Sharif University of Technology

CE Department

Course: Stochastic Processes

PS. 1

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(a)

Let A be the event that 5 coin tosses appear heads. by the law of total probability:

$$\mathbb{P}[A] = \int_0^1 \mathbb{P}[A|p]\mathbb{P}[p]\,dp = \int_0^1 p^5 \times 1\,dp = \frac{p^6}{6}\Big|_0^1 = \frac{1}{6}$$

(b)

Let A be the event that 4 first coin tosses are heads and let B be the event that the fifth toss is also a head. we want to calculate:

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[A]}$$

Again by using the law of total probability:

$$\mathbb{P}[A \cap B] = \int_0^1 \mathbb{P}[A \cap B|p]\mathbb{P}[p] = \int_0^1 p^5\,dp = \frac{p^6}{6}\Big|_0^1 = \frac{1}{6}$$

$$\mathbb{P}[A] = \int_0^1 \mathbb{P}[A|p]\mathbb{P}[p] = \int_0^1 p^4\,dp = \frac{p^5}{5}\Big|_0^1 = \frac{1}{5}$$

thus,

$$\mathbb{P}[B|A] = \frac{5}{6}$$

2

First we need to find $f_{X_1+X_2}$, this function can be derived by applying convolution.

$$Z = X_1 + X_2 \quad , \quad f_Z(z) = \int_{-\infty}^{\infty} f_{X_1,X_2}(\tau, z-\tau)\,d\tau = \lambda_1\lambda_2\, e^{-\lambda_2 z} \int_0^z e^{(\lambda_2-\lambda_1)\tau}\,d\tau$$

Now we should solve for two cases.

(1)

Suppose $\lambda_1 = \lambda_2 = \lambda$ then:

$$f_Z(z) = \lambda^2 e^{-\lambda z} z \sim Gamma(2, \lambda)$$

$$f_{X_1|Z}(x|z) = \frac{f_{X_1Z}(x,z)}{f_Z(z)} = \frac{f_{X_1X_2}(x, z-x)}{f_Z(z)} = \frac{\lambda^2 e^{-\lambda x} e^{-\lambda(z-x)}}{\lambda^2 e^{-\lambda z} z}$$

$$f_{X_1|Z=2}(x) = \frac{\lambda^2 e^{-\lambda x} e^{-\lambda(2-x)}}{\lambda^2 e^{-\lambda 2} 2} = \frac{1}{2}$$

which is equal to:

$$\lim_{\lambda_1-\lambda_2 \to 0} \frac{(\lambda_1-\lambda_2)e^{-(\lambda_1-\lambda_2)x}}{1-e^{-2(\lambda_1-\lambda_2)}}$$

(2)

Now suppose that $\lambda_1 \neq \lambda_2$:

$$f_Z(z) = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 z} - e^{-\lambda_2 z})$$

$$f_{X_1|Z}(x|z) = \frac{f_{X_1Z}(x,z)}{f_Z(z)} = \frac{f_{X_1X_2}(x, z-x)}{f_Z(z)} = \frac{(\lambda_1 - \lambda_2)e^{-(\lambda_1-\lambda_2)x}}{1 - e^{-z(\lambda_1-\lambda_2)}}$$

$$f_{X_1|Z=2}(x) = \frac{(\lambda_1 - \lambda_2)e^{-(\lambda_1-\lambda_2)x}}{1 - e^{-2(\lambda_1-\lambda_2)}}$$

now we can calculate the expected value:

$$\mathbb{E}[X_1|Z=2] = \frac{(\lambda_1 - \lambda_2)}{1 - e^{-2(\lambda_1-\lambda_2)}}\int_0^2 xe^{-(\lambda_1-\lambda_2)x}\,dx = \frac{(2\lambda_2 - 2\lambda_1 - 1)\,\mathrm{e}^{2\lambda_2} + \mathrm{e}^{2\lambda_1}}{(\lambda_2 - \lambda_1)\left(\mathrm{e}^{2\lambda_2} - \mathrm{e}^{2\lambda_1}\right)}$$

3

Let X be the number of tosses until 2 heads or 2 tails appear. and also $\mathbb{P}[Heads] = p$ then:

$$\mathbb{P}[X=k] = \begin{cases} 0 \qquad k < 2 \\ p^{n+1}(1-p)^{n-1} + (1-p)^{n+1}p^{n-1} \qquad k = 2n \\ p^{n+1}(1-p)^{n} + (1-p)^{n+1}p^{n} \qquad k = 2n+1 \end{cases}$$

then $\mathbb{E}[X]$ can be derived by:

$$\mathbb{E}[X] = \sum_{n=1}^{\infty}(2n)(p^{n+1}(1-p)^{n-1} + (1-p)^{n+1}p^{n-1}) + \sum_{n=1}^{\infty}(2n+1)(p^{n+1}(1-p)^{n} + (1-p)^{n+1}p^{n})$$

$$= 2\frac{p^2+q^2}{(1-pq)^2} + 2\frac{p^2q+q^2p}{(1-pq)^2} + \frac{p^2q+q^2p}{1-pq}$$

(a)

this part doesn't make any sense.

(b)

by Markov's inequality:

$$\mathbb{P}[X \leq 9] \geq 1 - \frac{\mathbb{E}[X]}{9} == 2\frac{p^2+q^2}{9(1-pq)^2} + 2\frac{p^2q+q^2p}{9(1-pq)^2} + \frac{p^2q+q^2p}{9(1-pq)}$$

for special case $p = \frac{1}{2}$:

$$\mathbb{P}[X \leq 9] \geq 1 - \frac{3}{9} = \frac{2}{3}$$

4

$$\lambda = 1 \quad , \quad X_i \sim Pois(1) \quad , \quad Z = \sum_{i=1}^{n} X_i \sim \mathcal{N}(n, n)$$

Also sum of independent poissons is also a poisson.

$$Z = \sum_{i=1}^{n} X_i \sim Pois(n)$$

Next we calculate $\mathbb{P}[Z \leq n]$ in two different ways.

$$Z \sim \mathcal{N}(n, n) \to \mathbb{P}[Z \leq n] = 1/2 \quad \mu \text{ is normal r.v's median}$$

$$Z \sim Pois(n) \to \mathbb{P}[Z \leq n] = \sum_{i=0}^{n} \frac{e^{-n} n^i}{i!}$$

thus,

$$\lim_{n \to \infty} \sum_{i=0}^{n} \frac{e^{-n} n^i}{i!} = \frac{1}{2}$$

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$$Var(Z|Y) = \mathbb{E}[Z^2|Y] - \mathbb{E}[Z|Y]^2 = \mathbb{E}[(XY)^2|Y] - \mathbb{E}[XY|Y]^2$$

$$\mathbb{E}[(XY)^2|Y] = \int_{-\infty}^{\infty} x^2 y^2 \frac{f_{XY}(x,y)}{f_Y(y)}\, dx = y^2 \int_{-\infty}^{\infty} x^2 \frac{f_{XY}(x,y)}{f_Y(y)}\, dx = Y^2 \mathbb{E}[X^2|Y]$$

$$\mathbb{E}[XY|Y] = \int_{-\infty}^{\infty} xy \frac{f_{XY}(x,y)}{f_Y(y)}\, dx = y \int_{-\infty}^{\infty} x \frac{f_{XY}(x,y)}{f_Y(y)}\, dx = Y\mathbb{E}[X|Y]$$

$$\to Var(Z|Y) = Y^2\mathbb{E}[X^2|Y] - Y^2\mathbb{E}[X|Y]^2 = Y^2(\mathbb{E}[X^2|Y] - \mathbb{E}[X|Y]^2) = Y^2\ Var(X|Y)$$

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(a)

$$Cov(Y, X) = Cov(X^2, X) = \mathbb{E}[X^3] - \mathbb{E}[X^2]\mathbb{E}[X]$$

note that $\mathbb{E}[X^3] = \mathbb{E}[X] = 0$ because of negative values of x:

$$\mathbb{E}[X^3] = \int_{-1}^{1} \frac{x^3}{2} = 0 \quad \mathbb{E}[X] = \int_{-1}^{1} \frac{x}{2} = 0$$

$$Cov(Y, X) = 0$$

(b)

There is no conflict. $Cov(X, Y) = 0$ implies that there is no linear correlation between X and Y. But independency is a stronger condition. So in conclusion $Cov(X, Y) = 0$ doesn't always imply independency.

7

Let X_i be the number of beads taken starting from $(i-1)^{th}$ unique number seen until the i^{th} unique number is seen. In this way total number of beads taken (X) is given by:

$$X = \sum_{i=1}^{n} X_i \quad , \quad X_i \sim Geo(\frac{n-i+1}{n})$$

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n(\frac{1}{n} + \cdots + \frac{1}{1})$$

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(a)

By Chebyshev's inequality:

$$\mathbb{P}[|X-\mu| \geq a] \leq \frac{\sigma^2}{a^2}$$

Take X to be the mean of i.i.d random variables with defined mean and variance μ and σ^2.

$$X = \frac{1}{n} \times \sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

now for every $a > 0$ we can write:

$$\lim_{n\to\infty} \mathbb{P}[|X-\mu| \geq a] \leq \lim_{n\to\infty} \frac{\sigma^2}{na^2} = 0 \to \lim_{n\to\infty} \mathbb{P}[|X-\mu| \geq a] = 0$$

(b)

Resulting from the weak law of large numbers, sample mean is the interval $[\mu - a, \mu + a]$ with probability 1. In other words

$$\lim_{n\to\infty} \mathbb{P}[\mu - a \leq X \leq \mu + a] = 1$$

Now by letting a take smaller and smaller values we have

$$\lim_{n\to\infty, a\to\infty} \mathbb{P}[\mu - a \leq X \leq \mu + a] = 1$$

$$\lim_{n\to\infty} X \to \mu$$