Derivative of $\sin x$, Algebraic Proof

A specific derivative formula tells us how to take the derivative of a specific function: if $f(x) = x^n$ then $f'(x) = nx^{n-1}$. We'll now compute a specific formula for the derivative of the function $\sin x$.

As before, we begin with the definition of the derivative:

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

You may remember the following angle sum formula from high school:

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

This lets us untangle the x from the Δx as follows:

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin(x)}{\Delta x}.$$

We can simplify this expression using some basic algebraic facts:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\Delta x \to 0}\left[\frac{\sin x \cos \Delta x - \sin x}{\Delta x} + \frac{\cos x \sin \Delta x}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0}\left[\frac{\sin x(\cos \Delta x - 1)}{\Delta x} + \frac{\cos x \sin \Delta x}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0}\left[\sin x\left(\frac{\cos \Delta x - 1}{\Delta x}\right) + \cos x\left(\frac{\sin \Delta x}{\Delta x}\right)\right] \\
\frac{d}{dx}\sin x &= \lim_{\Delta x \to 0} \sin x\left(\frac{\cos \Delta x - 1}{\Delta x}\right) + \lim_{\Delta x \to 0} \cos x\left(\frac{\sin \Delta x}{\Delta x}\right)
\end{aligned}$$

We now have two familiar functions – $\sin x$ and $\cos x$ – and two ugly looking fractions to deal with. The fractions may be familiar from our discussion of removable discontinuities.

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} &= 0 \\
\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} &= 1.
\end{aligned}$$

Using these (as yet unproven) facts,

$$\lim_{\Delta x \to 0} \sin x\left(\frac{\cos \Delta x - 1}{\Delta x}\right) + \lim_{\Delta x \to 0} \cos x\left(\frac{\sin \Delta x}{\Delta x}\right)$$

simplifies to $\sin x \cdot 0 + \cos x \cdot 1 = \cos x$. We conclude:

$$\frac{d}{dx}\sin x = \cos x$$