Stochastic Processes November 7, 2022
Sharif University of Technology CE 40-695
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Time: 25 mins

Name: Std. Number:

Quiz 3 (LTI + Power Spectrum) Solutions

Questions

1. We have a white noise process $X(t)$ with spectral density $S_X(\omega) = N_0$. Let $Y(t)$ a random process satisfying the following equation:

$$2\frac{d}{dt}Y(t) + 3Y(t) = X(t)$$

Find $R_Y(\tau)$ the autocorrelation function of Y.

Solution:
Taking Fourier transform of the equation:

$$2(j\omega)Y(\omega) + 3Y(\omega) = X(\omega)$$

So we have:

$$\begin{aligned} Y(\omega) &= \frac{1}{2(j\omega)+3}X(\omega) \\ \rightarrow S_Y(\omega) &= \frac{1}{|2(j\omega)+3|^2}S_X(\omega) = \frac{1}{4\omega^2+9} \\ \rightarrow R_X(\tau) &= \frac{1}{12}e^{-\frac{3}{2}|\tau|} \end{aligned}$$

2. Let $X(t)$ be a random process with the autocorrelation function as defined bellow:

$$R_X(\tau) = \sum_{k=1}^{N} \cos(k\tau)e^{\beta k}$$

such that $\beta > 0$. Consider the following filter:

$$H(\omega) = e^{-\lambda|\omega|}$$

Where $\lambda > \frac{\beta}{2}$. Find the limit of average power of the random process after the application of this filter, when $N \rightarrow \infty$.

Solution:

$$
\begin{aligned}
S_X(\omega) &= \sum_{k=1}^{N} \pi(\delta(\omega-k)+\delta(\omega+k))e^{\beta k} \\
\to S_Y(\omega) &= |H(\omega)|^2 S_X(\omega) = \sum_{k=1}^{N} \pi(\delta(\omega-k)+\delta(\omega+k))e^{-(2\lambda-\beta)k} \\
\to E[Y(t)^2] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi(\delta(\omega-k)+\delta(\omega+k))e^{-(2\lambda-\beta)k}d\omega \\
&= \sum_{k=1}^{N} e^{-(2\lambda-\beta)k} \to \frac{1}{1-e^{-(2\lambda-\beta)}} - 1 = \frac{e^{-(2\lambda-\beta)}}{1-e^{-(2\lambda-\beta)}}
\end{aligned}
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