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## **ROBOT DYNAMICS ALGORITHMS**

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by

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# Preface

The purpose of this book is to present computationally efficient algorithms for calculating the dynamics of robot mechanisms represented as systems of rigid bodies. The efficiency is achieved by the use of recursive formulations of the equations of motion, i.e. formulations in which the equations of motion are expressed implicitly in terms of recurrence relations between the quantities describing the system. The use of recursive formulations in dynamics is fairly new, so the principles of their operation and reasons for their efficiency are explained.

Three main algorithms are described: the recursive Newton-Euler formulation for inverse dynamics (the calculation of the forces given the accelerations), and the composite-rigid-body and articulated-body methods for forward dynamics (the calculation of the accelerations given the forces). These algorithms are initially described in terms of an un-branched, open-loop kinematic chain -- a typical serial robot mechanism. This is done to keep the descriptions of the algorithms simple, and is in line with descriptions appearing in the literature. Once the basic algorithms have been introduced, the restrictions on the mechanism are lifted and the algorithms are extended to cope with kinematic trees and loops, and general constraints at the joints. The problem of simulating the effect of contact between a robot and its environment is also considered. Some consideration is given to the details and practical problems of implementing these algorithms on a computer.

The algorithms are presented using a six-dimensional vector notation called spatial notation, which is similar to the use of screw coordinates to represent vector and tensor quantities. This notation greatly simplifies the

analysis of rigid-body dynamics by reducing the number and size of the equations involved. It simplifies the process of formulating the algorithms and allows the finished algorithms to be expressed clearly and concisely.

The reader is assumed to have a knowledge of vectorial mechanics. A knowledge of Lagrangian mechanics would be helpful, but is not necessary. No prior knowledge of robot dynamics, or the dynamics of any other kind of rigid-body system, is required.

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My thanks to John Hallam and Robert Gray, who read through the draft version of this book and made many useful comments; and also to Magdalena Müller, who took on the arduous task of formatting the equations.

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## **ROBOT DYNAMICS ALGORITHMS**

NOTE: Where Greek letters have been used to denote vectors or matrices, it was intended that they should appear in a bold typeface to distinguish them from scalars. Unfortunately this has not happened, and it is possible that some confusion may arise where the same letter has been used to denote both a vector and a scalar.