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Pseudo-Monotone Complementarity Problems in Hilbert Space[1]

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Abstract. In this paper, some existence results for a nonlinear complementarity problem involving a pseudo-monotone mapping over an arbitrary closed convex cone in a real Hilbert space are established. In particular, some known existence results for a nonlinear complementarity problem in a finite-dimensional Hilbert space are generalized to an infinite-dimensional real Hilbert space. Applications to a class of nonlinear complementarity problems and the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions are given.

Key Words. Nonlinear complementarity problems, variational inequality problems, pseudo-monotone mappings, monotone mappings, weakly coercive mappings.

1. Introduction

Let H be a real Hilbert space with inner product $<\cdot,\cdot>$ and norm $\|\cdot\|$. A nonempty subset K of H is said to be a cone if $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$. Let K be a closed convex cone in H with dual cone K^*, that is,

$$K^* = \{u \in H \,|\, \langle u, x \rangle \geq 0, \forall x \in K\}.$$

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Let f be a mapping from K into H. We consider the following nonlinear complementarity problem (NCP): find $x \in K$ such that

$$f(x) \in K^* \quad \text{and} \quad \langle x, f(x) \rangle = 0.$$

Such problems were introduced by Karamardian (Ref. 1) and have been extensively studied in the literature. See, e.g., Refs. 1–15 and the references therein. Another problem that is closely related to NCP is the following variational inequality problem (VIP): find $\bar{x} \in K$ such that

$$\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0, \qquad \text{for all } x \in K.$$

The VIP has also been extensively investigated both in finite- and infinite-dimensional spaces. See, e.g., Refs. 8, 16–20. The paper by Harker and Pang (Ref. 8) provides an excellent survey on developments of the VIP in finite-dimensional Euclidean spaces. In Ref. 1, Lemma 3.1, Karamardian has shown that, if K is a closed convex cone, then both VIP and NCP have the same solution set. Therefore, one approach to studying NCP is by studying VIP over closed convex cones. The purpose of this paper is to use this approach to prove some existence results for a nonlinear complementarity problem involving a pseudo-monotone mapping over an arbitrary closed convex cone in a real Hilbert space. Nonlinear complementarity problems involving pseudo-monotone mappings in Hilbert spaces have not yet been investigated except for those in the finite-dimensional case. See, e.g., Ref. 13. In Section 2, we shall prove some necessary and sufficient conditions for existence of solutions to variational inequality problems. Then in Section 3, we give some existence results for solutions to nonlinear complementarity problems by combining the existence results given in Section 2 with the aforementioned Lemma of Karamardian. Finally, in Section 4, we shall consider some applications to a class of nonlinear complementarity problems studied in Ref. 12 and the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions.

2. Existence Results for Variational Inequality Problems

Let K be a nonempty subset of H. A mapping $f: K \to H$ is said to be continuous on finite-dimensional subspaces if it is continuous on $K \cap U$ for every finite-dimensional subspace U of H with $K \cap U \neq \varnothing$. The mapping f is said to be pseudo-monotone (in the sense of Karamardian, Ref. 13) if, for any x and y in K,

$$\langle x - y, f(y) \rangle \geq 0 \text{ implies } \langle x - y, f(x) \rangle \geq 0,$$

and f is said to be monotone if

$$\langle x-y, f(x)-f(y)\rangle \geq 0, \qquad \text{for all } x, y \in K.$$

It is easy to see that, if f is monotone, then it is pseudo-monotone, but not conversely. The mapping f is said to be strictly monotone if the above inequality is strict whenever x and y are distinct. The mapping f is said to be dissipative if $-f$ is monotone. For other types of generalized monotone mappings, we refer readers to Refs. 21 and 22. In Ref. 21, Karamardian and Schaible introduce various types of generalized monotone mappings starting with pseudo-monotone mappings in Ref. 13, whereas in Ref. 22 they present characterizations of differentiable generalized monotone mappings in Ref. 21 and, as a special case, of affine-linear generalized monotone mappings.

For any real number x, $|x|$ denotes the absolute value of x. For $K \subset H$, $\operatorname{int}(K)$, $\partial(K)$, and K^c denote the interior, boundary, and complement of K, respectively. For $K, B \subset H$, $\operatorname{int}_K(B)$ and $\partial_K(B)$ denote the relative interior and relative boundary of B in K, respectively. The set $K \backslash B$ denotes the complement of B in K. A subset of a Hilbert space is said to be solid if it has a nonempty interior. A cone K is said to be pointed if $K \cap -K = \{0\}$. Unless specified otherwise, the topology of a Hilbert space mentioned in this paper refers to the norm topology.

Lemma 2.1. Let K be a closed convex subset in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Then $x \in K$ is a solution of

$$\langle u-x, f(x)\rangle \geq 0, \qquad \text{for all } u \in K, \tag{1}$$

if and only if

$$\langle u-x, f(u)\rangle \geq 0, \qquad \text{for all } u \in K. \tag{2}$$

Proof. Suppose that $x \in K$ is a solution of (2). Let $u \in K$ be arbitrary and, for $0 < \lambda \leq 1$, let

$$x_\lambda = \lambda u + (1-\lambda)x.$$

Then $x_\lambda \in K$, and from (2) we have

$$\lambda\langle u-x, f(x_\lambda)\rangle \geq 0.$$

Hence

$$\langle u-x, f(x_\lambda)\rangle \geq 0. \tag{3}$$

Let $\lambda \to 0$. Then $x_\lambda \to x$ along a line segment. By the continuity on finite-dimensional subspaces, $f(x_\lambda)$ converges to $f(x)$ as λ goes to 0. It follows from (3) that

$$\langle u-x, f(x)\rangle \geq 0.$$

Therefore x is a solution of (1).

Conversely, suppose that $x \in K$ is a solution of (1). Since f is pseudo-monotone, we have

$$\langle u-x, f(u)\rangle \geq 0, \qquad \text{for all } u \in K.$$

Thus x is a solution of (2). $\square$

Remark 2.1. Lemma 2.1 is also true for a hemicontinuous pseudo-monotone operator on a reflexive Banach space.

The following existence theorem is a key result for the remainder of this paper.

Theorem 2.1. Let K be a closed, bounded, and convex subset in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Then there exists $\bar{x} \in K$ such that $\langle x-\bar{x}, f(\bar{x})\rangle \geq 0$ for all $x \in K$.

Proof. Let U be any finite-dimensional subspace of H with $K \cap U \neq \varnothing$, and let P_U be the orthogonal projection of H onto U. Let $f_U = P_U \circ f$, the composition of P_U and f. By a result of Hartman and Stampacchia (Ref. 17, Lemma 3.1), there exists $x_U \in K \cap U$ such that

$$\langle u-x_U, f(x_U)\rangle \geq 0, \qquad \text{for all } u \in K \cap U. \tag{4}$$

Let Λ be the family of all finite-dimensional subspaces U of H with $K \cap U \neq \varnothing$, and let

$$K_U = \{x_V \mid U \subset V \in \Lambda\}.$$

For $U \in \Lambda$, let $\overline{K_U}^{\omega}$ be the weak closure of K_U. Then the family $\{\overline{K_U}^{\omega} \mid U \in \Lambda\}$ has the finite intersection property. Indeed, for $U, V \in \Lambda$, let $W \in \Lambda$ be such that $U \cup V \subset W$. Then $\varnothing \neq K_W \subset K_U \cap K_V$. Since K is closed, bounded, and convex, it is weakly compact. Also, since $\overline{K_U}^{\omega} \subset K$ for all $\bigcap \in \Lambda$, it follows that $\bigcap_{U \in \Lambda} \overline{K_U}^{\omega} \neq \varnothing$.

Let $\bar{x} \in \bigcap_{U \in \Lambda} \overline{K_U}^{\omega}$. Suppose that $u \in K$ is arbitrary, and let $U \in \Lambda$ contain u. Since K_U is bounded and $\bar{x} \in \overline{K_U}^{\omega}$, there exists a sequence $\{x_n\} \subset K_U$ which converges to $\bar{x}$ weakly. By Lemma 2.1 and (4), we have

$$\langle u-x_n, f(u)\rangle \geq 0, \qquad \text{for all } n.$$

The function $\langle u-x, f(u)\rangle$ is weakly continuous in x. Letting n go to infinity, we have

$$\langle u-\bar{x}, f(u)\rangle \geq 0, \qquad \text{for all } u \in K.$$

Hence by Lemma 2.1 again, we have

$$\langle u-\bar{x}, f(\bar{x})\rangle \geq 0, \qquad \text{for all } u \in K. \qquad \square$$

Remark 2.2. In 1966, Hartman and Stampacchia published their celebrated existence result for the VIP which asserts that, if f is continuous, H is a finite-dimensional Euclidean space, and K is a nonempty compact convex set, then the VIP has a solution (Ref. 17, Lemma 3.1). Since then, many extensions of this result have been derived. See, e.g., Ref. 18, Theorem 1.4, p. 84 for the corresponding result in a reflexive Banach space assuming the monotonicity of operators, and Ref. 20, Corollary, p. 187) for a general variational inequality result in locally convex spaces. The latter result was further extended by Théra (Ref. 19, Theorem) in Banach spaces. See also Ref. 16, Theorem 2.6, p. 133, where monotonicity can be replaced by pseudomonotonicity (not in the sense of Karamardian), as the proof shows.

Next, we give some necessary and sufficient conditions for the existence of solutions to the variational inequality problem for unbounded sets.

Theorem 2.2. Let K be a closed convex subset of the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Then the following statements are equivalent:

(i) There exists $\bar{x} \in K$ such that

$$\langle x-\bar{x}, f(\bar{x})\rangle \geq 0, \qquad \text{for all } x \in K. \tag{5}$$

(ii) There exist $u \in K$ and constant $r > \|u\|$ such that $\langle x-u, f(x)\rangle \geq 0$, for all $x \in K$ with $\|x\| = r$.

(iii) There exists $r > 0$ such that the set $\{x \in K \mid \|x\| \leq r\}$ is nonempty and such that, for each $x \in K$ with $\|x\| = r$, there exists $u \in K$ with $\|u\| < r$ and $\langle x-u, f(x)\rangle \geq 0$.

(iv) There exists a closed, solid, convex set E in H such that $\varnothing \neq K \cap E$ is bounded and, for each $x \in K \cap \partial(E)$, there exists $u \in K \cap \operatorname{int}(E)$ such that $\langle x-u, f(x)\rangle \geq 0$.

(v) There exists a nonempty closed, bounded, convex subset B of K with $\operatorname{int}_K(B) \neq \varnothing$ that satisfies the following condition: for each $x \in \partial_K(B)$, there exists $u \in \operatorname{int}_K(B)$ such that $\langle x-u, f(x)\rangle \geq 0$.

Proof.

(i) implies (ii). Let $\bar{x}$ be a solution of (5). Then by choosing $r>0$ such that $\|\bar{x}\|<r$ and letting $u=\bar{x}$, (ii) follows from the pseudo-monotonicity of the mapping f.

(ii) implies (iii). This is obvious.

(iii) implies (iv). Let

$$E=\{x\in H \mid \|x\|\le r\}.$$

Then (iv) follows from (iii) immediately.

(iv) implies (v). Let $B=K\cap E$. Then B is a nonempty closed, bounded, and convex subset of K. First we claim that

$$K\cap \operatorname{int}(E)\subset \operatorname{int}_K(B) \quad \text{and} \quad \partial_K(B)\subset K\cap \partial(E).$$

Suppose $x\in K\cap \operatorname{int}(E)$. Then there exists an open set O such that $s\in O\subset E$. Then $A=K\cap O$ is open in K and $x\in A$. Since $A\subset B$, we have $A\subset \operatorname{int}_K(B)$. Therefore $x\in \operatorname{int}_K(B)$. Hence

$$K\cap \operatorname{int}(E)\subset \operatorname{int}_K(B).$$

Next, suppose $x\in\partial_K(B)$. Let A be any neighborhood of x in K. Then $A\cap B\neq\varnothing$ and $A\cap(K\backslash B)\neq\varnothing$. Then

$$\varnothing\neq(A\cap K\cap E)\subset(A\cap E).$$

Also

$$\begin{aligned} A\cap(K\backslash B) &= A\cap(K\cap(E^c\cup K^c)) \\ &= A\cap((K\cap E^c)\cup(K\cap K^c)) \\ &= A\cap K\cap E^c. \end{aligned}$$

So $A\cap E^c\neq\varnothing$. If A is any neighborhood of x, then A is also a neighborhood of x in K. Thus $A\cap E\neq\varnothing$ and $A\cap E^c\neq\varnothing$. Therefore $x\in K\cap\partial(E)$. Hence

$$\partial_K(B)\subset K\cap\partial(E).$$

Now, let $x\in\partial_K(B)$. Since $\partial_K(B)\subset K\cap\partial(E)$, $x\in K\cap\partial(E)$. Then by (iv), there exists $u\in K\cap\operatorname{int}(E)\subset\operatorname{int}_K(B)$ such that $\langle x-u, f(x)\rangle\ge 0$. Hence (v) follows.

(v) implies (i). By Theorem 2.1 there exists $\bar{x}\in B$ such that

$$\langle x-\bar{x}, f(\bar{x})\rangle\ge 0, \qquad \text{for all } x\in B. \tag{6}$$

For arbitrary $x\in K$, there are two possibilities.

(a) $\bar{x}\in\operatorname{int}_K(B)$. There exists $0<\lambda<1$ such that $\lambda x+(1-\lambda)\bar{x}\in B$. Then by (6), we have $\lambda\langle x-\bar{x}, f(\bar{x})\rangle\ge 0$. Thus $\langle x-\bar{x}, f(\bar{x})\rangle\ge 0$.

(b) $\bar{x} \in \partial_K(B)$. By the condition in (v), there exists $u \in \mathrm{int}_K(B)$ such that

$$\langle \bar{x} - u, f(\bar{x}) \rangle \geq 0.$$

Therefore by (6) we have $\langle \bar{x} - u, f(\bar{x}) \rangle = 0$. Now choose $0 < \lambda < 1$ such that

$$\lambda x + (1 - \lambda) u \in B.$$

By (6) we have

$$\begin{aligned} 0 &\leq \langle \lambda (x - u) + u - \bar{x}, f(\bar{x}) \rangle \\ &= \lambda \langle x - u, f(\bar{x}) \rangle \\ &= \lambda \langle x - \bar{x}, f(\bar{x}) \rangle. \end{aligned}$$

So again $\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0$. Hence

$$\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0, \qquad \text{for all } x \in K.$$

Therefore $\bar{x}$ is a solution to (5). □

In finite-dimensional spaces, the result that (ii) or (iii) implies (i) in Theorem 2.2 without the assumption of pseudo-monotonicity of f is given in Ref. 14, Theorems 2.3, 2.4.

3. Existence Results for Pseudo-Monotone Complementarity Problems

In this section, we obtain some existence results on solutions to nonlinear complementarity problems by combining Theorem 2.2 and the following lemma.

Lemma 3.1. Let K be a closed convex cone in the Hilbert space H, and let f be a mapping from K into H. Then $\bar{x} \in K$ satisfies

$$\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0, \qquad \text{for all } x \in K,$$

if and only if

$$f(\bar{x}) \in K^* \quad \text{and} \quad \delta \bar{x}, f(\bar{x}) \rangle = 0.$$

Proof. This is a special case of Ref. 1, Lemma 3.1. □

By Theorem 2.2 and Lemma 3.1, we have the following necessary and sufficient conditions for the existence of a solution to a nonlinear complementarity problem.

Theorem 3.1. Let K be a closed convex cone in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Then the following statements are equivalent:

(i) There exists $\bar{x} \in K$ such that $f(\bar{x}) \in K^*$ and $\langle \bar{x}, f(\bar{x}) \rangle = 0$.
(ii) There exist $u \in K$ and constant $r > \|u\|$ such that $\langle x-u, f(x) \rangle \geq 0$ for all $x \in K$ with $\|x\| = r$.
(iii) There exists $r > 0$ such that, for each $x \in K$ with $\|x\| = r$, there exists $u \in K$ with $\|u\| < r$ and $\langle x-u, f(x) \rangle \geq 0$.
(iv) There exists a closed, solid, convex set E in H such that the set $K \cap E$ is nonempty and bounded and, for each $x \in K \cap \partial(E)$, there exists $u \in K \cap \operatorname{int}(E)$ such that $\langle x-u, f(x) \rangle \geq 0$.
(v) There exists a nonempty closed, bounded, and convex subset B of K with nonempty $\operatorname{int}_K(B)$ satisfying the following condition: for each $x \in \partial_K(B)$, there exists $u \in \operatorname{int}_K(B)$ such that $\langle x-u, f(x) \rangle \geq 0$.

Before we give some other important sufficient conditions for the existence of a solution to NCP implied by Theorem 3.1, we recall some definitions.

Let K be a subset of a real Hilbert space H. Let f be a mapping from K into H. Then f is said to be coercive if

$$\langle x, f(x) \rangle / \|x\| \to \infty, \qquad \text{as } \|x\| \to \infty \text{ and } x \in K,$$

and f is said to be weakly coercive if

$$\langle x, f(x) \rangle \to \infty, \qquad \text{as } \|x\| \to \infty \text{ and } x \in K.$$

The mapping f is said to be α-monotone if there exists an increasing function $\alpha : [0, \infty) \to [0, \infty)$, with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$, such that

$$\langle x-y, f(x)-f(y) \rangle \geq \|x-y\| \alpha(\|x-y\|), \qquad \text{for all } x, y \in K.$$

If $\alpha(r) = kr$ for some $k > 0$, then f is said to be strongly monotone.

Theorem 3.2. Let K be a closed convex cone in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Then there exists $\bar{x} \in K$ such that

$$f(\bar{x}) \in K^* \quad \text{and} \quad \langle \bar{x}, f(\bar{x}) \rangle = 0$$

under each of the following conditions:

(i) $\lim_{\|x\| \to \infty, x \in K} \langle x, f(x) \rangle > 0$,
(ii) f is weakly coercive,
(iii) f is coercive.

Proof.

(i) By the condition, there exists an $r>0$ such that $\langle x, f(x)\rangle \geq 0$ for all $x \in K$ with $\|x\|=r$. Therefore, the result follows from Theorem 3.1 (ii).

Parts (ii) and (iii) follow directly from (i).

Remark 3.1. If instead f is assumed to be monotone and hemicontinuous, then the analogue of Theorem 3.2 follows directly from Ref. 11, Theorem 3.6, p. 52.

Corollary 3.1. Let K be a closed convex cone in the real Hilbert space H. Let f be a mapping from K into H which is continuous on finite-dimensional subspaces. Then there exists a unique solution to NCP under each of the following conditions:

(i) f is strictly monotone and weakly coercive,
(ii) f is strongly monotone,
(iii) f is α-monotone.

Proof. The existence of a solution to NCP follows from Theorem 3.2 directly. The uniqueness of the solution can be proved by the same argument as that in Ref. 14, Corollary 3.2. □

We note that Corollary 3.1 (ii) extends a result of Nanda and Nanda (Ref. 15, Theorem) where f is assumed to be strongly monotone and Lipschitzian.

Corollary 3.2. Let K be a closed convex cone in the real Hilbert space H. Let f be a monotone and coercive mapping from K into H which is continuous on finite-dimensional subspaces. Then for each $u \in H$, there exists $\bar{x} \in K$ such that

$$f(\bar{x})-u \in K^* \quad \text{and} \quad \langle \bar{x}, f(\bar{x})-u\rangle=0.$$

Proof. Let $g: K \rightarrow H$ be defined by $g(x)=f(x)-u$ for all $x \in K$. Since f is monotone and coercive, g is also monotone and coercive. Therefore by Theorem 3.2(ii), there exists $\bar{x} \in K$ such that $g(\bar{x}) \in K^*$ and $\langle \bar{x}, g(\bar{x})\rangle=0$. Hence the result follows. □

Remark 3.2. Corollary 3.2 is the famous Hartman–Stampacchia–Browder theorem. See, e.g., Ref. 18, Corollary 1.8, p. 87.

We note that the conclusion of Corollary 3.2 may not hold if f is simply assumed to be weakly coercive. For example, let $H = \mathbb{R}$, the set of real numbers,

$$K = \{x \in \mathbb{R} | x \geq 0\},$$

and f be a constant mapping from K into H with $f(x) = 1$ for all $x \in K$. Then f is weakly coercive. But it is clear that, for each $u > 1$, there exists no $\bar{x} \in K$ such that

$$f(\bar{x}) - u \in K^* \quad \text{and} \quad \langle \bar{x}, f(\bar{x}) - u \rangle = 0.$$

We also note that, when H is finite-dimensional, the monotonicity of f in Corollary 3.2 is not needed. See, e.g., Ref. 14, Theorem 3.1.

Lemma 3.2. Let K be a pointed solid closed convex cone in the real Hilbert space H, and let $v \in \mathrm{int}(K^*)$. Then for any $\alpha > 0$, the set $S = \{x \in K | \langle x, v \rangle \leq \alpha\}$ is weakly compact.

Proof. It is clear that the set S is weakly closed. Let

$$S^\circ = \{y \in H | \langle x, y \rangle \leq 1, \forall x \in S\}$$

be the polar of S. Since K^* has nonempty interior in the norm topology, S° has nonempty interior in the norm topology. Then by Ref. 23, Proposition 2.4, p. IV.5, S° has nonempty interior in the Mackey topology. Since for each $x \in S$ we have $-\alpha \leq \langle x, v \rangle \leq 0$, therefore S is weakly compact by Ref. 24, Theorem 1. □

Remark 3.3. As a referee has pointed out, Lemma 3.2 is immediate for when $\mathrm{int}(K^*) \neq \varnothing$, it follows that K is well based.

The following result is a generalization of a result of Karamardian (Ref. 13, Theorem 4.1) to infinite-dimensional spaces.

Theorem 3.3. Let K be a pointed solid closed convex cone in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Suppose that there exists $u \in K$ such that $f(u) \in \mathrm{int}(K^*)$. Then there exists $\bar{x} \in K$ such that $f(\bar{x}) \in K^*$ and $\langle \bar{x}, f(\bar{x}) \rangle = 0$.

Proof. If $\langle u, f(u) \rangle = 0$, then we are done. So we may assume that $\langle u, f(u) \rangle > 0$. Let U be any finite-dimensional subspace of H with $u \in U$, and let P_U be the orthogonal projection of H onto U. Let $f_U = P_U \circ f$. Since

$f(u)\in\operatorname{int}(K^*)$, for each $x\in K\cap U$ with $x\neq 0$ we have

$$\langle x, f_U(u)\rangle=\langle x, f(u)\rangle>0.$$

Therefore $f_U(u)\in\operatorname{int}((K\cap U)^*)$ by Ref. 13, Lemma 2.1 (i). Then by Ref. 13, Theorem 4.1, there exists $x_U\in K\cap U$ such that

$$f_U(x_U)\in(K\cap U)^* \quad \text{and} \quad \langle x_U, f_U(x_U)\rangle=0.$$

By Lemma 3.1, it follows that

$$\langle x-x_U, f(x_U)\rangle\geq 0, \qquad \text{for all } x\in K\cap U.$$

Let $\Lambda(u)$ be the family of all finite-dimensional subspaces U of H with $u\in U$, and let

$$K_U=\{x_V \mid U\subset V\in\Lambda(u)\}.$$

Let

$$S=\{x\in K \mid \langle x, f(u)\rangle\leq\langle u, f(u)\rangle\}.$$

If $x\in K\backslash S$, then $\langle x-u, f(u)\rangle>0$. Therefore $\langle x-u, f(x)\rangle>0$, and consequently $K_U\subset S$ for all $U\in\Lambda(u)$. For $U\in\Lambda(u)$, let $\overline{K_U}^\omega$ be the weak closure of K_U. Then the family $\{\overline{K_U}^\omega \mid U\in\Lambda(u)\}$ has the finite intersection property. Since S is weakly compact by Lemma 3.2 and $\overline{K_U}^\omega\subset S$ for all $U\in\Lambda(u)$, it follows that $\bigcap_{U\in\Lambda(u)}\overline{K_U}^\omega\neq\emptyset$. Let $\bar{x}\in\bigcap_{U\in\Lambda(u)}\overline{K_U}^\omega$. By the same argument as that in the proof of Theorem 2.1, we have

$$\langle x-\bar{x}, f(\bar{x})\rangle\geq 0, \qquad \text{for all } x\in K.$$

Hence by Lemma 3.1,we have

$$f(\bar{x})\in K^* \quad \text{and} \quad \langle\bar{x}, f(\bar{x})\rangle=0. \qquad \square$$

Remark 3.4. Results similar to Theorem 3.3 in topological vector spaces have been proved in Ref. 2, Theorem 4 and Ref. 4, Theorem 10, respectively.

We note that, by the same argument as that in the proof of Theorem 3.3, a similar existence result for variational inequality problems can be obtained as follows (cf. Ref. 8, Theorem 3.4).

Let K be a closed convex set in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Suppose that there exists $u\in K$ such that $f(u)\in\operatorname{int}(K^*)$. Then there exists a vector $\bar{x}\in K$ such that

$$\langle x-\bar{x}, f(\bar{x})\rangle\geq 0, \qquad \text{for all } x\in K,$$

where K^* is defined exactly as in the case where K is a convex cone.

4. Some Applications

In this section, we shall consider some applications of existence results established in Section 3 to a class of nonlinear complementarity problems studied in Ref. 12, and in particular, to the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions.

Let $K \subset H$ be a closed convex cone, and let $L_1, L_2 : K \to H$ be two mappings. The nonlinear complementarity problem (NCP) (T, K) is to find $x \in K$ such that

$$T(x) \in K^* \quad \text{and} \quad \langle x, T(x) \rangle = 0,$$

where

$$T(x) = x - L_1(x) + L_2(x), \qquad \text{for each } x \in K.$$

Such problems were studied in Ref. 12 and were used as mathematical models for mechanical problems. By employing Theorem 3.2(i), we have the following existence result for (NCP)(T, K).

Theorem 4.1. Suppose the mapping T is pseudo-monotone and is continuous on finite-dimensional subspaces. If $\lim_{\|x\| \to \infty, x \in K} \langle x, T(x) \rangle > 0$, then there exists a solution to (NCP)(T, K).

Remark 4.1. In Theorem 4.1, the closed convex cone K need not be necessarily pointed, and the mapping T need not be the one-sided Gâteaux directional derivative of a functional defined on K (cf. Ref. 12, Theorem 3.1).

Remark 4.2. Problem (NCP)(T, K) will have a unique solution if the mapping $-L_1$ is dissipative and the mapping L_2 is monotone since in this case the mapping T is strongly monotone. The uniqueness of the solution of problem (NCP)(T, K) was not discussed in Ref. 12.

Next, we consider another application to the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions. We adopt the notations used in Ref. 12.

Let Ω be a thin elastic plate whose thickness is supposed to be constant and which rests without friction on a flat rigid support. The material is also supposed to be homogeneous and isotropic. Mathematically, Ω may be identified as a bounded open connected subset of $\mathbb{R}^2$. The plate Ω is assumed to be clamped on $\gamma_1 \subset \Omega$ and simply supported on $\gamma_2 = \gamma \backslash \gamma_1$, where γ is the boundary of Ω which is supposed to be sufficiently regular.

Suppose that a lateral variable load λL_α, where λ is positive and increasing representing the magnitude of lateral loading, is applied to the boundary of Ω. Consider the Sobolev space

$$H^2(\Omega)=\{u\in L^2(\Omega)\,|\,\partial u/\partial x_i,\ \partial^2 u/\partial x_i\,\partial x_j\in L^2(\Omega),\ \forall i,j=1,2\},$$

equipped with the norm $\|\cdot\|_{H^2(\Omega)}$, and let E be the closed subspace of $H^2(\Omega)$ defined by

$$E=\{z\in H^2(\Omega)|z_{|\gamma}=0,\text{ and }\partial z/\partial n_{|\gamma_1}=0,\text{ a.e.}\},$$

where n denotes the normal to γ exterior to Ω and $\partial(\cdot)/\partial n$ denotes the normal exterior derivative. We may equip E with the inner product defined by a continuous bilinear form on $E\times E$ such that the associated norm is equivalent to the initial norm $\|\cdot\|_{H^2(\Omega)}$ (see Ref. 12). For a fixed λ, the post-critical equilibrium state of the plate subjected to unilateral conditions is governed by the following variational inequality problem: Find $x\in K$ such that

$$\langle z-w,\ w-\lambda L(w)+C(w)\rangle\geq 0,\qquad \text{for all } z\in K. \tag{7}$$

Here,

$$K=\{z\in E\,|\,z\geq 0,\text{ a.e. on }\Omega\}$$

represents the admissible vertical displacements of the plate; $\lambda\geq 0$ is exactly the intensity of the lateral load λL_α; L is a self-adjoint linear compact operator defined by the nature of the load applied on the boundary of Ω; and C is a bounded nonlinear continuous operator connected with the expansive properties of the plate. In practice, the mapping C is also assumed to be a Gâteaux derivative of a nonlinear functional Φ such that $\Phi(0)=0$ (see, e.g., Ref. 25). In this case, $w=0$ is a trivial solution to problem (7). The existence of a nonzero solution depends on the forces of magnitude λ. See Ref. 12 for a detailed discussion on this issue. For more detailed discussion on the problem (7), we refer readers to Refs. 12 and 24 and the references therein.

Since K is a closed convex cone, by Lemma 3.1, problem (7) is equivalent to problem (NCP)(T, K), where

$$T(x)=x-\lambda L(x)+C(x),\qquad \text{for all } x\in K.$$

Therefore, by Theorem 4.1, we have the following existence result for problem (7), and hence the existence of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions.

Theorem 4.2. Suppose that the mapping T is pseudo-monotone and also suppose that $\lim_{\|x\|\to\infty,\, x\in K}\langle x, T(x)\rangle>0$. Then there exists a solution to problem (7).

Finally, by Remark 4.2, we have the following result concerning the uniqueness of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions:

Theorem 4.3. Suppose that the mapping L is dissipative and the mapping C is monotone. Then there exists a unique solution to problem (7).

Remark 4.3. Other results in Section 3 can also be employed to obtain existence results for problem (NCP)(T, K) and problem (7).

Remark 4.4. Physically, if the intensity λ of the lateral load λL_α applied on the boundary of Ω exceeds a critical value, called the critical load which is specific for each plate Ω, then the plate deflects out of its plane, and we say that it buckles. We may interpret that assumptions of Theorem 4.3 as follows. If the forces of the lateral load λL_α are dissipative and the expansive properties of the plate are monotone, then there is a unique equilibrium.

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