Sharif University of Technology

CE Department

Course: Stochastic Processes

PS. 5

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$$m_1 = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] \quad , \quad m_2 = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i^2]$$

$$m_1 = \mu \quad , \quad m_2 = \mu^2 + \sigma^2$$

Now we need to find the inverse functions.

$$\to \mu = m_1 = \sum_{i=1}^{n} X_i$$

$$\to \sigma = \sqrt{m_2 - m_1^2} = \sqrt{(\sum_{i=1}^{n} X_i^2) - (\sum_{i=1}^{n} X_i)^2}$$

2

$$\theta_{\text{MLE}} = \text{Argmax}_{\theta} f(x|\theta)$$

2.1 (a)

$$f(x|\theta) = \frac{1}{\theta^n} \quad \text{given that } \forall_i X_i \leq \theta$$

Note that for any $\theta < \text{Max}(x) \; : \; f(x|\theta) = 0$. And because $f(x|\theta)$ is strictly decreasing on $\mathbb{R}^+$ Then $\forall \theta' \; : \; f(x|\theta = \text{Max}(x)) > f(x|\theta')$.
Therefore for any θ:

$$f(x|\theta) < f(x|\text{Max}(x))$$

So $\text{Max}(x)$ is the answer to our MLE estimation.

2.2 (b)

Note that just like the previous section $\forall \theta \leq \text{Max}(x) \; : \; f(x|\theta) = 0$. Furthermore we can show that the MLE estimation doesn't exist.

2.2.1 Proof

By now we know that $f(x|\theta) = 0 | \theta \in [0, \text{Max}(x)]$. So the MLE estimation must be in the interval $(\text{Max}(x), \infty)$. Suppose that $\theta_{\text{MLE}} \in (\text{Max}(x), \infty)$ is the MLE estimation. Define θ_1 as $\theta_1 = \frac{\text{Max}(x) + \theta_{MLE}}{2}$.
We have:

$$\theta_1 \in (\text{Max}(x), \infty) \quad , \quad \theta_1 < \theta_{\text{MLE}}$$

$$\to f(x|\theta_1) > f(x|\theta_{\text{MLE}})$$

Therefore the MLE estimation doesn't exist.

2.3 (c)

First we need to find the likelihood function.

$$\mathcal{L}(\theta|x) = f(x|\theta) = \begin{cases} 1 & \text{if } \forall_i X_i \leq \theta + 1 \quad \text{AND} \quad \forall_i \theta \leq X_i \\ 0 & \text{O.W} \end{cases}$$

Note that for any $\text{Max}(x) - 1 \leq \theta \leq \text{Min}(x)$ the likelihood function of θ is equal to 1. So any θ in the interval $[\text{Max}(x) - 1, \text{Min}(x)]$ can be a MLE estimator.

2.4 (d)

Note that $\mathcal{L}(\theta_1, \theta_2|x) = 0$ if $\theta_1 > \text{Min}(x)$ or $\theta_2 < \text{Max}(x)$. So $\theta_1 < \text{Min}(x)$ and $\theta_2 > \text{Max}(x)$.

$$\mathcal{L}(\theta|x) = \frac{1}{\theta_2 - \theta_1}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{1}{(\theta_1 - \theta_2)^2}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{-1}{(\theta_2 - \theta_1)^2}$$

Gradient of $\mathcal{L}$ can never be zero. So the optimal value for $\mathcal{L}$ must exist on boundary values of θ_1, θ_2.

$$\theta_{1_{\text{MLE}}} = \text{Min}(x) \quad , \quad \theta_{2_{\text{MLE}}} = \text{Max}(x)$$

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3.1 (a)

MLE for a: (just like 2-a)

$$\hat{a}_{\text{MLE}} = \text{Max}(x)$$

MLE for η:

$$\mathcal{L}(\eta|x) = f(x|\eta) = \frac{1}{\eta^n} e^{\frac{-1}{\eta} \sum_{i=1}^n x_i}$$

$$\text{Log } \mathcal{L}(\eta|x) = -n\log(\eta) - \frac{1}{\eta} \sum_{i=1}^n x_i$$

$$\frac{\partial \text{ Log } \mathcal{L}}{\partial \eta} = -\frac{n}{\eta} + \frac{1}{\eta^2} \sum_{i=1}^n x_i = 0$$

$$\hat{\eta}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

MLE for μ and σ:

$$\mathcal{L}(\mu, \sigma^2|x) = f(x|\mu, \sigma^2) = (\frac{1}{\sqrt{2\pi}})^n \frac{1}{\sigma^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2}$$

$$\text{Log } \mathcal{L}(\mu, \sigma^2|x) = \text{Const. } - n\text{Log}(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\frac{\partial \text{Log } \mathcal{L}}{\partial \mu} = \frac{1}{\sigma^2}(\mu - x_i) = 0$$

$$\rightarrow \hat{\mu}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n}$$

$$\frac{\partial \text{Log } \mathcal{L}}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i - \mu)^2 = 0$$

$$\rightarrow \hat{\sigma^2}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

Substitude μ with $\hat{\mu}$:

$$\hat{\sigma^2}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu}_{\text{MLE}})^2$$

3.2 (b)

3.2.1 (a)

$$\mathbb{E}[\hat{a}] = \mathbb{E}[\text{Max}(x)]$$

First we need to find the CDF of Max(x). Also note that for a positive valued random variable X the expected value can be written as:

$$\mathbb{E}[X] = \int_0^\infty S(x)\,dx \quad , \quad S(x) = 1 - F(x)$$

$$F_{\hat{a}}(y) = \mathbb{P}[\forall_i x_i \le y] = F_x(y)^n = (\frac{y}{a})^n$$

$$\mathbb{E}[\hat{a}] = \int_0^a (1 - (\frac{y}{a})^n)\,dy = a - \frac{a}{n+1} = \frac{n}{n+1}a$$

So $\hat{a}$ is an biassed estimator.

3.2.2 (η)

$$\hat{\eta} = \frac{n}{n}\mathbb{E}[x_i] = \mathbb{E}[x_i]$$

$$\mathbb{E}[x_i] = \int_0^\infty \frac{1}{\eta}e^{\frac{-x}{\eta}}\,dx = \eta$$

$$\rightarrow \mathbb{E}[\hat{\eta}] = \eta$$

$\hat{\eta}$ is an unbiassed estimator.

3.2.3 (μ)

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}[x_i] = \mu$$

$\hat{\mu}$ is unbiassed.

3.3 (c)

$$\mathbb{E}[\hat{\sigma^2}] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n}x_i^2 - \frac{2}{n}\sum_{i=1}^{n}(x_i\bar{x}) + \frac{1}{n}\bar{x}^2]$$
$$= \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n}x_i^2 - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j]$$
$$= (\mu^2 + \sigma^2) - \mu^2 - \frac{\sigma^2}{n}$$
$$= \frac{n-1}{n}\sigma^2$$

σ^2 is a biassed estimator. Then the following estimator will be unbiassed.

$$\hat{\sigma}_2^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

3.4 (d)

We know that:

$$\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}(\hat{\theta}) + \mathrm{Bias}^2(\hat{\theta})$$

for MLE estimator:

$$\mathrm{MSE}(\hat{\sigma^2}_{\mathrm{MLE}}) = \mathrm{Var}(\hat{\sigma^2}_{\mathrm{MLE}})$$

Now we need to calculate $\mathrm{Var}(\sigma^2)$. Note that sum of squarred standard normal random variables has a Chi-squarred distribution with $n-1$ degrees of freedom.

$$\mathrm{Var}(\frac{n-1}{\sigma^2}\hat{\sigma}_2^2) = 2(n-1)$$
$$\to \mathrm{Var}(\hat{\sigma}_2^2) = \frac{2\sigma^4}{n-1}$$
$$\mathrm{Var}(\frac{n}{\sigma^2}\hat{\sigma}^2) = 2(n-1)$$
$$\to \mathrm{Var}(\hat{\sigma}^2) = \frac{2(n-1)\sigma^4}{n^2}$$
$$\mathrm{MSE}(\hat{\sigma}^2) - \mathrm{MSE}(\hat{\sigma}_2^2) = \frac{2(n-1)\sigma^4}{n^2} + \frac{\sigma^4}{n^2} - \frac{2\sigma^4}{n-1}$$
$$= -\frac{2(2n-1)\sigma^4}{n^2(n-1)} + \frac{\sigma^4}{n^2} = -\frac{(-3n+3)\sigma^4}{n^2(n-1)} < 0$$

So the original estimator has a lower MSE.

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We need to find the following two things. First an unbiassed estimator and second a complete statistic.

$$\mathcal{L}(\theta|x) = f(x|\theta) = \frac{1}{(2\theta)^n} \quad , \quad \text{given that } \forall_i \quad X_i <= \theta \text{ And } X_i >= -\theta$$

$$= \frac{1}{(2\theta)^n} \quad \forall_i \quad |X_i| <= \theta = \frac{1}{(2\theta)^n}\mathcal{I}(\text{Max}(|X_i|) <= \theta)$$

By factorization $\text{Max}(|X_i|)$ is a sufficient statistic. Note that it's also a complete statistic.

$$W = \text{Max}(|X_i|)$$

$$F_W(w) = \mathbb{P}[\forall_i - w \leq X_i \leq w] = (\frac{w}{\theta})^n$$

$$f_W(w) = n\frac{w^{n-1}}{\theta^n}$$

Now suppose there exists some function g such that $\mathbb{E}[g(\text{Max}(|X_i|))] = 0$.

$$\mathbb{E}[g(W)] = \int_0^\theta g(w)\frac{nw^{n-1}}{\theta^n}\,dw = 0$$

$$\rightarrow g(\theta)\theta^{n-1} = 0 \rightarrow \mathbb{P}[g(\theta) = 0] = 1$$

So by now we know that $\text{Max}(|X_i|)$ is a complete statistic. Now we need to find a unbiassed estimator. Let's first find the MLE estimator.

$$\mathcal{L}(\theta|x) = f(x|\theta) = \frac{1}{(2\theta)^n} \quad , \quad \text{given that } \forall_i \quad X_i <= \theta \text{ And } X_i >= -\theta$$

Just like section 2-1, for any $\theta \leq \text{Max}(|X_i|) \ : \ \mathcal{L} = 0$ and for any $\theta > \text{Max}(|X_i|) \ : \ \mathcal{L}(\theta) < \mathcal{L}(\text{Max}(|X_i|))$. Therefore the MLE estimation of is $\hat{\theta}_{\text{MLE}} = \text{Max}(|X_i|)$. We previously derived its density function.

$$\mathbb{E}[\hat{\theta}_{\text{MLE}}] = \int_0^\theta n\frac{x^n}{\theta^n}\,dx = \frac{n}{n+1}\theta$$

So $\hat{\theta} = \frac{n+1}{n}\text{Max}(|X_i|)$ is an unbiassed estimator. Now we can find the UMVUE by conditioning on complete statistic.

$$\theta_{\text{UMVUE}} = \mathbb{E}[\frac{n+1}{n}\text{Max}(|X_i|)|\text{Max}(|X_i|)] = \frac{n+1}{n}\text{Max}(|X_i|)$$

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For i.i.d $X_1, \ldots, X_n$ The Cramer-Rao is achieved by $T(x)$ iff the derivative of log likelihood funciton can be factorized as:

$$a(\theta)[T(X) - g(\theta)] = \frac{\partial}{\partial\theta}\log\mathcal{L}$$

5.1 (a)

$$\mathcal{L} = \theta^n (\prod_i X_i)^{\theta-1}$$

$$\log \mathcal{L} = n\log(\theta) + (\theta-1)\sum_{i=1}^{n}\log(X_i)$$

$$\frac{\partial}{\partial\theta}\log \mathcal{L} = \frac{n}{\theta} + \sum_{i=1}^{n}\log(X_i)$$

So $\sum_{i=1}^{n}\log(X_i)$ is the UMVUE of $-\frac{n}{\theta}$ and reaches the bound.

5.2 (b)

The density function should be as:

$$f(x|\theta) = \frac{\log(\theta)}{\theta-1}\theta^x$$

$$\mathcal{L} = (\frac{\log(\theta)}{\theta-1})^n \theta^{\sum_{i=1}^{n} X_i}$$

$$\log \mathcal{L} = n\log(\frac{\log(\theta)}{\theta-1}) + \sum_{i=1}^{n} X_i \log(\theta)$$

$$\frac{\partial}{\partial\theta}\log \mathcal{L} = \frac{n}{\theta\log(\theta)} - \frac{n}{\theta-1} + \frac{\sum_{i=1}^{n} X_i}{\theta}$$

So $\sum_{i=1}^{n} X_i$ is the UMVUE for $\frac{n\theta}{\theta-1} - \frac{n}{\log(\theta)}$.