$

$$= \mathbb{P}[f(t_1 + kT + \tau) \le x_1, \dots, f(t_n + kT + \tau) \le x_n] = \mathbb{P}[f(t_1 + \tau) \le x_1, \dots, f(t_n + \tau) \le x_n]$$

$$= F(x_1, \dots, x_n, t_1, \dots, t_n)$$

Lemma 2

If $f : \mathbb{R} \to \mathbb{R}$ is a periodic function with period T and $X(t) = f(t + \tau), \quad \tau \sim \text{Unif}(0, T)$, then:

$$F(x_1, \dots, x_n, t_1, \dots, t_n) = F(x_1, \dots, x_n, 0, t_2 - t_1, \dots, t_n - t_1)$$

Proof:

First suppose that $\forall_i \; 0 \le t_i < T$. If they are not, by Lemma 1 you can choose t_i's that are in the specified range.

Then without loss of generality suppose that $t_1 \le \min(t_2, \dots, t_n)$. Note that this assumption doesn't affect the result. because you can always swap random variables in the CDF:

$$F(x_1, x_2, t_1, t_2) = F(x_2, x_1, t_2, t_1)$$

We can also define the Indicator function as below:

$$I\{A\} = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } \bar{A} \end{cases}$$

Which A is an event.

Now we can start proving the lemma:

$$F(x_1, \dots, x_n, t_1, t_2, \dots, t_n) = \mathbb{P}\{X(t_1) \le x_1, X(t_2) \le x_2, \dots, X(t_n) \le x_n\}$$

$$= \mathbb{P}\{f(t_1+\tau) \le x_1, f(t_2+\tau) \le x_2, \dots, f(t_n+\tau) \le x_n\} = \int_0^T I\{f(t_1+\tau) \le x_1, f(t_2+\tau) \le x_2, \dots, f(t_n+\tau) \le x_n\} f_\tau(\tau)\, d\tau$$

$$= \int_0^{T-t_1} I\{f(t_1 + \tau) \le x_1, f(t_2 + \tau) \le x_2, \dots, f(t_n + \tau) \le x_n\} \frac{1}{T}\, d\tau$$

$$+ \int_{T-t_1}^{T} I\{f(t_1 + \tau) \le x_1, f(t_2 + \tau) \le x_2, \dots, f(t_n + \tau) \le x_n\} \frac{1}{T}\, d\tau$$

In the first integral substitute $u = t_1 + \tau$ and in the second integral substitute $u = t_1 + \tau - T$:

$$= \int_{t_1}^{T} I\{f(u) \le x_1, f(u + t_2 - t_1) \le x_2, \dots, f(u + t_n - t_1) \le x_n\} \frac{1}{T}\, du$$

$$+ \int_0^{t_1} I\{f(u + T) \le x_1, f(u + t_2 - t_1 + T) \le x_2, \dots, f(u + t_n - t_1 + T) \le x_n\} \frac{1}{T}\, du$$

$$= \int_{t_1}^{T} I\{f(u) \le x_1, f(u + t_2 - t_1) \le x_2, \dots, f(u + t_n - t_1) \le x_n\} \frac{1}{T}\, du$$

$$+ \int_0^{t_1} I\{f(u) \le x_1, f(u + t_2 - t_1) \le x_2, \dots, f(u + t_n - t_1) \le x_n\} \frac{1}{T}\, du$$

$$= \int_0^{T} I\{f(0 + u) \le x_1, f(t_2 - t_1 + u) \le x_2, \dots, f(t_n - t_1 + u) \le x_n\} \frac{1}{T}\, du$$

$$= F(x_1, x_2, \dots, x_n, 0, t_2 - t_1, \dots, t_n - t_1)$$

Now we can use these lemmas to prove that $X(t) = Y(t+T),\ T \sim \text{Unif}(0, T_0)$ is SSS. By lemma 2:

$$F(x_1, x_2, \ldots, x_n, t_1, \ldots, t_n) = F(x_1, x_2, \ldots, x_n, 0, t_2 - t_1, \ldots, t_n - t_1)$$

$$F(x_1, x_2, \ldots, x_n, t_1 + c, t_2 + c, \ldots, t_n + c) = F(x_1, x_2, \ldots, x_n, 0, t_2 - t_1, \ldots, t_n - t_1)$$

$$\rightarrow F(x_1, x_2, \ldots, x_n, t_1, \ldots, t_n) = F(x_1, x_2, \ldots, x_n, t_1 + c, t_2 + c, \ldots, t_n + c)$$

Which holds for any n. Which means that $X(t)$ is SSS.

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$$x(t) = Y + Zt \quad \text{and} \quad Y, Z \sim \mathcal{N}(1, 1)$$

First start by calculating $\eta_x(t)$:

$$\eta_x(t) = \mathbb{E}[x(t)] = \mathbb{E}[Y + Zt] = 1 + t$$

Then we calculate R_x:

$$R_x(t_1, t_2) = \mathbb{E}[x(t_1)x(t_2)] = \mathbb{E}[(Y + Zt_1)(Y + Zt_2)] = \mathbb{E}[Y^2] + (t_1 + t_2)\mathbb{E}[Y]\mathbb{E}[Z] + t_1 t_2 \mathbb{E}[Z^2]$$

we have:

$$\mathbb{E}[Y] = \mathbb{E}[Z] = 1 \qquad \mathbb{E}[Y^2] = \mathbb{E}[Z^2] = 2$$

$$\rightarrow R_x(t_1, t_2) = 2 + (t_1 + t_2) + 2t_1 t_2$$

Finally we can calculate C_x:

$$\rightarrow C_x(t_1, t_2) = R_x(t_1, t_2) - \eta_x(t_1)\eta_x(t_2) = 2 + (t_1 + t_2) + 2t_1 t_2 - (1 + t_1)(1 + t_2) = 1 + t_1 t_2$$

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$$x(t) = Acos(wt) + Bsin(wt) \quad \text{and} \quad A, B \sim \mathcal{N}(0, 1)$$

We should check for η_x being constant and $R_x(t_1, t_2)$ being a function of $t_1 - t_2$. First we find $\eta_x(t)$:

$$\eta_x(t) = \mathbb{E}[x(t)] = \mathbb{E}[Acos(wt)] + \mathbb{E}[Bsin(wt)] = 0 + 0 = 0$$

First condition is met. Now we calculate $R_x(t_1, t_2)$:

$$R_x(t_1, t_2) = \mathbb{E}[x(t_1)x(t_2)]$$

$$= \mathbb{E}[A^2](cos(wt_1)cos(wt_2)) + \mathbb{E}[B^2](sin(wt_1)sin(wt_2)) + \mathbb{E}[A]\mathbb{E}[B](sin(wt_1)cos(wt_2) + cos(wt_1)sin(wt_2))$$

$$= \mathbb{E}[A^2](cos(wt_1)cos(wt_2)) + \mathbb{E}[B^2](sin(wt_1)sin(wt_2)) \quad \text{because} \quad \mathbb{E}[A] = \mathbb{E}[B] = 0$$

$$= cos(wt_1)cos(wt_2) + sin(wt_1)sin(wt_2) = cos(w(t_1 - t_2)) = R_x(t_1 - t_2) = R_x(\tau)$$

Now we can conclude that x is WSS.

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First start by finding $\eta_Y(n)$:

$$\eta_Y(n) = \mathbb{E}[y(n)] = \mathbb{E}[\sum_{i=1}^{n} X_i] = n\sum_{i=1}^{n}\mathbb{E}[X_1] = 0$$

With $\eta_Y(n) = 0$ we have $C_Y(m,n) = R_Y(m,n)$ so: (suppose for now $m \leq n$)

$$R_Y(m,n) = \mathbb{E}[y(m)y(n)] = \sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}[X_iX_j]$$

note that:

$$\mathbb{E}[X_iX_j] = \begin{cases} 0 & \text{if} \quad i \neq j \\ 4 & \text{if} \quad i = j \end{cases}$$

thus,

$$C_Y(m,n) = R_Y(m,n) = 4m$$

Now if not $m \leq n$ then:

$$C_Y(m,n) = R_Y(m,n) = min(m,n) \times 4$$