Моделирование Фьючерсов и Опционов на акции

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Agenda

- Introduction
  - What is Equity Future/Forward Call/Put Options?
  - How do they trade? OTC vs Exchange traded.
  - What are they used for?

- Modelling Equity Future/Forward
  - The concept of Fair Theoretical Value (FV), replication principle.
  - How to calculate a single cash flow FV? What does it mean?
  - How to calculate a Future’s FV? What does it really mean? Theory vs Practice.

- Modelling Equity Option
  - Trader’s point of view on FV
  - Trader’s P&L of a single Call option
  - Black-Sholes Model, where does it come from?

- Typical (market making) trading desk
  - Daily operation
  - Business model overview

- Conclusion

What is a Forward (contract)?

You agree today to buy some asset at an agreed price at a future date.

- The agreed price fixed today is called the **forward price**
- The future date at which the exchange happens is called **maturity**
- The asset in question is called **underlying**
- No money changes hands until maturity
- It is tailor made, traded OTC (Over-The-Counter) – forward starts with a signature!
- relatively illiquid
- http://www.contract-template.org/forward-contract-example.html

What is a Future (contract)?

Same thing but Exchange Traded

- Highly standardized: standardized underlying, maturity and quantity
- Daily margining! Money change hands every day until maturity
- Very liquid – future starts with a click!
- http://www.cmegroup.com/trading/equity-index/us-index/sandp-500_contract_specifications.html

Go to Excel

What is a Put Option (contract)?

If you sell a put option you agree to buy some asset at a predefined price at a future date <u>if the option buyer asks you to</u>

- The agreed price fixed today is called the **strike**
- The future date is called **maturity**
- The asset in question is called **underlying**
- It's effectively an insurance against the price fall:
  - if you bought a put option for you the price cannot go below the strike
  - In other words: you get compensated the difference between the strike and the market price if the market price goes below the strike

$$payoff_{put}(S_T) = \begin{cases} 0, S_T \geq K \\ K - S_T, S_T < K \end{cases}$$

Where:
S_T is price of the underlying at maturity
$payoff_{put}(S_T)$ is the payoff of the put option, paid to the put buyer at maturity

Quick notations: $payoff_{put}(S_T) = Max(K - S_T, 0)$ or$payoff_{put}(S_T) = (K - S_T)^+$

What is it used for? (1/4)

Business/Industry client

- You are a farmer. It's Dec 17, you sell your crops in Sep 18.
- You pay people every month (assume constant sum). You get paid in Sep18 when you sell crops.
- On average year you are have 10% profit. On a good year you're 20% up. On a bad year you're 5% down.
- There is one bad year every 5 years on average.
- How do you survive the bad year?

Strategy 1: put some profits aside every year to cover up the bad year.
Strategy 2: start a business that has a good year when farming has a bad year – diversification!
Strategy 3: buy an insurance – a put option – notice it is the same thing as strategy 2, but easier to execute.
Strategy 4: each year buy a forward contract

What would you do? What do you take into consideration?
Which strategy is the most effective in this case?
Is this strategy always achievable?
Is it important to have developed financial markets?

What is it used for? (2/4)

Retail client

- You are an individual. You work for company XXX.
- You see the management is going to ruin the company. You'll lose your job.
- You see that your company is traded at par with the industry leaders.

What do you do?
Strategy 1: try influencing the management
Strategy 2: start looking for another job
Strategy 3: Strategy 2 + buy put options
Strategy 4: ???

How about you're not sure if you company is actually going to be ruined but you think it's definitely not going to do as well as company YYY?
Is it important to have developed financial markets?

What is it used for? (3/4)

Pension fund

- You are a pension fund manager.
- You have $20b under management. You need to do something with it.
- If you lost it you're fired.
- If other pension funds do much better than you, you're fired.

What do you do?
Strategy 1: Put money into a saving's account? No. It's too much. Return is too small. Risk is too high!
Strategy 2: Trend following with a great degree of diversification
Strategy 3: Specially designed products with limited, no downside and unlimited upside

Is it important to have developed financial markets?

What is it used for? (4/4)

Hedge fund

- You are a hedge fund manager.
- You have $2b under management.
- You need to do something with it and deliver high returns!
- If other hedge funds do much better than you, you're fired.

What do you do?
Strategy 1: ???

- Say you think no matter what happens Europe with its Brexit, poor economics and migrants will be more volatile than the US? What can you do?

Is it important to have developed financial markets?

Fair Theoretical Value = Replication + Absence of Arbitrage

Consider a single Cash Flow with amount CF at some future date $\mathrm{t} = T$.
What is the Fair Price of this cash flow?

Well, assume one can invest/borrow money at rate r using some cash account
then one can *replicate* this cash flow by

1. Putting $\frac{CF}{1+rT}$ into the cash account today
2. and withdrawing it at time T: the withdrawn amount will be $\frac{CF}{1+rT}(1+\mathrm{rT}) = \mathrm{CF}$

So I shall say that Fair Value of this future cash flow is $\mathrm{FV} = \frac{CF}{1+rT}$!
Why? Replication: I see that I need exactly FV to create this future cashflow.

Alternative reasoning: Say I can buy this cashflow today for $X < FV$. I'll borrow FV from the bank now (the future cash flow will cover that debt) then I pay X and pocket FV-X, that's is my pure profit. This is called arbitrage. So X doesn't sound fair…

Similarly if I can sell at X > FV, I can pocket $X - FV$, then use FV to replicate the CashFlow.
I conclude that the only "Fair" price is FV.

Forward (assuming no counterparty risk) / Future FV

Forward: underlying price today is S_{t_0} interest rate is r. Assume there are no dividends/borrow rates.

As a forward seller I need to deliver 1 unit of stock at time T. How can I replicate that?

1. Borrow S_{t_0} now
2. Use that to buy 1 unit of stock (now)
3. At maturity I still hold the stock so I can deliver that but I will owe $S_{t_0}(1 + rT)$ to the bank
4. Let's set the forward price to that so that forward buyer pays off my debt!

$$FV\big(Forward(t_0, T)\big) = S_{t_0}(1 + rT)$$

Borrow: usually it's possible to "lend" stock units at a certain rate called borrow. In this case your replication will be slightly cheaper.

How about dividends, say there is a dividend of size B at time t_d: $t_0 < t_d < T$?
Not a problem - let's do the same reasoning

1. Borrow S_{t_0} now
2. Use that to but 1 unit of stock (now)
3. At time t_d I will receive the dividend as I hold 1 unit of stock; put it into the cash account.
   How much do I owe the bank now? Well it's $S_0(1 + rt_d) - d$ (i.e. accrued debt – the dividend)
4. At maturity I deliver the unit of stock I hold to the forward buyer and I owe to the bank
   $$(S_0(1 + rt_d) - d)(1 + r(T - t_d))$$
5. Let's set forward price to that so that the buyer pays off my debt!

What is a Call Option (contract)?

If you sell a call option you agree to sell some asset at a predefined price at a future date <u>if the option buyer asks you to</u>

- Similar to put but this time It's effectively an insurance against the price rise:
  - if you bought a call option for you the price cannot go above the strike
  - In other words: you get compensated the difference between the strike and the market price if the market price goes above the strike

$$payoff_{call}(S_T) = \begin{cases} S_T - K, S_T \geq K \\ 0, S_T < K \end{cases}$$

Where:

S_T is price of the underlying at maturity

$Payoff_{call}(S_T)$ is the payoff of the call option, paid to the call buyer at maturity

Notice that $payoff_{call}(S_T) - payoff_{put}(S_T) = S_T - K$

we'll get back to it…

Denote call with strike K and maturity T by **Call(K,T)**

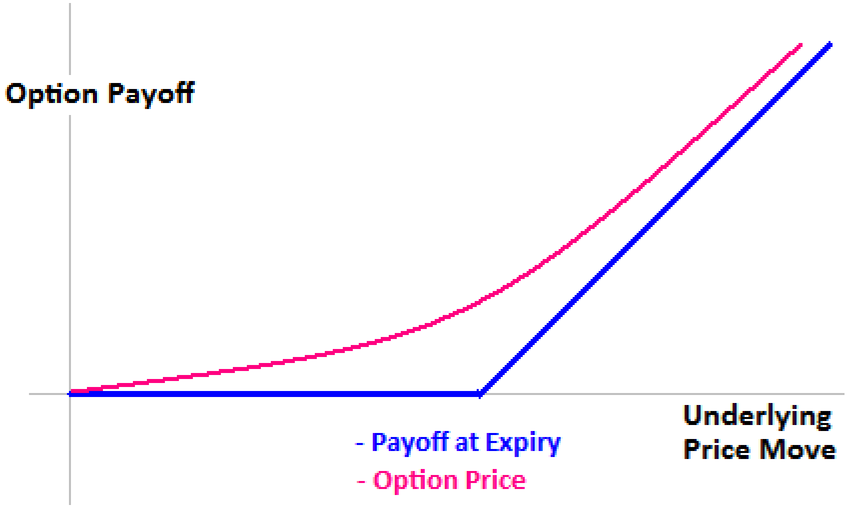
Call Option Trader

You arrive at a trading desk. It's your first day! You receive a request to sell one call option on SPX
Your quants implemented a pricing function for call options. What do you do?

Denote call price with und spot S strike K and maturity T at time ***t*** by
$\boldsymbol{C(S,t,K,T)}$

You're not familiar with the pricing functions -> do some sanity checks first

1. Option with maturity T=0 should be equal to $Call(S_T) = Max(S_T - K, 0)$
2. Option with very high strike should be equal to 0
3. Option with very low strike should be equal to… forward, which you can roughly calculate on a piece of paper!
4. Option with higher strike should be cheaper
5. If spot goes up, option price should go up
6. Option with higher volatility should be more expensive
7. Option with zero volatility should be, assuming rates are close to zero, equal to $Max\left(S_{t_0} - K, 0\right)$ also called intrinsic value
8. From 6 and 7 I conclude that option should be more expensive than it's intrinsic value -> like on the graph
9. Butterfly option $(C(K - \Delta K) - 2C(K) + C(K + \Delta K))$ should be positive



Call Option Trader

Ok, sanity checks passed and you sold the option at the price given by the pricing function. Ok in your portfolio you now hold some cash from the client (the option premium) and a short position in the option you sold. What do you do next?

If you don't do anything at maturity you might have to pay to the client a HUGE payoff if the spot ends up higher than the strike! You'll lose money and get fired ;(How about you buy something to offset your option position!

Strategy 1: buy the same call option from somebody else cheaper! That's an ideal situation, it is not always possible.

Strategy 2: hmm what else can you buy? It has to be something as similar to call option as possible. How about you buy the underlying? It's not too bad as we know underlying goes up -> call option price goes up! Underlying goes down -> call option price goes down! Fantastic!

Call Option Trader

Ok, so you decide to but the underlying to offset your call position. How much should you buy?

So if my und moves from S to $S + \Delta S$ my call price will move from $\mathrm{C}(S)$ to $C(S + \Delta S)$

I want to buy some amount **X** of my underlying so that the move in stock does not affect my total position i.e. I want

$$\boldsymbol{X} \times \mathrm{S} - C(S) = \boldsymbol{X} \times (\mathrm{S} + \boldsymbol{\Delta}\mathbf{S}) - C(S + \Delta S)$$

Or in other words my **X** must be

$$X = \frac{C(S + \Delta S) - C(S)}{\Delta S} \approx \frac{\partial C(S)}{\partial S}$$

$\Delta(S) = \frac{\partial C(S)}{\partial S}$ is called Delta risk in jargon, it shows how many units of stock you need to hold at any given point in time to offset a short call option position.

Fantastic! You had a great first day: you sold a call option, you bought X of the underlying, you go home knowing that you portfolio is “flat” i.e. tomorrow your P&L should be around zero.

Call Option Trader

You're back to the office the next day. Your second day. You look at your book. What do you see?

1. Spot has moved a bit.
2. You have a small negative (or positive) PnL. Where is it coming from? $Call(S, K, T)$ is a non linear function in S so your linear hedge cannot offset it perfectly.
3. Next you notice that Delta (risk) is now slightly different. Why? Well $\frac{\partial C\ (S)}{\partial S}$ is also a function of S.
4. Wait a sec, so that means the hedge you bought yesterday no longer going to offset your call option position today? Right! You have to adjust your delta hedge.

You need to calculate today's Delta and adjust your position in stock. Done?
Great, now your portfolio is flat again.

Ok, second day's gone by: you had some small PnL due to non-linearity of the call option and you had to readjust your delta hedge, you go home knowing that you portfolio is "flat" again…

Call Option Trader

Day 252 in the office. You've been getting small PnL every day and every day you've been readjusting your delta hedge. The option had just expired. Have you made any money? Not obvious.

Let's look in more detail on your daily PnL:

$$\Delta PnL_{t+dt} = \Delta_t(dS) - \big(C(S_t + dS, t + dt) - C(S_t, t)\big) - r\Delta_t S_t dt + rCdt$$

$$C(S_t + dS, t + dt) - C(S_t, t) \approx \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S}(dS)^2$$

So

$$\Delta PnL_{t+dt} = -r\Delta_t S_t - \frac{\partial C}{\partial t} dt - \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S}(dS)^2 + rC$$

Pricing function C is designed so that on average PnL is zero

$$E[\Delta PnL_{t+dt}] = -r\Delta_t S_t - \frac{\partial C}{\partial t} dt - \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S} E[(dS)^2] + rC = 0 \Rightarrow -r\Delta_t S_t - \frac{\partial C}{\partial t} dt + \mathrm{rC} = \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S} E[(dS)^2]$$

So

$$\Delta PnL_{t+dt} = \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S}\left(E[(dS)^2] - (dS)^2\right)$$

Call Option Trader

Now we see that the small PnL you were observing is this

$$\Delta PnL_{t+dt} = \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S} S^2 \left(\frac{E[(dS)^2]}{S^2} - \left(\frac{dS}{S}\right)^2 \right)$$

In our first lecture on Equities we noticed that

$$\frac{\Delta S}{S} = \mu_t \Delta t + \sigma_t \sqrt{\Delta t} N_t$$

So $E[\Delta S^2] = S^2 \sigma_t^2 \Delta t$

That means

$$\Delta PnL_{t+dt} = \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S} S^2 \left(\sigma_t^2 \Delta t - \left(\frac{dS}{S}\right)^2 \right)$$

In other words daily Pnl is proportional to the difference between implied variance and realized variance.

Black Sholes

Pricing function C is designed so that on average PnL is zero

$$E[\Delta PnL_{t+dt}] = -r\Delta_t S_t dt - \frac{\partial C}{\partial t}dt - \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S}E[(dS)^2] + rCdt = 0$$

BS assumption

$$\frac{\Delta S}{S} = \mu\Delta t + \sigma\sqrt{\Delta t}N_t$$

So $E[(dS)^2] = S^2\sigma^2\mathrm{dt}$

So

$$-r\Delta_t S_t dt - \frac{\partial C}{\partial t}dt + \mathrm{rCdt} = \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S}S^2\sigma^2\mathrm{dt}$$

Or if I simplify

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S}S^2\sigma^2 + rS\frac{\partial C}{\partial S} - rC = 0$$

That's Black's Equation!

Black Sholes: how to get the crazy formula nobody remembers?

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S \partial S} S^2 \sigma^2 + rS\frac{\partial C}{\partial S} - rC = 0$$

There is a mathematical theorem saying that the solution of the above equation can be found as a mathematical expectation:

$$C = E[Max(S_T - K, 0)]e^{-rT}$$

Where

$$\frac{\mathrm{d}S}{S} = \boldsymbol{r}\mathrm{d}t + \sigma\sqrt{\Delta t}N_t$$

In finance we say we need to take expectation of payoff in so-called risk-neutral measure and then discount it back to option inception date.

Home work

1. Review the slides.

2. You have a Black-Sholes pricing function in the .py file that comes with the lecture.
   Imagine you're the trader from the lecture, do all the sanity check, make sure it works as expected!

3. You have daily PnL function and also some Monte-Carlo code.
   You can simulate the path of the trade and figure out what total PnL will be on that path. Check it is zero on average. Explain what it means for the market making business model. Notice that PnL can be negative on some paths – why is that?

4. Check that Black-Sholes price is equal to the Monte-Carlo price! You'll need the MC pricing function from the previous lecture on Equites.

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