



Sharif University of Technology

CE Department

Course: Stochastic Processes

PS. 5

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$$m_1 = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] \quad , \quad m_2 = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i^2]$$

$$m_1 = \mu \quad , \quad m_2 = \mu^2 + \sigma^2$$

Now we need to find the inverse functions.

$$\rightarrow \mu = m_1 = \sum_{i=1}^n X_i$$

$$\rightarrow \sigma = \sqrt{m_2 - m_1^2} = \sqrt{\left(\sum_{i=1}^n X_i^2\right) - \left(\sum_{i=1}^n X_i\right)^2}$$

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$$\theta_{\text{MLE}} = \text{Argmax}_{\theta} f(x|\theta)$$

2.1 (a)

$$f(x|\theta) = \frac{1}{\theta^n} \quad \text{given that } \forall_i X_i \leq \theta$$

Note that for any $\theta < \text{Max}(x) : f(x|\theta) = 0$. And because $f(x|\theta)$ is strictly decreasing on \mathbb{R}^+ Then $\forall \theta' : f(x|\theta = \text{Max}(x)) > f(x|\theta')$.

Therefore for any θ :

$$f(x|\theta) < f(x|\text{Max}(x))$$

So $\text{Max}(x)$ is the answer to our MLE estimation.

2.2 (b)

Note that just like the previous section $\forall \theta \leq \text{Max}(x) : f(x|\theta) = 0$. Furthermore we can show that the MLE estimation doesn't exist.

2.2.1 Proof

By now we know that $f(x|\theta) = 0 | \theta \in [0, \text{Max}(x)]$. So the MLE estimation must be in the interval $(\text{Max}(x), \infty)$. Suppose that $\theta_{\text{MLE}} \in (\text{Max}(x), \infty)$ is the MLE estimation. Define θ_1 as $\theta_1 = \frac{\text{Max}(x) + \theta_{\text{MLE}}}{2}$. We have:

$$\theta_1 \in (\text{Max}(x), \infty) \quad , \quad \theta_1 < \theta_{\text{MLE}}$$

$$\rightarrow f(x|\theta_1) > f(x|\theta_{\text{MLE}})$$

Therefore the MLE estimation doesn't exist.

2.3 (c)

First we need to find the likelihood function.

$$\mathcal{L}(\theta|x) = f(x|\theta) = \begin{cases} 1 & \text{if } \forall_i X_i \leq \theta + 1 \quad \text{AND} \quad \forall_i \theta \leq X_i \\ 0 & \text{O.W} \end{cases}$$

Note that for any $\text{Max}(x) - 1 \leq \theta \leq \text{Min}(x)$ the likelihood function of θ is equal to 1. So any θ in the interval $[\text{Max}(x) - 1, \text{Min}(x)]$ can be a MLE estimator.

2.4 (d)

Note that $\mathcal{L}(\theta_1, \theta_2|x) = 0$ if $\theta_1 > \text{Min}(x)$ or $\theta_2 < \text{Max}(x)$. So $\theta_1 < \text{Min}(x)$ and $\theta_2 > \text{Max}(x)$.

$$\begin{aligned} \mathcal{L}(\theta|x) &= \frac{1}{\theta_2 - \theta_1} \\ \frac{\partial \mathcal{L}}{\partial \theta_1} &= \frac{1}{(\theta_1 - \theta_2)^2} \\ \frac{\partial \mathcal{L}}{\partial \theta_2} &= \frac{-1}{(\theta_2 - \theta_1)^2} \end{aligned}$$

Gradient of \mathcal{L} can never be zero. So the optimal value for \mathcal{L} must exist on boundary values of θ_1, θ_2 .

$$\theta_{1\text{MLE}} = \text{Min}(x) \quad , \quad \theta_{2\text{MLE}} = \text{Max}(x)$$

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3.1 (a)

MLE for a: (just like 2-a)

$$\hat{a}_{\text{MLE}} = \text{Max}(x)$$

MLE for η :

$$\mathcal{L}(\eta|x) = f(x|\eta) = \frac{1}{\eta^n} e^{\frac{-1}{\eta} \sum_{i=1}^n x_i}$$

$$\text{Log } \mathcal{L}(\eta|x) = -n \log(\eta) - \frac{1}{\eta} \sum_{i=1}^n x_i$$

$$\frac{\partial \text{Log } \mathcal{L}}{\partial \eta} = -\frac{n}{\eta} + \frac{1}{\eta^2} \sum_{i=1}^n x_i = 0$$

$$\hat{\eta}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

MLE for μ and σ :

$$\mathcal{L}(\mu, \sigma^2|x) = f(x|\mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sigma^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2}$$

$$\text{Log } \mathcal{L}(\mu, \sigma^2 | x) = \text{Const.} - n \text{Log}(\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$$

$$\frac{\partial \text{Log } \mathcal{L}}{\partial \mu} = \frac{1}{\sigma^2} (\mu - x_i) = 0$$

$$\rightarrow \hat{\mu}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^n$$

$$\frac{\partial \text{Log } \mathcal{L}}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^n (x_i - \mu)^2 = 0$$

$$\rightarrow \hat{\sigma}_{\text{MLE}}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2$$

Substitute μ with $\hat{\mu}$:

$$\hat{\sigma}_{\text{MLE}}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu}_{\text{MLE}})^2$$

3.2 (b)

3.2.1 (a)

$$\mathbb{E}[\hat{a}] = \mathbb{E}[\text{Max}(x)]$$

First we need to find the CDF of $\text{Max}(x)$. Also note that for a positive valued random variable X the expected value can be written as:

$$\mathbb{E}[X] = \int_0^\infty S(x) dx \quad , \quad S(x) = 1 - F(x)$$

$$F_{\hat{a}}(y) = \mathbb{P}[\forall_i x_i \leq y] = F_x(y)^n = \left(\frac{y}{a}\right)^n$$

$$\mathbb{E}[\hat{a}] = \int_0^a \left(1 - \left(\frac{y}{a}\right)^n\right) dy = a - \frac{a}{n+1} = \frac{n}{n+1}a$$

So \hat{a} is an biased estimator.

3.2.2 (η)

$$\hat{\eta} = \frac{n}{n} \mathbb{E}[x_i] = \mathbb{E}[x_i]$$

$$\mathbb{E}[x_i] = \int_0^\infty \frac{1}{\eta} e^{-\frac{x}{\eta}} dx = \eta$$

$$\rightarrow \mathbb{E}[\hat{\eta}] = \eta$$

$\hat{\eta}$ is an unbiased estimator.

3.2.3 (μ)

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}[x_i] = \mu$$

$\hat{\mu}$ is unbiased.

3.3 (c)

$$\begin{aligned}
\mathbb{E}[\hat{\sigma}^2] &= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right] = \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n x_i^2 - \frac{2}{n} \sum_{i=1}^n (x_i \bar{x}) + \frac{1}{n} \bar{x}^2\right] \\
&= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n x_i^2 - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n x_i x_j\right] \\
&= (\mu^2 + \sigma^2) - \mu^2 - \frac{\sigma^2}{n} \\
&= \frac{n-1}{n} \sigma^2
\end{aligned}$$

σ^2 is a biased estimator. Then the following estimator will be unbiased.

$$\hat{\sigma}_2^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

3.4 (d)

We know that:

$$\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta}) + \text{Bias}^2(\hat{\theta})$$

for MLE estimator:

$$\text{MSE}(\hat{\sigma}_{\text{MLE}}^2) = \text{Var}(\hat{\sigma}_{\text{MLE}}^2)$$

Now we need to calculate $\text{Var}(\hat{\sigma}^2)$. Note that sum of squared standard normal random variables has a Chi-squared distribution with $n-1$ degrees of freedom.

$$\begin{aligned}
\text{Var}\left(\frac{n-1}{\sigma^2} \hat{\sigma}_2^2\right) &= 2(n-1) \\
\rightarrow \text{Var}(\hat{\sigma}_2^2) &= \frac{2\sigma^4}{n-1} \\
\text{Var}\left(\frac{n}{\sigma^2} \hat{\sigma}^2\right) &= 2(n-1) \\
\rightarrow \text{Var}(\hat{\sigma}^2) &= \frac{2(n-1)\sigma^4}{n^2} \\
\text{MSE}(\hat{\sigma}^2) - \text{MSE}(\hat{\sigma}_2^2) &= \frac{2(n-1)\sigma^4}{n^2} + \frac{\sigma^4}{n^2} - \frac{2\sigma^4}{n-1} \\
&= -\frac{2(2n-1)\sigma^4}{n^2(n-1)} + \frac{\sigma^4}{n^2} = -\frac{(-3n+3)\sigma^4}{n^2(n-1)} < 0
\end{aligned}$$

So the original estimator has a lower MSE.

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We need to find the following two things. First an unbiased estimator and second a complete statistic.

$$\begin{aligned}\mathcal{L}(\theta|x) = f(x|\theta) &= \frac{1}{(2\theta)^n} \quad , \quad \text{given that } \forall_i \quad X_i \leq \theta \text{ And } X_i \geq -\theta \\ &= \frac{1}{(2\theta)^n} \quad \forall_i \quad |X_i| \leq \theta = \frac{1}{(2\theta)^n} \mathcal{I}(\text{Max}(|X_i|) \leq \theta)\end{aligned}$$

By factorization $\text{Max}(|X_i|)$ is a sufficient statistic. Note that it's also a complete statistic.

$$W = \text{Max}(|X_i|)$$

$$\begin{aligned}F_W(w) &= \mathbb{P}[\forall_i -w \leq X_i \leq w] = \left(\frac{w}{\theta}\right)^n \\ f_W(w) &= n \frac{w^{n-1}}{\theta^n}\end{aligned}$$

Now suppose there exists some function g such that $\mathbb{E}[g(\text{Max}(|X_i|))] = 0$.

$$\begin{aligned}\mathbb{E}[g(W)] &= \int_0^\theta g(w) \frac{nw^{n-1}}{\theta^n} dw = 0 \\ &\rightarrow g(\theta)\theta^{n-1} = 0 \rightarrow \mathbb{P}[g(\theta) = 0] = 1\end{aligned}$$

So by now we know that $\text{Max}(|X_i|)$ is a complete statistic. Now we need to find a unbiased estimator. Let's first find the MLE estimator.

$$\mathcal{L}(\theta|x) = f(x|\theta) = \frac{1}{(2\theta)^n} \quad , \quad \text{given that } \forall_i \quad X_i \leq \theta \text{ And } X_i \geq -\theta$$

Just like section 2-1, for any $\theta \leq \text{Max}(|X_i|)$: $\mathcal{L} = 0$ and for any $\theta > \text{Max}(|X_i|)$: $\mathcal{L}(\theta) < \mathcal{L}(\text{Max}(|X_i|))$.

Therefore the MLE estimation of is $\hat{\theta}_{\text{MLE}} = \text{Max}(|X_i|)$. We previously derived its density function.

$$\mathbb{E}[\hat{\theta}_{\text{MLE}}] = \int_0^\theta n \frac{x^n}{\theta^n} dx = \frac{n}{n+1} \theta$$

So $\hat{\theta} = \frac{n+1}{n} \text{Max}(|X_i|)$ is an unbiased estimator. Now we can find the UMVUE by conditioning on complete statistic.

$$\theta_{\text{UMVUE}} = \mathbb{E}\left[\frac{n+1}{n} \text{Max}(|X_i|) | \text{Max}(|X_i|)\right] = \frac{n+1}{n} \text{Max}(|X_i|)$$

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For i.i.d X_1, \dots, X_n The Cramer-Rao is achieved by $T(x)$ iff the derivative of log likelihood function can be factorized as:

$$a(\theta)[T(X) - g(\theta)] = \frac{\partial}{\partial \theta} \log \mathcal{L}$$

5.1 (a)

$$\mathcal{L} = \theta^n \left(\prod_i X_i \right)^{\theta-1}$$

$$\log \mathcal{L} = n \log(\theta) + (\theta - 1) \sum_{i=1}^n \log(X_i)$$

$$\frac{\partial}{\partial \theta} \log \mathcal{L} = \frac{n}{\theta} + \sum_{i=1}^n \log(X_i)$$

So $\sum_{i=1}^n \log(X_i)$ is the UMVUE of $-\frac{n}{\theta}$ and reaches the bound.

5.2 (b)

The density function should be as:

$$f(x|\theta) = \frac{\log(\theta)}{\theta - 1} \theta^x$$

$$\mathcal{L} = \left(\frac{\log(\theta)}{\theta - 1} \right)^n \theta^{\sum_{i=1}^n X_i}$$

$$\log \mathcal{L} = n \log\left(\frac{\log(\theta)}{\theta - 1}\right) + \sum_{i=1}^n X_i \log(\theta)$$

$$\frac{\partial}{\partial \theta} \log \mathcal{L} = \frac{n}{\theta \log(\theta)} - \frac{n}{\theta - 1} + \frac{\sum_{i=1}^n X_i}{\theta}$$

So $\sum_{i=1}^n X_i$ is the UMVUE for $\frac{n\theta}{\theta-1} - \frac{n}{\log(\theta)}$.
