

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$$

While calculating the derivatives of $\cos(x)$ and $\sin(x)$, Professor Jerison said that

$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$. This is true, but in order to be certain that our derivative formulas are correct we should understand *why* it's true.

As in the discussion of $\sin(\theta)/\theta$, our explanation involves looking at a diagram of the unit circle and comparing an arc with length θ to a straight line segment. (Remember that θ is measured in radians!) As shown in Figure 1, the vertical distance between the endpoints of the arc is $\cos \theta$, and the horizontal distance between the ends of the arc is $1 - \cos \theta$.

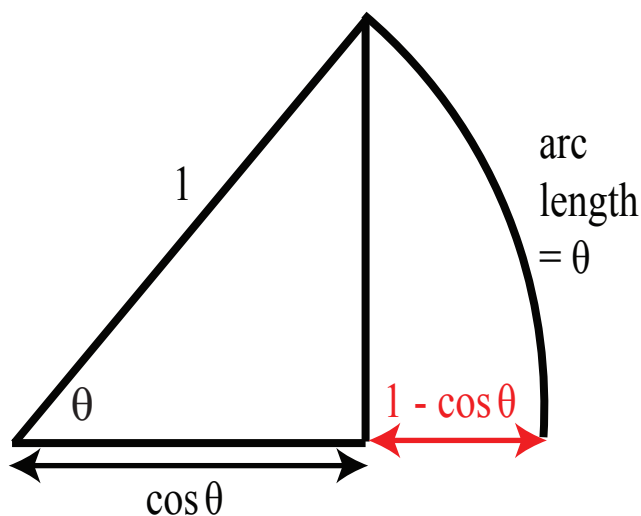


Figure 1: Same figure as for $\frac{\sin x}{x}$ except that the horizontal distance between the edge of the triangle and the perimeter of the circle is marked

From Fig. 2 we can see that as $\theta \rightarrow 0$, the horizontal distance $1 - \cos \theta$ between endpoints of the arc (what Professor Jerison calls “the gap”) gets much smaller than the length θ of the arc. Hence, $\frac{1 - \cos \theta}{\theta} \rightarrow 0$.

If you find this hard to believe it may be helpful to use a calculator to verify that if x is small, $1 - \cos x$ is much smaller. You might also study the graph of $y = 1 - \cos x$ near $x = 0$ or use a web application to compare the distance $1 - \cos \theta$ to the arc length θ for very small angles θ .

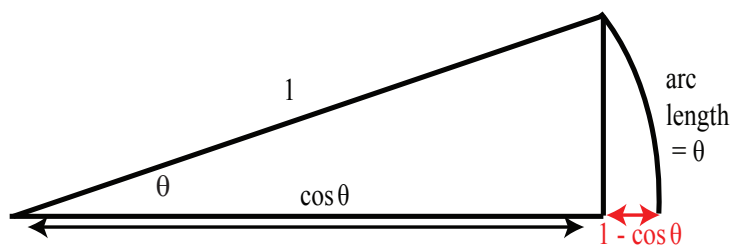


Figure 2: The sector in Fig. 1 as θ becomes very small