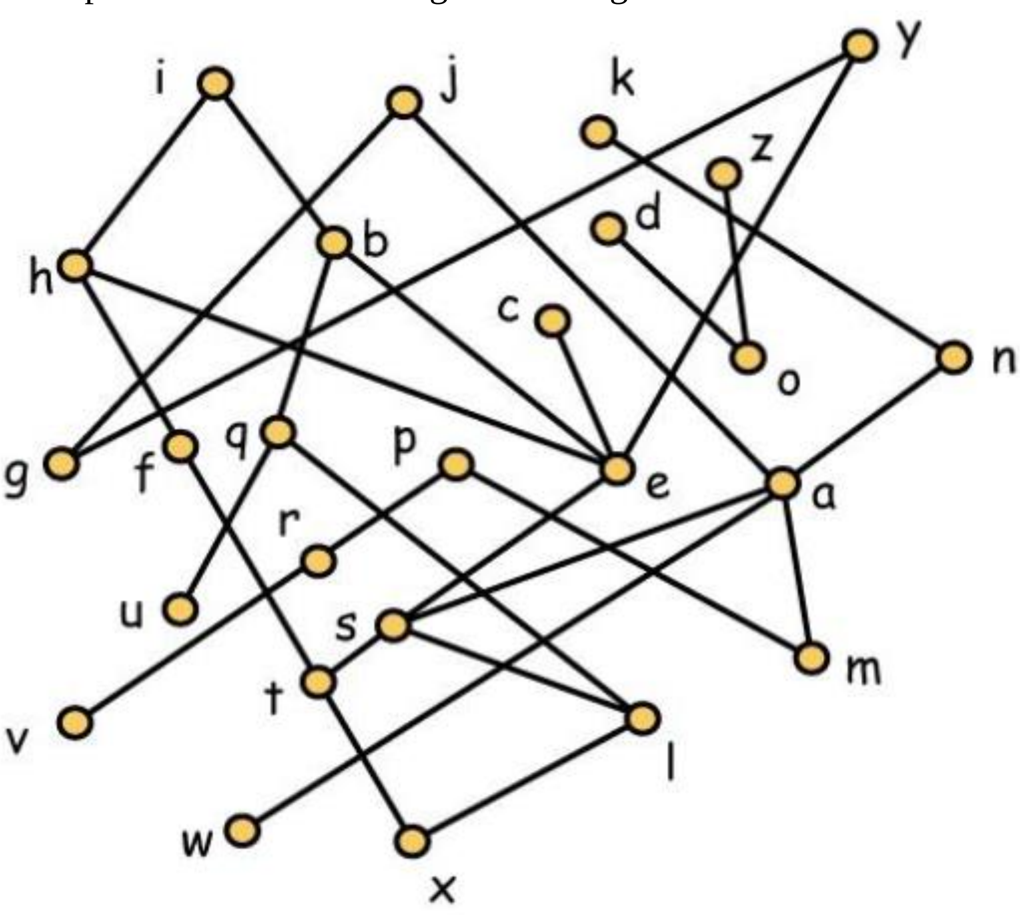
Tutorial on Posets, Lattices, and Hall's Matching Theorem

(Practice Problems: Jongsma, Calvin, "Discrete Mathematics: Chapter 7, Posets, Lattices, & Boolean Algebra" (2016) (pdf mailed with slides). *Faculty Work: Comprehensive List.* Paper 427 Section 7.1, Exercise Set 7.1: (47 problems) Posets, page nos 7.07 to 7.11)

1. Consider poset with the following Hasse Diagram:



   (a) Determine the width of the above poset.
   (b) Suppose the width of the above poset is w. Give w chains that partition the twenty-six elements, namely {a, b, c, d, e, f, …, x, y, z} of the above poset. (extra

questions:) How many such distinct partitions are possible? What properties the collection of these distinct partitions satisfy?

(c) Determine the height of the above poset.

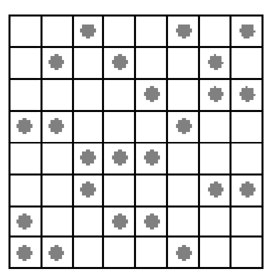
(d) Suppose the height of the above poset is h. Give h antichains that partition the twenty-six elements, namely {a, b, c, d, e, f, …, x, y, z} of the above poset. (extra questions:) How many such distinct partitions are possible? What properties the collection of these distinct partitions satisfy?

2. State and prove (a) Mirsky's theorem, and (b) Dilworth's theorem for finite posets. (Extra question:) Are these theorems applicable for infinite posets?

3. State and prove Hall's Matching theorem (also called as Hall's Marriage Theorem. State how it can be used for proving Dilworth's theorem for finite posets.

4. We are given two square sheets of paper with area 2003. Suppose we divide each of these papers into 2003 polygons, each of area 1. (The divisions for the two pieces of papers may be distinct.) Then we place the two sheets of paper directly on top of each other. Show that we can place 2003 pins on the pieces of paper so that all 4006 polygons have been pierced. (Hint: Use Hall's Matching Theorem).

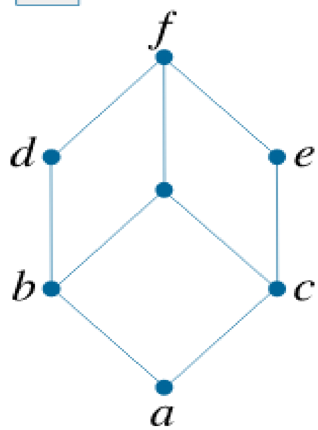
5. Let n ∈ {1, 2, . . . , 8}. Consider an 8 × 8 chessboard with the property that on each column and each row there are exactly n pieces. Prove that we can choose 8 pieces such that no two of them are in the same row or same column. (Hint: Use Hall's Matching Theorem).



Case (of 8 x 8 Chessboard) with n = 3

6. Which of the following Hasse Diagrams (also called as Order Diagrams) represent lattice?

1



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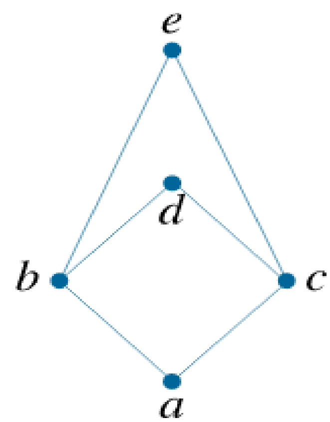

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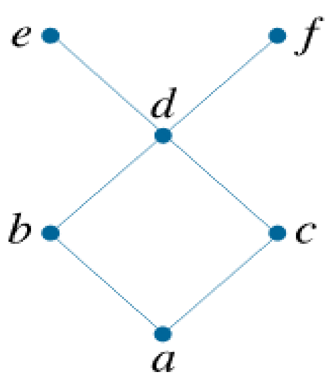

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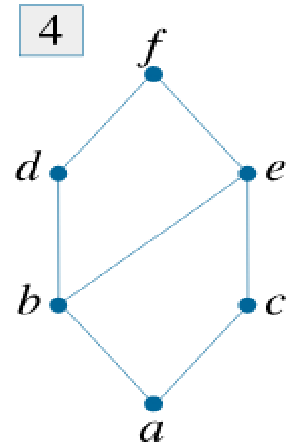

7. Show that the following lattices (order diagrams given) are not distributive: