Derivative of a Sum

One of our examples of a general derivative formula was:

$$(u+v)'(x) = u'(x) + v'(x).$$

(Remember that by $(u+v)(x)$ we mean $u(x)+v(x)$.)

In other words, the derivative of the sum of two functions is just the sum of their derivatives. We'll now prove that this is true for any pair of functions u and v, provided that those functions have derivatives. Since we don't know in advance what functions u and v are, we can't use any specific information about the functions or the slopes of their graphs; all we have to work with is the formal definition of the derivative.

When we apply the definition of the derivative to the function $(u+v)(x)$ we get:

$$(u+v)'(x) = \lim_{\Delta x \to 0} \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x}$$

Since $(u+v)(x)$ is just $u(x)+v(x)$,

$$(u+v)'(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x}.$$

Combining like terms, we see that:

$$(u+v)'(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x) + v(x+\Delta x) - v(x)}{\Delta x}$$

or:

$$(u+v)'(x) = \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x} \right\}.$$

Because u and v are differentiable (and therefore continuous), the limit of the sum is the sum of the limits. Therefore:

$$(u+v)'(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{v(x+\Delta x) - v(x)}{\Delta x}.$$

The two limits above match the definition of the derivatives of u and v, so we've shown that $(u+v)'(x) = u'(x) + v'(x)$.