Sharif University of Technology

CE Department

Course: Stochastic Processes

PS. 3

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1.1

Their sum need not be WSS. Suppose two WSS processes with cross-correlation given as:

$$R_{xy}(t_1, t_2) = e^{-|t_1+t_2|}$$

$$z(t) = x(t) + y(t) \to R_z(t_1, t_2) = \mathbb{E}[(x(t_1) + y(t_1))(x(t_2) + y(t_2))] = R_x(t_1 - t_2) + R_y(t_1 - t_2) + 2e^{-|t_1+t_2|}$$

They need to be jointly WSS for their sum to be WSS.

1.2

$$x(t) = At + b \quad , \quad \mu_x = b$$

Define S_T as below:

$$S_T = \frac{1}{2T}\int_{-T}^{T} At\ dt + b = b$$
$$\mathrm{Var}(S_T) = 0$$

Mean-ergodic.

1.3

$$R_x(\tau) = \mathscr{F}^{-1}\{\frac{5}{6} \times \frac{6}{9+w^2}\} = \frac{5}{6}e^{-3|\tau|} \quad , \quad \mu_x = 0$$

Define S_T as below:

$$S_T = \frac{1}{T}\int_0^T x(t)\ dt =$$
$$\mathrm{Var}(S_T) = \mathbb{E}[\frac{1}{T^2}\int_0^T\int_0^T x(\alpha)x(\beta)\ d\alpha\ d\beta] = \frac{1}{T^2}\int_0^T\int_0^T e^{-3|\tau|}\ d\alpha\ d\beta$$
$$\lim_{T\to\infty}\mathrm{Var}(S_T) = 0 \quad \text{(Like in Q3)}$$

This process is mean-ergodic.

2

$$\mu_y = \mu_x \int_{-\infty}^{\infty} h(\alpha)\ d\alpha$$

Also

$$\int_{-\infty}^{\infty} h(\alpha)\ d\alpha = \int_{-\infty}^{\infty} h(\alpha)e^{-j\alpha\times 0}\ d\alpha = H(0) = \pm 1$$

In either case:

$$\mu_y = 0$$

To get R_y we must first compute S_y:

$$S_y(w) = S_x(w)|H(w)|^2 \quad , \quad |H(w)| = \sqrt{1+w^2} \ , \ |w| \leq 4\pi$$

$$S_x(w) = \int_{-\infty}^{\infty} e^{-|\tau|-jw\tau} d\tau = \frac{1}{1+jw} + \frac{1}{1-jw} = \frac{2}{1+w^2}$$

$$\rightarrow S_y(w) = \frac{2}{1+w^2} \times (1+w^2) = 2 \quad , \ |w| \leq 4\pi$$

$$R_y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(w)e^{jw\tau}\ dw = \frac{1}{2\pi}\int_{-4\pi}^{4\pi} 2e^{jw\tau}\ dw = \frac{1}{j\pi\tau}e^{jw\tau}\Big|_{-4\pi}^{4\pi} = \frac{e^{4\pi j\tau} - e^{-4\pi j\tau}}{j\pi\tau}$$

$$= \frac{2\mathrm{Sin}(4\pi\tau)}{\pi\tau}$$

Now for $R_y(0)$ first note that in the limit $\lim_{\tau\rightarrow 0}$ it is clear that $R_y(0) = 8$. another way would be:

$$R_y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(w)\ dw = \frac{1}{2\pi}\int_{-4\pi}^{4\pi} 2\ dw = \frac{8\pi}{\pi} = 8$$

3

First let's suppose that we want $y(t)$ to be mean-ergodic. define S_T as:

$$S_T = \frac{1}{T}\int_0^T [x(t)+A]\ dt = A + \frac{1}{T}\int_0^T x(t)\ dt$$

$$y(t) \text{ mean-ergodic } \leftrightarrow \lim_{T\rightarrow\infty} \mathrm{Var}(S_T) = 0$$

$$\mathrm{Var}(S_T) = \mathbb{E}[(A+\frac{1}{T}\int_0^T x(t)\ dt)^2 - 2(\mu_x+\mu_A)(A+\frac{1}{T}\int_0^T x(t)\ dt) + (\mu_x+\mu_A)^2] = \frac{1}{T^2}\int_0^T\int_0^T e^{|\alpha-\beta|}\ d\alpha d\beta + \sigma_A^2 - \mu_x^2$$

define I_T as:

$$I_T = \frac{1}{T^2}\int_0^T\int_0^T e^{-|\alpha-\beta|}\ d\alpha d\beta = \frac{1}{T^2}\int_0^T\int_\beta^T e^{\alpha-\beta}\ d\alpha d\beta + \frac{1}{T^2}\int_0^T\int_0^\beta e^{\beta-\alpha}\ d\alpha d\beta = 2\times\frac{e^{-T}+T-1}{T^2}$$

$$\lim_{T\rightarrow\infty} I_T = 0$$

$$\rightarrow \mathrm{Var}(S_T) = \sigma_A^2 - \mu_X^2$$

Thus,

$$y(t) \text{ is mean-ergodic } \leftrightarrow \sigma_A^2 = \mu_x^2$$

4

First note that for both y_1 and y_2:

$$\mu_{y_1} = \mu_{y_2} = \mu_x = 0$$

$$\rightarrow C_{y_1y_2}(\tau) = R_{y_1y_2}(\tau)$$

We already know how to calculate R_{xy_2}:

$$R_{xy_2}(\tau) = R_x(\tau) \star h_2^*(-\tau)$$

Now we can add y_1 to the autocorrelation function.

$$R_{y_1y_2}(t_1,t_2) = \mathbb{E}[y_1(t_1)y_2(t_2)] = \mathbb{E}[\int_{-\infty}^{\infty} x(\alpha)h_1(t_1-\alpha)y_2(t_2)\ d\alpha] = \int_{-\infty}^{\infty} \mathbb{E}[x(\alpha)y_2(t_2)]h_1(t_1-\alpha)\ d\alpha$$

$$= \int_{-\infty}^{\infty} R_{xy_2}(\alpha-t_2)h_1(t_1-\alpha)\ d\alpha = \int_{-\infty}^{\infty} R_{xy_2}(\alpha)h_1(t_1-t_2-\alpha)\ d\alpha = (R_{xy_2}\star h_1)(t1-t2) \rightarrow (R_{xy_2}\star h_1)(\tau)$$

Thus,

$$R_{y_1y_2}(\tau) = R_x(\tau) \star h_2^*(-\tau) \star h_1(\tau)$$

Both h_1 and h_2 can be computed using the corresponding system function.

$$h_1(t) = \mathscr{F}^{-1}\{H_1(jw)\}$$

$$h_2(t) = \mathscr{F}^{-1}\{H_2(jw)\}$$

Also now with $R_{y_1y_2}$ we can have $S_{y_1y_2}$:

$$S_{y_1y_2}(w) = \mathscr{F}\{R_{y_1y_2}\} = \mathscr{F}\{R_x(\tau)\} \times \mathscr{F}\{h_2^*(-\tau)\} \times \mathscr{F}\{h_1(\tau)\} = S_x(w)H_2^*(w)H_1(w)$$

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The average power of the output y is given by:

$$\mathbb{E}[y^2(t)] = R_y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(w)\ dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(w)|H(w)|^2\ dw \qquad (H(w) = \frac{1}{13-w^2+4jw})$$

$$\rightarrow R_y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_x(w)}{(13-w^2)^2+(4w)^2}\ dw$$

We also know that the average power of $x(t)$ is 10. So we have the following constraint on $S_x(w)$:

$$\int_{-\infty}^{\infty} S_x(w)\ dw = 20\pi$$

So the problem is equal to the following optimization problem:

$$S_x^* = \text{argmax}_{S_x} \int_{-\infty}^{\infty} \frac{S_x(w)}{(13-w^2)^2+(4w)^2}$$

$$\text{s.t} \quad \int_{-\infty}^{\infty} S_x(w)\ dw = 20\pi$$

Let's first write S_x as sum of delta functions and also note that $S_x(w) \geq 0$ for every w.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(w-\alpha)\delta(\alpha)\ d\alpha\ dw$$

Now we can see each $S_x(w-\alpha)$ as a coefficient for the delta funciton at point α with a constraint that all these coefficients must add up to 20π.

Note that $(13-w^2)^2+(4w)^2$ has two global minimums at $\pm\sqrt{5}$

Claim:

Maximum average power can be obtained by setting $S_x(w)$ to be equal to $20\pi\delta(w-\sqrt{5})$.

Proof:

The denominator $(13-w^2)^2+(4w)^2$ is always positive and has a minimum value of 144. So we can write:

$$0 < \frac{1}{(13-w^2)^2+(4w)^2} \le \frac{1}{144}$$

$$\rightarrow \int_{-\infty}^{\infty} \frac{S_x(w)}{(13-w^2)^2+(4w)^2}\,dw \;\le\; \int_{-\infty}^{\infty} \frac{S_x(w)}{144}\,dw \;=\frac{5\pi}{36}$$

This is true because both S_x and the denominator have non-negative values.

So the highest possible average power that we can get from $y(t)$ is $\frac{5}{72}$.

$$\hat{S_x} = 20\pi\delta(w-\sqrt{5})$$

$$\rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{S_x(w)}}{(13-w^2)^2+(4w)^2}\,dw = \frac{5}{72}$$

So $\hat{S_x(w)}$ has the optimal average power. Note that the answer is not unique, $20\pi\delta(w+\sqrt{5})$ gives the same result.

6

Suppose that $X(w)$ and $Y(w)$ are the fourier transforms of inputs $x(t)$ and $y(t)$.

$$\mathscr{F}\{y'(t)+2y(t)\} = (jw+2)Y(w) = \mathscr{F}\{x(t)\} = X(w) \rightarrow H(w) = \frac{1}{jw+2} = \frac{2-jw}{4+w^2}$$

Now let's compute S_x

$$S_x(w) = \int_{-\infty}^{\infty} (\delta(\tau)+4e^{-|\tau|})e^{-jw\tau}\,d\tau = 1+\frac{8}{1+w^2} = \frac{9+w^2}{1+w^2}$$

$$\rightarrow S_{xy}(w) = S_x(w)H^*(w) = \frac{9+w^2}{1+w^2}\times\frac{2+jw}{4+w^2}$$

Now we can derive R_{xy} from impulse response:

$$h(t) = \mathscr{F}^{-1}\{\frac{1}{jw+2}\} = e^{-2t}u(t)$$

$$R_{xy}(\tau) = R_x(\tau)\star h^*(-\tau)$$

7

First we need to find S_x.

$$S_x(w) = \int_{-\infty}^{\infty} R_x(\tau)e^{-jw\tau}\ d\tau = \int_{-\infty}^{\infty} (25 + 4e^{-|\tau|})e^{-jw\tau}\ d\tau$$

$$= 50\pi\delta(w) + 4[\int_{0}^{\infty} e^{-jw\tau-2\tau}\ d\tau\ + \int_{-\infty}^{\infty} e^{-jw\tau+2\tau}\ d\tau\ ] = 50\pi\delta(w) + \frac{16}{4+w^2}$$

Also suppose that $X(w)$ and $Y(w)$ are the fourier transforms of $x(t)$ and $y(t)$ then:

$$\mathscr{F}\{y(t)\} = 2\mathscr{F}\{x(t)\} + 3jw\mathscr{F}\{x(t)\} \rightarrow Y(w) = (2+3jw)X(w) \rightarrow H(w) = 2+3jw$$

Then S_y is given by:

$$S_y(w) = S_x(w)|H(w)|^2 = (4+9w^2)[50\pi\delta(w) + \frac{16}{4+w^2}]$$

P.S: Derivative operator is an LTI system and it's system function is given by $H(w) = jw$.

Proof:

$$\text{Impulse response } x(t) = \delta(t) \rightarrow y(t) = \delta'(t) \rightarrow h(t) = \delta'(t)$$

$$H(w) = \int_{-\infty}^{\infty} \delta'(t)e^{-jwt}\ dt = \int_{-\infty}^{\infty} jw\delta(t)e^{-jwt}dt = jw$$

8

8.1

First lets expand $y[k]$.

$$\begin{gathered} y[n] = \\ x[n] \quad + 0\times x[n-1] + x[n-2] - \frac{1}{2}x[n-3] \\ +\frac{1}{2}x[n-1] + \frac{0}{2}\times x[n-2] + \frac{1}{2}x[n-3] - \frac{1}{4}x[n-4] \\ +\frac{1}{4}x[n-2] + \frac{0}{4}\times x[n-3] + \frac{1}{4}x[n-4] - \frac{1}{8}x[n-5] \\ +\frac{1}{8}x[n-3] + \frac{0}{8}\times x[n-4] + \frac{1}{8}x[n-5] - \frac{1}{16}x[n-6] \\ \vdots \end{gathered}$$

Summing on the diagonals we have:

$$y[n] = x[n] + \frac{1}{2}x[n-1] + \frac{5}{4}x[n-2] + \sum_{i=3}^{\infty}\frac{1}{2^i}x[n-i]$$

Inputting $\delta[n]$ as x:

$$h[k] = 0 \quad \text{for } k < 0$$

$$h[0] = 1 \quad , \quad h[1] = \frac{1}{2} \quad , \quad h[2] = \frac{5}{4}$$

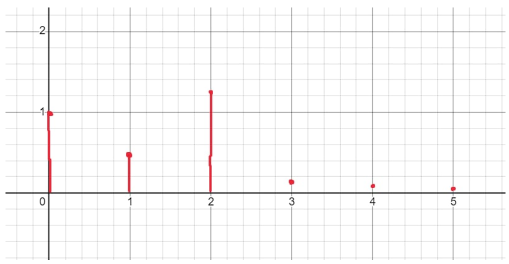
$$h[k] = \frac{1}{2^k} \qquad \text{for } k \geq 3$$

Then the fourier transform of h is given by:

$$\mathscr{F}\{h\} = \sum_{k=-\infty}^{\infty} h[k]e^{-jwk} = 1 + \frac{1}{2}e^{-jw} + \frac{5}{4}e^{-2jw} + \sum_{k=3}^{\infty} \frac{e^{-jwk}}{2^k} = 1 + \frac{1}{2}e^{-jw} + \frac{5}{4}e^{-2jw} + \frac{e^{-3jw}}{4(2 - e^{-jw})}$$

8.2

Graph of h is as below.



8.3

$$R_{xy}(L) = R_x[L] \star h^*[-L] = (\delta[L] + \delta[|L| - 1]) \star (h[-L])$$

$$= \sum_{k=-\infty}^{\infty} h[-k]\delta[L-k] + \sum_{k=-\infty}^{\infty} h[-k]\delta[|L-k| - 1] = h[-L] + h[-L-1] + h[-L+1]$$