

Contents

1	Concepts	1
1.1	Definitions	1
1.1.1	Events	1
1.1.2	Streams	2
1.1.3	Functions	2
1.1.4	Nodes	3
1.1.5	TeSSLa Evaluation Engine	4
1.1.6	State	5
1.1.7	Transitions	5
1.1.8	Run	6
1.2	Behaviour of different schedules without timing functions	8
1.2.1	Synchronous Evaluation Engines	8
1.2.2	Asynchronous evaluation	10
1.3	Equalitys of different schedules without timing functions	10
1.3.1	Equality of synchronous Systems	11
1.3.2	Equality of synchronous and asynchronous schedules	12
1.4	Behaviour with Timing functions	13
1.5	Equalitys with Timing functions	13
1.6	Parallel computation	13

Concepts

In this Chapter different ways to evaluate TeSSLa specifications are given and their equivalence is shown. To do so in Section 1.1 building blocks for evaluation approaches are defined, which are then used in later Sections to define behaviour of them and show their equivalence.

1.1 Definitions

While the TeSSLa specification itself defines a set of semantics, for this Thesis we will slightly alter some of it and add some new definitions based on them. This is necessary to reason about the specifics how the evaluation engine is built (Note that TeSSLa doesn't define an operational semantic, therefore we will define our own) and how it behaves.

1.1.1 Events

Events are the atomic unit of information that all computations are based on. There are three Types of Events: input, output and internal Events.

The Set of all Events is denoted as E . Each event carries a value, which can be *nothing* or a value of a Type (types are formally defined in the TeSSLa specification but aren't important for this thesis), a timestamp and the stream it's perceived on (e.g. a function call of a specific function or the name of an output stream).

The value of an event can be queried with the function v , its timestamp with π and its Stream with ς .

$E_i \subset E$ is the Set of all input events, their Stream corresponds to a specific trace. $E_o \subset E$ is the Set of all output events, their Stream is specified by an output name of the TeSSLa specification. $E_n \subset E$ is the Set of all internal events. Internal events are mostly an implementation detail, which denote steps of computation inside the runtime. The Stream of internal events is implicitly given by the node that produces the stream of the Event. Note that E_i, E_o, E_n are pairwise disjoint and $E_i \cup E_o \cup E_n = E$.

1.1.2 Streams

Streams are a collection of Events with some added characteristics. While Events are the atomic unit of information, Streams represent the sequence of related Events over time.

There are two kinds of Streams: Signals, which carry values at all times, and EventStreams, which only hold values at specific times. EventStreams can be described by a sequence of Events. Signals can be described by a sequence of changes, where a change notes that the value of a Signal changed at a specific timestamp. The only difference between a Signal and an EventStream is that Signals always have a value while an EventStream may return \perp , which denotes, that no event happened at that time. Because of the similarity of Signals and EventStreams in the following we will mainly reason about EventStreams, but most things can also be applied to Signals.

Formally a Stream σ can be represented as a Sequence of Events $\langle e_1, \dots, e_n \rangle$ where $\pi(e_i) < \pi(e_{i+1}) \forall i < n \in \mathbb{N}$. The Set of all Streams Σ is defined as all possible finite sequences of Events $\Sigma = \{\sigma \mid \sigma \in E^*\}$. An input Stream σ_i is a stream consisting only of input Events, the set of all input Streams is $\Sigma_i = \{\sigma_i \mid \sigma_i \in E_i^*\}$. Output and internal Streams are defined analogous. To get the Event of a Stream σ at a timestamp t it can be queried like a function: $\sigma(t) = e$ with $\pi(e) = t$. When working with Signals, the function will return the latest Event that happened at or before t while an EventStream may return \perp .

Furthermore Streams hold the timestamp to which they have progressed, which can be equal or greater than the timestamp of the last Event happened on them.

1.1.3 Functions

A TeSSLa specification consists of functions. Functions generate new Streams by applying an operation on other Streams. TeSSLa itself defines a syntax to write a specification, a set of types and a standard library of functions, but an implementation is free to choose the functions it supports.

An example function is $add(S_D, S_D) \rightarrow S_D$: It takes two Signals, which have to hold values of some numerical type, and produces a signal which holds values of the same type. The produced stream can either be assigned to a named identifier (think: a variable) or directly be given to another function (function composition).

Functions can be divided into three categories: pure, unpure and timing. Pure Functions, also called stateless, are evaluated only on the values their inputs have at the timestamp they are evaluated, therefore they don't have to *remember* anything

about earlier events. Unpure, or stateful, Functions are evaluated over the whole input stream, meaning they can look at all Events that happened on its inputs before the time of evaluation and also at all its previous output events. E.g. a Function *eventCount* has to *remember* how many events already happened on its input stream and increment that counter on every new event. Timing Functions are evaluated not only on the value of Events but also on their timestamp and can also manipulate it: While non timing functions will consume events at a specific timestamp and emit an event with that timestamp, timing functions can emit Events with a different timestamp.

Timing Functions complicate the reasoning about schedules and causality and therefore aren't included in Section 1.2. In Section ?? the conclusions of earlier sections will be extended to include timing functions.

1.1.4 Nodes

Nodes are the atomic unit of computation for the evaluation of a TeSSLa specification. A Node implements a single Function, e.g.: there is an *AddNode* which takes two input Signals and produces a new Signal. Therefore a Node is the concrete implementation of a function in a runtime for TeSSLa specifications. Each Node has a set of inputs, which are either input or internal Streams, and one output, which is either an internal or an output Stream.

Nodes use a FIFO queue, provided by the Erlang platform, to process new received Events in multiple steps:

1. Add the new Event to the inputs
2. Check if a new output Event can be produced (see Section 1.1.4)
3. If so, compute all timestamps, where new Events might be computed and
 - a) Compute the Events, add them to the History as new outputs
 - b) Distribute the updated output to all successors
4. Else wait for another input

Determination of processable Events

Based on the asynchronous nature of Nodes, Events from different Streams can be received out of order. E.g. if a Node C is a child of Node A and B, it can receive Events from Node A at timestamps t_1, t_2, t_3, t_4 before receiving an event with timestamp t_1 from Node B. Therefore a Node can not compute its output upto a

timestamp unless it has informations from all predecessors that they did progress to that timestamp. When Node C receives the first four Events from Node A, it will only add them to its inputs but won't compute an output. When it finally receives the first Event from Node B it can compute all Events upto t_1 . To do so it will compute *change timestamps*: The union of all timestamps where an Event occurred on any input between the timestamp of the last generated output and the minimal progress of all inputs.

To see why this is necessary lets assume that Node C will receive a new Event from Node B with timestamp t_4 : All inputs have progressed to t_4 , but on the stream from Node A there are changes between t_1 (where the last output was generated) and t_4 , therefore the *change timestamps* are t_2, t_3, t_4 and the Node will have to compute its output based on the values of the streams at that timestamps.

1.1.5 TeSSLa Evaluation Engine

Because Functions in TeSSLa specifications itself depend on other Functions, and these dependencies have to be circle free, the specification can be represented as a DAG@. This DAG can be directly translated into an evaluation Engine for that specification: The Nodes of the DAGs are Nodes representing the functions and the Edges are the input and output Streams between the Nodes.

Definition 1. Equality of evaluation Engines: *Two evaluation Engines are called equal, if they produce the same Relationship between inputs and outputs for all possible inputs.*

Definition 2. Equality of evaluation Engines for one Input: *Two evaluation Engines are called equal for an input, if they produce the same output for that input.*

Definition 3. Enabledness of Nodes: *A Node in the DAG is called enabled when it has at least one Event buffered on all of its inputs, meaning it is able to perform a computation and produce a new output. In the context of a Run enabledness means, that a Transition can be taken which represents the computation of the enabled Node.*

Definition 4. Independence of Nodes: *A Node in the DAG is called independent of another Node, if it is no descendant of that Node.*

Lemma 1. *A Node which is enabled Stays enabled until it is scheduled*

Proof. TODO

□

To evaluate a specification over Traces, the Evaluation Engine has process the Events that were traced. To do so the Nodes have to run their computations until no more Events are present (or the specification found an error in the trace). This

leads to the question in which order Nodes should be scheduled to perform their computation. While some schedules are simply not rational (think of unfairness and causality) there are many different schedules that are feasible. It has to be proven that a chosen schedule produces the correct conclusions for a specification, else the Evaluation Engine is not valid.

In this Thesis it is shown that multiple schedules will lead to the same conclusions and therefore an implementation of an Evaluation Engine is free to choose between them.

1.1.6 State

All TeSSLa evaluation Engines have to hold a State, which encodes information necessary to continue the evaluation. The State of a whole Evaluation Engine is made up of the States of it's Nodes.

Each Node has a State which holds the History of its input Streams and its output Stream. One part of the State of every Node is the History of a Node, which holds all Events received from it's parents, called it's inputs, and all produced Events of the Node, called it's output.

To distinguish between the two types of States, we will call the State of the whole Engine the *global State* and the State of a Node the *Node State*.

Formally the State of a Node is a tuple of the inputs (a Set of internal and input Streams) and the output (an internal or an output Stream). The Set of all valid States of a Node is $N = (\Sigma_i \cup \Sigma_n)^* \times (\Sigma_n \cup \Sigma_o)$. Based on this, the Set of all valid States is $S = N^*$. A global State can be queried like $S_1(i) = n_i$ to yield the State of the Node at that position.

Everytime a Node is scheduled it changes it State by:

- Updating it's inputs (Adding a new Event to the Input Streams)
- Generating a new Event if possible and add it to it's outputs

1.1.7 Transitions

A Transition describes what happens when the evaluation Engine switches State by performing a Step: The consumption of one Event from each input of a Node and the optional generation of one or more new output Events of that Node. To look at it in another way: A Transition is the computation of a Node, therefore when

we say ‘Node A is scheduled’ we mean that a Transition is taken which models the computation of that Node.

Formally Transitions are a Relation between two Sets of Events. E.g. the Transition $\tau = (\{e_1, e_2\}, \{e_3\})$ specifies that two Events were consumed by a Node (thereby adding them to its inputs) and one Event was produced based on them (which is added to the output). The Set of all Transitions is

$$T = \{(e, e') | e \subseteq (E_i \cup E_n), e \neq \emptyset, e' \subseteq (E_n \cup E_o), \forall e_i, e_j \in e' : \varsigma(e_i) = \varsigma(e_j)\}$$

where the Node which caused the Transition is the one creating the Stream $\varsigma(e_i)$.

The empty Transition, meaning no input was consumed and no output produced, is labeled with λ . Note that all Transitions, except the empty one, have to consume at least one Event (therefore no Events can be created from nowhere) and that it’s possible that no Event was produced based on the consumed Events (think of a *FilterNode*). Furthermore it’s theoretically possible to create multiple Events in one Transition. This makes only sense in the context of Timing Nodes, because else the generated Events would have the same timestamp, which is forbidden by the definition of Streams. With Timing Nodes one could for example implement an *EchoNode*, which duplicates an input after a specified amount of Time.

Definition 5. A Transition is called independent of another, if the Node of which it represents the computation, is independent of the Node of the other Transition.

1.1.8 Run

A Run of an Evaluation Engine is a Sequence of Transitions and States. The first element of the sequence is the empty Transition and the initial State of the evaluation Engine. It is a representation of the steps the Engine takes to evaluate a specification over input Streams. The Run $\langle (\lambda, s_0), (\tau_1, s_1) \rangle$ means, that the Engine was in it’s initial State, took the transition τ_1 and thereby reached the state s_1 .

Definition 6. Equality of Runs: Two Runs are called equal if they have the same State at their last position.

Because the State can be built from the Transitions that were taken, equality can also be defined over Transitions.

Lemma 2. If the Set of Transitions of two Runs are equal, the Runs are equal.

Proof. Let R_1, R_2 be the Runs of two Engines with the same Set of Transitions T' . Let S_1 be the final State of the Run R_1 and S_2 of R_2 . If the two global States weren’t equal, there would have to be at least one i with $S_1(i) \neq S_2(i)$, meaning the same

Node has to have a different State in both Engines. Let n_1, n_2 be the States of one such Node in both Engines. If the two States are different, one of them has to contain at least one Event on an input or output that isn't on the same Stream in the other State. Let that Event be e_d . To be added to the State, there has to be a Transition $\tau = (e, e')$ with $e_d \in e \vee e_d \in e'$. This Transition has to be in T' , which means it was taken by both Engines, therefore e_d is in the History of the Node in both Runs. \square

Definition 7. Distance of Runs: *The Distance δ of a Run R_1 to a Run R_2 is a pair $\delta(R_1, R_2) = (x, y)$, where x is the index before the first position where the two Runs differ and y is the number of Steps between the index of the first difference and the position where R_2 takes the Transition that R_1 took after step x . The Distance of Runs is compared element-wise: $(x, y) < (x', y') \leftrightarrow ((x > x') \vee (x = x' \wedge y < y'))$.*

Example 1. *Let*

$$R_1 = \langle (\lambda, S_0), (\tau_1, S_1), (\tau_2, S_2), (\tau_3, S_3), (\tau_4, S_4), (\tau_5, S_5), (\tau_6, S_6) \rangle$$

$$R_2 = \langle (\lambda, S_0), (\tau_1, S_1), (\tau_2, S_2), (\tau_5, S'_3), (\tau_4, S'_4), (\tau_6, S'_5), (\tau_3, S'_6) \rangle$$

$$R_3 = \langle (\lambda, S_0), (\tau_1, S_1), (\tau_2, S_2), (\tau_3, S_3), (\tau_5, S''_4), (\tau_4, S''_5), (\tau_6, S''_6) \rangle$$

Then is

$$\delta_{1,2} = \delta(R_1, R_2) = (3, 3), \delta_{1,3} = \delta(R_1, R_3) = (4, 1)$$

$$\delta_{2,1} = \delta(R_2, R_1) = (3, 2), \delta_{2,3} = \delta(R_2, R_3) = (3, 1)$$

$$\delta_{3,1} = \delta(R_3, R_1) = (4, 1), \delta_{3,2} = \delta(R_3, R_2) = (3, 2)$$

And $\delta_{1,3} < \delta_{2,3} < \delta_{2,1} < \delta_{2,1}$.

Lemma 3. *When a Node A is independent of a Node B, than it has no influence on the enabledness of Node A if Node B is scheduled before or after it.*

Proof. As already shown in Lemma 1, if Node A is enabled, it stays enabled until it's scheduled, so the computation of B can't change A from enabled to disabled. It remains to show, that the computation of Node B can't change Node A from disabled to enabled. A Node becomes enabled when on all of its input at least one new Event is present. New Events on an input are added, when a parent Node performs computation. Because A is independent of B, no input of A is the output of B. Furthermore B is no parent of any predecessor of A, therefore no predecessors input is the output of B. Because of that, when B is scheduled, no Event can be added to the input of any of A's predecessors or A itself. So if A wasn't enabled earlier because it hasn't a new Event at one input, the computation of B can't lead to adding that Event. Therefore it doesn't at which position B is scheduled in regard to the enabledness of A. \square

1.2 Behaviour of different schedules without timing functions

For a first step we specify and compare behaviours of different approaches to evaluate TeSSLa specifications without timing Functions. Without timing Functions, all Nodes work only on values or the presence of Events. This leads to behaviours that can be easily reason about, as seen in the next sections.

All Systems to evaluate TeSSLa specifications we will look at are based on the described structure in Section 1.1.5. While there may be other approaches to evaluation, a DAG based approach seems to fit most naturally and focusing on one structure makes comparing Systems easier.

Each evaluation Engine will work in steps, where each step is synonymous with an index in the Run of the System. Therefore at each Step one Node is scheduled to perform it's operation, represented as the Transition in the Run. The Transition will encode one of the following three Things that can happen:

- An input Event can be consumed by a source in the DAG, which generates an internal Event that is propagated to it's children.
- An internal Node, which has at least one new input buffered on all of its input queues, can perform its computation and generate a new internal Event, which is propagated to the children of that node, which therefore can compute in the next step.
- An output node, which has at least one new input buffered on all of its input queues, can produce a new output.

Evaluation Engines are free in the way they are scheduling their Nodes, only limited by causality (no Event can be consumed before it's produced). In the following evaluation Engines are classified by their scheduling approaches.

1.2.1 Synchronous Evaluation Engines

The first class of evaluation Engines are synchronous ones. They are characterized by a specific, fixed schedule. The scheduling algorithm is as follows: Choose a reversed topological order of the Nodes once and assign numbers to them in that order. Then always schedule the enabled Node with lowest Number.

Obviously for many DAGs there is no unique reversed topological order, therefore one can be chosen by the evaluation engine.

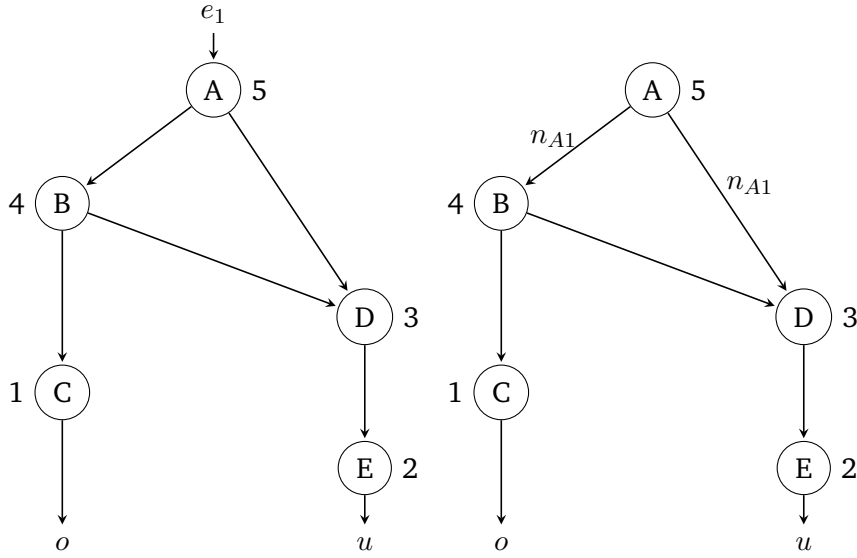


Fig. 1.1: Visualization of a simple asynchronous system with a reversed topological order.

This schedule ensures that no Node is scheduled which has a successor that can be scheduled, therefore Events are *pushed* through the DAG towards an output Node as fast as possible.

Definition 8. Valid evaluation Engine: *An evaluation Engine is called valid, if it is equal to a synchronous evaluation Engine.*

Figure 1.1 visualizes a synchronous evaluation Engine. It shows two DAG representations of an evaluation Engine where the Nodes A to E are labeled in a reversed topological order and o and u represents the output Streams with that name. The left System is in its initial State and an input event e_1 is present and can be consumed by the input Node A. When a Node is chosen to compute by the scheduler, only Node A is enabled, therefore it is scheduled. The right system is the representation of the next step: Node A has consumed the external event and produced an internal event n_{A1} which is propagated to all its children: Node B and D. In the next step Node B would be scheduled, because it has the lowest number of any node that can compute (actually it's the only node that can compute at all, because D has to wait for the event from B). After B was scheduled, it would have produced the internal Event n_{B1} which would then be distributed to Nodes C and D.

The complete Run of the synchronous Engine for one Input is the following, where the States are not further defined:

$$\begin{aligned} & \langle (\lambda, S_0), ((\{n_{A1}\}, \{n_{B1}\}), S_1), ((\{n_{B1}\}, \{o_1\}), S_2), \\ & ((\{n_{A1}, n_{B1}\}, \{n_{D1}\}), S_3), ((\{n_{D1}\}, \{u_1\}), S_4) \rangle \end{aligned}$$

If there were more than one input Event, at this point Node A would be scheduled again. It would consume the next input and the following Nodes would be scheduled in the same order as before, extending the Run in an obvious way.

1.2.2 Asynchronous evaluation

An asynchronous evaluation Engine one with a fair, but not fixed schedule.

In contrast to the synchronous evaluation Engine it has no fixed schedule, the only requirement is that the schedule is fair. Therefore predecessors of enabled Nodes can perform multiple computations before their children are scheduled and Events are not *pushed* through the DAG as fast as possible.

more
later

1.3 Equality of different schedules without timing functions

Based on the described behaviours of the approaches we now can proof the equality of different Schedules for the same evaluation Engine for a TeSSLa specification.

Lemma 4. *Two evaluation Engines are equal for an input, if their Runs are equal for that input.*

Proof. Since two Runs are equal if they have the same last State, and all Events which were produced are stored in the State, during both Runs the same output Events had to be generated or else their State would differ. \square

As defined by Definition 8 any evaluation Engine has to be equal to a synchronous to be valid.

The Equality is shown in two steps: First in Section 1.3.1 it is shown, that all possible synchronous Engines for a specification are equal, so there is only one valid Evaluation. Afterwards in Section 1.3.2 it is shown that any asynchronous evaluation Engine is equal to a synchronous one.

1.3.1 Equality of synchronous Systems

When given a series of input Events, two synchronous evaluation Engines for a specification with different schedules will have different Runs. But both will produce all outputs that can be produced after consuming one specific input before the next Input is consumed as reasoned in Section 1.2.1. Also both Runs will obviously have the same length (both Engines are the same DAG, so they have the same number of Nodes), let that length be l .

To proof the equality of both Engines we have to proof the equality of their Runs. To show the equality we will show that there is always another Run, which is equal to the second one, that has a smaller Distance from the first one. If such a closer Run always exist, we will show that the Run with Distance $l, 0$ to the first Run, which has to be the first Run itself, is also an equal Run to the second Run.

Theorem 1. Equality of different synchronous evaluation Engines: *Two synchronous evaluation Engines for a specification with different schedules are equal.*

Proof. Let $\vec{e} = (e_1, e_2, \dots, e_x)$ be the input Events both Engines receive. Furthermore let R_1, R_2 be the Runs of the two Engines for a given TeSSLa specification.

Because each TeSSLa specification contains only a finite amount of Functions and works on finite traces, the Runs also have to be finite.

If the two Runs aren't equal, they must have a Distance which is bigger than $(l, 0)$. Let $[R_2]$ be the Set of all Runs that are equal to R_2 . All of those Runs will also have a Distance from R_1 which is bigger than $(l, 0)$. Select one Run $R'_2 \in [R_2]$ which has a minimal Distance from R_1 . Let $(d, k) = \delta(R_1, R'_2)$.

This means that at Step d the Run R'_2 has taken a different Transition than Run R_1 . Let the Transitions the Runs have taken be τ_1 for R_1 and τ_2 for R'_2 . Run R'_2 will take Transition τ_1 at step $d + k$ (as per the definition of the Distance). Obviously the two Transitions have to be independent of each other, else they couldn't have been taken in different order by the two Runs.

If $k > 1$ there will be a Transition $\tau'_2 \neq \tau_1$ which is taken by the Run R'_2 at Step $d + (k - 1)$. While this Transition τ'_2 must also be taken in the first Run as per Lemma 1, it's not possible, that it was taken before τ_1 , because then the two runs wouldn't have been the same upto the point where τ_1 was taken. Therefore τ_1 has to be independent of τ'_2 , and because τ'_2 was scheduled by the second Run before τ_1 both Transitions are independent of each other.

As of Lemma 3 which one of them is taken first can't change the enabledness of the Node of the second Transition. Therefore there is a Run R''_2 which is exactly the same to R'_2 , except that the Transitions τ_1, τ'_2 are scheduled the other way around.

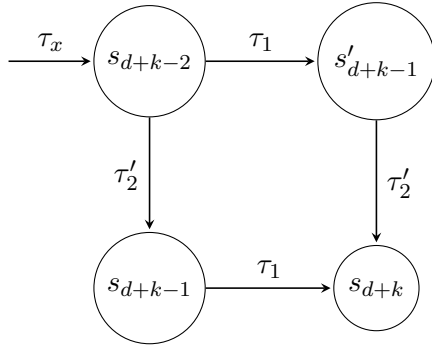


Fig. 1.2: Commutativity Diagram of Node scheduling

Figure 1.2 visualizes how changing the order of the two Transitions can't change the global State of the Engine after both were taken. Therefore the Runs R'_2 and R''_2 are equal and the Distance to R_1 is $d, k - 1$, which contradicts the initial statement that R'_1 was a Run with a minimal Distance. This means that there is an equal Run to R_2 which has at most the Distance $(d, 1)$.

If $k = 1$ the Transition τ'_2 from the previous case is equal to τ_2 . Based on the same reasoning there also exist a Run R''_2 which is exactly the same to R'_2 except that the order of τ_2 and τ_1 is changed, and which is also equal to R'_2 and to R_2 . This Run has the Distance $(d, 0)$ from R_1 . This obviously doesn't make Sense: The first element of the Distance is the last Step where both Runs are equal, the second element describes how many Steps afterwards the differing Transition was taken. But if it was taken right in the Step after the last equal Step, there is no difference at that position, so the Difference of R_1 and R''_2 can be at most $(d + 1, x), x \in \mathbb{N}_{>0}$. This also contradicts our initial Statement that R'_2 was the Run with the smallest Distance from R_1 which is equal to R_2 .

Combined we can now say, that there is no lower bound on the Distance of equal Runs of R_2 to R_1 , therefore the Run with the Distance $(l, 0)$ also has to be equal to R_2 .

□

1.3.2 Equality of synchronous and asynchronous schedules

When the Nodes of A aren't scheduled in reversed topological order, the system can consume Inputs before producing all outputs based on the last consumed Input. Therefore the reordering of runs has to be performed over wider parts of the run.

1.4 Behaviour with Timing functions

1.5 Equalitys with Timing functions

1.6 Parallel computation

List of Figures

1.1	Visualization of a simple asynchronous system with a reversed topological order.	9
1.2	Commutativity Diagramm of Node scheduling	12

List of Tables

