Sharif University of Technology

CE Department

Course: Stochastic Processes

PS. 4

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1.1 a

First we need to find $S_x(w)$:

$$S_x(w) = \int_{-\infty}^{\infty} 2e^{-\frac{|\tau|}{\alpha}} e^{-jw\tau}\, d\tau$$

$$= \int_{-\infty}^{0} 2e^{-\frac{-\tau}{\alpha}} e^{-jw\tau}\, d\tau + \int_{0}^{\infty} 2e^{-\frac{\tau}{\alpha}} e^{-jw\tau}\, d\tau$$

$$= \frac{2}{\frac{1}{\alpha} + jw} + \frac{2}{\frac{1}{\alpha} - jw} = \frac{\frac{4}{\alpha}}{\frac{1}{\alpha^2} + w^2}$$

Define $a = \frac{1}{\alpha}$:

$$S_x(w) = \frac{4a}{a^2 + w^2}$$

We can substitute a later.

Then we need to find $|H(w)|^2$:

$$H(w) = 1 + \frac{e^{-2jw}}{2} \quad , \quad H^*(w) = 1 + \frac{e^{2jw}}{2}$$

$$\rightarrow |H(W)|^2 = (1 + \frac{e^{-2jw}}{2})(1 + \frac{e^{2jw}}{2}) = \frac{5}{4} + \frac{e^{-2jw}}{2} + \frac{e^{2jw}}{2}$$

Knowing that $S_y(w) = S_x(w)|H(w)|^2$:

$$S_y(w) = \frac{5}{2} \times \frac{2a}{a^2 + w^2} + e^{-2jw} \times \frac{2a}{a^2 + w^2} + e^{2jw} \times \frac{2a}{a^2 + w^2}$$

Note that:

$$\mathscr{F}\{\delta(a + \tau)\} = e^{ajw} \quad \rightarrow \quad \delta(a + \tau) = \mathscr{F}^{-1}\{e^{ajw}\}$$

And for positive a:

$$\mathscr{F}\{e^{-a|\tau|}\} = \frac{2a}{a^2 + w^2} \quad \rightarrow \quad e^{-a|\tau|} = \mathscr{F}^{-1}\{\frac{2a}{a^2 + w^2}\}$$

Then R_y is given by:

$$R_y(\tau) = \frac{5}{2} e^{-a|\tau|} + \delta(\tau + 2) * e^{-a|\tau|} + \delta(\tau - 2) * e^{-a|\tau|}$$

$$\rightarrow R_y(\tau) = \frac{5}{2} e^{-a|\tau|} + e^{-a|\tau+2|} + e^{-a|\tau-2|}$$

1.2 b

In an LTI system defined by impulse response $h(t)$, output signal in a given point is a linear combination of the input signal.

$$y(t) = \int_{-\infty}^{\infty} h(t - \alpha) x(\alpha)\, d\alpha$$

Here we can see $h(t - \alpha)$ as the coefficient of $x(\alpha)$ in $y(t)$. Then because any linear summation of jointly normal random variables is also jointly normal, then any subset of y is jointly normal. Thus $y(t)$ is a gaussian process.

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2.1 a

Define Φ_3 as:

$$\Phi_3(x) = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \end{bmatrix}$$

Then

$$\Phi_3(x)^T \Phi_3(y) = \Phi_1(x)^T \Phi_1(y) + \Phi_2(x)^T \Phi_2(y) = K_3(x, y)$$

So there exists Φ s.t $K_3(x, y) = \Phi(x)^T \Phi(y)$ so k_3 is a valid kernel.

2.2 b

First suppose that Φ_1 and Φ_2 are as below:

$$\Phi_1 = \begin{bmatrix} \Phi_{11} \\ \vdots \\ \Phi_{1n} \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} \Phi_{21} \\ \vdots \\ \Phi_{2m} \end{bmatrix}$$

Then define $\Phi_3 \in \mathcal{R}^{m \times n}$ as:

$$\Phi_3(x) = \begin{bmatrix} \Phi_{11}(x) \begin{bmatrix} \Phi_{21}(x) \\ \vdots \\ \Phi_{2m}(x) \end{bmatrix} \\ \vdots \\ \Phi_{1n}(x) \begin{bmatrix} \Phi_{21}(x) \\ \vdots \\ \Phi_{2m}(x) \end{bmatrix} \end{bmatrix}$$

$$\rightarrow \Phi_3(x)^T \Phi_3(y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \Phi_{1i}(x)\Phi_{1i}(y)\Phi_{2j}(x)\Phi_{2j}(y) = (\sum_{i=1}^{n} \Phi_{1i}(x)\Phi_{1i}(y))(\sum_{j=1}^{m} \Phi_{2j}(x)\Phi_{2j}(y))$$
$$= \Phi_1(x)^T \Phi_1(y) \times \Phi_2(x)^T \Phi_2(y) = K_3(x, y)$$

Then $k_1 \times k_2$ is also valid.

2.3 c

From (b) we know that if k_1 and k_2 are valid kernels then $k_1 \times k_2$ is also valid. By induction for any $n \geq 1$ if k_1 is valid then k_1^n is valid. Also note that if k_1 is valid then for a positive c, ck_1 is also valid. We can prove this by defining $\Phi = \sqrt{c}\Phi_1$.

Also from (a) we know that $\sum k_i$ is a valid kernel. Then the following kernel would be valid:

$$1 + \sum_{i=1}^{n} \frac{k_1^i}{i!} = e^{k_1}$$

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3.1 a

Define $N(t_1, t_2)$ as the events between t_1 and t_2 and $N(t_1)$ as the events between 0 and t_1. First suppose that $t_1 \leq t_2$.

$$\mathrm{Cov}(N(t_1), N(t_2)) = \mathbb{E}[N(t_1) \times (N(t_1) + N(t_1, t_2))] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)] = \mathbb{E}[N^2(t_1)] + \mathbb{E}[N(t_1, t_2)N(t_1)] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)]$$

$$= \lambda t_1 + \lambda^2 t_1^2 + \lambda^2 (t_2 - t_1) t_1 - \lambda^2 t_2 t_1 = \lambda t_1$$

For the general case:

$$\mathrm{Cov}(t_1, t_2) = \lambda \mathrm{min}(t_1, t_2)$$

It is clear that this process is not W.S.S. Additionally $\mu_N(t) = \lambda t$.

3.2 b

Without Loss of generality suppose that $t_1 \leq t_2$:

$$\mathbb{P}[N(t_1) = n, N(t_2) = m] = \mathbb{P}[N(t_1) = n, N(t_1, t_2) = m - n] = e^{-\lambda t_1} \times \frac{(\lambda t_1)^n}{n!} \times e^{-\lambda(t_2 - t_1)} \frac{(\lambda(t_2 - t_1))^{(m-n)}}{(m-n)!}$$

$$= e^{-\lambda t_2} \times \frac{(\lambda t_1)^n}{n!} \times \frac{(\lambda(t_2 - t_1))^{(m-n)}}{(m-n)!}$$

3.3 c

Just like the previous section,

$$\mathbb{P}[N(t_0) = a_0, \ldots, N(t_n) = a_n] = \mathbb{P}[N(t_0) = a_0, N(t_0, t_1) = a_1 - a_0, \ldots, N(t_{n-1}, t_n) = a_n - a_{n-1}]$$

$$= e^{-\lambda t_0} \times \frac{(\lambda t_0)^{a_0}}{a_0!} \times e^{-\lambda(t_1 - t_0)} \frac{(\lambda(t_1 - t_0))^{(a_1 - a_0)}}{(a_1 - a_0)!} \cdots \times e^{-\lambda(t_n - t_{n-1})} \frac{(\lambda(t_n - t_{n-1}))^{(a_n - a_{n-1})}}{(a_n - a_{n-1})!}$$

$$= e^{-\lambda t_n} \times \frac{(\lambda t_0)^{a_0}}{a_0!} \times \frac{(\lambda(t_1 - t_0))^{(a_1 - a_0)}}{(a_1 - a_0)!} \cdots \times \frac{(\lambda(t_n - t_{n-1}))^{(a_n - a_{n-1})}}{(a_n - a_{n-1})!}$$

3.4 d

Now substitute $a_0 = a, a_n = a + nk$ and $t_i = t + in$:

$$f(a_0, \ldots, a_n) = e^{-\lambda(t+n^2)} \frac{(\lambda t)^a}{a!} \times \frac{(\lambda n)^{(a_1 - a_0)}}{(a_1 - a_0)!} \cdots \times \frac{(\lambda n)^{(a+nk-a_{n-1})}}{(a + nk - a_{n-1})!}$$

$$= e^{-\lambda t}(e^{-\lambda n})^n \times \frac{(\lambda n)^{nk}}{(a_1 - a)! \ldots (a + nk - a_{n-1})!}$$

Now define $d_i = a_i - a_{i-1}$ ($i \in \{1, \ldots n\}$ and $d_i \in \mathcal{N} \cup \{0\}$) such that $\sum_i d_i = nk$. Then we have:

$$\text{Argmax}_{a_1,\ldots,a_{n-1}} f(a, a_1, \ldots, a_{n-1}, a + nk) \propto \text{Max}_{d_1,\ldots,d_n} \frac{1}{\prod_i d_i!} \quad \text{s.t} \quad \sum_i d_i = nk$$

So we need to find:

$$\text{Argmax}_{d_1,\ldots,d_n} \frac{1}{\prod_i d_i!} \quad , \quad \sum_i d_i = nk$$

Or:

$$\text{Argmin}_{d_1,\ldots,d_n} \prod_i d_i! \quad , \quad \sum_i d_i = nk$$

3.4.1 Claim:

The minimum occurs in $d_1 = d_2 = \cdots = k$.

3.4.2 Proof:

Suppose that $d_i < k < d_j$ for some $i \neq j$. (Note that if no such i, j exist then all d_i's are equal to k). Define the function that we want to minimize as g. $g(d_1, \ldots d_n) = \prod_i d_i!$. Then it's trivial to see that $\frac{g(d_1,\ldots d_i+1,\ldots,d_j-1,\ldots d_n)}{g(d_1,\ldots d_i,\ldots,d_j,\ldots d_n)} = \frac{d_i+1}{d_j} < 1$. This means that the minimizer of the function g can not happend where there exists i, j such that $d_i < k < d_j$. Meaning that for all d_i, $d_i = k$.

So we have:

$$a_i = a + ik$$

3.5 e

$$N(4) = N(0,3) + N(3,4) \quad , \quad N(0,3) \sim \text{Pois}(3\lambda, N(3,4) \sim \text{Pois}(\lambda)$$

$$N(5) - N(3) = N(3,4) + N(4,5) \quad , \quad N(3,4) \sim \text{Pois}(\lambda, N(4,5) \sim \text{Pois}(\lambda)$$

Conditioning on $N(3,4)$ we can write:

$$\mathbb{P}[N(4) < 3, N(5) - N(3) > 3] = \mathbb{P}[N(0,3) < 3]\mathbb{P}[N(4,5) > 3]\mathbb{P}[N(3,4) = 0]$$

$$+\mathbb{P}[N(0,3) < 2]\mathbb{P}[N(4,5) > 2]\mathbb{P}[N(3,4) = 1] + \mathbb{P}[N(0,3) < 1]\mathbb{P}[N(4,5) > 1]\mathbb{P}[N(3,4) = 2]$$

$$= (\sum_{k=0}^{2} e^{-3\lambda} \frac{(3\lambda)^k}{k!})(1 - \sum_{k=0}^{3} e^{-\lambda} \frac{(\lambda)^k}{k!})(e^{-\lambda})(e^{-\lambda}) + (\sum_{k=0}^{1} e^{-3\lambda} \frac{(3\lambda)^k}{k!})(1 - \sum_{k=0}^{2} e^{-\lambda} \frac{(\lambda)^k}{k!})(e^{-\lambda})(e^{-\lambda}(\lambda))$$

$$+(\sum_{k=0}^{0} e^{-3\lambda} \frac{(3\lambda)^k}{k!})(1 - \sum_{k=0}^{1} e^{-\lambda} \frac{(\lambda)^k}{k!})(e^{-\lambda})(e^{-\lambda} \frac{(\lambda)^2}{2!})$$

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4.1 a

$$2X_t \sim \mathcal{N}(0,0)$$

$$\to \mathbb{P}[X=0]=1$$

$$\mathrm{Cov}(X_t,X_s)=\frac{\mathrm{Cov}(X_t+X_s,X_t+X_s)-\mathrm{Cov}(X_t,X_t)-\mathrm{Cov}(X_s,X_s)}{2}$$

Note that $X_t+X_s \sim \mathcal{N}(0,\sqrt{|t-s|})$.

$$\mathrm{Cov}(X_t,X_s)=\frac{\sqrt{|t-s|}-0-0}{2}=\frac{\sqrt{|t-s|}}{2}$$

4.2 b

Because $X_{t_1},\dots,X_{t_n}$ are jointly normal then the pdf is given by:

$$X_{t_1},\dots,X_{t_n} \sim \mathcal{N}(0,\Sigma)$$

$$\to f_{X_{t_1},\dots,X_{t_n}}(x_1,\dots,x_n)=\frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}e^{\frac{1}{2}x^T\Sigma^{-1}x}$$

And the Sigma matrix is given by:

$$\Sigma_{ij}=\mathrm{Cov}(X_{t_i},X_{t_j})$$

4.3 c

This process doesn't exist because if a random variable takes with c with probability 1, then it should be independent of any other random variables. Here for every t, X_t takes value 0 with probability 1. So it should be independent of any other $X_{t'}$ but we saw that $\mathrm{Cov}(X_t,X_{t'})\neq 0$ for $t\neq t'$. So this process can not exist.

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5.1 a

Define T_1 as the time of the first maintenance ($T_1 \sim \exp(\mu)$). Then:

$$\mathbb{P}[\text{Break down before first maintenance}]=\mathbb{P}[T_1>h]=1-(1-e^{\mu h})=e^{-\mu h}$$

5.2 b

Define N as the number of maintenances before break down. Also define X as the time before the first maintenance.

$$\mathbb{E}[X|N=n] = \mathbb{E}[h + \sum_{i=1}^{n} T_i] \quad , \quad T_i \sim \exp(\mu)$$

$$\mathbb{E}[X|N=n] = h + \frac{n}{\mu}$$

Now with conditional expectation we can find $\mathbb{E}[X]$. Note that N has geometric distribution. Because each T_i is greater than h with probability $= e^{-\mu h}$ which is a bernouli trial. So the number of maintenances before a break down is $\text{Geo}(e^{-\mu h})$

$$\mathbb{E}[X] = \mathbb{E}_N[\mathbb{E}_X[X|N]] = \mathbb{E}_N[h + \frac{N}{\mu}] = h + \frac{\mathbb{E}[N]}{\mu}$$

We also know that $\mathbb{E}[\text{Geo}(p)] = \frac{1-p}{p}$. Then:

$$\mathbb{E}[X] = h + \frac{1-e^{-\mu h}}{\mu e^{-\mu h}}$$

5.3 c

Define $X_{\text{on}}^{(i)}$ as the i'th time interval that the machine is on and $X_{\text{off}}^{(i)}$ as the i'th interval the machine is off. X_{on} is the total time that the machine is on and X_{off} is the total time that the machine is off. Suppose that a total of n off's and on's have happened:

$$\text{Proportion of on } = \frac{\mathbb{E}[X_{\text{on}}]}{\mathbb{E}[X_{\text{on}} + X_{\text{off}}]} = \frac{\mathbb{E}[X_{\text{on}}^{(1)}]}{\mathbb{E}[X_{\text{on}}^{(1)} + X_{\text{off}}^{(1)}]}$$

From the previous section (b) we know that:

$$\mathbb{E}[X_{\text{on}}^{(1)}] = h + \frac{1-e^{-\mu h}}{\mu e^{-\mu h}}$$

Also $X_{\text{off}}^{(1)} \sim \exp(\lambda)$.

$$\mathbb{E}[X_{\text{off}}^{(1)}] = \frac{1}{\lambda}$$

$$\text{Proportion of on } = \frac{\mathbb{E}[X_{\text{on}}^{(1)}]}{\mathbb{E}[X_{\text{on}}^{(1)} + X_{\text{off}}^{(1)}]} = \frac{h + \frac{1-e^{-\mu h}}{\mu e^{-\mu h}}}{h + \frac{1-e^{-\mu h}}{\mu e^{-\mu h}} + \frac{1}{\lambda}}$$

6

Suppose that we are in car 1 (C1) and we are currently in the queue to enter station 1. Car 2 in already station 1. Define the following random variables.

$X_1 =$ time spent by C1 in station 1. $X_2 =$ time spent by C1 in station 2. $Y_1 =$ time spend by C2 in station 1. $Y_2 =$ time spent by C2 in station 2.

$$X_1, Y_1 \sim \exp(\lambda_1) \quad , \quad X_2, Y_2 \sim \exp(\lambda_2)$$

First we must wait for Y_1 so that we can enter station 1. Then we must wait for $\text{Max}(X_1, Y_2)$ to enter station 2. (Max because by that time both C1 and C2 must have finished their works). Then we must wait for another X_2 so that we can exit.

The total time spent for C1 is given by:

$$T = Y_1 + \text{Max}(X_1, Y_2) + X_2$$

Distribution of Y_1 and X_2 is known. We need to find the distribution of $\text{Max}(X_1, Y_2)$:

$$Z = \text{Max}(X_1, Y_2)\ ,\ F_Z(z) = \mathbb{P}[X_1 \leq z]\mathbb{P}[Y_2 \leq z] = (1 - e^{-\lambda_1 z})(1 - e^{-\lambda_2 z}) = 1 - e^{-\lambda_1 z} - e^{-\lambda_2 z} + e^{-(\lambda_1+\lambda_2)z}$$

$$f_Z(z) = \lambda_1 e^{-\lambda_1 z} + \lambda_2 e^{-\lambda_2 z} - (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)z}$$

$$\mathbb{E}[Z] = \int_0^\infty x(\lambda_1 e^{-\lambda_1 x} + \lambda_2 e^{-\lambda_2 x} - (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)x})\, dx$$

Note that:

$$\int_0^\infty x(\lambda e^{-\lambda x})\, dx = \frac{1}{\lambda}$$

Then:

$$\mathbb{E}[Z] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

$$\rightarrow \mathbb{E}[T] = \frac{2}{\lambda_1} + \frac{2}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

7

7.1 a

G_t is the time difference between t and the last event. D_t is the time difference between t and the next event. $(G_t < x, D_t \leq y)$ means that the time difference between t and the last event is less than x and the time difference between t and the next event is less than or equal to y. This means that at least one event has occured in the time interval $[t - x, t]$ and at least one event has occured in the time interval $[t, t + y]$. This means that $N_{t-x} < N_t < N_{t+y}$. As said earlier $(G_t < x, D_t \leq y)$ means that at least one event has occured in the time interval $[t - x, t]$ and at least one event has occured in the time interval $[t, t + y]$.

$$\mathbb{P}[G_t < x, D_t \leq y] = \mathbb{P}[N(t - x, t) > 0, N(t, t + y) > 0] = \mathbb{P}[N(t - x, t) > 0]\mathbb{P}[N(t, t + y) > 0]$$

$$= (1 - e^{-\lambda x})(1 - e^{-\lambda y})$$

7.2 b

If for a given t, $G_t = t$ ($T_{N_t} = 0$) holds this means that no events have occured in the interval $[0, t]$. (define $T_0 = 0$). Then $(G_t = t, D_t \leq y)$ means that no events have occured in $[0, t]$ and at least one event has occured in $[t, t + y]$.

$$\mathbb{P}[G_t = t, D_t \leq y] = \mathbb{P}[N(0, t) = 0, N(t, t + y) > 0] = (e^{-\lambda t})(1 - e^{-\lambda y})$$

7.3 c

As discussed, $(D_t \leq y)$ means that at least one event happens in the interval $[t, t+y]$.

$$\mathbb{P}[D_t \leq y] = \mathbb{P}[N(t, t+y) > 0] = 1 - e^{-\lambda y} = F_{D_t}(y)$$

$$f_{D_t}(y) = \lambda e^{-\lambda y} \to D_t \sim \exp(\lambda)$$

7.4 d

$(G_t \leq x)$ for $x \leq t$, is equivalent to at least one event happening in the interval $[t-x, t]$.

$$F_{G_t}(x) = \mathbb{P}[G_t \leq x] = \mathbb{P}[N(t-x, t) > 0] = 1 - e^{-\lambda x}$$

For $x > t$, $F_{G_t}(x) = 1$. Because by definition G_t is always less than t.

$$f_{G_t}(x) = \frac{\partial(1 - e^{-\lambda x} + e^{-\lambda x} u(x-t))}{dx} = (\lambda e^{-\lambda x})(1 - u(x-t)) + e^{-\lambda x}\delta(x-t)$$

7.5 e

$$T_1 \sim \exp(\lambda) \quad , \quad X = \text{Min}(T_1, t)$$

$$F_X(x) = \begin{cases} 1 & \text{if } x \leq t \\ 1 - e^{-\lambda x} & \text{O.W} \end{cases}$$

$$F_X(x) = 1 - e^{-\lambda x} + e^{-\lambda x} u(x-t)$$

$$\to X \sim G_t$$

7.6 f

First consider $x < t$. We've shown that $\mathbb{P}[G_t < x, D_t < y] = \mathbb{P}[N(t-x, t) > 0, N(t, t+y) > 0]$ which equals $\mathbb{P}[N(t-x, t) > 0]\mathbb{P}[N(t, t+y) > 0]$. This holds because of poisson properties.

$$\mathbb{P}[N(t-x, t) > 0]\mathbb{P}[N(t, t+y) > 0] = \mathbb{P}[G_t < x]\mathbb{P}[D_t < y]$$

So G_t and D_t are independent.

7.7 g

$$\mathbb{E}[G_t] = \mathbb{E}[\text{Min}(T1, t)] = \int_0^\infty \text{Min}(T, t)\lambda e^{-\lambda T}\, dT$$

$$= \int_0^t T\lambda e^{-\lambda T}\, dT + t\lambda \int_t^\infty e^{-\lambda T}\, dT = \frac{1 - e^{-\lambda t}}{\lambda}$$

Also note that $G_t + D_t = T_{N_{t+1}} - T_{N_t} \sim \exp(\lambda)$:

$$\mathbb{E}[G_t + D_t] = \frac{1}{\lambda}$$