Topic 1: Varrify Invariant Theorem with some example.

Soln:

Theorem: If transformation equation of $ax^2+2hxy+by^2=0$ is $a_1x^2+2h_1xy+b_1y^2=0$ when the direction of the axes turned through an angle θ, without changing of the origin show that $a+b$ and $ab-h^2$ are invarient. i.e: $a+b=a_1+b_1$ and $ab-h^2=a_1b_1-h_1^2$.

Proof:

Given equation is $ax^2+2hxy+by^2=0$

now rotate the axes by angle θ,

$\therefore x = x_1\cos\theta - y_1\sin\theta$

$\therefore y = x_1\sin\theta + y_1\cos\theta$

$\therefore ax^2+2hxy+by^2=0$

$\Rightarrow a(x_1\cos\theta - y_1\sin\theta)^2 + 2h(x_1\cos\theta - y_1\sin\theta)(x_1\sin\theta + y_1\cos\theta) + b(x_1\sin\theta + y_1\cos\theta)^2 = 0$

$\Rightarrow a(x_1^2\cos^2\theta - 2x_1y_1\sin\theta\cos\theta + y_1^2\sin^2\theta) + 2h(x_1^2\sin\theta\cos\theta + x_1y_1\cos^2\theta - x_1y_1\sin^2\theta - y_1^2\sin\theta\cos\theta) + b(x_1^2\sin^2\theta + 2x_1y_1\sin\theta\cos\theta + y_1^2\cos^2\theta) = 0$

$\Rightarrow ax_1^2\cos^2\theta - 2ax_1y_1\sin\theta\cos\theta + ay_1^2\sin^2\theta + 2hx_1^2\sin\theta\cos\theta + 2hx_1y_1\cos^2\theta - 2hx_1y_1\sin^2\theta - 2hy_1^2\sin\theta\cos\theta + bx_1^2\sin^2\theta + 2bx_1y_1\sin\theta\cos\theta + by_1^2\cos^2\theta = 0$

$\Rightarrow x_1^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) + 2x_1y_1(-a\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta + b\sin\theta\cos\theta) + y_1^2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta) = 0$

by comparing with our given equation $a_1x^2 + 2h_1xy + b_1y^2 = 0$

$\therefore a_1 = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$

$\therefore b_1 = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$

$\therefore h_1 = -a\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta + b\sin\theta\cos\theta$

Now,

$a_1+b_1 = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta + a\sin^2\theta - 2h\sin\theta\cos\theta$
$\quad + b\cos^2\theta$

$= a(\sin^2\theta + \cos^2\theta) + b(\sin^2\theta + \cos^2\theta)$

$\therefore a_1+b_1 = a+b$ $\qquad [\because \sin^2\theta + \cos^2\theta = 1]$

Now,

$2a_1 = 2a\cos^2\theta + 2h.2\sin\theta\cos\theta + 2b\sin^2\theta$

$= 2a(1-\sin^2\theta) + 2h.\sin 2\theta + 2b\sin^2\theta$ $\qquad [\because \cos^2\theta = 1-\sin^2\theta$, $\because 2\sin\theta\cos\theta = \sin 2\theta]$

$= 2a - 2a\sin^2\theta + 2b\sin\theta^2 + 2h\sin 2\theta$

$= 2a + 2\sin^2\theta(b-a) + 2h\sin 2\theta$

$= 2a + (1-\cos 2\theta)(b-a) + 2h\sin 2\theta$ $\qquad [\because 2\sin^2\theta = 1 - \cos 2\theta]$

$= 2a + b - a - b\cos 2\theta + a\cos 2\theta + 2h\sin 2\theta$

$= (a+b) + \cos 2\theta(a-b) + 2h\sin 2\theta$

Similarly,

$2b_1 = (a+b) - \cos 2\theta(a-b) + 2h\sin 2\theta$

$2h_1 = 2h\cos 2\theta - \sin 2\theta(a-b)$

$\therefore 4a_1b_1 = (a+b)^2 - \{\cos 2\theta\,(a-b) + 2h\sin 2\theta\}^2$

$\therefore 4h_1^2 = 4h^2\cos^2 2\theta - 4h(a-b)\sin 2\theta\cos 2\theta + (a-b)^2\sin^2 2\theta$

$\therefore 4(a_1b_1 - h_1^2) = (a+b)^2 - (a-b)^2\cos^2 2\theta - 4h^2\sin^2 2\theta -$
$4h(a-b)\sin 2\theta\cos 2\theta - 4h^2\cos^2 2\theta +$
$4h(a-b)\sin 2\theta\cos 2\theta - (a-b)^2\sin^2 2\theta$

$= (a+b)^2 - (a-b)^2(\sin^2 2\theta + \cos^2 2\theta) -$
$4h^2(\sin^2 2\theta + \cos^2 2\theta)$

$= \{(a+b)^2 - (a-b)^2\} - 4h^2$

$\Rightarrow 4(a_1b_1 - h_1^2) = 4ab - 4h^2$

$\therefore a_1b_1 - h_1^2 = ab - h^2$

(proved.)

Topic : 02 - Show the method how we can eleminate the first degree term of a second degree equation.

Q. Remove the first degree terms from the equation $x^2 + 2xy + 3y^2 + 2x - 4y - 1 = 0$.

Sol^n:

Let, first degree terms will be removed if we transfrom or shift origin in (α, β).

Given equation: $x^2 + 2xy + 3y^2 + 2x - 4y - 1 = 0$ ——— (1)

Transformed equation of (1) will be,

$$(x_1+\alpha)^2 + 2(x_1+\alpha)(y_1+\beta) + 3(y+\beta)^2 + 2(x_1+\alpha) - 4(y_1+\beta) - 1 = 0$$

$$\Rightarrow x^2 + 2x\alpha + \alpha^2 + 2xy + 2x\beta + 2y\alpha + 2\alpha\beta + 3y^2 + 6y\beta + 3\beta^2$$
$$+ 2x + 2\alpha - 4y - 4\beta - 1 = 0$$

$$\Rightarrow x^2 + 2xy + y^2 + 2x(\alpha+\beta+1) + 2y(\alpha+3\beta-2) +$$
$$(\alpha^2 + 2\alpha\beta + 3\beta^2 + 2\alpha - 4\beta - 1) = 0 \text{ ——— (2)}$$

$\therefore$ first degree term will be removed if

$$\alpha + \beta + 1 = 0 \text{ ——— (3)}$$
$$\alpha + 3\beta - 2 = 0 \text{ ——— (4)}$$

Now, $\{(4) - (3)\} \Rightarrow \alpha + 3\beta - 2 - \alpha - \beta - 1 = 0$

$$\Rightarrow 2\beta - 3 = 0$$
$$\therefore \beta = 3/2$$

putting β in equation (3),

$$\alpha + \frac{3}{2} + 1 = 0$$
$$\therefore \alpha = -5/2$$

Now from equation (2)

$$x^2+2xy+y^2+\frac{25}{4}-\frac{30}{4}+\frac{27}{4}-\frac{10}{2}-\frac{12}{2}-1=0$$

$$\Rightarrow x^2+2xy+y^2=\frac{13}{2}$$

$\therefore$ Elemated. first degree term equation is

$$x^2+2xy+y^2=\frac{13}{2}$$

There is another way to remove first degree term from asecond degree equation.

Given equation,

$$x^2+2xy+3y^2+2x-4y-1=0$$

Comparing with $ax^2+2hxy+by^2+2gx+2fy+c=0$ equation,

$$a=1,\ b=3,\ c=-1,\ h=1,\ g=1,\ f=-2$$

Now the formula of origin shift (α,β) is

$$(\alpha,\beta)=\left(\frac{hf-bg}{ab-h^2},\ \frac{gh-af}{ab-h^2}\right)$$

$$=\left(\frac{1\times(-2)-3\times 1}{1\times 3-(1)^2},\ \frac{1\times 1-1\times(-2)}{1\times 3-(1)^2}\right)$$

$$=\left(-5/2,\ 3/2\right)$$

Now, first degree term removed equation,

$$x^2+2xy+3y^2=c_1 \quad \text{———} \quad (1)$$

where, $c_1=-(\alpha g+\beta f+c)$

$$=-\left(-5/2\cdot 1+3/2\cdot(-2)+(-1)\right)$$

$$\Rightarrow c_1 = -\left(-\frac{5}{2} - 4\right)$$
$$= -\left(-\frac{13}{2}\right)$$
$$= 13/2$$

putting $c_1 = 13/2$ in equation (1) we get first degree removed equation,

$$x^2 + 2xy + 3y^2 = \frac{13}{2}$$

(Ans.)

Topic 03: Explane pair of straight lines, plane, conic and space.

Pair of Straight lines:

A pair of straight line refers to two straight lines represented by a single second-degree equation in two variable x, y. This is typically representes as:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

If $l_1 x + m_1 y + n_1 = 0$ and $l_2 x + m_2 y + n_2 = 0$ represents two equation of straight lines then, the equation of pair of straight line is,

$$(l_1 x + m_1 y + n_1)(l_2 x + m_2 y + n_2) = 0$$

... condition: The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ canbe either pair of straight lines or conics. To determine whether this equation representes a pair of straight line or conics, if the determinants is zero.

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$\therefore$ If $a+b=0$; both lines are perpendicular

$\therefore$ If $h^2 = ab$; they are parallel.

Plane:

Plane is a two-dimentional flat surface that extends infinitely in the all directions. It is characterized by its width and length but has no thickness.

In geometry, a plane is defined by an infinite number of points and can be described using the equation of plane involving variables such as x, y, z in three dimentional space.

$$ax + by + cz + d = 0$$

Conics :

Conics are the curves formed by the intersection of a plane with a double-napped cone. Depending on the angle and the position of the plane, the intersection can produce different types of curves: circle, ellipse, parabolla or hyperbolla.

The general equation of second degree is:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

To determine whether this equation represents a conics or not we have to evaluate the, 'Δ'

$$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$$

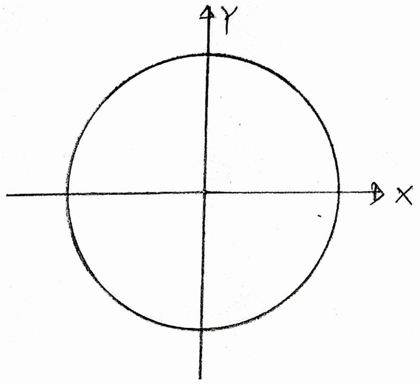
if $\Delta \neq 0$, the equation represents conics.

Conics can be circle, ellipse, parabolla or hyperboll. To determine exact type we have to dive deeper.

... circle :

condition - $a = b$ and $h = 0$

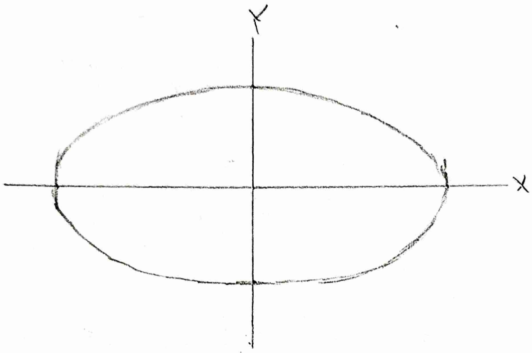
Standard equation - $x^2 + y^2 = a^2$



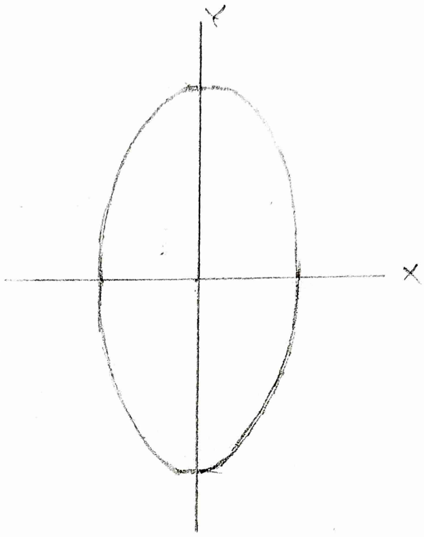
... Ellipse:

condition – $ab - h^2 > 0$

Standard Equation – $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$

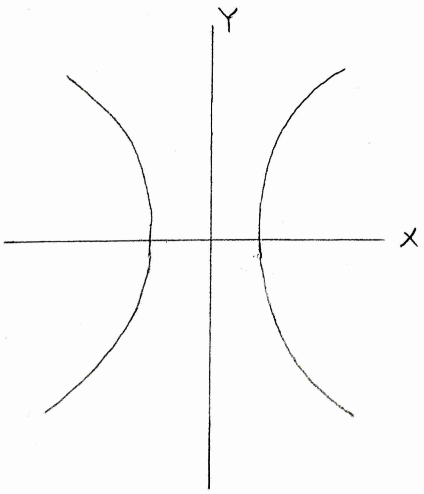


$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (b > a)$

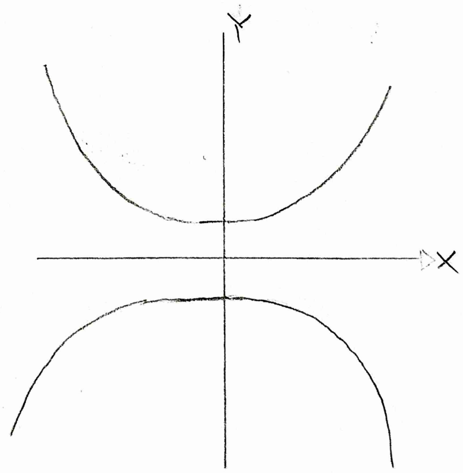
... Hyperbolla:

condition – $ab - h^2 < 0$

Standard Equation – $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



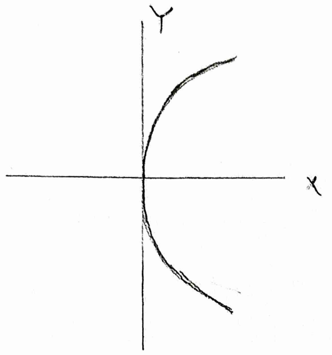
$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

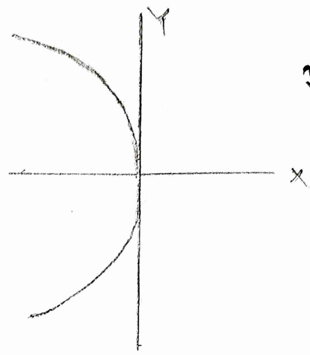
... parabola :

condition : $ab - h^2 = 0$

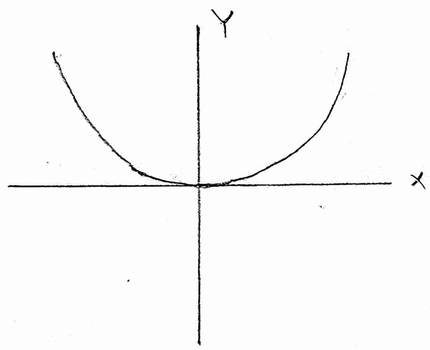
standard equation : $y^2 = 4ax$ or $x^2 = 4ay$

$y^2 = 4ax$
$(a > 0)$

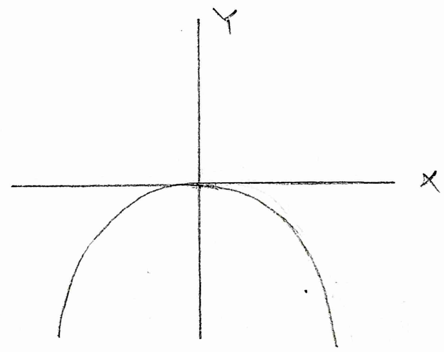




$y^2 = 4ax$
$(a < 0)$



$x^2 = 4ay \ (a > 0)$



$x^2 = 4ay \ (a < 0)$

Space :

Space refers to the system of describing points, lines, planes and shapes in a multidimentional framework using numerical co-ordinates.

- 1D - Space -

A single line where points are represented by one coordinate, x

- equation : $x = c$, where x and c, constant

2D space:

A flat surface where each point is represented by an ordered pair of coordinates (x, y).

- equation : $ax + by + c = 0$

3D space :

A three dimentional region where each point is represented by a triplet of coordinates, (x, y, z)

- equation : $ax + by + cz + d = 0$

Topic 04:

Evaluate: $x^2+y^2+3xy-6x+8y-12=0$

a) Intersection point

b) Angle between the lines

c) Bisectors equation

d) Test the equation represents pair of straight line or not.

sol^n: To evaluate those points we need a second degree equation first.

$\therefore$ Given equation: $x^2+y^2+3xy-6x+8y-12=0$

comparing with general equation.

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

$$a=1,\ b=1,\ c=-12,\ h=\frac{3}{2},\ g=-3,\ f=4$$

a) Intersection point:

- Intersection points represents as (α,β)

we know,

$$(\alpha,\beta)=\left(\frac{hf-bg}{ab-h^2},\ \frac{gh-af}{ab-h^2}\right)$$

$$=\left(\frac{3/2\cdot 4-1.(-3)}{1.1-(3/2)^2},\ \frac{(-3).3/2-1.4}{1.1-(3/2)^2}\right)$$

$$=\left(-36/5,\ 34/5\right)$$

b) Angle between the lines.

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

here,

$h = 3/2$

$a = 1$

$b = 1$

$$\Rightarrow \tan\theta = \frac{2\sqrt{(3/2)^2 - 1.1}}{1+1}$$

$$\Rightarrow \tan\theta = \sqrt{5}/2$$

$$\therefore \theta = 48.189°$$

C) Bisector equation:

From (a), $(\alpha, \beta) = \left(-\frac{36}{5}, \frac{34}{5}\right)$

$\therefore$ bisector is,

$$\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$$

$$\Rightarrow h\{(x-\alpha)^2 - (y-\beta)^2\} = (a-b)(x-\alpha)(y-\beta)$$

$$\Rightarrow \frac{3}{2}\left\{\left(x+\frac{36}{5}\right)^2 - \left(y-\frac{34}{5}\right)^2\right\} = (1-1)\left(x+\frac{36}{5}\right)\left(y-\frac{34}{5}\right)$$

$$\Rightarrow x^2 + 2.x.\frac{36}{5} + \frac{1296}{25} - y^2 + 2.y.\frac{34}{5} - \frac{1156}{25} = 0 \times 2/3$$

$$\Rightarrow x^2 - y^2 + \frac{72}{5}x + \frac{68}{5}y + \frac{28}{5} = 0$$

$$\Rightarrow 5x^2 - 5y^2 + 72x + 68y + 28 = 0$$

d) Test the equation represents pair of straight line or not.

If $\Delta = 0$, represents a second degree equation a pair of straight line.

$$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= 1.1.(-12) + 2.4.(-3).(3/2) - 1.4^2 - 1.(-3)^2 - (-12).(3/2)^2$$

$$= -46$$

$\therefore x^2 + y^2 + 3xy - 6x + 8y - 12 = 0$ equation represents a conics.